



Title	Radicals of group algebras
Author(s)	Tsushima, Yukio
Citation	Osaka Journal of Mathematics. 1967, 4(1), p. 179-182
Version Type	VoR
URL	https://doi.org/10.18910/7995
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

RADICALS OF GROUP ALGEBRAS

YUKIO TSUSHIMA

(Received December 9, 1966)

1. Introduction. Let k be a field of characteristic $p \neq 0$, G be a finite group whose order is divisible by p and H be its normal subgroup. By \mathfrak{R} and \mathfrak{R} we denote the radical of the group algebra kG and kH respectively. We know $\mathfrak{R} \subset \mathfrak{R}$ by the theorem of Clifford [1]. Hence $\mathfrak{L} = kG \cdot \mathfrak{R} = \mathfrak{R} \cdot kG$ is a two sided ideal of kG contained in \mathfrak{R} . We investigate in this note some properties between \mathfrak{R} and \mathfrak{L} , (especially when $[G:H] = p$) and also we show if G is p -solvable, $\mathfrak{R}^{p^n} = 0$, where p^n is the order of a p -Sylow subgroup of G . Throughout this note, we adhere to the above notation and the following conventions; modules are finitely generated left modules, $\otimes = \otimes_{kH}$, and for a positive integer e and a module M , eM means a direct sum of e copies of M . And finally, if M is a kG -module, M_H is the kH -module obtained by restricting the operators to kH .

The author is indebted to H. Nagao and M. Harada for their directions and for a generalization of his original result.

2. Lemma 1. *Let M be an irreducible kG -module. If kH -module N is a composition factor of M_H , then M is a composition factor of $N^G = kG \otimes N$.*

Proof. $\text{Hom}_{kG}(N^G, M) \cong \text{Hom}_{kH}(N, M)$. The right hand side is not 0, since N is a direct summand of M_H . So there is a kG -epimorphism $N^G \rightarrow M$, which shows our assertion.

Here we recall the theorem of Clifford [1].

Let N be any kH -module. A conjugate of N means $g \otimes N (\subset kG \otimes N)$, considered naturally as kH -module, where $g \in G$. The inertia group of N , denoted by $H^*(N)$, means $H^*(N) = \{g \in G \mid g \otimes N \cong N \text{ as } kH\text{-modules}\} \supset H$.

Let M be an irreducible kG -module and N be any irreducible kH -submodule of M_H . Then we have $M_H = e(N_1 \oplus N_2 \oplus \cdots \oplus N_r)$, where the N_i 's are non isomorphic conjugates of $N_1 = N$, $r = [G:H^*(N)]$, and e is a positive integer.

Lemma 2. *We use the above notation. If $H^*(N) = H$, then we have $N^G \cong M$, equivalently, if the inertia group of an irreducible kH -module N is H itself, then N^G is also irreducible.*

Proof. $r = [G:H]$ by the assumption. From lemma 1 $\dim N^G \geq \dim M$.

On the other hand, $\dim N^G = [G:H] \dim N$ and $\dim M = e \dim N = e[G:H] \dim N$. Therefore, we have $\dim N^G = \dim M$, that is $N^G \cong M$ and $e=1$.

Proposition 1. *If $[G:H]$ is prime to p , then $\mathfrak{S} = \mathfrak{R}$.*

Proof. It is well known that in this case kG is just a semisimple extension of kH . In other words, any kG -module is (kG, kH) -projective in the sense of Hochschild [5]. And so kG/\mathfrak{R} is also a semisimple extension of kH/\mathfrak{R} by [6]. However, kH/\mathfrak{R} is a semisimple algebra in an usual sense, so is kG/\mathfrak{R} . Therefore, $\mathfrak{S} = \mathfrak{R}$.

3. In the section, we assume k is a splitting field for kG and $[G:H] = p$. Hence for any kH -module N , its inertia group is H or G .

Lemma 3. *Let N be any irreducible kH -module. Then N^G is either irreducible or its composition factors are all isomorphic to each other, and the number of them is equal to p . More precisely, the former case holds if $H^*(N) = H$, and the latter holds if $H^*(N) = G$.*

Proof. Anyway, there exists an irreducible kG -module M such that N is a composition factor of M_H . If $H^*(N) = H$, then we have $N^G \cong M$ by lemma 2. If $H^*(N) = G$, then $M = eN$ (since $r=1$). Suppose M appears a times as a composition factor of N^G , then $a \neq 0$ from lemma 1.

We have $\dim N^G \geq a \dim M \geq ae \dim N$, that is $p \dim N \geq ae \dim N$. On the other hand, the group character of N^G , as is easily to be shown, is 0. However the distinct irreducible characters of G are linearly independent over k , since k is a splitting field for G . Hence we have $p | a$. Combining with the above inequality, we have $p \geq ae \geq p$, that is $a=p, e=1$ and $\dim N^G = p \dim NM$. This completes the proof.

REMARK. From the proof, we know for any irreducible kG -module M, M_H is either irreducible or its decomposed into a direct sum of non isomorphic irreducible kH -modules.

Now let $\{U_1 \cdots U_s, V_1 \cdots V_t\}$ be the full set of non isomorphic irreducible kH -modules in which we assume $H^*(U_i) = H$, and $H^*(V_j) = G$. Then we have $kH/\mathfrak{R} = \bigoplus \sum f_i U_i \oplus \sum h_j V_j$ and $f_i = \dim U_i, h_j = \dim V_j$. We put $kG/\mathfrak{R} = A$. Clearly $A \cong kG \otimes kH/\mathfrak{R}$ as kG -modules. Hence $A \cong f_1 U_1^G \oplus f_2 U_2^G \oplus \cdots f_s U_s^G \oplus h_1 V_1^G \oplus h_2 V_2^G \oplus \cdots h_t V_t^G$.

Proposition 2. *V_i^G is either indecomposable or completely reducible as an A -module.*

Proof. Since V_i^G is A -projective, we can decompose $V_i^G = Ae_1 \oplus Ae_2 \oplus \cdots Ae_n$, where $\{e_i\}$ are primitive orthogonal idempotents of A . From lemma 3, V_i^G has p number of the composition factors which are isomorphic to each other.

Especially we have $Ae_i \cong Ae_j$ for all i, j . So if Ae_i is irreducible, then we have $k=p$, and V_i^G is completely reducible. If this is not the case, each Ae_i has the same number of composition factors greater than one. Since p is a prime number, we have $k=1$. This completes our proof.

For a brevity of notations, we put $f_1U_1^G \oplus f_2U_2^G \oplus \dots \oplus f_sU_s^G = C_0$, $h_jV_j^G = C_j$, and $A \cong C_0 \oplus C_1 \oplus \dots \oplus C_t$. We identify each C_i with its isomorphic image in A .

Theorem 1.

- (1) C_0 is a semisimple algebra and each C_i is a block of A ($i \geq 1$).
- (2) A is a quasi-Frobenius algebra over k .
- (3) The composition factors of $\mathfrak{R}/\mathfrak{I}$ are those irreducible kG -modules which are also irreducible as kH -modules. Conversely any irreducible kG -module, say M , which is also irreducible as kH -module appears as composition factor of $\mathfrak{R}/\mathfrak{I}$ with multiplicity $(p-1) \dim M$.

Proof.

- (1) We know from lemma 3 and the remark, for $i \neq j$, C_i and C_j have no composition factor in common. Hence clearly C_i is a block of A for $i \geq 1$ and C_0 is a semisimple algebra.
- (2) For $i \geq 1$, C_i has only one irreducible module and C_0 is a semisimple algebra. hence our assertion is clear from the definition.
- (3) Since $\mathfrak{R}/\mathfrak{I}$ is the radical of A , it is contained in $C_1 \oplus C_2 \oplus \dots \oplus C_t$. So the first assertion is clear. Let M be an irreducible kG -module which is irreducible as kH -module. Then $M_H \cong V_i$ for some i . We have $\dim M = \dim V_i = h_i$. M appears $ph_i = p \dim M$ times as a composition factor of A . On the other hand, M appears $\dim M$ times in kG/\mathfrak{R} , since k is a splitting field for G . Hence M appears $(p-1) \dim M$ times between \mathfrak{I} and \mathfrak{R} .

Lemma 4. $\mathfrak{R}^p \subset \mathfrak{I}$

Proof. Let e be any primitive idempotent of A . Then by lemma 3 and Theorem 1(1), Ae has at most p number of composition factors. Hence we have $(\mathfrak{R}/\mathfrak{I})^p e = 0$ and since e is arbitrary, $(\mathfrak{R}/\mathfrak{I})^p = 0$, that is $\mathfrak{R}^p \subset \mathfrak{I}$.

4. Theorem 2. If G is p -solvable, then $\mathfrak{R}^{p^n} = 0$.

Proof. We may assume k is a splitting field for G . If G is a p -group of order p^n , our assertion is clear, since in this case $\dim \mathfrak{R}^n = p^n - 1$ by [2]. (or [3] p. 189) Generally, there exists a normal subgroup of G whose index is p or prime to p . Using proposition 1 and lemma 4, it is easy to prove the theorem by induction on the order of G .

REMARK. It will be necessary to remark that $\dim \mathfrak{R} \geq p^n - 1$ in general. We may also assume k is a splitting field for G , since in the group algebra the

radical is preserved by the extension of the coefficient field¹⁾. Then there exists a primitive idempotent e of kG such that $(kG)e/\mathfrak{N}e \cong k$. Hence $\dim \mathfrak{N} \geq \dim \mathfrak{N}e \geq \dim (kG)e - 1 \geq p^n - 1$.

OSAKA CITY UNIVERSITY

References

- [1] A.H. Clifford: *Representations induced in an invariant subgroup*, Ann. of Math. **38** (1937), 533–550.
- [2] S.A. Jennings: *The structure of the group ring of a p -group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185.
- [3] C.W. Curtis and I. Reiner: *Representation theory of finite groups and associative algebras*, Wiley, New York, 1962.
- [4] H. Cartan and S. Eilenberg: *Homological algebra*, Princeton, 1956.
- [5] G. Hochschild: *Relative homological algebra*, Trans. Amer. Math. Soc. **82** (1956), 246–269.
- [6] A. Hattori: *Semisimple algebra over a commutative ring*, J. Math. Soc. Japan **15** (1963), 404–419.

1) This is true in general if the structure constants are in a perfect field contained in the coefficient field.