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PERIODIC AUTOMORPHISMS OF SURFACES AND COBORDISM

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0. Introduction

In this paper, we work in the differential category. Unless otherwise stated, a surface is an oriented closed, possibly disconnected, surface, and an automorphism is an orientation preserving self-homeomorphism. An automorphism of a surface (F,f) is said to be *null-cobordant* if there is a compact oriented 3-manifold M equipped with an automorphism $(M,\hat{f})$, such that $\partial(M,\hat{f})=(\partial M,\hat{f}|_{\partial M})$ is equal to (F,f). We call this 3-manifold M the *null-cobordism* for (F,f). Two automorphisms of surfaces (F_1,f_1) and (F_2,f_2) are *cobordant* if $(F_1,f_1)\quad(-F_2,f_2)$ is null-cobordant. The cobordism classes form a group Δ_{2+} whose group law is induced by disjoint sum $\amalg$. Bonahon [B], Edmonds and Eving [EE] proved that Δ_{2+} is isomorphic to $\boldsymbol{Z}^{\boldsymbol{Z}}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{\boldsymbol{Z}}$. Bonahon asked the following question in his paper [B;section 9]

Given an automorphism of a surface (for instance presented as a product of Dehn twists), decide whether it is null-cobordant or not.

For the sake of characterizing null-cobordant automorphisms, we want to know, for arbitrary null-cobordant automorphism, what kind of 3-manifold can be constructed as its null-cobordism, and we want to get an explicitly constructed family of 3-manifolds in which, for any null-cobordant automorphism, we can find a null-cobordism of this automorphism. For example, if an automorphism of a 2-torus is null-cobordant then it bounds an automorphism of a solid torus ([B]). In this paper, we show that the same kind of things are true for other surfaces:

Theorem 1. *If an automorphism over a surface is null-cobordant, then this automorphism bounds an automorphism of a 3-manifold obtained by glueing 1-handles over disjoint union of orientable I-bundles over closed, possibly non orientable, surfaces, handlebodies, and trivalent manifolds (defined in section 3).*

Contents are as follows: in section 1, we review some results and terminologies in [B]. In section 2, we review some results on periodic maps, show that any periodic map *compresses* to a *trivalent map*, and introduce a graph which corresponds to a null-cobordant trivalent map. In section 3, we introduce a *trivalent manifold* which is a null-cobordism of a null-cobordant trivalent map, and construct hyperbolic structures on these manifolds. In section 4, we give a proof of Theorem 1. In section 5, we apply trivalent maps and trivalent graphs for another problem. Let $\Delta^P_{2+}(n)$ denote the group of periodic cobordism classes of automorphisms (F,f) with period n. Bonahon [B; Proposition 8.3] proved that $\Delta^P_{2+}(n) \cong \boldsymbol{Z}^{[(n-1)/2]}$ (here, [ ] means "integer part"). We show this fact explicitly with giving the basis of this abelian group in terms of trivalent maps.

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1. Preliminaries

In this section, we review some results and terminologies in [B]. The following was shown:

Lemma [B; Lemma 5.2]. *If (F,f) is null-cobordant, it bounds an automorphism $(M,\hat{f})$ with M irreducible.*

For an irreducible 3-manifold M, its boundary may be compressible. Hence, we want to extract compressing discs of boundaries from this 3-manifold. A terminology was defined:

DEFINITION. A 3-manifold V is a *compression body* for a surface F if V is an irreducible 3-manifold formed from $F\times I$ by adding 2- and 3-handles to $F\times\{1\}$, i.e. V is obtained by adding 2-handles along thin regular neighborhoods of disjoint simple closed curves in $F\times\{1\}$ and capping off any 2-sphere boundary components this creates with 3-balls. There exists a partition $\partial V=\partial_e V \amalg \partial_i V$, where $\partial_e V=F\times\{0\}$, $\partial_i V=\partial V-\partial_e V$. We call $\partial_e V$ the *exterior boundary* and $\partial_i V$ the *interior boundary*.

We construct compression bodies for ∂M embedded in M. There exist a great variety of compression bodies, but there is a "maximal" one. Namely,Bonahon showed:

Theorem [B; Theorem 2.1]. *Let M be an irreducible, three manifold. There exsts a compression body $V\subset M$ for ∂M,unique up to isotopy, such that $\overline{M-V}$ is ∂-irreducible (and irreducible).*

We call the compression body V given in this Theorem the *characteristic* compression body of M. For an irreducible and ∂-irreducible manifold M', Johannson [Jo], Jaco and Shalen [JS] showed:

Theorem [Jo], [JS]. *By a family of essential tori and annuli properly embedded in M', which are not parallel pair by pair. M' is decomposed into two factors,*
1) *a Seifert factor: this factor consists of Seifert fibered manifolds and I-bundles over surfaces*
2) *a simple factor: this factor is atoroidal and anannular, but does not have Seifert fiber structure or I-bundle structure and this decomposition is unique up to isotopy.*

Hence, an irreducible 3-manifold M is decomposed into three factors, a characteristic compression body, a Seifert part, and a simple factor, unique up to isotopy. Bonahon deeply investigated this decomposition, and showed:

Proposition A ([B;Proposition 5.1]). *If (F,f) is null-cobordant, it bounds an automorphism $(M^3,\hat{f})$ where M split into three pieces V, M_1 and M_P, preserved by $\hat{f}$, such that:*
(1) *V is a compression body for ∂M and $\overline{M-V}=M_I \cup M_P$.*
(2) *M_I is an orientable I-bundle over a closed, possibly non-orientable, surface.*
(3) *The restriction of $\hat{f}$ to M_P is periodic.*

In this paper, we study M_P, that is, for a given periodic null-cobordant automorphism (F_P,f_P), we construct, explicitly, a 3-manifold $\hat{M}$ such that there is a periodic automorphism $(\hat{M},\hat{f})$ whose restriction to the boundary $(\partial\hat{M},\hat{f}|_{\partial\hat{M}})$ is (F_P,f_P). In section 3, we will show that this $\hat{M}$ can be decomposed into hyperbolic 3-manifolds by essential tori. Hence, Theorem 1 is restated as follows:

Theorem 1′. *If (F,f)* is null-cobordant, it bounds an automorphism of an irreducible 3-manifold whose Seifert factor consists of an orientable I-bundle over a surface and whose simple factor is a trivalent manifold (defined in section 3).

2. Periodic automorphisms

An automorphism of a surface (F,f) is *periodic*, if there is positive integers n such that $f^n=\mathrm{id}_F$. The *period of* (F,f) is the smallest positive integer which satisfies the above condition. Let n be the period of (F,f). Denote $Fix_+f=\{x\in F|$ there exists a positive integer $m<n$ such that $f^m(x)=x\}$. For any periodic map (F,f), its orbit space F/f is defined by identifying x in F with $f(x)$, let $\pi_f: F\to F/f$ be the quotient map. For any component F_i of F, the *period of f in F_i* is the period of the map $f|_{\hat{F}_i}$, where $\hat{F}_i=\pi_f^{-1}(\pi_f(F_i))$. If all the components of F have the same period n, then (F,f) is the periodic map with the *total period n*. For any periodic map (F,f) with the total period n, denote $\pi_f(Fix_+f)$ by S_f and called

singular set of F/f and its elements are called *singular points.* Let O_i be any connected component of F/f and its elements are called *singular points.* Let O_i be any connected component of $F/f-S_f$, x_i be any point of O_i and $\tilde{x}_i$ be any point in F such that $\pi_f(\tilde{x}_i)=x_i$. Define the homomorphism R_f from $\bigoplus_i \pi_1(O_i,x_i)$ to $\boldsymbol{Z}_n$ as follows: Let λ be an element of $\pi_1(O_i,x_i)$, and let l be a loop representing λ. Let $\tilde{l}$ be a path which begins at $\tilde{x}_i$ and $\pi_f(\tilde{l})=l$, where $\pi_f|_{\tilde{l}}$ is injective. There exists a positive integer r smaller than or equal to n such that $f^r(\tilde{x}_i)$ is the terminal point of $\tilde{l}$. We define $R_f(\lambda)=r$. We note that this definition does not depend on the choice of the base points $\tilde{x}_i$ and the loops l and their lifts $\tilde{l}$ on F. Since $\boldsymbol{Z}_n$ is abelian, we can naturally define a homomorphism ρ_f from $H_1(F/f-S_f;\boldsymbol{Z})$ to $\boldsymbol{Z}_n$ induced by R_f. For any point s_j of S_f, let D_i be a disk in F/f, which include s in its interior and is sufficiently small such that no other points s_j $(i\neq j)$ is included in D_i. Define $I_f(s_i)=\rho_f([\partial D_i])$. We note that $I_f(s_i)$ is independent of the choice of D_i.

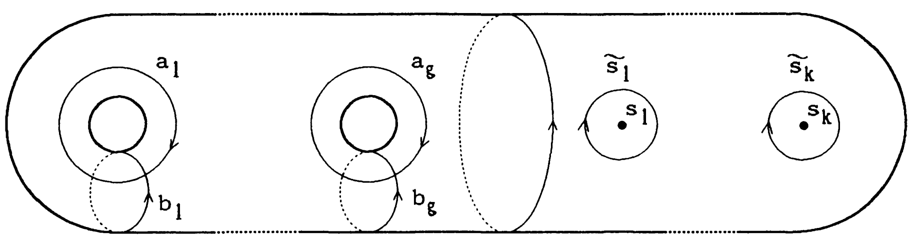


Fig. 1.

Let Σ_g be a connected surface of genus g, S a set of finite points in Σ_g. Denote by $\mathscr{P}_n(\Sigma_g,S)$ the set of the periodic map (F,f) with total period n such that $S_f=S$. A periodic map (F,f) with total period n is *(n,g,k)-periodic map,* if (F,f) is the element of $\mathscr{P}_n(\Sigma_g,S)$ where the number of the points of S is k. Two elements (F_1,f_1) and (F_2,f_2) of $\mathscr{P}_n(\Sigma_g,S)$ are equivalent if there exists an orientation preserving diffeomorphism $h:F_1\to F_2$ such that $h\circ f_1=f_2\circ h$. Denote the set of equivalent classes in $\mathscr{P}_n(\Sigma_g,S)$ by $P_n(\Sigma_g,S)$. We take a model for Σ_g in the 3-dimensional Euclidean space as shown in Figure 1. Let $Hom(H_1(\Sigma_g-S),\boldsymbol{Z}_n)^*$ be the set of homomorphisms ω from $H_1(\Sigma_g-S)$ to $\boldsymbol{Z}_n$ such that $\omega(\tilde{s}_i)\neq 0$ for every $\tilde{s}_i$. We say that two elements ω_1 and ω_2 of $Hom(H_1(\Sigma_g-S),\boldsymbol{Z}_n)^*$ are *$\mathscr{A}$-equivalent,* if there exists a homeomorphism h on (Σ_g,S) such that $\omega_1\circ h_*=\omega_2$ where h_* is the automorphism of $H_1(\Sigma_g-S)$ induced by $h|_{\Sigma_g-S}$. We denote by $Q_n(\Sigma_g,S)$ the set of the $\mathscr{A}$-equivalent class of $Hom(H_1(\Sigma_g-S),\boldsymbol{Z}_n)^*$. Yokoyama [Y] showed the following theorem.

Theorem B

I) *The map that associates with each* (F,f) in $P_n(\Sigma_g,S)$ *the homomorphism* $\rho_f: H_1(\Sigma_g - S) \to \boldsymbol{Z}_n$ *defines a one-to-one correspondence between* $P_n(\Sigma_g,S)$ *and* $Q_n(\Sigma_g,S)$.

II) *Any element of* $Q_n(\Sigma_g,S)$ *can be represented by homomorphism* ρ: $H_1(\Sigma_g - S) \to \boldsymbol{Z}_n$ *such that* $\rho(a_1)=m$, $\rho(b_1)=0$, $\rho(a_i)=\rho(b_i)=0$ $(i \geq 2)$ *and, for* $\theta_j = \rho(\tilde{s}_j)$, $1 \leq \theta_i \leq \theta_2 \leq \cdots \leq \theta_k < n$, $\theta_1 + \cdots + \theta_k \equiv 0 \pmod{n}$.

Corollary 2 [B; Lemma 8.2]. *If* $S_f = \phi$, *then* (F,f) *bounds a periodic automorphism of a disjoint union of handlebodies.*

Proof. Following from Theorem B, we can see that such a map is a composition of a transitive cyclic permutation of components of F and a rotation around the axis as in Figure 2. Since this map bounds an automorphism of a disjoint union of handlebodies, we get the result.

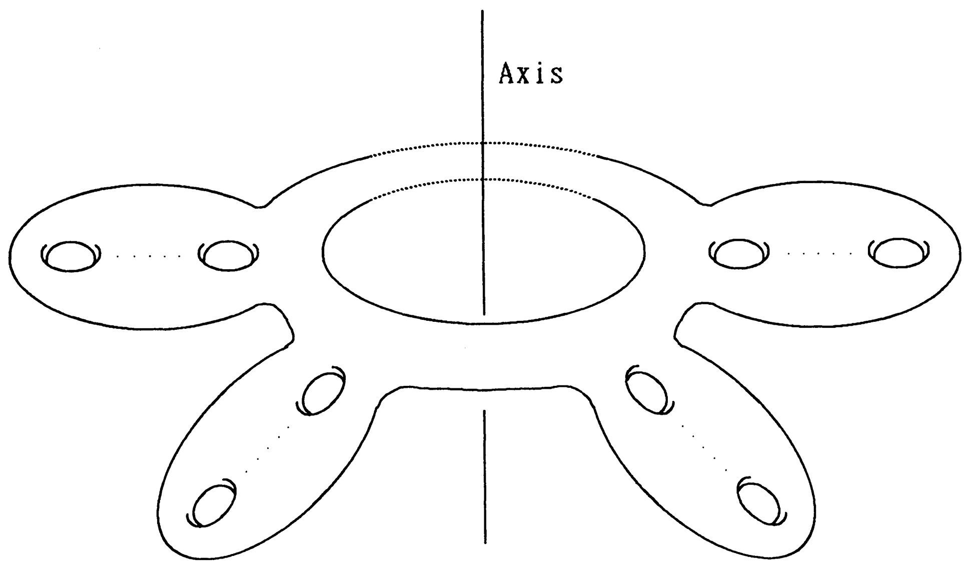


Fig. 2.

DEFINITION. A periodic map (F,f) is *trivalent map*, if it is a disjoint union of $(n,0,3)$-periodic maps, i.e. the orbit space F/f is a disjoint union of 2-spheres and each components have three singular points.

The *genus* of a trivalent map (F,f) is the sum of genera of all components of F. By Theorem B, there exists a unique element of $P_n(S^2,\{x_1,x_2,x_3\})$ represented by a trivalent map (F,f) under the condition that $\theta_i = I_f(x_i)$ $(i=1,2,3)$. Represent this map (F,f) by $\{\theta_1,\theta_2,\theta_3;n\}$. This map $\{\theta_1,\theta_2,\theta_3;n\}$ is independent of the choice of the order of $\theta_1,\theta_2,\theta_3$, as an element of $P_n(S^2,\{x_1,x_2,x_3\})$, we may assume $0<\theta_1\leq\theta_2\leq\theta_3<n$. Define $n_i=g.c.d.(\theta_i,n)$ $(i=1,2,3)$, $N=g.c.d.(n_1,n_2,n_3)$. Then N is the number of the components of F. The genus G of the trivalent map $\{\theta_1,\theta_2,\theta_3\,;n\}$ is given by the following formula

$$G=N+\{n-(n_1+n_2+n_3)\}/2$$

Here, we will give some examples of trivalent maps.

Example. Using the above formula, we classify all trivalent maps on surfaces with genera 0,1,2 up to equivalence.

0) *Trivalent maps on* 2-*sphere.*

There is no trivalent map on disjoint union of 2-spheres.

1) *Trivalent maps on* 2-*tori.*

There are 6 types of trivalent maps on a 2-torus; (1) $\{1,1,1\,;3\}$, (1′) $\{2,2,2\,;3\}$, (2) $\{1,1,2\,;4\}$, (2′) $\{2,3,3\,;4\}$, (3) $\{1,2,3\,;6\}$, (3′) $\{3,4,5\,;6\}$. Here, (1′) is the same as (1) but the orientation reversed, and (2′), (3′) are also. These maps are represented by 2×2 matrices; (1) $\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$ (2) $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ (3) $\begin{pmatrix}1&-1\\1&0\end{pmatrix}$. This is proved as follows: For these maps, $N=1$ and $\{n-(n_1+n_2+n_3)\}/2=0$ in the above formula for G. Therefore $n=n_1+n_2+n_3$. Divide this equation by n and replace n/n_i by m_i, then $1/m_1+1/m_2+1/m_3=0$. To satisfy this condition, (m_1,m_2,m_3) is one of (3,3,3), (2,3,6),(2,4,4). By the definition of N, n must be $l.c.m.(m_1,m_2,m_3)$. For each (m_1,m_2,m_3), we can reconstruct trivalent maps and get the result. On the disjoint union of 2-tori, trivalent maps whose orbit spaces are connected are constructed by combining the trivalent maps on one 2-torus with the cyclic transitive permutation of the components. For example there are 6 types of trivalent maps on a disjoint union of two 2-tori; (1) $\{2,2,2\,;6\}$, (1′) $\{4,4,4\,;6\}$, (2) $\{2,2,4\,;8\}$, (2′) $\{4,6,6\,;8\}$, (3) $\{2,4,6\,;12\}$, (3′) $\{6,8,10\,;12\}$.

2) *Trivalent maps on a genus* 2 *closed surface* Σ_2.

Trivalent map on Σ_2 is one of the following; (1) $\{1,2,2\,;5\}$, (1′) $\{3,3,4\,;5\}$, (2) $\{1,1,3\,;5\}$, (2′) $\{2,4,4\,;5\}$, (3) $\{2,5,5\,;6\}$, (3′) $\{1,1,4\,;6\}$, (4) $\{4,5,7\,;8\}$, (4′) $\{1,3,4\,;8\}$, (5) $\{1,4,5\,;10\}$, (5′) $\{5,6,9\,;10\}$, (6) $\{2,3,5\,;10\}$, (6′) $\{5,7,8\,;10\}$. This is proved by the two facts, (a) if a positive prime integer n is a period of a periodic map on a connected surface of genus g $(g\geq 2)$, then $n\leq 2g+1$ (it is a corollary of Riemann-Hurwitz Relation (see [FK])), (b) the greatest number of the period of the periodic map over a connected surface of genus g $(g\geq 2)$ is $2(2g+1)$ (see [H; Theorem 6]).

Definition. An automorphism of surface (F_1,f_1) *compresses* to (F_2,f_2), if there exists an automorphism of a compression body $(V,\hat{f})$ such that $(F_1,f_1)=(\partial_e V,\hat{f}|_{\partial_e V})$, $(F_2,f_2)=(-\partial_i V,\hat{f}|_{\partial_i V})$.

The following Theorem shows that trivalent maps are the essential parts of periodic maps.

Theorem 3. *Any periodic map compresses to a trivalent map.*

Proof. Let (F,f) be a periodic automorphism. For a simple closed curve l in $F/f-S_f$, let N be a thin regular neighborhood of l in $F/f-S_f$, and let $\coprod_j N_j=\pi_f^{-1}(N)$ be the decomposition into connected components. Cut the surface F along $\pi_f^{-1}(l)$, and denote $F^c=F-\coprod_j N_j$, then $(F^c,f|_{F^c})$ is a periodic map. A restriction of this map to the boundary, $(\partial F^c,f|_{\partial F^c})$, bounds a periodic map $((\coprod D_j)\coprod(\coprod D'_j),g)$, where $\partial D_j\coprod\partial D'_j=\partial N_j$ and $g|D_j$, $g|D'_j$ are rotations. We denote by s_j, s'_j the centers of these rotations. Let $\tilde F=F^c\cup_{\partial F^c}((\coprod D_j)\coprod(\coprod D'_j))$ where ∂F^c and $(\coprod\partial D_j)\coprod(\coprod\partial D'_j)$ are identified naturally. On this surface, we can obtain a periodic map $(\tilde F,\tilde f)$ such that $(F^c,\tilde f|F^c)=(F^c,f|F^c)$ and $((\coprod D_j)\coprod(\coprod D'_j),\tilde f|_{(\coprod D_j)\coprod(\coprod D'_j)})=((\coprod D_j)\coprod(\coprod D'_j),g)$. We say that $(\tilde F,\tilde f)$ is obtained from (F,f) by an *equivariant* 2-*surgery* along l. If $\rho_f(l)\neq 0$, then $S_{\tilde f}=S_f\cup\{s_j,s'_j\}$ and $I_{\tilde f}(s_j)=-I_{\tilde f}(s'_j)=\pm\rho_f(l)$. If $\rho_f(l)=0$ then $S_{\tilde f}=S_f$.

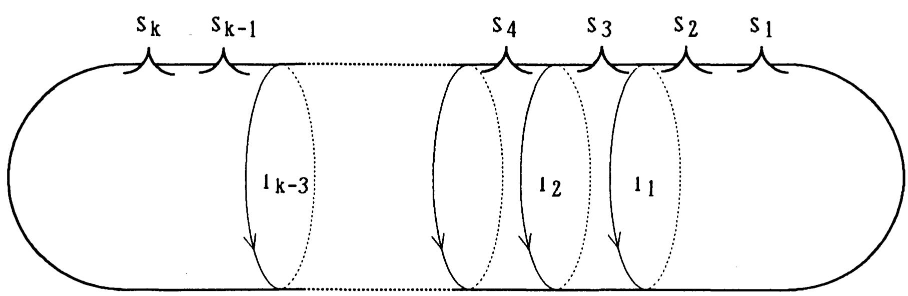


Fig. 3.

We can divide automorphism into parts which have total period n and n's are different each other, and discuss each parts. Therefore, we assume the periodic map (F,f) has the total period n. For each component O of F/f, let l be a simple closed curve as in Figure 1. Perform an equivariant 2-surgery along l and obtain a periodic automorphism (F',f'). This periodic automorphism (F',f') is a disjoint union of $(n,0,k)$-periodic maps and $(n,g,0)$-periodic maps. Thus, by Corollary 2, (F,f) compresses to a disjoint union of $(n,0,k)$-periodic maps. For an $(n,0,k)$-periodic map (F',f'), perform equivariant 2-surgeries along mutually disjoint simple closed curves $l_1,\cdots,l_{k-3}$ as in Figure 3 and obtain a periodic map (F'',f'')

which is a disjoint union of $(n,0,3)$- and $(n,0,2)$-periodic maps. Remark that, for each component of F''/f'', the number of singular points is either two or three, depending on the value $\rho_{f'}(l_i)$. An $(n,0,2)$-periodic map is a composition of a transitive cyclic permutation of components and rotations of 2-spheres whose axes are the lines through north poles to south poles. These maps bound periodic maps on 3-balls. This shows that an $(n,0,k)$-periodic map compresses to a disjoint union of $(n,0,3)$-periodic maps,i.e. trivalent maps, and finishes the proof.

A periodic map (F,f) is *periodic null-cobordant*, if there exists a periodic map $(M,\hat{f})$ of a 3-manifold M such that $\partial(M,\hat{f})=(F,f)$ and periodic maps (F_1,f_1), (F_2,f_2) are *periodic cobordant*, if $(F_1,f_1)\amalg(-F_2,f_2)$ is periodic null-cobordant. Remark that, for any periodic null-cobordant map (F,f), periods of f in each component of F may be different. Let $(M,\hat{f})$ be the null-conbordism of (F,f), for each component M_i of M, as is easy to see, the periods of f in each component of $F\cap\partial M_i$ are the same. Hence, for the sake of our investigation, it is sufficient to work on periodic maps with some total period. For any point x in F, let m be the smallest positive integer with $f^m(x)=x$. Then there exists an element ρ of $\boldsymbol{Q}/\boldsymbol{Z}$ such that f^m is locally conjugate to a rotation of angle $2\pi\rho$ around x where the conjugation is given by the orientation preserving local automorphism. Denote this ρ by $r(f,x)$.

Bonahon [B; Proposition 8.1] showed the following proposition.

Proposition C. *If (F,f) is a periodic map, (F,f) is periodic null-cobordant if and only if Fix_+f admits a partition into pairs $\{x_i,x_i'\}$ such that:*

(1) $r(f,x_i)+r(f,x_i')=0$.

(2) *For every i, $f(\{x_i,x_i'\})=\{x_j,x_j'\}$ for some j.*

The following lemma shows some relationship between $r(f,x)$ and $I_f(\pi_f(x))$:

Lemma 4. *Let (F,f) be a periodic map with the total period n. For two points x and x' in Fix_+f, $r(f,x)+r(f,x')=0$ if and only if $I_f(\pi_f(x))+I_f(\pi_f(x'))=0$.*

Proof. If the total period n is fixed, $r(f,x)$ and $I_f(\pi_f(x))$ are determined by each other, and this does not depend on the map f. Hence, it suffices to show the claim for $(n,0,2)$-periodic maps, in which case the statement is trivial.

We can restate Proposition C in terms of $I_f(*)$:

Lemma 5. *A periodic map (F,f) with the total period n is periodic null-cobordant if and only if S_f admits a partition into pairs $\{s_i,s_i'\}$ such that $I_f(s_i)+I_f(s_i')=0$.*

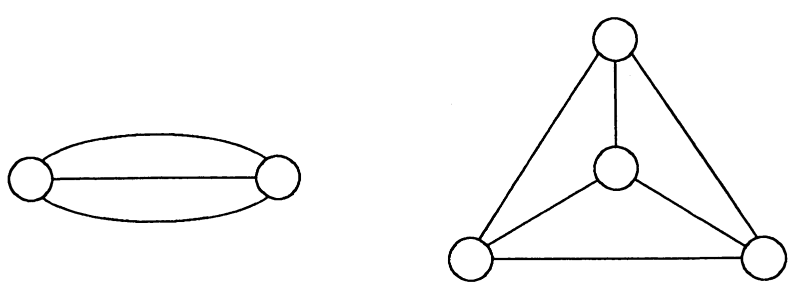
Proof. First, we see the sufficiency. Let $\{x_i,x_i'\}$ be the lift of $\{s_i,s_i'\}$,

$r(f, x_i)+r(f, x_i')=0$ by Lemma 4. By the definition of $r(f, *)$, $r(f, x)=r(f, f(x))$ for all x in $Fix_+ f$, therefore $r(f, f(x_i))+r(f, f(x_i'))=0$. We can see that a partition into pairs of S_f naturally induces a partition into pairs of $Fix_+ f$ which satisfies the condition mentioned in Proposition C.

Next, we see the necessity. Let $Fix_{++} f=\{x \in Fix_+ f \mid r(f, x) \neq 1/2\}$. Then this set admits a partition into pairs $\{x_i, x_i'\}$ following from Proposition C. The subset $S_{f+}=\pi_f(Fix_{++} f)$ of S_f admits a partition into pairs $\{s_i, s_i'\}$ such that $I_f(s_i)+I_f(s_i')=0$ following from Lemma 4. For each element s of $S_f - S_{f+}$, since any lift x of s satisfies $r(f, x)=1/2$, $I_f(f, x)$ is equal to $n/2 \in \boldsymbol{Z}_n$. For each element s_i of S_{f+}, let D_i be a small 2-disk in F/f around s_i such that they do not intersect each other. By the definition of $I_f(*)$, we can see $\rho_f(\sum_i [\partial D_i])=0$. For each element $s_j' \in S_f - S_{f+}$, let D_j' be a small 2-disk in F/f around s_j' as sbove. Then $\sum_j [\partial D_j'] = -\sum_i [\partial D_i]$ and it follows that $\rho_f(\sum_j [\partial D_j'])=0$. By the definition of $I_f(*)$, $\rho_f(\sum_j [\partial D_j'])=\sum_j I_f(s_j')$. Since $I_f(s_j')=n/2$, $S_f - S_{f+}$ consists of even number of points. The set $S_f - S_{f+}$ can admit a partition into pairs $\{s_i, s_i'\}$ such that $I_f(s_i)+I_f(s_i')=0$. Hence, S_f admits a partition into pairs which we need.

DEFINITION. For any periodic null-cobordant map (F, f) with total period n, define the set

$$P_f = \left\{ \{\{s_i, s_i'\}\}_i \;\middle|\; \begin{array}{l} \bigcup_i \{s_i, s_i'\} = S_f,\ \{s_i, s_i'\} \cap \{s_j, s_j\} = \phi \text{ for any } i \neq j, \\ \hfill \text{and } I_f(s_i)+I_f(s_i')=0 \end{array} \right\}$$



(i) (ii)

Fig. 4.

A *graph* Γ is a 1-dimensional finite CW-complex. A *vertex* of Γ is a 0-cell of Γ, an *edge* of Γ is an 1-cell of Γ. We call a graph Γ *trivalent* if, for each vertex, the number of edges which terminate at this vertex is three (here, remark that edges are not oriented). Clearly, the number of vertices of a trivalent graph is even. A graph Γ' is a *subgraph* of a graph Γ, if Γ' is the subcomplex of Γ. In Figure 4, we give two simple examples of trivalent graphs, which play central roles in this paper. A subgraph C of Γ is *circuit* over Γ if C is homeomorphic to S^1, and if the number of edges of C is l we call C a *l-circuit*. If the number of components of Γ is k and there exists an edge $e_1,\cdots,e_m$ such that $\Gamma - e_1 \cup \cdots \cup e_m$ have $k+1$ connected components, then Γ is said to be *m-splittable*, and the set $\{e_1,\cdots,e_m\}$ is called a *splitting edge set*. Let (F,f) be a periodic null-cobordant trivalent map, and $p \in P_f$. We can make a trivalent graph $\Gamma_{f,p}$ which corresponds to this map (F,f) and an element p of P_f, by identifying each component of F/f with the vertex of $\Gamma_{f,p}$ and each pair $\{s_i,s_i'\} \in p$ with the edge of $\Gamma_{f,p}$ which connect two vertices identified with two components of F/f including s_i and s_i'. Give an arbitrary orientation on each edge, if a terminal vertex of an oriented edge e corresponds to the component of F/f including s_i', then give a weight $I_f(s_i') \in \boldsymbol{Z}_n$ on this oriented edge. The weights on the graph $\Gamma_{f,p}$ depend on the orientation of edges, but we do not tell one from the others, that is, we regard the graphs in Figure 5 as the same weighted graphs.

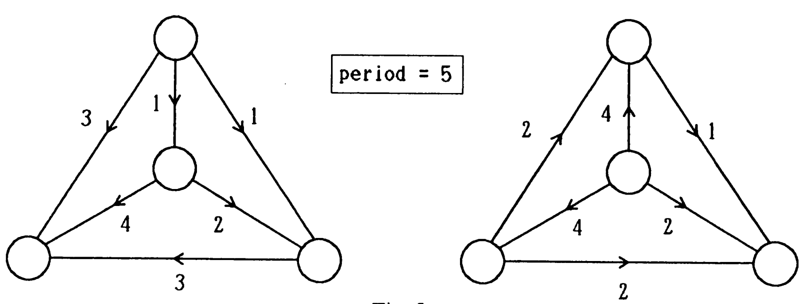


Fig. 5.

REMARK. Let $\Gamma_{f,p}$ be connected, $\{e_1,\cdots,e_m\}$ be splitting edge set, and Γ_1, Γ_2 be the components of $\Gamma_{f,p} - e_1 \cup \cdots \cup e_m$. Give an orientation of each e_i such that whose terminal vertex is in Γ_2, then the summation of weights given to $e_1,\cdots,e_m$ is 0 (we can prove this fact by the induction of the number of vertices). From this fact, we can see that if $\Gamma_{f,p}$ has two vertices then $\Gamma_{f,p}$ is as in Figure 4(i).

3. Trivalent manifolds and their geometry

Regard S^3 as a 1-point compactification of $\boldsymbol{R}^3$. Let $\boldsymbol{R}^3$ be the Euclidean 3-space. Let Γ be the set which consists of vertices and edges of a tetrahedra in $\boldsymbol{R}^3 \subset S^3$. This CW-complex Γ is the trivalent graph as in Figure 4(ii). Let $T = S^3$-regular neighborhood of vertices of Γ, and $(T,\hat{\Gamma}) = (T, T \cap \Gamma)$. $\hat{\Gamma}$ is four arcs properly embedded in T. Let $\{(T_i,\hat{\Gamma}_i)\}_i$ be the arbitrary number of copies of $(T,\hat{\Gamma})$, $\{\{S_k,S_k'\}\}_k$ be the pairing of connected components of $\bigcup_i \partial T_i$ such that

$\{S_k, S'_k\} \cap \{S_l, S'_l\} = \phi$ for any $k \neq l$ and there may be some components of $\bigcup_i \partial T_i$ which are not included in $\bigcup_k (S_k, S'_k)$. T can be regarded as a 3-ball removed three 3-balls. For a pair $\{S_k, S'_k\}$, let T_{i_k}, T_{j_k} be the two of T_i's which include S_k, S'_k as their boundary component. Put a mirror between T_{i_k}, T_{j_k} as in Figure 6. $(T_{i_k} \cup_{S_k = -S'_k} T_{j_k}, \hat{\Gamma}_{i_k} \cup \hat{\Gamma}_{j_k})$ is a pair of a 3-manifold and arcs properly embedded in this 3-manifold which given as a result of identification of S_k, S'_k given by using this mirror. Do the same thing for other pairs, then we have a pair $(\hat{T}, \hat{\hat{\Gamma}})$ of a 3-manifold and arcs properly embedded in this 3-manifold. Construct a cyclic branched covering $\tilde{T}$ of this 3-manifold $\hat{T}$ whose branch point set is $\hat{\hat{\Gamma}}$. We call this 3-manifold $\tilde{T}$ given as a result of this process a *trivalent manifold.*

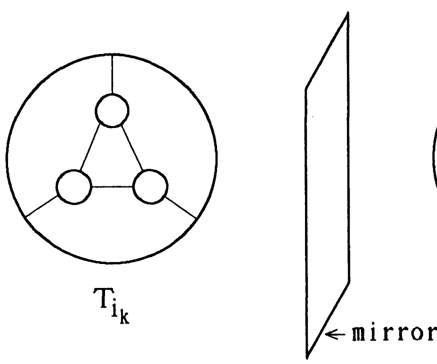


Fig. 6.

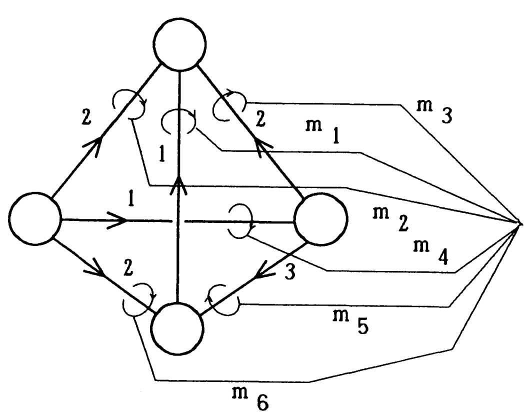


Fig. 7.

REMARK. The homeomorphism type of $\tilde{T}$ is depend not only on $(\hat{T},\hat{\hat{\Gamma}})$ but also on the type of cyclic branched covering.

EXAMPLE. Let (F,f) be a trivalent map of period 4, and embed a graph $\Gamma_{f,p}$ with weight into S^3 as indicated in Figure 7. T is a 3-manifold constructed from a 3-sphere with removing neighborhood of each vertices. Define $\hat{\Gamma}_{f,p}=\Gamma_{f,p}\cap T$. The fundamental group of a space $T-\hat{\Gamma}_{f,p}$ is generated by the loops $m_1,m_2,\cdots m_6$ given in Figure 7. (As a system of generators of this fundamental group, four of them is enough.) We define a homomorphism ρ from $\pi_1(T-\hat{\Gamma}_f,*)$ to $\boldsymbol{Z}_4$ by $\rho(m_1)=1$, $\rho(m_2)=1$, $\rho(m_3)=2$, $\rho(m_4)=1$, $\rho(m_5)=3$, $\rho(m_6)=2$, we can easily check the well-definedness of this homomorphism. Let $\hat{T}_0$ be the covering space of $T-\hat{\Gamma}_f$ whose fundamental group is $\ker\rho$. Let $\pi:\hat{T}\to T$ be the branched covering associated to the covering $\hat{T}_0\to T\to\hat{\Gamma}_f$. The covering transformation group of $\pi:\hat{T}\to T$ is $\boldsymbol{Z}_4$. The manifold T is a trivalent manifold, and a generator of this group $\hat{f}:\hat{T}\to\hat{T}$ satisfies $\partial(\hat{T},\hat{f})=(F,f)$.

Any 3-manifold M which is a cyclic branched covering space of T whose branch point set is $\hat{\Gamma}$ (denote this cyclic branched covering by $\pi:M\to T$), has a hyperbolic structure with geodesic boundaries or cusps. This structure can be constructed as follows:

For a connected component l of $\hat{\Gamma}$, let x be a point in l, and D be the regular neighborhood of x in T sufficiently small such that D does not include points in $\hat{\Gamma}-l$. Let $\tilde{D}$ be a component of $\pi^{-1}(D)$. Then, $\pi|_{\tilde{D}}:\tilde{D}\to D$ is a n-fold cyclic branched covering. This number does not depend on the choice of the point x in l, and the choice of $\tilde{D}$. We call this number n a *branching index* of l. For a periodic automorphism f on a surface F, by the same manner, we can define a *branching index* of $s\in S_f$. Here, we review the definition of a truncated tetrahedra [K]. Let L_1, L_2, L_3 and L_4 be geodesic planes in the 3-dimensional hyperbolic space $\boldsymbol{H}^3$, every two of which intersect each other, and every three of which intersect at infinity or do not intersect. For each three of them, say L_2, L_2 and L_3, which do not intersect, there is unique geodesic plane P_{123} which intersects with them perpendiculary [K; Lemma 2.1]. The domain D in $\boldsymbol{H}^3$ bounded by these L's and P's are called a *truncated tetrahedra*. The face of D which is a part of P's is called a *truncation face*. For a truncated tetrahedra, label the internal edges as in Figure 8 and denote the dihedral angle along the edges j by φ_j. The sufficient and necessary condition of φ_j's to the existence of a truncated tetrahedra whose dihedral angles are these numbers is

$$\begin{cases}\varphi_1+\varphi_2+\varphi_3\leq\pi\\ \varphi_1+\varphi_5+\varphi_6\leq\pi\\ \varphi_2+\varphi_4+\varphi_6\leq\pi\\ \varphi_3+\varphi_4+\varphi_5\leq\pi\end{cases}$$

[K; Lemma 2.3].

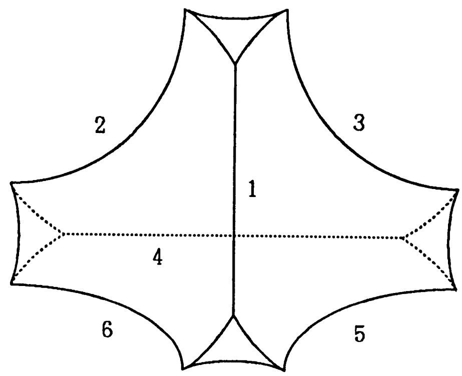


Fig. 8.

REMARK. In [K], the definition of a truncated tetrahedra is slightly different, namely the case which some three of L_1, L_2, L_3 and L_4 intersect at infinity is excludeed, but, here, to avoid complexity, we do not exclude this case. Of course, the above sufficient and necessary condition is a little different, however, we can prove this in the same manner as [K].

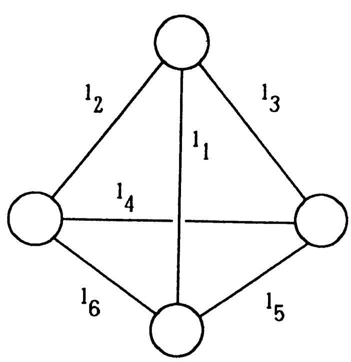


Fig. 9.

Label each component of $\hat{\Gamma}$ as in Figure 9. Let n_i be a branching index of l_i of the cyclic branched covering $\pi: M \to T$. Define $\varphi_i = \pi / n_i$, then φ_i's satisfy the above condition, because each boundary of T is an orbit space of a trivalent map which acts on the surface with genus more than 1. Therefore, we have a truncated tetrahedra whose dihedral angles are φ_i's. Make a double of this truncated tetrahedra along a surface which is not truncation face, then this define a hyperbolic orbifold structure on T whose singular locus is $\hat{\Gamma}$. Lift this hyperbolic orbifold structure to M. Since, for each component l of $\hat{\Gamma}$, the total of the dihedral angle around $\pi^{-1}(l)$is $(\pi/n_i\times 2)\times n_i=2\pi$, this define a hyperbolic structure on M.

Any trivalent manifold is constructed from a disjoint union of the above M's with identifying some components of boundaries in a way compatible with the structure of the branched covering. This identification is given as an isometry on the hyperbolic structure constructed above. Therefore, we can give a hyperbolic structure to any trivalent manifold. We showed the following:

Proposition 6. *Any trivalent manifold is a compact, irreducible sufficiently-large 3-manifold, by essential tori, decomposed into hyperbolic 3-manifolds with geodesic boundaries or cusps.*

As a corollary of this Proposition and a relative version of Gromov's Theorem [T; 6.5.4], we can see the following:

Corollary. *Any trivalent manifold is not a Seifert fibered space.*

Example. We will give a hyperbolic structure to a trivalent manifold $\hat{T}$ of the last example. Let $\boldsymbol{H}^3=\{(x,y,z)\in\boldsymbol{R}^3\mid z>0\}$ be the upper half space with the hyperbolic metric. The domain $D_{1/2}=\{(x,y,z)\in\boldsymbol{H}^3\mid 0\leq x\leq 1,\ 0\leq y\leq x,\ z\geq\sqrt{(x-1/2)^2+(y-1/2)^2}\}$ is a truncated tetrahedra. Make a double of $D_{1/2}$, then we get hyperbolic orbifold whose underlying space is T and whose singular locus is $\hat{\Gamma}$. Let G be the Kleinean group generated by

$$g_1=\begin{pmatrix}1&2\\0&1\end{pmatrix},\quad g_2=\begin{pmatrix}1&2i\\0&1\end{pmatrix},\quad h_1=\begin{pmatrix}1&0\\-i-1&1\end{pmatrix},\quad h_2=\begin{pmatrix}1&0\\i-1&1\end{pmatrix}$$

The fundamental domain of G is

$$\begin{aligned}D=&\{(x,y,z)\in\boldsymbol{H}^3\mid 0\leq x\leq 1,\ 0\leq y\leq 1,\ z\geq\sqrt{(x-1/2)^2+(y-1/2)^2}\}\\&\cup\{(x,y,z)\in\boldsymbol{H}^3\mid 0\leq x\leq 1,\ -1\leq y\leq 0,\ z\geq\sqrt{(x-1/2)^2+(y+1/2)^2}\}\\&\cup\{(x,y,z)\in\boldsymbol{H}^3\mid -1\leq x\leq 0,\ -1\leq y\leq 0,\ z\geq\sqrt{(x+1/2)^2+(y+1/2)^2}\}\\&\cup\{(x,y,z)\in\boldsymbol{H}^3\mid -1\leq x\leq 0,\ 0\leq y\leq 1,\ z\geq\sqrt{(x+1/2)^2+(y-1/2)^2}\}\end{aligned}$$

$\boldsymbol{H}^3/G$ is a hyperbolic 3-manifold with four cusps given from D by identifying $\{(x,y,z)\in D\mid x=1\}$ with $\{(x,y,z)\in D\mid x=-1\}$, $\{(x,y,z)\in D\mid y=1\}$ with $\{(x,y,z)\in D\mid y=1\}$, $\{(x,y,z)\in D\mid 0\leq x\leq 1,\ 0\leq y\leq 1,\ z\geq\sqrt{(x-1/2)^2+(y-1/2)^2}\}$ with $\{(x,y,z)\in D\mid -1\leq x\leq 0,\ -1\leq y\leq 0,\ z\geq\sqrt{(x+1/2)^2+(y+1/2)^2}\}$, $\{(x,y,z)\in D\mid 0\leq x\leq 1,\ -1\leq y\leq 0,\ z\geq\sqrt{(x-1/2)^2+(y+1/2)^2}\}$ with $\{(x,y,z)\in D\mid -1\leq x\leq 0, 0\leq y\leq 1, z\geq\sqrt{(x+1/2)^2+(y-1/2)^2}\}$. The interior of $\hat{T}$ is homeomorphic to $\boldsymbol{H}^3/G$. An element of isometry of H^3 given by

$$\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

induce an isomorphism $\hat{f}$ on H^3/G. This map $\hat{f}$ is a periodic map with period 4 and $(H^3/G,\hat{f})$ is periodic null-cobordism of (F,f) in the last example.

4. Proof of Theorem 1

In this section, we prove Theorem 1.

DEFINITION. The trivalent map (F,f) and $p \in P_f$ is simple piece if $\Gamma_{f,p}$ is one of the two types of trivalent graph given in Figure 4. If $\Gamma_{f,p}$ is Figure 4(i)(resp. Figure 4(ii)), (F,f) and p is called a simple piece of type I (resp. type II).

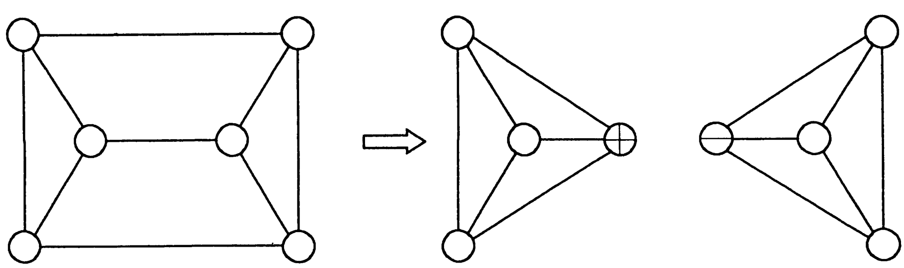
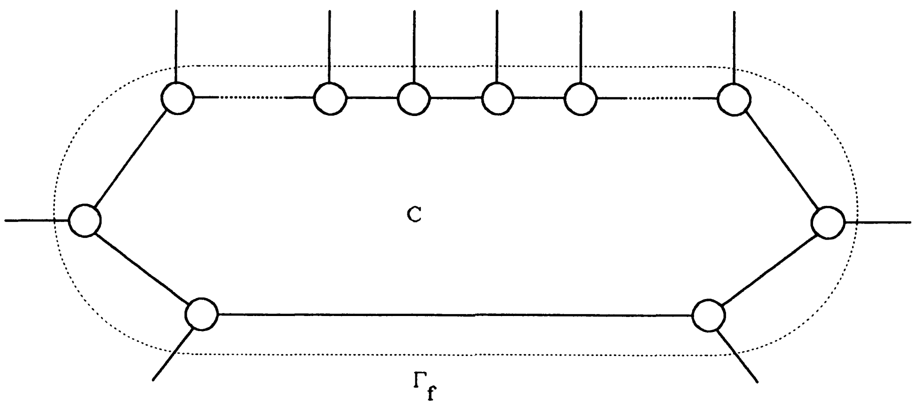


Fig. 10.

From here to the end of this paper, we write Γ_f instead of $\Gamma_{f,p}$ for the sake of avoiding complications of notation. But, remark that Γ_f is depend also on $p \in P_f$. Let (F,f) be a periodic null-cobordant trivalent map which corresponds to a graph Γ_f as in the left hand of Figure 10. We can modify the graph Γ_f to the disjoint union of two trivalent graphs $\Gamma_{f'}$, $\Gamma_{f''}$ by adding two vertices, where $\oplus = -\ominus = (\tilde{F},\tilde{f})$. Let (F',f'), (F'',f'') be trivalent maps corresponding to $\Gamma_{f'}$, $\Gamma_{f''}$ and let $(M',\hat{f}')$, $(M'',\hat{f}'')$ be periodic automorphisms which are periodic null-cobordisums of (F',f'), (F'',f''). Then the periodic automorphism $(M' \underset{\tilde{F}}{\cup} M'', \hat{f}' \cup \hat{f}'')$ gives a periodic null-cobordism of (F,f). Therefore, the periodic null-cobordism can be constructed by gluing periodic null-cobordisms of simple pieces of type II. The same holds for any periodic null-cobordant trivalent map (F,f).

Proposition 7. *Let (F,f) be any periodic null-cobordant trivalent map, then there is a disjoint union of trivalent manifolds and surface $\times I$ which is a periodic* *null-cobordism of* (F,f).

Proof. We prove this by induction on the number c of components of F/f. If $c=2$, this proposition follows from Remark at the end of section 1. If $c\geq 4$, let C be the circuit of Γ_f which has the minimal number of edges, say m (see Figure 11). If m is 2, then Γ_f can be modified into a disjoint union of Γ'_f with $c-2$ vertices and simple piece of type I (see Remark at the end of section 1). If m is more than or equal to 3, then we can modify Γ_f in the dotted circle so as to be the disjoint



Γ_f

Fig. 11.

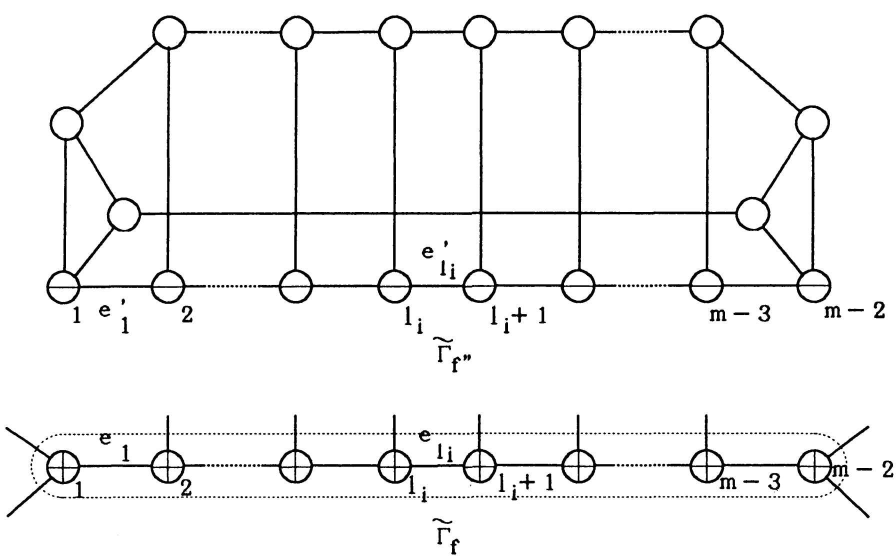
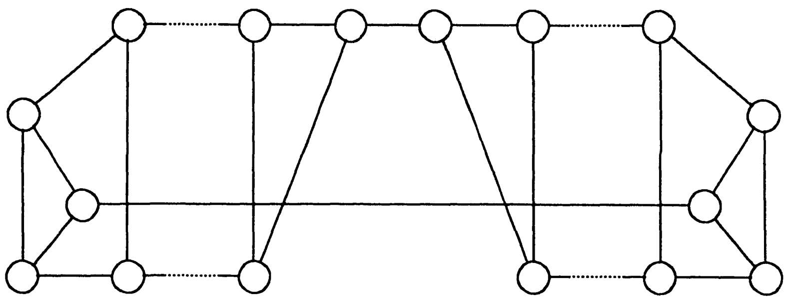
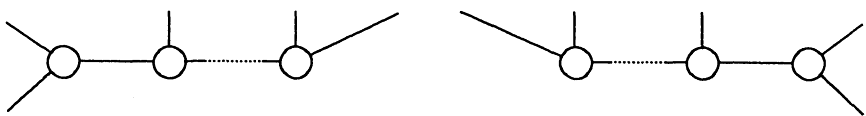


Fig. 12.

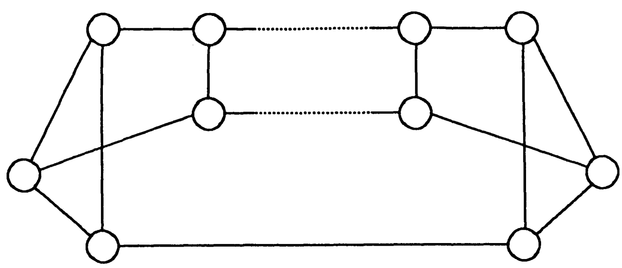


$\Gamma_{f''}$



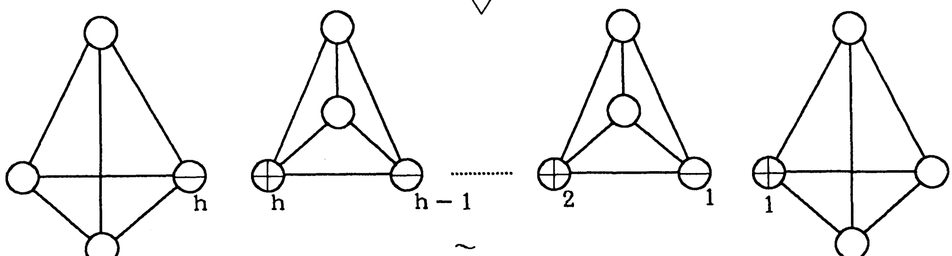
Γ_f

Fig. 13.



$\Gamma_{f'''}$

$\Downarrow$



$\widetilde{\Gamma}_{f'''}$

Fig. 14.

union of $\tilde{\Gamma}_{f'}$, and $\tilde{\Gamma}_{f''}$ by adding vertices with $\oplus_i = -\ominus_i$ and edges e_i, e_i' $(i=1,\cdots,m-2)$ as in Figure 12. Let $(\tilde{F}',\tilde{f}')$, $(\tilde{F}'',\tilde{f}'')$ be trivalent maps correspond to $\tilde{\Gamma}_{f'}$, $\tilde{\Gamma}_{f''}$. There may be edges whose end points have indices 0. Denote these edges by $e_{l_1},\cdots,e_{l_k},e_{l_1}',\cdots,e_{l_k}'$. Periodic maps $\oplus_{l_i}$, $\ominus_{l_i}$, $\oplus_{l_i+1}$, $\ominus_{l_i+1}$, $(i=1,\cdots,k)$ are $(n,0,2)$-periodic maps and bound periodic maps on 3-balls. Therefore, we cna remove these maps and get two graphs $\Gamma_{f'}$, $\Gamma_{f''}$ (see Figure 13). Let trivalent maps (F',f'') and (F'',f'') correspond to $\Gamma_{f'}$, $\Gamma_{f''}$. These trivalent maps (F',f''), (F'',f'') are periodic null-cobordant, and in a similar fashion as a discussion before the claim of this proposition, a periodic null-cobordism of (F,f) is constructed from periodic null-cobordisms of (F',f'') and (F'',f''). The trivalent graph $\Gamma_{f'}$ has fewer vertices than Γ_f, that is F'/f' has fewer components than F/f. By the assumption of induction, the periodic null-cobordism of (F',f'') can be constructed from periodic null-cobordisms of simple pieces. For the periodic map (F'',f''), by changing the pairing of $S_{f''}$, we can alter $\Gamma_{f''}$ to the disjoint union of trivalent graphs $\Gamma_{f'''}$ as in Figure 14. Let the periodic null-cobordant trivalent map (F''',f''') correspond to $\Gamma_{f'''}$. The trivalent graph $\tilde{\Gamma}_{f'''}$ is gotten from $\Gamma_{f'''}$ with adding $2h$ vertices $\oplus_1,\cdots,\oplus_h$, $\ominus_1,\cdots,\ominus_h$ where $\oplus_i=-\ominus_i$ $(i=1,\cdots,h)$. The periodic null-cobordant trivalent map corresponding to $\tilde{\Gamma}_{f'''}$ is a disjoint union of simple pieces of type II and a periodic null-cobordism of (F''',f''') is constructed from its periodic null-cobordism.

By Proposition A, Theorem 3, and Proposition 7, we can prove Theorem 1, and by Theorem 1 and Corollary of Proposition 6, we can prove Theorem 1′.

5. Periodic cobordism groups

Let $\Delta_{2+}^P(n)$ denote the subgroup of periodic cobordism classes of automorphisms (F,f) with the total period n. Bonahon [B; Proposition 8.3] proved that $\Delta_{2+}^P(n)\cong \boldsymbol{Z}^{[(n-1)/2]}$ (here [ ] means "integer part"). In this section, we give an explicit generator of this group by trivalent maps.

Theorem 8. *Let* $x_i=\{1,i,n-1-i;n\}$ $(i=1,\cdots,[(n-1)/2])$. *Then*

$$\Delta_{2+}^P(n)\cong \boldsymbol{Z}x_1\oplus\cdots\oplus \boldsymbol{Z}x_{[(n-1)/2]}.$$

Proof. Following from Theorem 3, any periodic map is periodic cobordant to a trivalent map. Therefore, trivalent maps generate $\Delta_{2+}^P(n)$ with the relations represented by trivalent graphs Γ_f.

Claim 1. $x_1,\cdots,x_{[(n-1)/2]}$ generate $\Delta_{2+}^P(n)$.

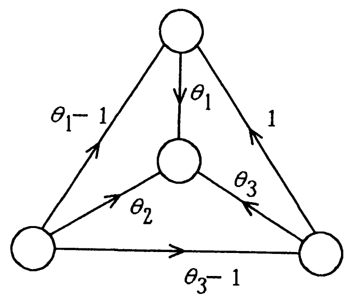


Fig. 15.

For any trivalent map $\{\theta_1,\theta_2,\theta_3;n\}$ (θ_1 is the least among θ_i's and $\theta_i\neq 1$), $\{\theta_1,\theta_2,\theta_3;n\}=\{\theta_1-1,\theta_2,\theta_3+1;n\}+\{1,\theta_3,n-\theta_3-1;n\}-\{1,\theta_1-1,n-\theta_1;n\}$ as elements of $\Delta^P_{2+}(n)$ (see Figure 15). By this formula, this claim is shown by induction on θ_1.

Claim 2. There is no relation among x_i's.

Let $\mathscr{F}^P_+(n)$ denote the set of oriented conjugacy classes of automorphisms (F,f), where f preserves the orientation of F and is periodic with the total period n. This set $\mathscr{F}^P_+(n)$ is the abelian group where the group law is induced by disjoint sum $\amalg$. Let the integer $v_c(f)$ be the number of points $x\in S_f$ such that $I_f(x)=c$. If the period n is an odd integer, we can define the homomorphism $\bar{\psi}$ from $\mathscr{F}^P_+(n)$ to $\boldsymbol{Z}^{[(n-1)/2]}$ by:

$$\bar{\psi}(F,f)=(v_a(f)-v_{n-a}(f))_{a=1,\cdots,[(n-1)/2]}.$$

Using Lemma 5, the homomorphism ψ from $\Delta^P_+(n)$ to $\boldsymbol{Z}^{[(n-1)/2]}$ is naturally induced from $\bar{\psi}$, and it is injective. Let ϕ be the natural surjective homomorphism from $\boldsymbol{Z}x_1\oplus\cdots\oplus\boldsymbol{Z}x_{[(n-1)/2]}$ to $\mathscr{F}^P_+(n)$. Then $\psi\circ\phi(x_1)=(2,-1,0,\cdots,0)$, $\psi\circ\phi(x_i)=(1,0,\cdots,0,\overset{(i)}{1},\overset{(i+1)}{-1},0,\cdots,0)$ $(i\neq 1,[(n-1)/2])$ and $\psi\circ\phi(x_{[(n-1)/2]})=(1,0,\cdots,0,2)$. If $\operatorname{Ker}\psi\circ\phi$ and $y=m_1x_1+m_2x_2+\cdots+m_{[(n-1)/2]}x_{[(n-1)/2]}$, then $\psi\circ\phi(y)=(2m_1+m_2+\cdots+m_{[(n-1)/2]}, m_2-m_1,m_3-m_2,\cdots,m_{[(n-1)/2]}-m_{[(n-1)/2]-1})=(0,\cdots,0)$. Therefore $y=0$ and $\psi\circ\phi$ is injective. So, ϕ is an isomorphism. If the period n is an even integer, we can define the homomorphism $\bar{\psi}$ from $\mathscr{F}^P_+(n)$to $\boldsymbol{Z}^{[(n-1)/2]}\oplus\boldsymbol{Z}_2$ by:

$$\bar{\psi}(F,f)=(v_a(f)-v_{n-a}(f))_{a=1,\cdots,[(n-1)/2]},\overline{v_{n/2}(f)}),$$

which induces the injective homomorphism ψ from $\Delta^P_+(n)$ to $\boldsymbol{Z}^{[(n-1)/2]}\oplus\boldsymbol{Z}_2$. Let ϕ be as above, then $\psi\circ\phi(x_1)=(2,-1,0,\cdots,0)$, $\psi\circ\phi(x_i)=(1,0,\cdots,0,\overset{(i)}{1},\overset{(i+1)}{-1},0,\cdots,0)$ $(i\neq 1,[(n-1)/2])$ and $\psi\circ\phi(x_{[(n-2)/2]})=(1,0,\cdots,0,1,1)$. We can see $\psi\circ\phi$ is injective as above. Therefore, ϕ is an isomorphism.

REMARK. The homomorphism ψ is originally given by Bonahon [B] in the proof of Proposition 8.3.

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