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ON MULTIPLY TRANSITIVE GROUPS

YUTAKA HIRAMINE

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1. Introduction

The known 4-fold transitive groups are $A_n$ $(n \geq 6)$, $S_n$ $(n \geq 4)$, $M_{11}$, $M_{12}$, $M_{23}$ and $M_{24}$. Let $G$ be one of these and assume $G$ is a $(4, \mu)$-group on $\Omega$ with $\mu \geq 4$. Here we say that $G$ is a $(k, \mu)$-group on $\Omega$ if $G$ is $k$-transitive on $\Omega$ and $\mu$ is the maximal number of fixed points of involutions in $G$. Let $t$ be an involution in $G$ with $|F(t)| = \mu$, then $G_F = G(F(t))$ is also a 4-fold transitive group. Here we set $F(t) = \{i \in \Omega | i^t = i\}$ and denote by $G(F(t))$, $G_F$, the global, pointwise stabilizer of $F(t)$ in $G$, respectively.

In this paper we shall prove the following

**Theorem 1.** Let $G$ be a 4-fold transitive group on $\Omega$. Assume that there exists an involution $t$ in $G$ satisfying the following conditions.

(i) $G$ is a $(4, \mu)$-group on $\Omega$ where $\mu = |F(t)|$.

(ii) $G_F$ is a known 4-fold transitive group; $A_n$ $(n \geq 6)$, $S_n$ $(n \geq 4)$ or $M_n$ $(n=11, 12, 23$ or $24)$.

Then $G$ is also one of the known 4-fold transitive groups.

This theorem is a generalization of the Theorem of T. Oyama of [10]: the case that $G_F = A_n$ $(n > 6)$, $S_n$ $(n > 4)$ or $M_{12}$ has been proved by T. Oyama and the case that $G_F = M_{23}$, $M_{24}$ by the author.

To consider the case that $G_F = M_{23}$ or $M_{24}$, we shall prove the following theorem in §3 and §4.

**Theorem 2.** Let $G$ be a $(1, 23)$-group on $\Omega$. If there exists an involution $t$ such that $|F(t)| = 23$ and $G_F = M_{23}$. Then we have

(i) If $P$ is a Sylow 2-subgroup of $G_{F(t)}$, then $P$ is cyclic of order 2 and $N_G(P) \cap g^{-1}Pg \leq P$ for any $g \in G$.

(ii) $|\Omega| = 69$ and $G$ is imprimitive on $\Omega$.

(iii) $O(G) = \{1\}$ and is an elementary abelian 3-group. If we denote by $\psi$ the set of $O(G)$-orbits on $\Omega$, then $|\psi| = 23$ and $G^\psi = M_{23}$.

It follows from this theorem that there is no $(3, 24)$-group such that for an involution $t$ fixing exactly twenty-four points $G_{F(t)} = M_{24}$. 
In the remainder of this section we introduce some notations: Let $G$ be a permutation group on $\Omega$. For $X \leq G$ and $\Delta \subseteq \Omega$, we define $F(X) = \{i \in \Omega | i^x = i \text{ for all } x \in X\}$, $X(\Delta) = \{x \in X | \Delta^x = \Delta\}$, $X_{\Delta} = \{x \in X | i^x = i \text{ for every } i \in \Delta\}$ and $X_{\Delta}^\Delta = X(\Delta)/X_{\Delta}$. If $p$ is a prime, we denote by $O^p(X)$, the subgroup of $X$ generated by all $p'$-elements in $X$ and by $O^p(X)$, the subgroup of $X$ generated by all $p$-elements in $X$. $I(X)$ is the set of involutions in $X$.

Other notations are standard (cf. [6], [13]).

2. Preliminaries

First we describe the various properties of $M_{23}$.

(i) $M_{23}$ is a 4-fold transitive group on twenty-three points $\{1, 2, \ldots, 23\}$ and a Sylow 2-subgroup of the stabilizer of four points in $M_{23}$ is of order $2^4$. It has a seven fixed points and acts regularly on the remaining points.

(ii) $M_{23}$ is a $(4, 7)$-group and has a unique conjugate class of involutions.

(iii) $M_{23}$ is a simple group and the outer automorphism group of it is trivial.

(iv) The centralizer of an involution $\omega$ in $M_{23}$ is a split extension of an elementary abelian normal subgroup $\mathcal{E}$ of order $2^4$ by a group $\mathcal{M}$ which is isomorphic to $GL(3, 2)$.

(v) The center of a Sylow 2-subgroup of $M_{23}$ is cyclic of order 2. Set $\mathcal{C} = C(\omega)$ and $F(\omega) = \Delta = \{1, 2, 3, 4, 5, 6, 7\}$. Then we have

(vi) $\mathcal{E}^{\Delta} = 1$ and $\mathcal{E}$ is regular on $\{8, 9, \ldots, 23\}$.

(vii) $\mathcal{M}$ is doubly transitive on $\Delta$.

(viii) $M_{23}^3 = A_4$ and $M_{23}(\Delta) = N(\mathcal{E})$.

(ix) $O(\bar{C}) = 1$, $O^2(\bar{C}) = \bar{C}$ and $O^7(\bar{C}) = \bar{C}$

We now prove the following lemmas.

Lemma 1. Let $P$ be a 2-group and $\phi$ an automorphism of $P$ of order 2. If $|C_P(\phi)| \leq 2^a$, then $|\Omega(\phi)/P'| \leq 2^b$.

Proof. Set $|\Omega(\phi)/P'| = 2^r$ and $Q/P' = \Omega(\phi)/P' \cap C(\phi)$. Then $|Q/P'| \geq 2^{1/2b}$ (cf. (2.7) of [8]). Since $[\phi, Q] \leq P'$, $(\langle \phi \rangle Q)' \leq P'$, whence $|\langle \phi \rangle Q : (\langle \phi \rangle Q)'| \geq 2^{1/2b+1}$. On the other hand $|C_P(\phi)| = |\langle \phi \rangle C_P(\phi)| \leq 2^{a+1}$ and so $|\langle \phi \rangle Q: (\langle \phi \rangle Q)'| \leq 2^{a+1}$ (cf. (2.8) of [8]). Thus $r \leq 2a$.

Lemma 2. Let $(G, \Omega)$ be a $(1, 23)$-group. Suppose there exists an involution $t$ such that $|F(t)| = 23$ and $G_F(t) = M_{23}$. If $P$ is a Sylow 2-subgroup of $G_{F(t)}$, then one of the following holds.

(i) $C_P(\phi)^{F(t)} = M_{23}$ and there is an involution $u$ in $N_G(P) - P$ satisfying $u^G \cap P = \phi$.

(ii) $N_G(P)^{F(t)} = M_{23}$ and $N_G(P) \cap g^{-1}P g \leq P$ for every $g \in G$.

Proof. Since $G(F(t)) = N_G(P)G_{F(t)}$, we have $N_G(P)^{F(t)} = M_{23}$. Suppose that
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N_G(P) ∩ g^{-1}Pg ≤ P for some g in G. Since F(P) + F(g^{-1}Pg), there is an involution u in g^{-1}Pg satisfying (i). As |F(u^F(P))| = 7 (cf. (ii) of §2) and |F(u)| = 23, |((Ω−F(P)) ∩ F(u))| = 16 and so |C_F(u)| ≤ 16 by the semi-regularity of P on Ω−F(P). By Lemma 1, |Ω_F(P)P'| ≤ 2^k. Since |GL(n, 2)| is not divisible by the prime 23 when 1 < w < 8, O_{23}^*(N_G(P)) is a normal subgroup of N_G(P) contained in C_G(P) by Theorem 5.1.4 and 5.2.4 of [6]. Thus we obtain C_G(P)^F(P) ∼ M_{23}.

According as the lemma, the proof of Theorem 2 is divided into two cases.

3. Case (i)

In this section, we prove that the case (i) does not occur.

(3.1) The following hold.

(i) P is cyclic of order 2 and so we can choose P such that P = 〈t〉.

(ii) N_G(P) = C_G(t) ∩ O^*(C_G(t)).

(iii) Set O^*(C_G(t)) = L(t). Then L(t)/O(L(t)) ∼ M_{23}, O(L(t))^F(t) = 1, t ∈ {g^2 | g ∈ G} and L(t) has a unique conjugate class of involutions.

(iv) Let s be an involution of L(t), then s is an involution of M_{23} and C_G(L(t)) < C_G(P), hence s is a central involution.

Proof. Since P is a Sylow 2-subgroup of N_G(P), Z(P) is a unique Sylow 2-subgroup of C_G(P). Consider C_G(P) = Z(P) ∪ 〈t〉 and C_G(L(t)) = Z(P) ∪ L(t) because O^∗(C_G(P)) = C_G(P). Let C_G(L(t)) = Z(P) ∪ L(t) be the normal series of C_G(P) such that C_G(L(t)) = Z(P) ∪ L(t) and L(t) is a normal subgroup of C_G(L(t)) < C_G(P).

Hence if u is an involution satisfying (i) of Lemma 2 there are an element v in I(P) ∪ {1} and w in I(L) with u = vω. Clearly C_i(w) = C_i(ω) ∩ C_i(w)/O(C_i(w)) where O(C_i(w)) = O(L) ∩ C_i(w) (cf. (ix) of §2). We denote O(C_L(w)) = H. Then C_L(w)/H is isomorphic to C_i(w)H = E/H ∙ M/H such that E/H = E^F(P) ∼ E_{16}, E^F(P) ∩ F(u) = 1, C_i(w)^F(P) ∩ F(w) = M^F(P) ∩ F(w) = M/H ∼ GL(3, 2), E is a normal subgroup of C_i(w) and E^F(P) ∩ M^F(P) = 1. By the fact that u is conjugate to some element of P, G^F(u) ∼ M_{23} and it follows that either y^F(u) = 1 or y^F(u) is an involution for y in I(E). If y^F(u) = 1, then F(y) ⊆ F(u). If y^F(u) is an involution, |F(y^F(u))| = 7 and so F(y) ∩ F(u) = F(u) ∩ F(P) because F(u) ∩ F(P) ⊆ F(y) ∩ F(u) and |F(u^F(P))| = |F(u^F(P))| = 7.

We argue F(y) ∩ F(P) = F(u) ∩ F(u) for any y in I(E). Suppose F(y) ⊆ F(u). Since |F(y)| ≤ 23, F(y) = F(u) and hence ⟨y, u⟩ is contained in a Sylow 2-subgroup of G^F(u) and so y^F ∩ P = ∅. Since G^F(y) = M_{23} |P, y⟩ = 1, F(P) ∩ F(y) = F(P) ∩ F(u) and P is semi-regular on Ω−F(P), we have P ∼ P^F(y) and P is an elementary abelian 2-group of order at most 16. Hence any element which is
conjugate to some element of \( P - \{1\} \) is not a square of any element in \( G \). But the element \( y \) in \( L \) is a square of some element in \( L \) because \( L/O(L) \cong M_{23} \) and (ii) of §2, which is a contradiction. This shows that \( F(u) \cap F(y) = F(u) \cap F(P) \) for any \( y \) in \( I(E) \).

Set \( \Delta = F(u) - F(P) = F(u) - F(y) \). Since \( |F(u) - F(P)| = |F(u) - (F(u) \cap F(P))| = 16 \) and a Sylow 2-subgroup \( T \) of \( E \) is isomorphic to \( E_{16} \), \( T \) acts regularly on \( \Delta \).

We argue \( |P| = 2 \). Suppose \( |P| > 4 \). Then \( |C_P(v)| > 4 \). Since \( C_P(v) \) is semi-regular on \( \Delta \) and \( [C_P(v), C_L(w)] = 1 \), we have \( O'(C_L(w))^a = 1 \). As \( E \triangleright O(C_L(w)), O(C_L(w))^3 = 1 \) and so by (ix) of §2, \( C_L(w)^3 = 1 \), a contradiction. Thus (i), (ii) and (iii) are proved.

Let \( s \) be an involution of \( L(t) \). Since \( t \) is not a square of any element of \( G \), \( t \) is not conjugate to \( s \) and \( u \) is of the form \( tw \) where \( w \) is an element in \( I(L(t)) \). On the other hand \( w \) is conjugate to \( s \) in \( L(t) \) by (iii) and so \( u \) is conjugate to \( ts \).

Hence \( t \) is conjugate to \( ts \). The four-group \( \langle t, s \rangle \) is the center of a Sylow 2-subgroup of \( C_G(t) \) by (v) of §2. Hence to complete the proof of (iv), we may assume \( t \) is not a central involution. Since \( \langle t, s \rangle \) contains a central involution and \( t \sim ts \), \( s \) must be a central involution. Thus (iv) is proved.

(3.2) Let notations be as in (3.1). Then

(i) If \( t_i \in \mathbb{C}, u_i \in I(G) \) and \( [t_i, u_i] = 1 \), then \( t_i = u_i \) or \( |F(t_i) \cap F(u_i)| = 7 \).

(ii) There exist an involution \( s \) in \( L(t) \) and a four-group \( \{u_i | 0 \leq i \leq 3\} \) of \( L(t) \) satisfying the following.

\[
\begin{align*}
u_0 = 1, \quad [s, u_i] = 1, \quad F(tu_i) \cap F(u_j) = F(t) \cap F(\langle u_i, u_j \rangle) & \text{ if } 0 \leq i, j \leq 3 \text{ and } j \neq 0.
\end{align*}
\]

Set \( F(t) \cap F(\langle u_i, u_j \rangle) = \Delta \). Then \( |\Delta| = 7 \) and \( |F(s) \cap \Delta| = 3 \).

Proof. By (ii) and (iii) of (3.1), (i) is obvious.

Let \( w, E \) and \( M \) be as in the proof of (3.1) and \( s \) an involution in \( M \). Let \( T \) be a Sylow 2-subgroup of \( E \) normalized by \( s \). Since \( T \) is isomorphic to \( E_{16} \), there is a subgroup \( \{1, u^2, 2, 3\} \) of \( T \) centralized by \( s \) (cf. Lemma 1). By (vi) of §2, \( |F(T) \cap F(t)| = 7 \) and \( T \) is regular on \( F(t) - F(T) \) and so \( |F(t) \cap F(\langle u_i, u_j \rangle)| = |\Delta| = 7 \). Since \( F(tu_i) \cap F(u_j) \) contains \( \Delta \), \( F(tu_i) \cap F(u_j) = \Delta \) follows from (i).

By (viii) of §2, \( |F(s) \cap (F(t) \cap F(T))| = 3 \), hence \( |F(s) \cap \Delta| = 3 \).

(3.3) Let \( s, \{u_0, u_1, u_2, u_3\} \) be as in (ii) of (3.2). For \( t_i \in \mathbb{C} \) and \( s_i \in I(L(t_i)) \), we set \( L(t_i) \cap C(s_i) = L(t_i, s_i) \). Then we have

(i) Set \( \Gamma_i = F(tu_i) \cap F(s) \) and \( N_i = L(tu_i, s) \) \((0 \leq i \leq 3)\), then \( |\Gamma_i| = 7, F(s) \supseteq \bigcup_{i=0}^{3} \Gamma_i \cap \Gamma_i \cap \bigcap_{i=0}^{3} \Gamma_i \cap (k \neq l), \bigcap_{i=0}^{3} \Gamma_i \cap (k \neq l) = 3 \text{ and } N_i/O(N_i) = N_i^f(tu_i) \cong \mathbb{C} \).

(ii) There exist subgroups \( E_i, M_i \) of \( N_i \) for each \( i \in \{0, 1, 2, 3\} \) such that \( N_i/O(N_i) = E_i/O(N_i), M_i/O(N_i) \supseteq E_i/O(N_i), E_i/O(N_i) = E_{16} \), \( M_i/O(N_i) \cong GL(3, 2), E_i \cong 1, N_i^f = M_i^f, E_i \cong GL(3, 2) \text{ and } M_i \) is doubly transitive.

Proof. By the choice of \( s \) and \( u_i \) \((0 \leq i \leq 3)\), (i) is clear. Since \( tu_i \) is con-
jugate to \( t \) for each \( i \), we can define \( E_i \) and \( M_i \) in exactly the same way as \( E \) and \( M \) mentioned in the proof of (3.1). From this, (ii) immediately follows.

(3.4) Let notations be as in (3.1), (3.2) and (3.3). Then

(i) There is a \( C_G(s) \)-orbit \( \Lambda \) on \( F(s) \) with \( F(s) \supseteq \Lambda \supseteq \bigcup_{i=0}^{3} \Gamma_i \).
(ii) \( |\Lambda| = 19, 21 \) or \( 23 \) and \( |F(s)| = 19, 21 \) or \( 23 \).
(iii) If \( k \in \Lambda \), then \( C_G(s)_k \) has an orbit on \( \Lambda - \{k\} \) of length at least 18.
(iv) If \( |\Lambda| = 19 \), then \( C_G(s)_\Lambda \approx A_{19} \) or \( S_{19} \).

Proof. Since \( N_i \leq C_G(s) \) and \( N_i^{s_i} \) is doubly transitive for \( i \) with \( 0 \leq i \leq 3 \), (i) follows immediately from (i) of (3.3). By assumption, \( |F(s)| \leq 23 \) and obviously \( |\bigcup_{i=0}^{3} \Gamma_i| = 19 \), hence \( 19 \leq |\Lambda| \leq 23 \). On the other hand \( \Lambda \supseteq \Gamma_0 = F(<t, s>) \), so \( |\Lambda| \) is odd. Thus (ii) holds. To prove (iii), we may assume \( k \in \bigcap_{i=0}^{3} \Gamma_i \). Since \( (N_i)_k \leq C_G(s)_k \) and \( (N_i)_k \) is transitive on \( \Gamma_i - \{k\} \), we have (iii).

Now suppose \( |\Lambda| = 19 \). Then \( C_G(s)_\Lambda \) is primitive and \( N_i^{s_i} \approx GL(3, 2) \). Hence \( C_G(s)_\Lambda \) possesses an element of order 7. By Theorem 13.10 of [13], \( C_G(s)_\Lambda \approx A_{19} \) holds and (3.4) is proved.

(3.5) Let notations be as in (3.1)—(3.4). There exists a Sylow 2-subgroup \( Q \) of \( G_{F(i)} \) such that \( s \in Z(Q) \) and \( t \in N_G(Q) \). Let \( \Gamma \) be the \( G_{F(i)} \)-orbit containing \( \Lambda \). Then

(i) \( F(Q) = F(s) \), \( G^{F(i)} = N_G(Q)^{F(i)} \) and \( |\Gamma| = 19, 21 \) or \( 23 \).
(ii) If \( k \in \Gamma \), then \( N_G(Q)_k \) has an orbit on \( \Gamma - \{k\} \) of length at least 18.
(iii) If \( |\Gamma| = 19 \), then \( N_G(Q)^\Gamma \approx A_{19} \) or \( S_{19} \).
(iv) If \( |\Gamma| = 21 \), then \( N_G(Q)^\Gamma \approx A_{21} \) or \( S_{21} \).
(v) If \( |\Gamma| = 23 \), then \( N_G(Q)^\Gamma \approx A_{23} \) or \( S_{23} \).

Proof. Let \( T \) be a Sylow 2-subgroup of \( C_G(s) \) containing \( t \). As \( s \) is a central involution by (iv) of (3.1) and \( C_G(s) \leq G(F(s)) \), \( T \) is a Sylow 2-subgroup of \( G(F(s)) \). Set \( Q = T \cap G_{F(i)} \). Then \( Q \) satisfies the condition of (3.5). Now we prove (i)—(v). (i), (ii) and (iii) follow immediately from (3.4).

To prove (iv), first we argue that \( N_G(Q)^\Gamma \) is primitive. If \( |\Lambda| = 19 \), \( C_G(s)_\Lambda \) possesses an element of order 19 by (iv) of (3.4), hence \( N_G(Q)^\Gamma \) is primitive. Therefore we may assume \( |\Lambda| = |\Gamma| = 21 \) and we argue that \( C_G(s)_\Lambda \) is primitive. Suppose \( C_G(s)_\Lambda \) is imprimitive. Let \( B_i \) be a nontrivial block of \( C_G(s)_\Lambda \), then by (iii) of (3.4) we have \( |B_i| = 3 \). Let \( \Pi = \{B_1, B_2, \ldots, B_j\} \) be a complete system of blocks. Since \( N_i \) is transitive on \( \Pi \) and \( [N_i, tu_i] = 1 \), \( tu_i \) fixes all blocks in \( \Pi \). Hence \( F(tu_i) \cap B_j \neq \phi \) for every \( l \) with \( 1 \leq l \leq 7 \). On the other hand \( |F(tu_i) \cap \Lambda| = 7 \), hence \( |F(tu_i) \cap B_1| = 1 \). From this \( (tu_i)_j^{s_i} = (u_i, u_j)^{s_i} = 1 \) for any \( i, j \in \{0, 1, 2, 3\} \). If \( F(Q) \neq \Lambda \), then \( |F(Q) - \Lambda| = 2 \) and so \( (tu_i, tu_j)^{s_i} = (u_i, u_j)^{s_i} = 1 \) where \( \Lambda_i = F(Q) - \Lambda \). Hence \( F(\langle u_i, u_j \rangle) = F(Q) = F(s) \), which is contrary to (ii) of (3.2). Thus \( N_G(Q)^\Gamma \) is primitive.
Next we shall show that we may assume $E_0^{F(0)} = 1$. Since $M_2^{F_2} = GL(3, 2)$ and $M_0^{F(0)}$ possesses an element of order 7. We may assume this element has no fixed point on $\Gamma$, for otherwise we obtain $N_0^G(0)^{\Gamma} \geq A_{21}$ by Theorem 13.10 of [13]. Hence an arbitrary $M_0^{F(0)}$-orbit on $\Gamma$ has length 7 or 14 and so $O(M_0)^{\Gamma} = 1$ holds because $M_0/O(M_0) = M_0^{F_0} = GL(3, 2)$. Hence $O(M_0)^{F(0)} = 1$. Set $\Gamma - F(t) = \Delta_0$. Then $\Delta_0 = \Gamma - \Gamma_0$ and $|\Delta_0| = 14$. Since the element of $M_0^{F(0)}$ of order 7 as above and the element $t$ have no fixed point on $\Delta_0$, $<t> \times N_0$ is transitive on $\Delta_0$. It follows from $N_0 > E_0$ that the orbits of $<t> \times E_0$ on $\Delta_0$ form a complete system of blocks of $<t> \times N_0$. We denote this $\Pi = \{B_1, \ldots, B_r\}$. Since $O(M_0) = O(N_0)$, $O(M_0)^{F(0)} = 1$ and $E_0/O(N_0) = E_{15}$, we have $<t> \times E_0$ is a 2-group on $\Delta_0$. Hence $|B_i| = 2$ and $r = 7$. By (i) of (3.2), $F(s) \cap F(tv) = F(s) \cap F(t)$ and so $\Delta_0 \cap F(tv) = \phi$. Hence $v^{\pi k} = t^k$ for each $B_i$ with $1 \leq k \leq 7$, which implies $E_0^{v^\pi} = 1$. If $F(Q) \not\equiv \Gamma$, then $|F(Q) - \Gamma| = 2$. Since $F(Q) - \Gamma \cap F(tv) = F(s) \cap F(tv) - \Gamma = \phi$ for every $v \in I(E_0)$, we get $v^{F(Q) - \Gamma = t^{F(Q) - \Gamma}} t^{F(Q) - \Gamma} = 1$. Thus $E_0^{F(Q)} = 1$.

We denote $L(t)^{F(0)} = L(t)$. Since $L(t)/O(L(t)) < G(F(t))$, we have $(L(t) \cap O(E_0))^{v_0} = A_7$ by (viii) of §2. Hence $L(t) \cap O(L(t))^{v_0} = A_7$ and so if $T$ is a Sylow 2-subgroup of $E_0$, we have $N_{L(t)}(T)^{v_0} = A_7$. We note that $F(T) = F(Q)$ because $E_0^{F(Q)} = 1$ and $L(t)$ has a unique conjugate class of involutions. So we have $N_{L(t)}(T) \subseteq G(F(Q))$. Let $y_0$ be a 5-element of $N_{L(t)}(T)$ such that the order of $y_0^{F_0}$ is 5. Since $y_0 \in G(F(Q)) \cap G(\Gamma_0)$, we get $y_0 \in G(\Gamma) \cap G(\Gamma_0)$. Therefore $|F(y_0)|^{\Gamma} \geq 6$. As $N_0^G(0)^{F(0)}$ is primitive, it follows from Theorem 13.10 of [13] that $N_0^G(0)^{F(0)} \geq A_{21}$. Thus (iv) is proved.

Finally we prove (v). If $|\Gamma| = 23$, $F(Q) = \Gamma$. Since $G^F \geq N_1^F$ and $N_1^F$ involves the group isomorphic to $GL(3, 2)$, $G^F$ is not solvable. Hence by the result of [11], we have $G^F = M_{23}, A_{23}$ or $S_{23}$. If $G^F = N_0^G(0)^{F(0)} = M_{23}$, we can apply (iii) of (3.1) to $s$ and obtain $s \notin \{g^{s} g \in G\}$, which is contrary to (iv) of (3.1). (Here we note that $I(L(t)) \subseteq v^G$ and hence (i) of Lemma 2 occurs with respect to $s$.)

(3.6) Let notations be as in (3.5). We set $N = C_0(0) \cap F(Q) = \Gamma$ and $N = C_0(0)^{F}$, where $\psi = F(Q) - \Gamma$ if $F(Q) = \Gamma$. Then $N^F \geq A_{1\Gamma}$.

Proof. Since $|\Gamma \cap F(t)| = 7$, by (i) of (3.2) $C_0(t)$ acts semi-regularly on $F(t) - \Gamma$ and so $|C_0(t)| \leq 16$. Hence $|\Omega t(0)/\Omega(Q)| \leq 2^8$ by Lemma 1. Since $GL(n, 2)$ is a 19'-group when $1 \leq n \leq 8$, $O^G(N_0(0)^F)$ is a normal subgroup of $N_0^G(0)$ contained in $C_0(0)$ by Theorem 5.1.4 and 5.2.4 of [6]. Hence $C_0(0)^{F} \geq A_{1\Gamma}$ by (iii), (iv) and (v) of (3.5), so that $N^F \geq A_{1\Gamma}$ because $|\psi| \leq 4$.

(3.7) We have now a contradiction in the following way.

Let notations be as in (3.1) — (3.6). Set $H = <t> \times N$. We denote $H^F = H$. Since $|F(0)| = 7$ and by (3.6) $N \geq A_{1\Gamma}$, there exists in $N$ an element $v$ such that the order of $v$ is 5, $[F(v)] = 1$ and $v^F = 1$. We may assume $v$ is a 5-element.
Cleary \( v \) normalizes \(< t > N_T \). Since \( Z(Q) \) is a unique Sylow 2-subgroup of \( N_T \), \(< t > Z(Q) \) is a Sylow 2-subgroup of \(< t > N_T \). By the Frattini argument there is a 5-element \( w \) in \( N \) such that \( \vartheta = \bar{w} \) and \( w \) normalizes \(< t > Z(Q) \). It follows from \( Z(Q) \leq Z(N) \) that \( w \) stabilizes a normal series \( < t > Z(Q) \triangleright Z(Q) \triangleright 1 \). By Theorem 5.3.2 of [6], \( w \) centralizes \(< t > Z(Q) \) and hence \( w \in L(t, s) \). Since \( F(t) \cap F(s) = F(t) \cap \Gamma, w^{F(t) \cap F(s)} = w^{F(t) \cap \Gamma} \neq 1 \). Hence \( L(t, s)^{F(t) \cap F(s)} \cong GL(3, 2) \) has a nontrivial 5-element, a contradiction.

4. Case (ii)

In this section we shall prove that if the case (ii) of Lemma 2 holds, then \((G, \Omega)\) is an imprimitive group of degree 69 and has properties listed in the conclusion of Theorem 2. From now on we assume the involution \( t \) is contained in \( P \) because \( P \) is an arbitrary Sylow 2-subgroup of \( G \).

4.1) \( O(G) \neq 1 \).

Proof. Let \((G, \Omega)\) be a minimal counterexample to (4.1). Since \( |G: N_G(P)| = 1 \), there is a Sylow 2-subgroup \( S \) of \( G \) such that \( S \triangleright P \). Set \( H = G(F(t)) \). If \( t \in H^S \) for some \( g \in G \), then \( t^g \in H \) and \( (t^g)^{-1} \in S \) for some \( h \in H \) because \( S \) is a Sylow 2-subgroup of \( H \). Since \( N_0(P) \cap P^g \leq P \), \( F(t^g) = F(P) = F(t) \), hence \( (t^g)^{-1} h \in H \), which implies \( g \in H \). Consequently \( t \in H^S \) if and only if \( g \in H \). If \( t_1(\neq t) \) is an involution in \( t^G \cap C(t) \), then as above \( t_1 \in H_{F(t)} \) and so \( t_1 \in I(H_{F(t)}) \). Hence \( (t_1 t)^S \in H \) if and only if \( g \in H \).

Thus we can apply Theorem 3.3 of [1] to \( t, H \) and \( G \). Set \( < t^G > = L \). Since \( O(G) = O_d(G) = 1 \), the 2-rank of any nontrivial characteristic subgroup of \( L \) is at least 2 by the Theorem of Brauer-Suzuki ([3]) and Theorem 7.6.1 of [6]. Hence \( H \cap L \) is strongly embedded in \( L \). By the Theorem of Bender ([2]), \( L^w \) is a simple group isomorphic to \( PSL(2, q), Sz(q) \) or \( PSU(3, q) \) for \( q = 2^n > 4 \). Here \( L^w \) is the last term of the derived series of \( L \). Set \( L^w = N \). We note that \( N \) is a normal subgroup of \( G \) and \( |N: N \cap H| \geq 5 \).

Since \( G^{F(t)} = N^{F(t)} \) and \( G^{F(t)} = M_{23} \), we have \( N^{F(t)} = M_{23} \) or 1. Suppose \( N^{F(t)} = M_{23} \). Since \( N \neq M_{23} \), we have \( N \leq G(F(t)) \). If \( \rho | N_{F(t)} | = \text{odd} \), \( G = < t > N \) and \( P = < t > \) by the minimality of \( G \). By the Glauberman's \( Z \)-theorem ([5]), \( G \triangleright < t > O(G) = < t > \), a contradiction. If \( \rho | N_{F(t)} | \) is even, by the minimality of \( G \), \( G = N \). Since \( N \) has a unique conjugate class of involution, \( I(N_0(P)) \leq I(P) \) by the assumption (ii) of Lemma 2. Hence \( S/P \) is an elementary abelian 2-group (cf. section 3 of [2]), which is contrary to \( N_0(P)^{F(P)} \neq M_{23} \).

Now we suppose \( N^{F(t)} = 1 \). Since \( N \cap P \neq 1 \) and \( NC_6(N) = N \times C_6(N) \), the assumption (ii) of Lemma 2 forces \( |C_6(N)|_{F(t)} \) is odd. Hence if \( |C_6(N)| \) is even, \( C_6(N)^{F(t)} = 1 \) and so \( C_6(N)^{F(t)} = M_{23} \) because \( M_{23} = G^{F(t)} \triangleright C_6(N)^{F(t)} \). Obviously \( C_6(N) \leq G(F((N \cap P)^t)) = G(F(t)^t) \) for any \( g \in G \). Therefore \( \{F(t)^t \mid g \in G \} \) forms a complete system of blocks of \( G \) on \( \Omega \) and an involution of \( C_6(N) \)
has exactly seven fixed points on each block. But \((G, \Omega)\) is a \((1, 23)\)-group and hence \(|\{F(t)^g \mid g \in G\}| = 3\), which implies \(|N: N \cap H| = 3\), a contradiction. Thus we have \(G(N) = 1\). From this \(G/N\) is isomorphic to a subgroup of outer automorphism group of \(N\). Hence \(G/N\) is solvable \(([12])\) and so \(G = N\). Thus \(N^F = M_{23}\), a contradiction.

4(2) \(P\) is cyclic or generalized quaternion.

Proof. Suppose that \(P\) contains a four-group \(Q\). Then \(O(G) = \langle C_{G(G)}(x) \rangle \) by Theorem 5.3.16 of \([6]\) and \(O(G) \leq G(F(P)) = G(F(t))\). Since \(O(G)^F = G^F = M_{23}, O(G)^F = 1\). Hence \(O(G) \leq G(F(t))\), so that \(O(G) = 1\), which is contrary to (4.1). Thus \(P\) is cyclic or generalized quaternion.

Let us note that the automorphism group of \(P\) is a \([2, 3]\)-group. Hence \(N^F = C_G(P)^F = M_{23}\). By the similar argument as in the first paragraph of the proof of (3.1), we have

\[(4.3) \quad C_G(P)^F = M_{23}, \quad C_G(P) = Z(P) \times O^2(C_G(P)). \]

Then \(L^F = L/O(L) = M_{23}\).

By the Feit-Thompson theorem \(([4])\), \(O(G)\) is solvable. Hence we have

\[(4.4) \quad N\text{-group of } P.\]

Let \(N\) be a minimal normal subgroup of \(G\) contained in \(O(G)\). Then \(N\) is an elementary abelian \(p\)-group for some odd prime \(p\).

\[(4.5) \quad |G| = 23.\]

(i) \(L\) normalizes \(K\) and \(K \unlhd G(F(t))\).

(ii) \(X = (t') N\) and \(\Gamma = G(t)\) where \(G(F(t))\). Then \(\Gamma \supseteq F(t), |\Gamma| > 23\) and \(|\Gamma|\) is odd.

Proof. Since \(M_{23}\) is regular, so that \(|K\cap C(t)| = 23\) and by the definition of \(K, |K^t \cap C(t^t)| = 1\), a contradiction. Thus \(r \neq 1\).

We consider the action of \(X\) on the set \(\Xi\). Since \(K_{\pi} = 1, [t, L] = 1\) and \(X\)
is transitive on $\Pi$, we have $t^n=1$ and $L$ is transitive on $\Pi$. Hence for $\Delta_i, \Delta_j \in \Pi$, there is an element $x \in L$ such that $(\Delta_i)^x = \Delta_j$. Then $|F(t) \cap \Delta_i| = |(F(t) \cap \Delta_i)^y| = |F(t) \cap \Delta_i|$, so that $|F(t)| = |\Delta_i \cap F(t)| \times r$ for any $\Delta_i \in \Pi$. Hence $|\Delta_i \cap F(t)| = 1$ and $r=23$. Since $F(O(L)) \supseteq F(t), O(L)^n = 1$ and $X_n = \langle x \rangle O(L) K$. Thus (i) holds.

Let $y \in I \langle t \rangle \times L$ and $y=t$. Then $y^n \neq 1$ and by (ii) of §2, $|F(y^n)| = 7$. Since $X_n = \langle t \rangle O(L) K, L^n \cap C(y^n) = (C_L(y))^n$. By (vii) of §2, $L^n \cap C(y^n)$ is transitive on $F(y^n)$. Therefore as above we obtain (ii).

Since $23 \geq |F(y) \cap \Gamma| = |F(y^n)| \times m(y) = 7 \times m(y)$, we have $m(y) \leq 3$. By (ii) of (4.5), $|\Gamma|$ is odd and so $m(y)$ is odd. Thus (iii) holds.

(4.7) Let $s \in I(L)$. Then the following hold.

(i) $m(s) = 3$ and $|F(s) \cap \Gamma| = 21$.

(ii) If $\Delta \in F(s^n)$, then $F(s) \supseteq \Delta$. Moreover $|\Delta| = 3$ and $N$ is an elementary abelian 3-group.

(iii) $F(s) \subseteq \Gamma$ and $|F(s)| = 21$.

Proof. Suppose $m(s) \neq 3$. Then by (iii) of (4.6) $m(s) = 1$. Since $K^s$ is regular for any $\Delta \in \Pi$, if $\Delta \in F(s^n)$, $s^\Delta$ inverts $K^s$. Hence $(ts)^s$ centralizes $K^s$ and so $F(ts) \supseteq \Delta$ and $m(ts) = |\Delta|$. Since $|\Delta| = 1$, by (iii) of (4.6) we have $|\Delta| = m(ts) = 3$. Therefore by (iii) of (4.6) $|F(ts) \cap \Gamma| = 21$. Since $L/O(L) \simeq M_{23}$, $s^\Delta$ is an even permutation. Furthermore $|F(s) \cap \Gamma| = 7$ because $m(s) = 1$. On the other hand $|\Gamma| = |\Delta| \times 23 = 69$ and $s^\Delta$ is an odd permutation, a contradiction. Thus (i) holds.

Since $|F(s) \cap \Gamma| = 21$ and $s^\Delta$ is an even permutation, $t^n$ is an odd permutation because $|F(t) \cap \Gamma| = 23$. Hence $(ts)^s$ is an odd permutation and so $m(ts) = 1$ and $(ts)^s$ inverts $K^s$ for $\Delta \in F(s^n)$ and $|(ts)^n| = 21$. Therefore $s^\Delta = (ts)^s$ centralizes $K^s$ and $F(s) \supseteq \Delta$, so that $m(s) = |\Delta| = 3$. Hence $K$ and $N$ are elementary abelian 3-groups, so (ii) holds.

Since $L^{F(t)} = L/O(L) \simeq M_{23}$, by (vi) of §2, there exists a four-group $\langle s_1, s_2 \rangle$ of $L$ such that $F(s_1) \cap F(t) = F(s_2) \cap F(t)$. Since $L$ has a unique conjugate class of involutions (cf. (ii) of §2), $m(s_1) = m(s_2) = m(s_1 s_2) = 3$. Hence $F(s_1) \cap \Gamma = F(s_2) \cap \Gamma = F(s_1 s_2) \cap \Gamma$ and $|F(s_1) \cap \Gamma| = 21$. To prove (iii) it will suffice to show that $|F(s_1)| = 21$. Assume $|F(s_1)| = 21$. Then $|F(s_1)| = 23$ and $|F(s_1) \cap (\Omega - \Gamma)| = 2$. Since $L/O(L) \simeq M_{23}$, we have $C_L(s_1)/O(C_L(s_1)) \simeq \overline{C}$ by the property of $M_{23}$. $C_L(s_1)$ acts on $F(s_1) \cap (\Omega - \Gamma)$ and $O(\overline{C}) = \mathbb{C}$ by (ix) of §2, hence $C_L(s_1)$ acts trivially on $F(s_1) \cap (\Omega - \Gamma)$. Therefore $F(s_1) = F(s_2) = F(s_1 s_2)$ and $|F(s_2)| = 23$. By Theorem 5.3.16 of [6], $N = \langle C_N(s) | 1 \neq s \in \langle s_1, s_2 \rangle \rangle$ and hence $N$ acts on $F(s_1)$. From this $3 | |F(s_1)|$, a contradiction. Thus (iii) holds.

(4.8) The following hold.

(i) $O(G)$ is an elementary abelian 3-group.

(ii) $G$ is imprimitive on $\Omega$ and the length of an $O(G)$-orbit is three. $|P| = 2$.  

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(iii) \(|\Omega|=69\). Let \(\psi\) be the set of \(O(G)\)-orbits on \(\Omega\). Then \(|\psi|=23\) and \(G^\psi=M_{23}\).

Proof. Since \(L^F=\langle L/O(L)\rangle=M_{23}\), there exist two subgroups \(\langle s_1, s_2\rangle, \langle s_3, s_4\rangle\) of \(L\) satisfying the following (cf. §2). \(\langle s_1, s_2\rangle \subset \langle s_3, s_4\rangle \cong E_6, F(s_1) \cap F(t) = F(s_2) \cap F(t) = F(s_3) \cap F(t) = F(s_4) \cap F(t), |(F(s_1) \cap F(t)) \cap (F(s_3) \cap F(t))| = 3\). By (ii) and (iii) of (4.7), we have \(\Gamma \not\supset F(s_1) = F(s_2) = F(s_3), |F(s_1)| = 21, |F(s_2)| = 21\) and \(|F(s_3)| = 21\) and \(|F(s_4)| = 9\).

On the other hand \(O(G) = \langle C_{O(G)}(t) | 1 \neq s \in \langle s_1, s_2\rangle = \langle C_{O(G)}(s) | 1 \neq s \in \langle s_3, s_4\rangle \rangle \) by Theorem 5.3.16 of [6]. Hence \(O(G)\) acts on \(F(s_1)\) and \(F(s_3)\), so that also on \(F(s_1) \cap F(s_3)\). Therefore the length of an \(O(G)\)-orbit is three because it is a common divisor of 9 and 21. From this \(O(G)\) is an elementary abelian 3-subgroup and by (4.2) \(P\) is cyclic of order 2. Thus (i) and (ii) hold.

Let \(\psi\) be the set of \(O(G)\)-orbits on \(\Omega\). Since \(\psi \supseteq \Pi, \Pi = F(t^\psi)\) and \(X^n = M_{23}\), we have \(G^n \geq M_{23}\). If \(G^n = M_{23}\), then \(G^n \geq A_{23}\) by the result of [11]. But if \(S\) is as in (4.1), the order of \(S/P\) is equal to that of a Sylow 2-subgroup of \(M_{23}\), a contradiction. Hence \(G^n = M_{23}\).

Now we suppose \(t^\psi \neq \Pi\). Then \(t^\psi = 1\) and \(G^\psi\) satisfies (ii) of Lemma 2. On the other hand \(O(G^\psi) = 1\), which is contrary to (4.1), so (iii) holds.

5. Proof of Theorem 1

The proof of Theorem 1 is obtained in the following way: By the Theorem of Oyama and his lemma of [10], it will suffice to consider the case that \(G^F(t)\) is isomorphic to \(M_{11}, M_{23}\) or \(M_{24}\). Since \(G\) is 4-fold transitive on \(\Omega, G^F(t) \neq M_{23}\) and \(M_{24}\) by Theorem 2. Hence we consider the case that \(G^F(t) = M_{11}\).

Suppose that \(G^F(t) = M_{11}\). Let \(P\) be a Sylow 2-subgroup of \(G^F(t)\) and \(S\) a Sylow 2-subgroup of a stabilizer of four points of \(\Omega\) in \(G\) such that \(S \geq P\). Then \(N_G(P) \leq G(F(P))\), hence \(N_G(P) F^F(t) = 1\) by the structure of \(M_{11}\), so \(F(N_G(P)) = F(t)\). Since \(P\) is a Sylow 2-subgroup of \(G^F(t), N_G(P) = P\), which forces \(S = P\), hence \(|F(S)| = 11\). By the Theorem of [9], \(G^o = M_{11}\), a contradiction.

References


Y. Hiramine: "On transitive groups in which the maximal number of fixed points of involutions is five," to appear.


