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INVERSE LIMITS OF POLYNOMIAL RINGS

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Introduction

In this note we deal with the question as to whether the inverse limit of one-variable polynomial rings is polynomial again. More precisely, working in the realm of the pro-affine algebra theory [1, 2], we look at the pro-affine algebra $\tilde{A} := \lim_{\leftarrow,i} A_i[T_i]$ given over $A := \lim_{\leftarrow,i} A_i$ and ask if that algebra $\tilde{A}$ is isomorphic to $A[\mathbb{Y}]$ for a suitable choice of variable $\mathbb{Y}$. Our first answer is that $\tilde{A}$ is always locally polynomial over A (*see* Th. 1), and our second answer is that $\tilde{A} \simeq A[\mathbb{Y}]$ if the inverse system of the units of A_i's satisfies a 'uniformized' Mittag-Leffler condition (*see* Th. 2).

After developing and proving these two theorems, we conclude this note with presentation of some examples in §3, originally made up by David Wright during the discussion sessions by him, N. Mohan Kumar and the present author in St. Louis, August 2002. Back in Japan in the fall of the same year the author was able to prove Theorem 2 through further study of these examples.

The author wishes to record here his heartfelt thanks to the two friends just mentioned, as well as to M. Miyanishi who initially suggested the main question to us informally in July 2002 and to R.V. Gurjar together with whom the author made a first analysis of the question.

1. The Problem

Referring the reader to [1, 2] for the basics on pro-affine algebras, we consider a strongly-reduced pro-affine algebra A over a field K. This means, in effect, that A is a commutative topological K-algebra admitting a representation as $A = \lim_{\leftarrow,i\in\mathbb{N}} A_i$, where all A_i's are reduced and discrete algebras over K forming a surjective inverse system $A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_i \leftarrow \cdots$ indexed by $\mathbb{N} = \{0, 1, 2, \ldots\}$. For each $i \in \mathbb{N}$, consider a one-variable polynomial ring $A_i[T_i]$, and suppose given maps $\phi_i \colon A_i[T_i] \longrightarrow A_{i-1}[T_{i-1}]$ such that $(A_i[T_i], \phi_i)_{i\in\mathbb{N}}$ forms a surjective inverse system compatible with $(A_i)_{i\in\mathbb{N}}$. Namely, we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots \longleftarrow & A_{i-1}[T_{i-1}] & \longleftarrow & A_i[T_i] & \longleftarrow & A_{i+1}[T_{i+1}] & \longleftarrow & \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \\ \cdots \longleftarrow & A_{i-1} & \longleftarrow & A_i & \longleftarrow & A_{i+1} & \longleftarrow & \cdots \end{array} \tag{1}$$

in which all horizontal arrows are surjections and the vertical ones are the canonical inclusions.

We now ask our basic question due to M. Miyanishi:

(Q) Given (1), is $\lim_{\leftarrow}(A_i[T_i]) \simeq (\lim_{\leftarrow} A_i)[\mathbb{Y}] \simeq A[\mathbb{Y}]$? (Is the inverse limit of polynomial algebras again a polynomial algebra?)

As data for (Q), we have the maps $\phi_i\colon A_i[T_i] \to A_{i-1}[T_{i-1}]$ ($\forall i \in \mathbb{N}$) which satisfy, for all $i > 0$, $A_{i-1}[\phi_i(T_i)] = A_{i-1}[T_{i-1}]$[1], *i.e.*, $\phi_i(T_i) = e_{i-1}T_{i-1} + b_{i-1}$ with $e_{i-1} \in \mathfrak{U}(A_{i-1}), b_{i-1} \in A_{i-1}$.[2] Note that the selection of the variable $\phi_i(T_i)$ is completely arbitrary and free of choices for other $\phi_j(T_j)$'s. At this point we carry out our first reduction as follows:

STEP 0. Let $T_0' := T_0$.

STEP i. If $T_0', \dots, T_{i-1}'$ have been set, and if $\phi_i(T_i) = e_{i-1}'T_{i-1}' + b_{i-1}'$ with $e_{i-1}', b_{i-1}' \in A_{i-1}$ and e_{i-1}' a unit, then choose any $b_i' \in A_i$ such that $\phi_i(b_i') = b_{i-1}'$ and let $T_i' := T_i - b_i'$.

One can see then that $\phi_i(T_i') = \phi_i(T_i - b_i') = e_{i-1}'T_{i-1}' + b_{i-1}' - \phi_i(b_i') = e_{i-1}'T_{i-1}' + b_{i-1}' - b_{i-1}' = e_{i-1}'T_{i-1}'$. So, we do Step 0, then Step 1, Step 2 and so on, replacing T_i with T_i' at each step as we climb up. We then obtain a new series of variables $T_0' = T_0, T_1', \dots, T_i', T_{i+1}', \dots$ such that $A_i[T_i'] = A_i[T_i]$ and $\phi_i(T_i') = e_{i-1}'T_{i-1}'$ for all $i > 0$. This implies that, in the surjective inverse system $(A_i[T_i], \phi_i)_{i\in\mathbb{N}}$ for the question (Q) above, one can assume from the outset that $\phi_i(T_i) = e_{i-1}T_{i-1}$. We will assume this hereafter.

It is now clear, then, that an affirmative answer to (Q) boils down to having a series of units $\{v_i \in \mathfrak{U}(A_i)\colon i \in \mathbb{N}\}$ such that, if we set $Y_i := v_iT_i$ then, $\forall i, \phi_i(Y_i) = Y_{i-1}$. When such v_i's have been gotten, we have $\phi_i(Y_i) = \phi_i(v_iT_i) = \phi_i(v_i)e_{i-1}T_{i-1} = Y_{i-1} = v_{i-1}T_{i-1}$, so that

$$\phi_i(v_i) = e_{i-1}^{-1}v_{i-1} \quad \text{for all } i > 0. \tag{2}$$

Conversely, if a series of units v_i's have been found satisfying (2), then (Q) is answered "yes" as $\lim_{\leftarrow} A_i[T_i] \simeq \lim_{\leftarrow} A_i[Y_i] \simeq (\lim_{\leftarrow} A_i)[\mathbb{Y}]$ with $\mathbb{Y} := (Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_{i-1} \leftarrow Y_i \leftarrow \cdots)$, where $Y_i := v_iT_i$ for all $i \in \mathbb{N}$.

[1]We then say $\phi_i(T_i)$ is a *variable for* $A_{i-1}[T_{i-1}]$.

[2]Here as elsewhere, for any ring R, we denote its group of units by $\mathfrak{U}(R)$.

Theorem 1. *Let* $A = \lim_{\leftarrow} A_i$ *be a strongly-reduced pro-affine algebra over a field* K. *Suppose given a surjective inverse system*

$$A_0[T_0] \leftarrow A_1[T_1] \leftarrow A_2[T_2] \leftarrow \cdots \tag{3}$$

of polynomial algebras $A_i[T_i]$*'s over* $A_0 \leftarrow A_1 \leftarrow \cdots$, *and also given any open prime* $P \subset A$, *where* $P = (P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_i \leftarrow \cdots)$ *with all* P_i *prime in* A_i. *Let* $S_i := A_i - P_i$. *Then, we get a surjective inverse system of polynomial algebras over local rings*:

$$(S_0^{-1}A_0)[T_0] \leftarrow (S_1^{-1})[T_1] \leftarrow \cdots \leftarrow (S_i^{-1}A_i)[T_i] \leftarrow \cdots \tag{4}$$

and its limit, $\lim_{\leftarrow}(S_i^{-1}A_i)[T_i]$, *is isomorphic to* $(\lim_{\leftarrow}(S_i^{-1}A_i))[\mathbb{Y}] = A_P[\mathbb{Y}]$ *for a suitably chosen variable* $\mathbb{Y}$.

Proof. First observe that, in general, the question (Q) is answerable as "yes" in case the map $A_i \to A_{i-1}$ induces a surjection $\mathfrak{U}(A_i) \to \mathfrak{U}(A_{i-1})$ for all $i > 0$. Indeed, suppose $A_i[T_i] \to A_{i-1}[T_{i-1}]$ is given by the assignment $T_i \mapsto e_{i-1}T_{i-1}$ with $e_{i-1} \in \mathfrak{U}(A_{i-1})$ for each i. Then, starting at any level, say $i-1$, and with any choice of $v_{i-1} \in \mathfrak{U}(A_{i-1})$, say $v_{i-1} = 1 \in A_{i-1}$, one can solve the equation (2) successively for $v_i \in \mathfrak{U}(A_i)$, then for v_{i+1} and so on, *ad infinitum*. Then, as we saw just above, putting $Y_i := v_i T_i$ for all $i \in \mathbb{N}$ and $\mathbb{Y} := (Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_{i-1} \leftarrow Y_i \leftarrow \cdots)$ gives an affirmative solution for (Q) in the present instance.

To complete the proof we have only to remember that, for all $i > 0$, the local homomorphism $S_i^{-1}A_i \to S_{i-1}^{-1}A_{i-1}$ of local rings clearly maps the units of the first ring *onto* those of the second. $\square$

REMARK. The result we saw just now says, geometrically, that the morphism of ind-affine schemes $\mathfrak{Sp}(\lim_{\leftarrow,i}(A_i[T_i])) \to \mathfrak{Sp}(\lim_{\leftarrow,i} A_i) = \mathfrak{Sp}(A)$ is locally a product $\mathfrak{Sp}(A_P) \times \mathbb{A}^1$ above each point $P \in \mathfrak{Sp}(A)$.

2. Stability and uniform stability

In analyzing the situation as outlined in §1 and in studying various examples of which some are to be found in §3 below, we see that (Q) is a question of units of the A_i's and their inverse images.

With the case $U_i = \mathfrak{U}(A_i)$ in mind, we consider more generally the following inverse system of groups:

$$U_0 \overset{\mu_1}{\leftarrow} U_1 \leftarrow \cdots \leftarrow U_{i-1} \overset{\mu_i}{\leftarrow} U_i \leftarrow \cdots \tag{5}$$

where the μ_i's are not necessarily surjective. For each pair of integers $i \geq 0$, $q \geq 0$

we define two types of subgroups of U_i:

(6) $$U_{i,q} := \mu_{i+1} \circ \cdots \circ \mu_{i+q}(U_{i+q}), \quad U_{i,\infty} := \bigcap_{q=1}^{\infty} U_{i,q},$$

where $U_{i,0} := U_i$ is to be understood. The inverse system (5) is said to be *stable*[3] if, for each $i \geq 0$, there exists a $q = q(i) \geq 0$ such that $U_{i,q} = U_{i,\infty}$. In this case, any integer $r \geq q$ is said to be *in the stable range* at level i, and then, clearly, $U_{i,r} = U_{i,\infty}$ holds.

In the stable case, there results a *surjective* inverse system of groups:

(7) $$U_{0,\infty} \xleftarrow{\mu_1'} U_{1,\infty} \longleftarrow \cdots \longleftarrow U_{i-1,\infty} \xleftarrow{\mu_i'} U_{i,\infty} \longleftarrow \cdots .$$

We need to consider next a stronger notion of "uniform stability". Namely, the inverse system (5) will be called *uniformly stable* if there exists an integer $q \geq 0$ such that $U_{i,q} = U_{i,\infty}$ for all $i \in \mathbb{N}$. Note that uniform stability with $q = 0$ is the same thing as the surjectivity of the inverse system.

From here on, it will be assumed always that, for all $i \in \mathbb{N}$, $U_i = \mathcal{U}(A_i)$ and $\mu_i = \phi_i|_{U_i}$. Let us say in the present note that the K-algebra inverse system $(A_i)_{i\in\mathbb{N}}$ is *stable or uniformly stable for units* if the inverse system (5) of units $U_i = \mathcal{U}(A_i)$ arising from it is stable or uniformly stable, respectively.

Theorem 2. *Let $A = \lim_{\leftarrow,i} A_i$ be a strongly-reduced pro-affine algebra over a field K. Suppose given a surjective inverse system over A*

(8) $$A_0[T_0] \xleftarrow{\phi_1} A_1[T_1] \longleftarrow \cdots \longleftarrow A_{i-1}[T_{i-1}] \xleftarrow{\phi_i} A_i[T_i] \longleftarrow \cdots$$

of polynomial algebras. Assume that the inverse system $(A_i)_{i\in\mathbb{N}}$ is uniformly stable for units. Then,

$$\lim_{\leftarrow,i}(A_i[T_i]) \simeq (\lim_{\leftarrow,i} A_i)[\mathbb{Y}] = A[\mathbb{Y}]$$

for a suitable choice of $\mathbb{Y} = (Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_i \leftarrow \cdots)$ where the Y_i for each $i \in \mathbb{N}$ is a variable for $A_i[T_i]$.

Proof. Letting $U_i := \mathcal{U}(A_i)$ and $\mu_i := \phi_i|_{U_i}$ for all i, we consider the inverse system (5) of unit groups. Let us begin by introducing an *ad hoc* notation as follows: For any $z \in U_j$ and $r \in \mathbb{N}$, define

(9) $$z^{(r)} := \mu_{j-r+1} \circ \cdots \circ \mu_{j-1} \circ \mu_j(z) \text{ for } r > 0; \; z^{(0)} := z,$$

[3] This is the well-known Mittag-Leffler Condition.

so that $z^{(r)} \in U_{j-r}$ in all cases. Let $q \geq 0$ be the level at which the inverse system $(U_i)_{i\in\mathbb{N}}$ achieves the uniform stability. The case $q = 0$ being trivial, we may and shall assume $q > 0$. So, $U_{i,q} = U_{i,\infty}$ for all $i \in \mathbb{N}$, and $z^{(q)} \in U_{j-q,\infty}$ if $z \in U_j$. Now write, for each $i > 0$, $\phi_i(T_i) = e_{i-1}T_{i-1}$ with $e_{i-1} \in U_{i-1}$ as given, and define $u_i \in U_{i,\infty}$ and $v_i' \in U_i$ by the formulae

$$u_i := e_{i+q}^{(q)}, v_i' := e_{i+q-1}^{(q-1)} e_{i+q-2}^{(q-2)} \cdot \cdots \cdot e_{i+1}^{(1)} e_i^{(0)}. \tag{10}$$

Finally, for each $i \in \mathbb{N}$, let us define $Y_i' := v_i' T_i$, a variable in $A_i[T_i]$. Then,

$$\begin{aligned}\phi_i(Y_i') &= \phi_i(v_i' T_i) = e_{i+q-1}^{(q)} e_{i+q-2}^{(q-1)} \cdot \cdots \cdot e_{i+1}^{(2)} e_i^{(1)} \cdot e_{i-1}^{(0)} T_{i-1} \\ &= e_{(i-1)+q}^{(q)} (e_{(i-1)+(q-1)}^{(q-1)} \cdots e_{(i-1)+2}^{(2)} e_{(i-1)+1}^{(1)} e_{i-1}^{(0)}) T_{i-1} \\ &= u_{i-1}(v_{i-1}' T_{i-1}) = u_{i-1} Y_{i-1}'.\end{aligned}$$

This calculation shows that, with respect to the new series of variables $Y_i' = v_i' T_i$'s, the critical coefficients u_i's occurring as $Y_i' \mapsto u_{i-1}Y_{i-1}'$ all belong to $U_{i,\infty}$'s. With reference to the equation system (2), one sees at once that (2) is solvable for $v_i \in U_{i,\infty}$ when $e_{i-1} = u_{i-1} \in U_{i-1,\infty}$ at each level $i > 0$. It follows that a final choice may be made for a variable Y_i in $A_i[Y_i'] = A_i[T_i]$ for each i such that $\phi_i(Y_i) = Y_{i-1}$, so that $\lim_{\leftarrow} A_i[Y_i] \simeq A[\mathbb{Y}]$ with $\mathbb{Y} =: (Y_0 \leftarrow \cdots \leftarrow Y_i \leftarrow \cdots)$. □

3. Examples

We conclude the present note with three examples. They were all originally made up by David Wright as explained in the *Introduction*. However, the author has considerably changed the arguments that follow these examples; in particular, the discussion below for Example 2 is entirely new, even in its direction.

EXAMPLE 1 (D. Wright). Here is an example of how Theorem 2 above works. Let

$$A_i := K[x_i, x_i^{-1}, t_1, t_1', t_2, t_2', \ldots, t_i, t_i'] \text{ for all } i \in \mathbb{N}, \tag{11}$$

and let $\phi_i' : A_i \to A_{i-1}$ be given by the rule of assignments:

$$x_i \mapsto 1, x_i^{-1} \mapsto 1, t_1 \mapsto x_{i-1}, t_1' \mapsto x_{i-1}^{-1}, t_j \mapsto t_{j-1}, t_j' \mapsto t_{j-1}' \ (2 \leq \forall j \leq i). \tag{12}$$

It is then easily seen that $\mathcal{U}(A_i) = K^\times \times \langle x_i \rangle$, with $\langle x_i \rangle$ meaning the multiplicative group $\simeq \mathbb{Z}$ generated by x_i, and that $\phi_i'(\mathcal{U}(A_i)) = K^\times$ for all $i > 0$. So, the surjective system $(A_i, \phi_i)_{i\in\mathbb{N}}$ is uniformly stable for units with $U_{i,1} = U_{i,\infty}$ for all $i \in \mathbb{N}$ and $q = 1$. Now extend ϕ_i' to $\phi_i : A_i[T_i] \to A_{i-1}[T_{i-1}]$ by choosing any unit $e_i \in \mathcal{U}(A_i)$ and defining $\phi_i(T_i) := e_{i-1}T_{i-1}$ for all $i > 0$. The resulting pro-affine K-algebra $\lim_{\leftarrow}(A_i[T_i])$ may then be 'untwisted' following the recipe given in the proof of Theorem 2, as follows: Since $q = 1$, the formula (10) gives $u_i = \mu_{i+1}(e_{i+1}) \in U_{i,\infty}$

and $v_i' = e_i$, so we define $Y_i' := e_i T_i$. Then, it follows that $\phi_i(Y_i') = \phi_i(e_i T_i) = u_{i-1}e_{i-1}T_{i-1} = u_{i-1}Y_{i-1}'$. Since all u_i's belong to $U_{i,\infty}$, the equations (2) with the u_i's replacing the e_i's are successively solvable as i goes up, starting with $v_0 := 1$ and each $v_{i-1} \in \mathcal{U}(A_{i-1})_{i-1,\infty}$ producing a $v_i \in U_{i,\infty}$. We now let $Y_i := v_i Y_i' = v_i e_i T_i$ and we then get $\lim_{\leftarrow}(A_i[T_i]) = \lim_{\leftarrow}(A_i[Y_i]) \simeq A[\mathbb{Y}]$ with $\mathbb{Y} := (Y_0 \leftarrow \cdots \leftarrow Y_i \leftarrow \cdots)$.

EXAMPLE 2 (D. Wright). We now give an example of a pro-affine algebra A not uniformly stable for units (and not even just stable for units), over which $\lim_{\leftarrow} A_i[T_i]$ may or may not be an $A[\mathbb{Y}]$ according as how the units e_i's are chosen to define the map $T_i \mapsto e_{i-1}T_{i-1}$ and consequently the A-algebra structure. Let K be a field, let $A_0 := K$ and let $A_i := K[t_1, t_1', \ldots, t_i, t_i', x_i, x_i^{-1}]$ for all $i > 0$, with t_j's, t_j''s and x_j's indeterminates for all $j > 0$. Construct a pro-affine K-algebra $A = \lim_{\leftarrow,i} A_i$ through defining

$$\phi_i'\colon A_i = K[t_1, t_1', \ldots, t_i, t_i', x_i, x_i^{-1}] \longrightarrow A_{i-1} = K[t_1, t_1', \ldots, t_{i-1}, t_{i-1}', x_{i-1}, x_{i-1}^{-1}]$$

for all $i > 0$ by means of the assignments:

$$t_j \mapsto t_j, t_j' \mapsto t_j' (1 \le \forall j \le i-1), t_i \mapsto x_{i-1}, t_i' \mapsto x_{i-1}^{-1}, x_i \mapsto x_{i-1}^2, x_i^{-1} \mapsto x_{i-1}^{-2}, \tag{13}$$

where we interpret $x_0 = x_0' = 1$. Since the unit group $\mathcal{U}(A_i) \simeq K^\times \times \langle x_i \rangle$, a typical unit looks like $rx_i^m \in \mathcal{U}(A_i)$, where $r \in K^\times$ and $m \in \mathbb{Z}$. It gets mapped by ϕ_i' to $rx_{i-1}^{2m} \in \mathcal{U}(A_{i-1})$, then to $rx_{i-2}^{4m} \in \mathcal{U}(A_{i-2})$, and so on. So, the image of $\mathcal{U}(A_i)$ becomes ever smaller as this group gets mapped down into $\mathcal{U}(A_j)$'s of lower indices j. It is then easy to see that the inverse system $A_0 \leftarrow \cdots \leftarrow A_i \leftarrow \cdots$ is *not* stable for units, much less uniformly so.

Now, over this same pro-affine algebra A one can build a variety of pro-affine algebras $\lim_{\leftarrow} A_i[T_i]$ by specifying $\phi_i\colon T_i \mapsto e_{i-1}T_{i-1} = x_{i-1}^{p_{i-1}}T_{i-1}$ with various choices for the sequence $\mathbf{p} = (p_0, p_1, \ldots, p_{i-1}, p_i, \ldots)$ of exponents p_j's all in $\mathbb{Z}$. And then one asks whether or not the equation (2) may be solved in succession as one climbs up on the levels i. Namely the question is,when $\mathbf{p}$ is given, whether or not the system of integer equations,

$$x_{i-1}^{2q_i} = x_{i-1}^{-p_{i-1}} \cdot x_{i-1}^{q_{i-1}}, \ \ \textit{i.e.},\ \ q_i = \frac{q_{i-1} - p_{i-1}}{2} \tag{14}$$

may be solved for q_i's as $i \to \infty$ starting with a suitable initial value for q_0.

As a first instance of this question, let $p_i := m$ for all $i \in \mathbb{N}$ with any integer m chosen and fixed. Then, by means of $\phi_i\colon A_i[T_i] \to A_{i-1}[T_{i-1}]$ defined by $T_i \mapsto x_{i-1}^m T_{i-1}$, one builds a pro-affine algebra $\tilde{A} := \lim_{\leftarrow,i}(A_i[T_i], \phi_i)$. Then, this apparently twisted algebra $\tilde{A}$, built over A which is non-stable for units, can nevertheless be straightened out and we find $\tilde{A} \simeq A[\mathbb{Y}]$ for an appropriate choice of $\mathbb{Y}$.

Indeed, one can easily solve for q_i the equation system (14) with $q_i = -m$ when $p_{i-1} = m$, $q_{i-1} = -m$. Then, letting $Y_i := x_i^{-m} T_i$ for all $i \geq 0$, we see that $\phi_i(Y_i) = \phi_i(x_i^{-m} T_i) = x_{i-1}^{-2m} \cdot x_{i-1}^m T_{i-1} = x_{i-1}^{-m} T_{i-1} = Y_{i-1}$, as asserted just above.

A somewhat more complex choice for the e_i's might be the case where $e_i := x_i^i$ for all $i \in \mathbb{N}$, or where $p_i = i$ for all $i \in \mathbb{N}$. In this instance, too, the untwisting is made by the solution set $q_i := -i + 2$ with initial value $q_0 = 2$. Then, indeed, $(q_{i-1} - p_{i-1})/2 = (-i + 3 - i + 1)/2 = (-2i + 4)/2 = -i + 2 = q_i$ shows $\mathbb{Y} = (Y_0, \ldots, Y_i, \ldots)$ with $Y_i := x_i^{-i+2} T_i$ makes our $\tilde{A} \simeq A[\mathbb{Y}]$.

These two cases show that the uniform stability for units (or, for that matter, even the plain stability for units) of A as in Theorem 2 above is not a necessary condition for the affirmative solution of (2).

There are, however, many other instances of assignments of integers p_i's for $e_i = x_i^{p_i}$ for which the resulting $\tilde{A} = \lim_{\leftarrow} A_i[T_i]$ is not $\simeq A[\mathbb{Y}]$ for any choice of $\mathbb{Y}$. One obvious example is the case of $p_i := 2^i$ for all $i \in \mathbb{N}$. In this instance, if we are to denote by $\nu_2(-)$ the 2-adic valuation of $\mathbb{Q}$, then

$$\nu_2(q_i) = \nu_2\left(\frac{q_{i-1} - p_{i-1}}{2}\right) < \mathrm{Min}(\nu_2(q_{i-1}), i - 1),$$

so that we have $\nu_2(q_i) < \nu_2(q_{i-1})$for all $i > 0$. This shows that we cannot keep getting *integer* solutions $q_0, q_1, \ldots, q_{i-1}, q_i, \ldots$. Therefore, this $\tilde{A}$ is not polynomial.

Other instances of this example in this negative direction may be worked out likewise.

EXAMPLE 3 (D. Wright). Let the base ring $B := \mathbb{Q}[X]$ and, for each $i, j \in \mathbb{N}$, let $S_i := \langle f_j : j \geq i \rangle$ (which is, by definition, the multiplicative monoid generated by the f_j's, $j \geq i$), where $f_j := X - j$. So, $S_0 \supset S_1 \supset \cdots \supset S_i \supset \cdots$. Now define, for each $i \in \mathbb{N}$,

$$A_i \stackrel{\text{def.}}{=} (S_i^{-1} B)[t_1, t_2, \ldots, t_{i-1}, t_i] \quad \text{(all } t_i\text{'s are variables)},$$

and further define $A_i \to A_{i-1}$ through the inclusion $S_i^{-1} B \subset S_{i-1}^{-1} B$ and the assignments $t_1 \mapsto t_1, \ldots, t_{i-1} \mapsto t_{i-1}, t_i \mapsto f_{i-1}^{-1}$. This gives a surjective $\mathbb{Q}$-map and a consequent inverse system $(A_i)_{i \in \mathbb{N}}$. (However, the A_i's are not algebraic over $\mathbb{Q}$.)

Now define for each $i > 0$ a map $\phi_i \colon A_i[T_i] \longrightarrow A_{i-1}[T_{i-1}]$ by setting $\phi_i(T_i) := f_{i-1} T_{i-1}$ over the map $A_i \to A_{i-1}$ given just above. We claim, then, that one cannot realize $\lim_{\leftarrow}(A_i[T_i], \phi_i)$ as $(\lim_{\leftarrow} A_i)[\mathbb{Y}]$. Because: if an appropriate series of units $(v_i \in \mathcal{U}(A_i))_{i \in \mathbb{N}}$ were to be found so that, letting $Y_i := v_i T_i (\forall i)$, one would get $\phi_i \colon Y_i \mapsto Y_{i-1}$, then these v_i's together with the given e_i's (where $e_i = f_i$ for all i) must provide a solution set of the equations (2). This, however, is impossible since, for each i, v_i is expressible like

$$v_i = r_i \cdot \frac{\prod(X - \alpha_j)}{\prod(X - \beta_k)} \quad \text{where } r_i \in \mathbb{Q}^\times; \text{ all } \alpha_j \geq i, \beta_k \geq i; \tag{15}$$

and, consequently, equation (2) for ϕ_{i+1} reads as

$$\phi_{i+1}(v_{i+1}) = \phi_{i+1}\left(r_{i+1} \cdot \frac{\prod(X - \alpha'_j)}{\prod(X - \beta'_k)}\right) = (X - i)^{-1} \cdot r_i \cdot \frac{\prod(X - \alpha_j)}{\prod(X - \beta_k)}, \tag{16}$$

where $r_{i+1} \in \mathbb{Q}^\times$ and all $\alpha'_j \geq i + 1$, $\beta'_k \geq i + 1$. Let us assume all rational functional expressions in (15), (16) are put in reduced form. Then, since the left-side fraction of (16) can have no $X - i$ as a fator, there must be exactly one i among the α_j's and none among the β_k's on the the right-side fraction. We deduce that $v_{i+1} = v_i/(X - i)$. Repeat the same reasoning on $i + 2$ and $i + 1$, and we get $v_{i+2} = v_{i+1}/(X - (i + 1))$, or $v_{i+2} = v_i/(X - i)(X - (i + 1))$, and so on. It follows that $v_i = r_i(X - i)(X - (i + 1)) \times (X - (i + 2)) \cdots (X - (i + q - 1))$ for some $q > 0$ and $v_{i+q} = r_i$, *as long as* we have stayed away from contradiction up to this point. But, come this far to the $(i+q)$-stage, one readily sees that nothing works for the choice of $v_{i+(q+1)}$, and this is the final contradiction. This proves our assertion in this example.

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