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ON A THEOREM OF ZARISKI - VAN KAMPEN TYPE AND ITS APPLICATIONS

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1. Introduction. Zariski constructed a method to calculate $\pi_1(\boldsymbol{P}^2-C)$, where $\boldsymbol{P}^2$ is the complex projective plane and C is a curve on it. In this paper, following the ideas of Zariski [5] and Van Kampen [4], we give a method to calculate $\pi_1(E-S)$, where E is a holomorphic line bundle over a complex manifold M and S is a hypersurface of E under certain conditions. Applying our method and the Reidemeister-Schreier method (see Rolfsen [3]), we can calculate the fundamental groups of regular loci of certain normal complex spaces. We give a few concrete examples in the final section.

This paper is a revised version of the author's master thesis [1]. The author would like to express his thanks to Professor M. Namba for his useful suggestions and encouragements and to Professor M. Sakuma whose suggestions about Lemma 1 (see section 2) was a great help to prove Main Theorem. He also expresses his thanks to the referee for useful comments.

2. Statement of Main Theorem. Let M be a connected -dimensional complex manifold and $\mu: E\rightarrow M$ be a holomorphic line bundle over M and S be a hypersurface of E. We assume that E and S satisfy the following conditions:

(1) $\mu: S\rightarrow M$ is a finite proper holomorphic map, where μ' is the ristriction of μ to $S(\mu'=\mu|s)$.

(2) There is a hypersurfase B of M such that $\mu'|_{S-\mu^{-1}(B)}: S-\mu^{-1}(B)\rightarrow M-B$ is an unbranched covering of degree .

(3) $(\mathrm{d}\mu')_p: T(S-\mu^{-1}(B))_p\rightarrow T(M-B)_{\mu'(p)}$ is isomorphic for every point $p \in S-\mu^{-1}(B)$.

Then we have a following lemma whose proof is given in section 4.

Lemma 1. $\mu|_{E-S-\mu^{-1}(B)}: E-S-\mu^{-1}(B)$ *is a continuous fiber bundle.*

We denote a standard fiber od $\mu: E\rightarrow M$ by $\hat{F}$ and that of $\mu|_{E-S-\mu^{-1}(B)}: E-S-\mu^{-1}(B)\rightarrow M-B$ by F. We assume that there is a continuous section $\xi: M\rightarrow E$ of $\mu: E\rightarrow M$ such that $\xi(M)\cap S=\phi$ (see Figure 1).

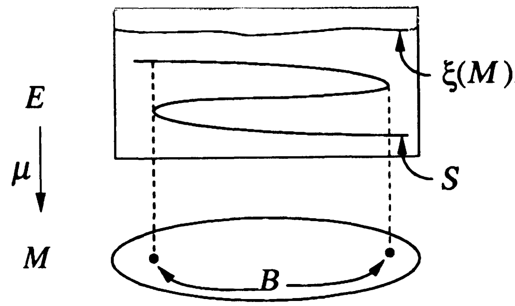

Figure 1

REMARK. Such a continuous sectinuous does not always exist. For example, if E is a negative line bundle and S is the image of the zero section, then there exists no such a continuous section.

In order to describe Main Theorem, we must prepare some more symbols. We choose $F \cap \xi(*)$ as a base point b_0 and we omit the base point hereafter. Since F can be identified with $C-\{n \text{ points}\}$, $\pi_1(F)$ is isomorphic to the n-th free group $F_n=\langle \gamma_1, \cdots, \gamma_n \rangle$ (see Figure 2).

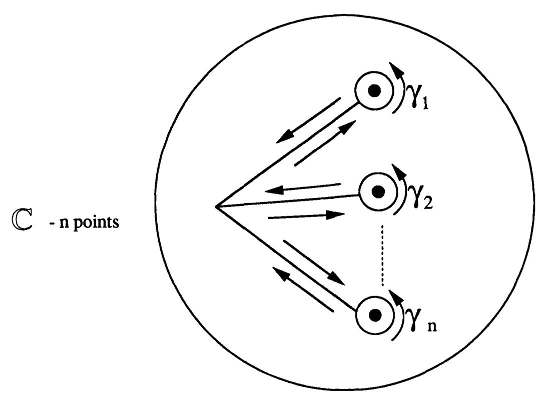

Figure 2

Let $\hat{Q}$ be the kernel of the surjective homomorphism

$$j_*: \pi_1(M-B) \rightarrow \pi_1(M),$$

induced from the injection $i: M-B \hookrightarrow M$. We assume that $\hat{Q}$ has a finite presentation as follows:

$$\hat{Q}=\langle \beta_1, \cdots, \beta_t | \square=1, \cdots, \square=1 \text{ (some relations) } \rangle.$$

Let $\theta: \pi_1(M-B)\to B_n$ be the braid monodromy representation of the continuous fiber bundle $\mu|_{E-S-\mu^{-1}(B)}$ in Lemma 1, where B_n is the n-th braid group:

$$B_n=\langle\sigma_1, \cdots, \sigma_{n-1}|[\sigma_i, \sigma_j]=1(|i-j|\geq 2),\ \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}(i=1, \cdots, n-2)\rangle.$$

We define a homomorphism $\varphi: B_n\to Aut(\pi_1(F))$ as follows:

$$\begin{cases}\varphi(\sigma_j)(\gamma_j) &=\gamma_j^{-1}\gamma_{j+1}\gamma_j\\ \varphi(\sigma_j)(\gamma_{j+1})&=\gamma_j\\ \varphi(\sigma_j)(\gamma_k) &=\gamma_k(\text{if } k\neq j, j+1).\end{cases}$$

Then we have the following theorem of Zariski-Van Kampen type:

Main Theorem. *If there is a continuous section ξ of $\mu: E\to M$ such that $\xi(M)\cap S=\phi$, then*

$$\pi_1(E-S)\cong\langle\gamma_1, \cdots, \gamma_n|\gamma_j=\varphi(\theta(\beta_k))(\gamma_j)(1\leq j\leq n, 1\leq k\leq t)\rangle\rtimes\pi_1(M).$$
(a semi-direct product)

Here $\beta_1, \cdots, \beta_k$ generate the kernel of the homomorphism $j_*: \pi_1(M-B)\to\pi_1(M)$ and $\gamma_1, \cdots, \gamma_n$ generate the image of the homomorphism $i_*: \pi_1(F)\to\pi_1(E-S-\mu^{-1}(B))$, where i_* is induced from injection $i: F\hookrightarrow E-Q-\mu^{-1}(B)$.

REMARK. In Main Theorem, the relations $\gamma_j=\varphi(\theta(\beta_k))(\gamma_j)$ are same as the usual monodromy relations, so it is not essential to facter the homomorphism

$$\pi_1(M-B)\longrightarrow Aut(\pi_1(F))$$

through the braid group.

Corollary. *Under the same assumptions in Main Theorem, assume moreover that M is simply connected $(i.e.\pi_1(M)=\{1\})$, then*

$$\pi_1(E-S)\cong\langle\gamma_1, \cdots, \gamma_n|\gamma_j=\varphi(\theta(\beta_k))(\gamma_j)(1\leq j\leq n, 1\leq k\leq t)\rangle.$$

3. Proof of Main Theorem. Since $\mu: E-S-\mu^{-1}(B)\to M-B$ is a continuous fiber bundle, there is the following exact sequence:

$$\begin{aligned}\cdots\to\pi_2(F)\xrightarrow{i_*}\pi_2(E-S-\mu^{-1}(B))\underset{\xi_*}{\overset{\mu_*}{\rightleftarrows}}\pi_2(M-B)\xrightarrow{\triangle}\\ \to\pi_1(F)\xrightarrow{i_*}\pi_1(E-S-\mu^{-1}(B))\xrightarrow{\mu_*}\pi_1(M-B)\to\\ \to\pi_0(F)\to\cdots\ (\text{exact}),\end{aligned}$$

where μ_* and ξ_* are the homomorphisms induced by μ and ξ respectively.

$\mu_*\circ\xi_*=id|_{\pi_2(M-B)}$, since $\mu\circ\xi=id|_{M-B}$. Therefore we have $\triangle=\triangle\circ\mu_*\circ\xi_*=0$. On the other hand $\pi_0(F)=\{1\}$, since F is connected. Hence we have the following exact sequence:

$$(1) \qquad 1 \to \pi_1(F) \overset{i_*}{\to} \pi_1(E-S-\mu^{-1}(B)) \underset{\xi_*}{\overset{\mu_*}{\rightleftarrows}} \pi_1(M-B) \to 1.$$

We denote $i_*\pi_1(F)$ by K, $\xi_*\pi_1(M-B)$ by H and $\pi_1(E-S-\mu^{-1}(B))$ by G. The short exact sequence means that:

$$G \cong K \rtimes H \text{ (a semi-direct product).}$$

Now let $B=B_1\cup\cdots\cup B_l$ be the irreducible decomposition of B and α_j be the meridian of B_j (see Figure 3).

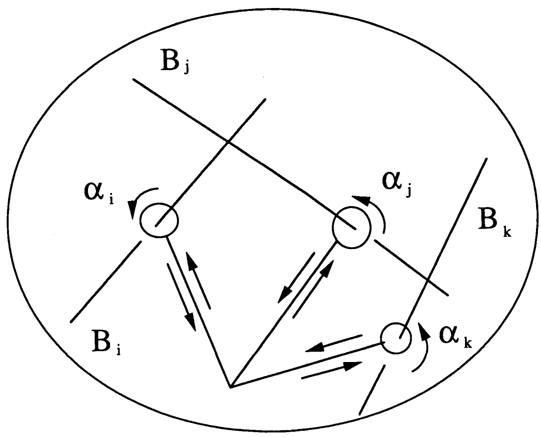

Figure 3

REMARK. Here we assume that B has a finite irreducible decomposition for simplicity. But even if B has an infinite irreducible decomposition the following argument is the same.

From a theorem of Van Kampen [4] (see also Namba [2] Cor.1.2.8), we have the following exact sequence:

$$1 \to \ll \alpha_1, \cdots, \alpha_l \gg^{\pi_1(M-B)} \to \pi_1(M-B) \to \pi_1(M) \to 1$$

where $\hat{Q} = \ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)}$ is the smallest normal subgroup of $\pi_1(M-B)$ which contains $\alpha_1, \cdots, \alpha_l$.

$\mu^{-1}(B)$ is a hypersurface of E, which has the irreducible decomposition

$$\mu^{-1}(B) = \mu^{-1}(B_1) \cup \cdots \cup \mu^{-1}(B_l).$$

$\xi_*(\alpha_j)$ is a meridian of $\mu^{-1}(B_j)$, for μ: $E \to M$ is a line bundle and so $d\mu$: $T_pM \to T_{\mu(p)}M$ is surjective. Then, from the theorem of Van Kampen again, we have the folloing exact sequence:

$$1 \to \ll \xi_*(\alpha_1), \ldots, \xi_*(\alpha_l) \gg^{G} \to G \to \pi_1(E-S) \to 1,$$

where $\ll \xi_*(\alpha_1), \ldots, \xi_*(\alpha_l) \gg^G$ is the smallest normal subgroup of G which contains $\xi_*(\alpha_1), \ldots, \xi_*(\alpha_l)$.

We denote $\ll \xi_*(\alpha_1), \ldots, \xi_*(\alpha_l) \gg^G$ by N, $\xi_*(\ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)})$ by Q and KN by R.

Then we can easily check that

(2) $$N \cap H = Q \text{ and } R \cap NH = N.$$

Consider the natural exact sequence

$$1 \to R/N \to G/N \to G/R \to 1.$$

Note that, by (1) and (2),

$$G/R = KH/R = (KN)(NH)/R = R(NH)/R \cong (NH)/(R \cap (NH)) = (NH)/N.$$

Hence, we have the exact sequence

(3) $$1 \to R/N \to G/N \xrightarrow{f} (NH)/N \to 1.$$

The homomorphism $g: (NH)/N \to G/N$ defined by

$$g: nh(modN) \mapsto h(modN)(n \in N, h \in H)$$

is well-defined and satisfies $f \circ g =$ the identity. Hence the exact sequence (3) splits, so

$$G/N \cong (R/N) \rtimes (NH/N) \text{ (a semi-direct product).}$$

We can easily check that

$$K \cap N = \ll a^{-1}qaq^{-1} | a \in K, q \in Q \gg^K,$$

where $\ll a^{-1}qaq^{-1} | a \in K, q \in Q \gg^K$ is the smallest normal subgroup of K which contains $\{a^{-1}qaq^{-1} | a \in K, q \in Q\}$. Furthermore, note that if K and Q are respectively generated by $\{a_1, \ldots, a_n\}$ and $\{q_1, \ldots, q_t\}$, then

$$K \cap N = \ll a_j^{-1} q_k a_j q_k^{-1} | 1 \leq j \leq n, 1 \leq k \leq t \gg^K.$$

We assume that $\ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)}$ has a finite presentation as follows:

$$\ll \alpha_1, \ldots, \alpha_l \gg^{\pi_1(M-B)} = < \beta_1, \ldots, \beta_t | \square = 1, \ldots, \square = 1 \text{ (some relations)} >.$$

Since $K = i_* \pi_1(F)$ is isomorphic to the n-th free group $< \gamma_1, \ldots, \gamma_n >$, we have:

$$K \cap N \cong \ll \gamma_j^{-1} \xi_*(\beta_k) \gamma_j \xi_*(\beta_k)^{-1} | 1 \leq j \leq n, 1 \leq k \leq t \gg^K.$$

Thus,

$$K/(K \cap N) \cong \langle \gamma_1, \ldots, \gamma_n | \gamma_j^{-1} \xi_*(\beta_k) \gamma_j \xi_*(\beta_k)^{-1} = 1 (1 \leq j \leq n, 1 \leq k \leq t) \rangle.$$

Since G/N is isomorphic to $\pi_1(E-S)$, R/N is isomorphic to $K/(K \cap N)$ and $NH/N \cong H/(N \cap H) = H/Q$ is isomorphic to $\pi_1(M)$, we have:

$$\pi_1(E-S)\cong(K/(K\cap N))\rtimes\pi_1(M) \text{ (a semi-direct product).}$$

Now useing φ and θ (defined in section 2), we have:

$$\xi_*(\beta_k)\gamma_j\xi_*(\beta_k)^{-1}=\varphi(\theta(\beta_k))(\gamma_j) \text{ (see Figure 4).}$$

Hence,

$$K/(K\cap N)\cong\langle\gamma_1, \ldots, \gamma_n|\gamma_j=\varphi(\theta(\beta_k))(\gamma_j)(1\leq j\leq n,\ 1\leq k\leq t)\rangle.$$

This completes the proof of Main Theorem.

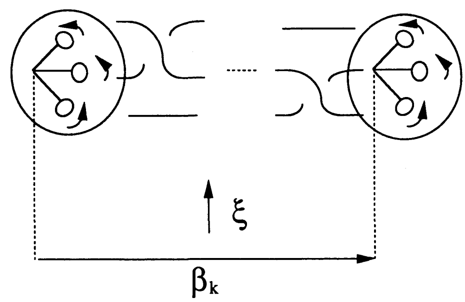

Figure 4

4. Proof of Lemma 1. (Due to M.Sakuma): For a given point $q\in M-B$, we can take a neiborhood U of q such that

(i) $\mu'^{-1}(U) \underset{homeomorphic}{\approx} \amalg_{i=1}^{n} \tilde{U}_i(\mu: \tilde{U}_i\xrightarrow{\sim}U)$ We write $\mu^{-1}(q)\cap\tilde{U}_i=\{\tilde{q}_i\}$.

(ii) The following diagram is commutative.

$$\begin{array}{l}\mu^{-1}(U)\xrightarrow{\sim}U\times \boldsymbol{C}\\ \quad\mu\searrow\circlearrowleft\nearrow P_1 \text{ (where } P_1: (p, z)\mapsto p)\\ \qquad U\end{array}$$

Here we define a map h_i: $\tilde{U}_i\to\boldsymbol{C}$ as follows:

$$h_i: \tilde{U}_i\to\mu^{-1}(U)\cong U\times\boldsymbol{C} \underset{projection}{\to} \boldsymbol{C}.$$

Then we can write $\tilde{U}_i$ as follows:

$$\tilde{U}_i=\{(x, h_i(x)\}\in U\times\boldsymbol{C}|x\in U\}.$$

We write $z_j=h_i(\tilde{q}_i)$, then there exists a positive number $\varepsilon>0$ such that
(1) $\mathrm{Im}h_i\subset\mathrm{Int}(D_\varepsilon(z_i))$, where $D_\varepsilon(z_i)$ is an ε-disk whose center is z_i and *Int* $(D_\varepsilon(z_i))$ is the interior of $D_\varepsilon(z_i)$.
(2) $D_\varepsilon(z_1)$, ..., $D_\varepsilon(z_n)$ are disjoint each other.
From Lemma 2 bellow, there exists a fiber preserving homeomorphism Φ such

that $\Phi(\tilde{U}_i)=U\times\{x\}$.

$$\begin{array}{ccc} \mu^{-1}(U)\cong U\times \boldsymbol{C} & \overset{\Phi}{\rightarrow} & U\times \boldsymbol{C} \\ \mu\searrow\circlearrowleft\downarrow P_1 & \circlearrowleft\nearrow P_1 & \\ & U & \end{array}$$

So we can take local coordinates of μ: $E-S-\mu^{-1}(B)\to M-B$. This shows Lemma 1.

Lemma 2. *Let D be an ε-disk of $\boldsymbol{C}$ whose center is the origin. Let U be the neiborhood of q as above. Let h: $U\to Int(D)$ be a continuous map such that $h(q)=0$, where $Int(D)$ is the interior of D. Put $\tilde{U}=\{(x, h(x))\in U\times Int(D)|x\in U\}\subset U\times Int(D)$. Then there exists a homeomorphism Ψ: $U\times D\tilde{\to}U\times D$ such that*

(i) $\Psi(\tilde{U})=U\times\{0\}$.

(ii) Ψ *is fiber preserving. (i.e. the folloing diagram is commutative.)*

$$\begin{array}{ccc} U\times D & \overset{\Psi}{\rightarrow} & U\times D \\ \searrow & \circlearrowleft & \nearrow \\ & U & \end{array}$$

(iii) $\Psi|_{U\times\partial D}$: $U\times\partial D\to U\times\partial D$ *is the identity map.*

Proof of Lemma 2. First we define a homeomorphism H_x: $D\to D$ for each point $x\in U$ as follows:

(i) $H_x(h(x))=0$.

(ii) $H_x|_{\partial D} \quad =id|_{\partial D}$.

(iii) H_x is extended to D with radial extention (see Figure 5).

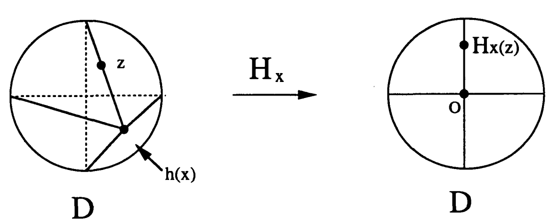

Figure 5

Second we define a homeomorphism Ψ: $U\times D\tilde{\to}U\times D$ as follows:

$$\Psi(x, z)=\Psi(x, Hx(z)).$$

Ψ satisfies the above conditions. (*q.e.d.*)

5. Case of Trivial Line Bundle. In Main Theorem, we assumed the existence of a continuous section ξ such that $\xi(M)\cap S=\phi$. In the case of the trivial line bundle we can prove the following proposition:

Proposition 1. *Let M be a connected complex manifold and μ: $E\rightarrow M$ be a trivial line bundle on M(i, e, $E=M\times \boldsymbol{C}$ and $\mu(p, z)=p$ for every point $(p, z)\in M\times \boldsymbol{C}$). Let $f_{\text{.}}, \ldots, f_n$ be holomorphic functions on M and S be the hypersurface of E defined by*

$$S=\{(p, z)\in E|z^n+f_1(p)z^{n-1}+\cdots+f_n(p)=0\}.$$

Then there is a continuous section ξ of μ: $E\rightarrow M$ such that $\xi(M)\cap S=\phi$.

Proof. We define a continuous function h: $M\rightarrow \boldsymbol{C}$ by

$$h(p)=|f_1(p)|+\cdots+|f_n(p)|+1.$$

We define a section ξ: $M\rightarrow E$ by

$$\xi(p)=(p, h(p)).$$

One can easily see that this section ξ of μ satisfies $\xi(M)\cap S=\phi$. In fact, if there is a point $p\in M$ such that $\xi(p)\in S$, then

$$\{h(p)\}^n+f_1(p)\{h(p)\}^{n-1}+\cdots+f_n(p)=0.$$

Since $h(p)\geq 1$

$$1=\frac{f_1(p)}{h(p)}-\frac{f_2(p)}{\{h(p)\}^2}-\cdots-\frac{f_n(p)}{\{h(p)\}^n}.$$

Hence

$$1\leq\frac{|f_1(p)|}{h(p)}+\frac{|f_2(p)|}{\{h(p)\}^2}+\cdots+\frac{|f_n(p)|}{\{h(p)\}^n}.$$

Since $\{h(p)\}^k\geq h(p)(k=1, 2, \ldots)$,

$$\begin{aligned}1&\leq\frac{|f_1(p)|}{h(p)}+\frac{|f_2(p)|}{h(p)}+\cdots+\frac{|f_n(p)|}{h(p)}\\&=\frac{|f_1(p)|+\cdots+|f_n(p)|}{|f_1(p)|+\cdots+|f_n(p)|+1}<1.\end{aligned}$$

A contradiction. (*q.e.d.*)

Let μ: $\boldsymbol{C}^{m+1}\rightarrow \boldsymbol{C}^m$ be the trivial line bundle on $\boldsymbol{C}^m$ defined by

$$\mu: (z_1, \ldots, z_m, z_{m+1})\rightarrow(z_1, \ldots, z_m).$$

Let S be the hypersurface of $\boldsymbol{C}^{m+1}$ defined by

$$S=\{(z_1, \dots, z_m, z_{m+1})\in \boldsymbol{C}^{m+1}|z_{m+1}{}^n+f_1(z)z_{m+1}{}^{n-1}+\cdots+f_n(z)=0\}\cdots(1),$$

where $z=(z_1, \dots, z_m)$ and $f_1(z), \dots, f_n(z)$ are polynomials.

By Corollary to Main Theorem and Proposition 1, we have

Theorem 1. *Let S be the hypersurface of $\boldsymbol{C}^{m+1}$ defined by* (1). *Then,*

$$\pi_1(\boldsymbol{C}^{m+1}-S)\cong\langle\gamma_1, \dots, \gamma_n|\gamma_j=\varphi(\theta(\beta_k))(\gamma_i), (1\leq j\leq n, 1\leq k\leq t)\rangle.$$

Furthermore, let $(X_0: X_1: \cdots : X_{m+1})$ be homogeneous coordinatss of $\boldsymbol{P}^{m+1}$ such that $(X_1/X_0, \cdots, X_{m+1}/X_0)=(z_1, \dots, z_{m+1})\in \boldsymbol{C}^{m+1}$ and $\bar{S}$ be the closure of S in $\boldsymbol{P}^{m+1}$. Then we have the following theorem of Zariski:

Theorem 2(Zariski [5]).
Suppose that $p_\infty=(0: \cdots : 0: 1)$ is not contained in $\bar{S}$. Then

$$\pi_1(\boldsymbol{P}^{m+1}-\bar{S})$$
$$\cong\langle\gamma_1, \dots, \gamma_n|\gamma_n\gamma_{n-1}\dots\gamma_1=1, \gamma_j=\varphi(\theta(\beta_k))(\gamma_j), (1\leq j\leq n, 1\leq k\leq t)\rangle.$$

Proof. Let H_∞ be the hypersurface of$\boldsymbol{P}^{m+1}$ defined by $H_\infty=\{X_0=0\}$, (*i.e.* hyperplane at infinity) and α be a meridian of H_∞ in $\boldsymbol{P}^{m+1}-\bar{S}-H_\infty$ (see Figure 6).

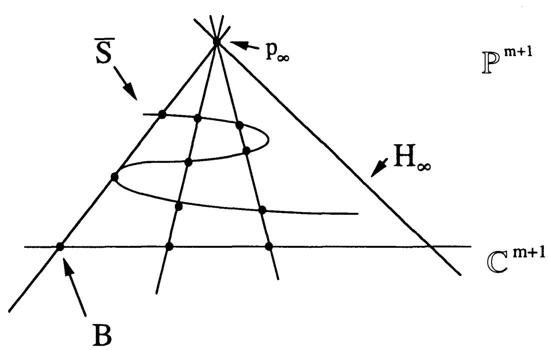

Figure 6

From the theorem of Van Kampen [4], we have the following exact sequence:

$$1\to\langle\!\langle\alpha\rangle\!\rangle^{\pi_1(C^{m+1}-S)}\to\pi_1(\boldsymbol{C}^{m+1}-S)\to\pi_1(\boldsymbol{P}^{m+1}-\bar{S})\to 1 \text{ (exact)}.$$

We can take α as $(\gamma_n\gamma_{n-1}\cdots\gamma_1)^{-1}$ in $\boldsymbol{C}^{m+1}-S$ (see Figure 7).
Thus,

$$\pi_1(\boldsymbol{P}^{m+1}-\bar{S})\cong\pi_1(\boldsymbol{C}^{m+1}-S)/\langle\!\langle\gamma_n\gamma_{n-1}\cdots\gamma_1\rangle\!\rangle^{\pi_1(C^{m+1}-S)}.$$

This shows Theorem 2.

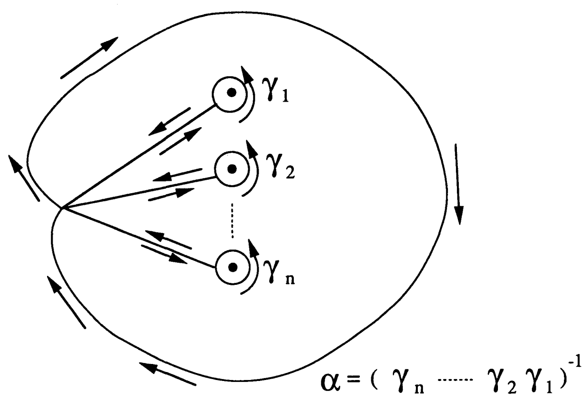

Figure 7

REMARK. A similar theorem to Theorem 1 holds for $\mu: \boldsymbol{B}^m(\varepsilon)\times \boldsymbol{B}^1(\varepsilon')\to\boldsymbol{B}^m(\varepsilon)$, where $\boldsymbol{B}^m(\varepsilon)$ is a m-dimensional complex ball; $\boldsymbol{B}^m(\varepsilon)=\{(z_1, \ldots, z_m)\in \boldsymbol{C}^m \| z_1|^2+\cdots+|z_m|^2<\varepsilon^2\}$. In this case, the existence of continuous section with a similar conditions to Theorem 1 is obvious.

6. Calculations of Fundamental Groups of Finite Branched Coverings

EXAMPLE 1.

Let X be the surface in $\boldsymbol{C}^3$ defined by

$$X=\{\lambda, x, y)\in \boldsymbol{C}^3|y^2=x(x-1)(x-\lambda)\}.$$

X has two isolated singular points at (0,0,0) and (1,1,0). Hence X is normal. Let $\pi: X\to\boldsymbol{C}^2$ be the projection map defined by

$$\pi(\lambda, x, y)=(\lambda, x).$$

Then π is a double branched covering of $\boldsymbol{C}^2$. The branch loucus S of π is a curve in $\boldsymbol{C}^2$ and is written as:

$$S=\{(\lambda, x)\in \boldsymbol{C}^2|x(x-1)(x-\lambda)=0\}.$$

According to Theorem 1, we can calculate $\pi_1(\boldsymbol{C}^2-S)$. Let $\mu: \boldsymbol{C}^2\to\boldsymbol{C}$ be the trivial line bundle on $\boldsymbol{C}$ defined by

$$\mu(\lambda, x)=\lambda.$$

The branch locus B of μ is $\{0, 1\}\subset\boldsymbol{C}$ and $\pi_1(\boldsymbol{C}-B)$ is isomorphic to the free group $\langle\beta_1, \beta_2\rangle$, where β_1 and β_2 are its generators and can be considered as the

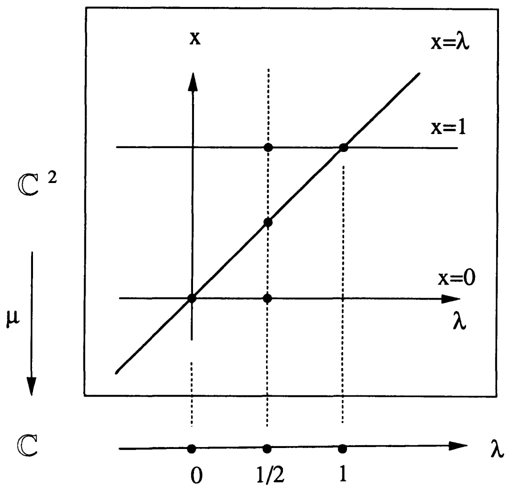

Figure 8

meridians of $\{0\}$ and $\{1\}$, respectively. We may take $q_0=\frac{1}{2}$ as a reference point of $\pi_1(C-B)$. In this case the standerd fiber F of $\mu|_{C-S-\mu^{-1}(B)}$ is $C-\{3\text{-points}\}$. We define γ_1, γ_2 and γ_3 as the meridians of $\left(\frac{1}{2}, 1\right)$,$\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$, respectively. The image of β_1 and β_2 by $\theta: \pi_1(C-B)\to B_3$ are σ_1^2 and σ_2^2, respectively. Then we have

$$\begin{aligned}\pi_1(C^2-S)&\cong\langle\gamma_1, \gamma_2, \gamma_3|\gamma_j=\varphi(\theta(\beta_k))(\gamma_j), j=1, 2, 3, k=1, 2\rangle\\ &\cong\langle\gamma_1, \gamma_2, \gamma_3|\gamma_2\gamma_3=\gamma_3\gamma_2, \gamma_1\gamma_2=\gamma_2\gamma_1\rangle.\end{aligned}$$

By using the Reidemeister- Schreier method (*c.f.* Rolfsen [3] P.315-P.316), we can calculate $\pi_1(\mathrm{Reg}X)$, where $\mathrm{Reg}X$ is the set of regular points of X. Since $\pi_1(C^2-B)$ is generated by three elements and since π is a double branched covering, we take the 3-th free group F_3 and the 5-th free group F_5. As in Figure 9, we take their generators $\{\gamma_1, \gamma_2, \gamma_3\}$ and $\{b_1, b_2, b_3, b_4, b_5\}$, respectively, where$\pi^{-1}(\gamma_1)=\{x_1, x_2\}$, $\pi^{-1}(\gamma_2)=\{y_1, y_2\}$, $\pi^{-1}(\gamma_3)=\{z_1, z_2\}$, and

$$\begin{aligned}b_1&=y_1x_1^{-1}\\ b_2&=x_1y_2x_2^{-1}x_1^{-1}\\ b_3&=z_1x_1^{-1}\\ b_4&=x_1z_2x_2^{-1}x_1^{-1}\\ b_5&=x_1x_2.\end{aligned}$$

Then we transfer the relation of $\pi_1(C^2-S)\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1}=1$ in the words of F_5:

$$y_1z_2y_2^{-1}z_1^{-1}=b_1b_4b_5b_5^{-1}b_2^{-1}b_3^{-1}=1$$

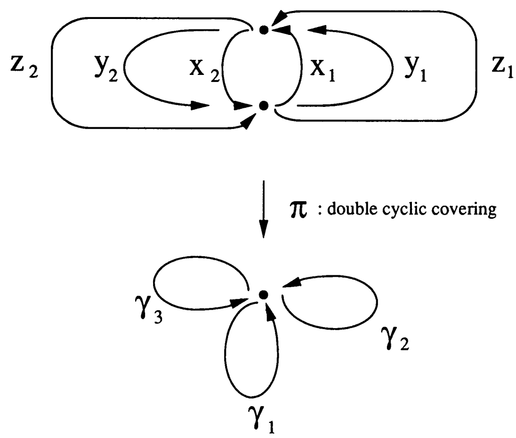

Figure 9

$$x_1 y_2 z_1 y_1^{-1} z_2^{-1} x_1^{-1} = b_2 b_5 b_3 b_1^{-1} b_5^{-1} b_4^{-1} = 1.$$

In a similar way, we transfer the relation $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}=1$ in the words of F_5:

$$x_1 y_2 x_2^{-1} y_1^{-1} = b_2 b_5 b_5^{-1} b_1^{-1} = 1$$
$$x_1 x_2 y_1 x_1^{-1} y_2^{-1} x_1^{-1} = b_5 b_1 b_5^{-1} b_2^{-1} = 1.$$

We also transfer the relations $\gamma_1^2=1$, $\gamma_2^2=1$, $\gamma_3^2=1$ since the ramification index of each irreducible component of S is equal to 2:

$$x_1 x_2 = b_5 = 1$$
$$y_1 y_2 = b_1 b_2 b_5 = 1$$
$$z_1 z_2 = b_3 b_4 b_5 = 1.$$

Putting $\alpha_1 = b_1$ and $\alpha_2 = b_4$, we have

$$\pi_1(\boldsymbol{Reg}\ X) \cong \langle \alpha_1, \alpha_2 | \alpha_1^2 = 1, (\alpha_1\alpha_2)^2 = 1 \rangle$$
$$\cong (\boldsymbol{Z}/2\boldsymbol{Z}) * (\boldsymbol{Z}/2\boldsymbol{Z}) \text{(free product)}.$$

EXAMPLE 2.

Let X be the hypersurfaces of $\boldsymbol{C}^4$ defined by

$$X = \{(x, y, z, w) \in \boldsymbol{C}^4 | w^n = z^2 - xy^2\} (n \geq 2).$$

The singular locus of X is the line $\{(x, y, z, w) \in X | y = z = w = 0\}$. Hence X is normal. Let $\pi : X \to \boldsymbol{C}^3$ be the projection map defined by:

$$\pi(x, y, z, w) = (x, y, z).$$

Then π is a cyclic branched covering of $\boldsymbol{C}^3$. The branch locus S of π is a surface in $\boldsymbol{C}^3$ and is written as:

$$S=\{(x, y, z)\in \boldsymbol{C}^3 | z^2-xy^2=0\} \text{ (the Cartan umbrella).}$$

According to Theorem 1, we can calculate $\pi_1(\boldsymbol{C}^3-S)$. The result is

$$\pi_1(\boldsymbol{C}^3-S)\cong\langle\gamma\rangle\text{(the free group).}$$

From the Reidemeister-Schreier method again, we have:

$\pi_1(RegX)\cong\{$*i.e.* $RegX$ is simply connected).

EXAMPLE 3.

Let X be the hypersurface of $\boldsymbol{C}^{m+2}$ defined by

$$X=\{(z_1, \ldots, z_{m+2})\in \boldsymbol{C}^{m+2} | z_{m+2}^2+z_{m+1}^2+g(z_1, \ldots, z_m)=0\},$$

where g is a polynomial which is not constant. The singular locus of X is at most $(m-1)$-dimensional. Hence X is normal. Let $\pi: X\rightarrow \boldsymbol{C}^{m+1}$ be the projection map defined by:

$$\pi(z_1, \ldots, z_{m+1}, z_{m+2})=(z_1, \ldots, z_{m+1}).$$

Then π is a branched covering of $\boldsymbol{C}^{m+1}$. The branch locus S of π is a hypersurface in $\boldsymbol{C}^{m+1}$ and is written as:

$$S=\{z_{m+1}^2+g(z_1, \ldots, z_m)=0\}.$$

By Theorem 1, $\pi_1(\boldsymbol{C}^{m+1}-S)$ can be written as:

$$\pi_1(\boldsymbol{C}^{m+1}-S)\cong\langle\gamma_1, \gamma_2 | \square=1, \ldots, \square=1\rangle.$$

From the Reidemeister-Schreier method again, we have:

$$\pi_1(RegX)\cong\begin{cases}\{1\} \text{ or} \\ \boldsymbol{Z}/q\boldsymbol{Z}(\exists\, q\in\boldsymbol{Z}) \text{ or} \\ \boldsymbol{Z}\end{cases}$$

(*i.e.* $\pi_1(RegX)$ is isomorphic to a cyclic group).

References

[1] T. Matsuno: *Normal singularities and fundamental groups*, Osaka Univ. master theses (in Japanese).

[2] M. Namba: *Branched coverings and algebraic functions*, Pitman Res. Notes Math. Ser., Vol.**161**, Longman Sci. Tech., Harlow,1987.

[3] D. Rolfsen: *Knots and links*, Publish or Perish, Berkeky, 1976.

[4] van Kampen: *On the fundamental group of an algebraic curves*, Amer.J.Math. **55** (1933), 255-260.

[5] O. Zariski: *Collected Papers* Vol **3**, MIT Press, Cambridge, 1973.

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