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On Quotient Rings

By Yuzo UTUMI

An extension ring S of a ring T is called a left quotient ring of T if for any two elements $x \neq 0$ and y of S there exists an element a of T such that $ax \neq 0$ and ay belongs to T . Let R be a ring without total right zero divisors. Then R has always a unique maximal left quotient ring, and moreover the maximal left quotient ring of a total matrix ring of finite degree over R is a total matrix ring of the same degree over the maximal left quotient ring of R .

A left ideal I of R is called an M -ideal if it contains every element x for which there exists a left ideal m of R satisfying the condition that (1) $mx \subseteq I$ and (2) R is a left quotient ring of m . When S is a left quotient ring of R , M -ideals of R and those of S correspond one-one in a definite way. A left ideal I of R is said to be complemented if there exists a left ideal I' such that I is a maximal one among left ideals which have zero intersection with I' . Every complemented left ideal is an M -ideal, but the converse is not true in general. In a ring without total right zero divisors, every M -ideal is complemented if and only if the ring has the zero left singular ideal. Another example of M -ideals is the annihilator left ideals. A sufficient condition for that every M -ideal of a ring with zero left singular ideal is an annihilator left ideal, is that the maximal left quotient ring coincides with the maximal right quotient ring.

Every semisimple I -ring has zero singular ideals and hence it has the left and the right maximal quotient rings. We discuss especially two types of semisimple I -rings, i.e., primitive rings with nonzero socle, and semisimple weakly reducible rings. Let P be a primitive ring with nonzero socle. Then the maximal left quotient ring of P is right completely primitive. Thus, the left and the right maximal quotient rings of P coincide if and only if P satisfies the minimum condition. Let W be a semisimple weakly reducible ring. The left and the right maximal quotient rings of W always coincide and is also semisimple weakly reducible. In particular, if W is plain then its maximal quotient ring is strongly regular. This implies that the (nilpotency) index of a total matrix ring of degree m over a semisimple I -ring of index n is mn .

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1. For any subset A of a ring S and any family B of right operators of S the set of all the elements in S satisfying $xB \subseteq A$ is denoted by $(A/B)^S$. In particular, when B consists of the right multiplications of all elements in a subset C of S we write it as $(A/C)^S$.

(1.1) Let R be a subring of a ring S . We say that S is a (*left*) *quotient ring* of R if for any pair of elements $x \neq 0$ and y in S there exists an element a in R such that $ay \in R$ and $ax \neq 0$. Notation: $S \geq R$.

We may also define a similar concept by a slightly weaker condition: We write $S(\geq)R$ if any nonzero $x \in S$ there is an element $a \in R$ such that $0 \neq ax \in R$. Of course, $S \geq R$ implies $S(\geq)R$. But the following example shows that the converse is false. Let K be a field and S the ring $K[x]/(x^4)$. We denote the subring of S generated by $\bar{1}$, \bar{x}^2 and \bar{x}^3 as R . Then $S(\geq)R$, while no $\bar{a} \in R$ satisfies $\bar{a}\bar{x} \in R$ and $\bar{a}\bar{x}^3 \neq 0$ simultaneously.

Our main object is the quotient ring in the sense of (1.1).

(1.2) *Let $S \geq R$. The only homomorphism of S into itself which leaves R invariant is the identity mapping.*

If $x\theta \neq x$ for some $x \in S$, there would exist an element $a \in R$ such that $ax \in R$ and $a(x\theta - x) \neq 0$. But then $a(x\theta) = (a\theta)(x\theta) = (ax)\theta = ax$. This contradiction shows that $x\theta = x$ for every $x \in S$.

(1.3) *Let $S \geq R$. An element x belongs to the center of S if it is commutative with every element in R .*

Assume $xy \neq yx$. Then $ay \in R$ and $a(xy - yx) \neq 0$ for some $a \in R$. $axy = xay = ayx$. This is a contradiction.

(1.4) *Let $S \geq R$. For any finite number of elements $x_1 \neq 0, x_2, \dots, x_n$ in S there exists an element $a \in R$ such that $ax_1, ax_2, \dots, ax_n \in R$ and $ax_1 \neq 0$.*

The assertion is evidently true if $n=1$. Let $n > 1$. We assume that $bx_1, bx_2, \dots, bx_{n-1} \in R$, $bx_1 \neq 0$ for some $b \in R$. Since $S \geq R$ there is $c \in R$ such that $cbx_n \in R$ and $cbx_1 \neq 0$. Therefore $cb, cbx_1, \dots, cbx_n \in R$ and $cbx_1 \neq 0$.

(1.5) *Let $S \geq R \geq T$. Then $S \geq R \geq T$ if and only if $S \geq T$.*

The "if" part is clear from the definition. To prove the "only if" part let $S \ni x (\neq 0), y$. Then $ax, ay \in R$ and $ax \neq 0$ for some $a \in R$. Hence $ca, cay \in T$ and $cax \neq 0$ for some $c \in T$. This implies $S \geq T$.

We denote by S^Δ the set of all left ideals I satisfying $S \geq I$.

(1.6) Let I be a left ideal of S . Then $I \in S^\Delta$ if (and only if) for any elements $x \neq 0$ and y in S there exists an element a in S such that $ay \in I$ and $ax \neq 0$.

In fact, it follows from the assumption that there is moreover an element $b \in S$ such that $ba \in I$ and $bax \neq 0$. Since $(ba)y = b(ay) \in I$ we have $S^\Delta \ni I$.

(1.7) If $S \geq R$ and $m \in S^\Delta$, then $S \geq R \cap m$, Rm .

Let $S \ni x (\neq 0)$, y . Then $ax \neq 0$, $ay \in m$ for some $a \in m$. Hence ba , $bay \in R$ and $bax \neq 0$ for some $b \in R$. We see that ba , $bay \in (R \cap m) \cap Rm$. Therefore $S \geq R \cap m$ and $S \geq Rm$.

(1.8) Let $S \geq R$ and let $m_x \in S^\Delta$ be preassigned to each $x \in R$. Then $\Sigma_{x \in R} m_x x \in S^\Delta$.

Let $S \ni x (\neq 0)$, y . Then $ay \in R$ and $ax \neq 0$ for some $a \in R$. We set $m = m_a \cap m_{ay}$. By (1.7), $m \in S^\Delta$. Hence $bax \neq 0$ for some $b \in m$ and then $ba \in m_a$, $bay \in m_{ay}$.

(1.9) Let $S^\Delta \ni R$, T . If θ is an S -left homomorphism of R into S then $(T/\theta)^R \in S^\Delta$.

Let $S \ni x (\neq 0)$, y . Then $ay \in R$, $ax \neq 0$ for some $a \in R$. Moreover, $b(a\theta)$, $b((ay)\theta) \in T$ and $bax \neq 0$ for some $b \in T$. Hence ba , $bay \in (T/\theta)^R$. Thus $(T/\theta)^R \in S^\Delta$.

This proof shows also the following

(1.10) Let $S \geq R$. If θ is an R -left homomorphism of R into S , then $(R/\theta)^R \in R^\Delta$.

(1.11) Let \bar{S} be a ring. The following conditions are equivalent:

- (1) There exists a ring T such that $S \geq T$ or $T \geq S$.
- (2) $S \geq S$.
- (3) S has no total right zero divisors, that is, $Sx = 0$ implies $x = 0$.

This is evident from the definition and (1.5).

By virtue of the above lemmas the R. E. Johnson's method¹⁾ for constructing the extended centralizer is verbatim applicable to our case.

Construction of \bar{S} . Let S be a ring such that $S \geq S$. Then S^Δ is non-void. We denote by \mathfrak{A}_S the set of all S -left homomorphisms each of which is defined on a left ideal in S^Δ and has values in S . The definition domain of $\theta \in \mathfrak{A}_S$ is denoted as M_θ . When $M_\theta = M_{\theta'}$, we define the addition by $x(\theta + \theta') = x\theta + x\theta'$. When $M_\theta \theta \subseteq M_{\theta'}$, the multiplication is defined by $x(\theta\theta') = (x\theta)\theta'$. For $\theta, \theta' \in \mathfrak{A}_S$ if there exists $I \in S^\Delta$ such that $I \subseteq M_\theta \cap M_{\theta'}$ and θ, θ' coincide on I , we say that θ and θ' are

1) See [8].

equivalent. Then this relation is reflexive, symmetric and transitive. We denote the equivalence class containing θ as $\bar{\theta}$ and the set of all the classes as \bar{S} . By (1.7), (1.9) it is easy to see that \bar{S} forms a ring in a natural way. For any $x \in S$ the right multiplication x_r belongs to \mathfrak{A}_S . We identify x with \bar{x}_r and regard \bar{S} as an extension ring of S .

(1.12) *If $x \in M_\theta$, then $x\theta = x\bar{\theta}$.*

This follows easily from that $y(x\theta) = (yx)\theta$ for every $y \in S$.

(1.13) $\bar{S} \geq S$.

In fact, let $\bar{\theta}, \bar{\varphi} \in \bar{S}$ and $\bar{\varphi} \neq 0$. By (1.7), $M_\theta \cap M_\varphi \in S^\Delta$. Hence $a\varphi \neq 0$ for some $a \in M_\theta \cap M_\varphi$. Then $a\bar{\theta} = a\theta \in S$ and $a\bar{\varphi} = a\varphi \neq 0$ by (1.12). This implies $\bar{S} \geq S$.

Theorem 1. *Let $T \geq S$. Then T is isomorphic, over S , to \bar{S} if and only if T satisfies either the following condition (1) or (2).*

In this case, we say that T is the (left) maximal quotient ring of S .

CONDITION (1). *For any $\theta \in \mathfrak{A}_S$ there are $x \in T$ and $m \in S^\Delta$ such that $m \subseteq M_\theta$ and $y\theta = yx$ for every $y \in m$.*

CONDITION (2). *If $R \geq S$, then there exists an isomorphism, over S , of R into T .*

Proof. To see the “only if” part it is sufficient to prove that \bar{S} satisfies these conditions. (1) is evident from (1.12). Let $R \geq S$. By (1.10), $(S/x)^S \in S^\Delta$ for every $x \in R$. Hence the right multiplication θ_x of x on $(S/x)^S$ belongs to \mathfrak{A}_S . Associating each $x \in R$ with $\bar{\theta}_x \in \bar{S}$ we obtain an isomorphism, over S , of R into \bar{S} . Therefore \bar{S} satisfies (2). If R satisfies the condition (1), this isomorphism is onto. This proves the first half of the “if” part of Theorem. Finally, let T satisfy (2). Then, since $\bar{S} \geq S$ by (1.13), \bar{S} is isomorphic, over S , into T . On the other hand, since $T \geq S$ and \bar{S} satisfies the condition (2), T is isomorphic, over S , into \bar{S} . Then product of these isomorphisms is the identity mapping of \bar{S} by (1.2). It follows from this that \bar{S} and T are isomorphic over S . This completes the proof.

The following (1.14)–(1.17) are easily proved by Theorem 1 and we omit their proofs.

(1.14) *If $T \geq S$, then $\bar{T} \simeq \bar{S}$ over S .*

(1.15) $\bar{\bar{S}} = \bar{S}$.

(1.16) *If $T \geq S$ and $T = \bar{T}$, then $T \simeq \bar{S}$ over S .*

(1.17) Every automorphism of S can be extended uniquely to that of \bar{S} .

2. (2.1) Let $\{S_\alpha\}$ be a family of rings with the property $S_\alpha \geq S_\alpha$ for every α . Then $\Sigma_\oplus^\circ \bar{S}_\alpha$ is the maximal quotient ring of $\Sigma_\oplus S_\alpha$, where Σ_\oplus° denotes the complete direct sum, while Σ_\oplus the (restricted) direct sum.

(1) First we note that if $\Sigma_\oplus^\circ T_\alpha \geq R$ and $T_\alpha \geq T_\alpha$ then $T_\alpha \geq R \cap T_\alpha$. In fact, let $T_\alpha \ni x (\neq 0)$, y . By the assumption, $bx \neq 0$ for some $b \in T_\alpha$. Hence $ab, aby \in R$ and $abx \neq 0$ for some $a \in R$. Then $ab, aby \in R \cap T_\alpha$. Therefore $T_\alpha \geq R \cap T_\alpha$. (2) Let $T_\alpha \geq R_\alpha$ for every α . Then it is easy to see that $\Sigma_\oplus^\circ T_\alpha \geq \Sigma_\oplus R_\alpha$. (3) We set $P = \Sigma_\oplus^\circ \bar{S}_\alpha$. Let $\theta \in \mathfrak{A}_P$ and denote its restriction to $M_\theta \cap \bar{S}_\alpha$ as $\bar{\theta}_\alpha$. Then $\bar{\theta}_\alpha \in \mathfrak{A}_{\bar{S}_\alpha}$ since $M_\theta \cap \bar{S}_\alpha \in \bar{S}_\alpha^\Delta$ by (1). By Theorem 1 there is $x_\alpha \in \bar{S}_\alpha$ such that $yx_\alpha = y\theta_\alpha$ for every $y \in M_\theta \cap \bar{S}_\alpha$. Hence $y \Sigma_\oplus^\circ x_\alpha = y\theta$ for every $y \in \Sigma(M_\theta \cap \bar{S}_\alpha)$. By (2), $\Sigma(M_\theta \cap \bar{S}_\alpha) \in P^\Delta$. Therefore it follows from Theorem 1 that $P = \bar{P}$ because of $P \geq \bar{P}$. By (2), $P \geq \Sigma S_\alpha$. Thus we see that $P \sim \Sigma \bar{S}_\alpha$ over ΣS_α by (1.16).

As a corollary of (2.1),

(2.2) If $\bar{S} = \alpha \oplus \alpha'$ where α and α' are two-sided ideals of \bar{S} , then α is the maximal quotient ring of $\alpha \cap S$.

From (1) of the proof of (2.1) we get $\alpha \geq \alpha \cap S$. Now $S = \bar{S} = \bar{\alpha} \oplus \bar{\alpha}'$. Hence $\alpha = \bar{\alpha}$. Owing to (1.16) this implies $\alpha \sim \overline{\alpha \cap S}$ over $\alpha \cap S$.

We use the notation R_n for the total matrix ring of degree n over a ring R .

(2.3) If $S \geq S$, then $(\bar{S})_n$ is the maximal quotient ring of S_n .

First, we assume that S has a unit element. (1) $S \geq T$ implies $S_n \geq T_n$. In fact, let $S_n \ni A_k = \Sigma a_{ij}^{(k)} e_{ij}$ for $k=0, 1$ and let $a_{pq}^{(0)} \neq 0$. Then there is $a \in T$ such that $aa_{pq}^{(0)} \neq 0$ and $aa_{pq}^{(1)} \in T$ ($\mu=1, \dots, m$). Hence $ae_{pp}, ae_{pp}A_1 \in T_n$ and $ae_{pp}A_0 \neq 0$. This shows $S_n \geq T_n$. (2) If $(S_n)^\Delta \ni R$, then $m_n \leq R$ for some $m \in S^\Delta$. In fact, we denote by m_k the set of all the elements of S each of which is a coefficient of a matrix in $R \cap S_n e_{kk}$. This is evidently a left ideal of S . Let $S \ni x (\neq 0)$, y . By the assumption there is a matrix $A = \Sigma a_{ij} e_{ij} \in R$ such that $Ae_{kk}, A(ye_{kk}) \in R$ and $A(xe_{kk}) \neq 0$. Hence $a_{ik}x \neq 0$ for some i . Since $a_{ik}, a_{ik}y \in m_k$, this implies $m_k \in S^\Delta$. By (1.7), $m = \bigcap m_k \in S^\Delta$. For any element $y \in m$ there exists a matrix $D \in R \cap S_n e_{kk}$ whose $(1, k)$ -coefficient is y . $ye_{jk} = e_{ji}D \in R$. Therefore $m_n \leq R$. (3) Let $\theta \in \mathfrak{A}_{S_n}$. By (2), $m_n \leq M$ for some $m \in S^\Delta$. For any $x \in m$ we denote $(xe_{1k})\theta = e_{11}(xe_{1k})\theta$ as $\Sigma_j (x\theta_{kj})e_{1j}$. Then θ_{kj} are S -left homomorphisms of m into S so that they belong to \mathfrak{A}_S . Hence there are $a_{kj} \in \bar{S}$ such that $x\theta_{kj} = xa_{kj}$ for every $x \in m$. Therefore, for every $\Sigma x_{ik}e_{ik} \in m_n$, $(\Sigma x_{ik}e_{ik})\theta = \Sigma_{ik} e_{i1}(x_{ik}e_{1k})\theta = \Sigma_{ik} e_{i1}(x_{ik}\theta_{kj})e_{1j} = \Sigma_{ij} (\Sigma_k x_{ik}a_{kj})e_{ij} = (\Sigma x_{ik}e_{ik})(\Sigma a_{ik}e_{ik})$. This shows that $(\bar{S})_n \sim (\bar{S}_n)$ over S_n .

since $(\bar{S})_n \geq S_n$ by (1). For S without unit element we denote by S' the subring of \bar{S} generated by S and the unit element of \bar{S} . Then $S \simeq S'$ over S by (1.14). Moreover, $(\bar{S}_n') \simeq (\bar{S}_n)$ over S_n since $S_n' \geq S_n$ by (1). From these facts it follows that $(\bar{S})_n \simeq (\bar{S}_n)$ over S_n as required.

3. In this section we shall consider some correspondence between ideals of a ring and those of its quotient ring.

Let $R \leq S$ and I be an R -left submodule of S . We denote by $\Delta_R^S I$ the set of all elements $x \in S$ satisfying $(I/x)^R \in R^\Delta$.

(3.1) $\Delta_R^S I$ is a left ideal of S containing I .

For, let $\Delta_R^S I \ni x$ and $S \ni y$. Since $(I/x)^R \in R^\Delta$ we see that $((I/x)^R/y)^R \in R^\Delta$ by (1.10). Now $((I/x)^R/y)^R yx \leq (I/x)^R x \leq I$. Hence $(I/yx)^R \in R^\Delta$, or $yx \in \Delta_R^S I$.

(3.2) $\Delta_R^S (I \cap I') = \Delta_R^S I \cap \Delta_R^S I'$.

This is easy to verify by (1.5), (1.7).

(3.3) $\Delta_R^S (I \cap R) \supseteq I$.

If $x \in I$, then $(R/x)^R \in R^\Delta$ by (1.10). This means $x \in \Delta_R^S (I \cap R)$ since $(R/x)^R = (R \cap I/x)^R$.

(3.4) $\Delta_R^S (Ix) \supseteq (\Delta_R^S I)x$ for every $x \in S$.

(3.5) Let $R \leq S \leq T$ and I be an S -left submodule of T . Then $\Delta_S^T I = \Delta_R^T I$.

(3.6) Let $R \leq S \leq T$. If I is an R -left submodule of S then $\Delta_S^T (\Delta_R^S I) = \Delta_R^T I$.

Since $\Delta_R^S I$ is a left ideal of S containing I , we see that $\Delta_S^T (\Delta_R^S I) = \Delta_R^T (\Delta_R^S I) \supseteq \Delta_R^T I$ by (3.2), (3.5). On the other hand, let $x \in \Delta_S^T (\Delta_R^S I)$ or $(\Delta_R^S I/x)^R \in R^\Delta$. If $y \in (\Delta_R^S I/x)^R$, then $yx \in \Delta_R^S I$; hence $(I/yx)^R \in R^\Delta$. It follows from (1.8) that $\Sigma (I/yx)^R y \in R^\Delta$, where Σ denotes the sum for all $y \in (\Delta_R^S I/x)^R$. Since $(\Sigma (I/yx)^R y)x \leq I$, this implies that $\Delta_R^T I \ni x$. Therefore $\Delta_S^T (\Delta_R^S I) \subseteq \Delta_R^T I$ and the equality holds.

Let $R \leq R$. A left ideal I is called a (left) M -ideal if $\Delta_R^R I = I$.

(3.7) The intersection of any collection of M -ideals in a ring is also an M -ideal.

Let I_α be M -ideals. By (3.2), $\Delta_R^R (\bigcap I_\alpha) \subseteq \Delta_R^R I_\alpha = I_\alpha$. Hence $\bigcap I_\alpha \subseteq \Delta_R^R (\bigcap I_\alpha)$ by (3.1). Thus $\Delta_R^R (\bigcap I_\alpha) = \bigcap I_\alpha$.

(3.8) Let $R \leq S$. Then $\Delta_R^S I$ is an M -ideal of S for every R -left submodule I of S .

In fact, $\Delta_S^S(\Delta_R^S \mathfrak{I}) = \Delta_R^S \mathfrak{I}$ by (3.6).

Theorem 2. *Let $R \leq S$. The Mappings $\mathfrak{I} \rightarrow \Delta_R^S \mathfrak{I}$ and $\mathfrak{J} \rightarrow \mathfrak{J} \cap R$ are mutually reciprocal and give a 1-1 correspondence between M -ideals \mathfrak{I} of R and \mathfrak{J} of S .*

Proof. If \mathfrak{I} is an M -ideal of R then $\Delta_R^S \mathfrak{I}$ is an M -ideal of S by (3.8). Clearly $\Delta_R^S \mathfrak{I} \cap R = \Delta_R^R \mathfrak{I} = \mathfrak{I}$ by the definition. On the other hand, if \mathfrak{J} is an M -ideal of S , then $\mathfrak{J} = \Delta_S^S \mathfrak{J} = \Delta_R^S \mathfrak{J} \supseteq \Delta_R^S (\mathfrak{J} \cap R) \supseteq \mathfrak{J}$ according to (3.5), (3.2) and (3.3). Hence $\mathfrak{J} = \Delta_R^S (\mathfrak{J} \cap R)$. Moreover, $\mathfrak{J} \cap R$ is an M -ideal of R since $\Delta_R^R (\mathfrak{J} \cap R) = \Delta_R^S (\mathfrak{J} \cap R) \cap R = \mathfrak{J} \cap R$.

(3.9) *Let $R \leq S$. If \mathfrak{I} is an M -ideal of R , then $\Delta_R^S \mathfrak{I}$ is the maximal left ideal of S of which intersection with S is \mathfrak{I} .*

From (3.3) we see that $\Delta_R^S \mathfrak{I} = \Delta_R^S (\mathfrak{J} \cap R) \supseteq \mathfrak{J}$ if $\mathfrak{J} \cap R = \mathfrak{I}$.

In the following we make mention of two special types of M -ideals, i. e., the left annihilator ideals and the complemented left ideals.

By $l_R(A)$ ($r_R(A)$), we mean the left (right) annihilator ideal of A in R .

(3.10) *If $R \leq R$, then every left annihilator ideal in R is an M -ideal.*

By (3.4), $(\Delta_R^R l(x))x \subseteq \Delta_R^R (l(x)x) = 0$ for every $x \in R$. Since $\Delta_R^R l(x) \supseteq l(x)$, we have $\Delta_R^R l(x) = l(x)$. According to (3.7), every left annihilator ideal is an M -ideal.

(3.11) *Let $R \leq S$. If \mathfrak{I} is a left annihilator ideal in R , then $\Delta_R^S \mathfrak{I}$ is also a left annihilator ideal in S .*

We assume $\mathfrak{I} = l_R(A)$. Then $l_S(A)$ is an M -ideal in S . Hence $l_S(A) \Delta_R^S (l_S(A) \cap R) = \Delta_R^S l_R(A) = \Delta_R^S \mathfrak{I}$ by Theorem 2.

We may define a *right quotient ring* in an obvious way.

(3.12) *Let S be a left and right quotient ring of R . If \mathfrak{J} is a left annihilator ideal in S , then $\mathfrak{J} \cap R$ is also a left annihilator ideal in R .*

Let $x \in r_R(\mathfrak{J} \cap R)$. Then $0 = \Delta_R^S ((\mathfrak{J} \cap R)x) \supseteq (\Delta_R^S (\mathfrak{J} \cap R))x \supseteq \mathfrak{J}x$ by (3.4), (3.3). Hence $x \in r_R(\mathfrak{J})$. Therefore $r_R(\mathfrak{J} \cap R) = r_R(\mathfrak{J})$. Similarly we see that $l_R(r_S(\mathfrak{J}) \cap R) = l_R(r_S(\mathfrak{J}))$. Thus $l_R(r_R(\mathfrak{J} \cap R)) = l_R(r_R(\mathfrak{J})) = l_R(r_R(\mathfrak{J}) \cap R) = l_R(r_S(\mathfrak{J})) = l_S(r_S(\mathfrak{J})) \cap R = \mathfrak{J} \cap R$ and our assertion is proved.

For given left ideal \mathfrak{I} of R a left ideal of R is called a *complement* of \mathfrak{I} if it is the maximal one among the left ideals having the zero intersections with \mathfrak{I} . We denote it by \mathfrak{I}^c . Of course, \mathfrak{I}^c is not uniquely determined by \mathfrak{I} . A left ideal which is a complement of some left ideal is called a *complemented left ideal*. We use the notation \mathfrak{I}^{cc} for $(\mathfrak{I}^c)^c$ containing \mathfrak{I} .

(3.13) *Let $R \leq R$. Any complemented left ideal of R is an M -ideal.*

In fact, $\Delta_R^R(I^c) = I^c$ since $\Delta_R^R(I^c) \cap I \subseteq \Delta_R^R(I^c) \cap \Delta_R^R I = \Delta_R^R(I^c \cap I) = 0$ and $I^c \subseteq \Delta_R^R(I^c)$.

(3.14) *Let $R \leq S$. If I is a complemented left ideal in R , then $\Delta_R^S I$ is also a complemented left ideal in S .*

We may assume that $I = I^c$. Clearly $\Delta_R^S I \cap \Delta_R^S(I^c) = \Delta_R^S(I \cap I^c) = 0$. On the other hand, if \mathcal{I}' is a left ideal of S such that $\mathcal{I}' \supset \Delta_R^S I$, then $\mathcal{I}' \cap R \supset \Delta_R^S I \cap R = I$ by (3.9) since I is an M -ideal in R by (3.13). Thus $\mathcal{I}' \cap \Delta_R^S(I^c) \supseteq (\mathcal{I}' \cap R) \cap I^c \neq 0$ since I^c is also an M -ideal in R . Therefore we have $\Delta_R^S I = (\Delta_R^S(I^c))^c$.

(3.15) *Let $R \leq S$. If \mathcal{I} is a complemented left ideal in S , then $\mathcal{I} \cap R$ is also a complemented left ideal in R .*

We assume that $\mathcal{I} = \mathcal{I}^c$. Let I' be a left ideal of R such that $\mathcal{I} \cap R \subseteq I'$ and $I' \cap (\mathcal{I}^c \cap R) = 0$. Then $\Delta_R^S I' \cap \mathcal{I}^c = \Delta_R^S I' \cap \Delta_R^S(\mathcal{I}^c \cap R) = 0$ and $\Delta_R^S I' \supseteq \Delta_R^S(\mathcal{I} \cap R) = \mathcal{I}$. Hence $\Delta_R^S I' = \mathcal{I}$. Thus $\mathcal{I} \cap R \supseteq I'$ by (3.1). Therefore $\mathcal{I} \cap R = I'$ and $\mathcal{I} \cap R = (\mathcal{I}^c \cap R)^c$.

4. In this section we discuss from our point of view the cose considered by R. E. Johnson [8].

A ring R is called a (left) C -ring if $R \leq R$ and every M -ideal of R is a complemented left ideal.

From (3.13), (3.14), (3.15) and Theorem 2 we obtain immediately the following proposition.

(4.1) *Let $R \leq S$. R is a C -ring if and only if S is a C -ring.*

We denote by R^Δ the set of all left ideals of R each of which has a nonzero intersection with every nonzero left ideal.

(4.2) *Let S be an extension ring of R . If every nonzero R -left submodule has a nonzero intersection with R , then $(R/x)^\Delta \in R^\Delta$ for every $x \in S$.*

Let I be a nonzero left ideal of R . If $l_R(x) \cap I \neq 0$, then evidently $(R/x)^R \cap I \neq 0$. And if $l_R(x) \cap I = 0$ we see that $Ix \neq 0$ and hence $Ix \cap R \neq 0$. This implies $(R/x)^R \cap I \neq 0$ again. Therefore $(R/x)^R \in R^\Delta$.

(4.3) *Let I be a left ideal of a ring R . If $x \in I^c$, then $(I/x)^R \in R^\Delta$.*

To see this let I' be any nonzero left ideal of R . First we assume that $(I^c + I'x) \cap I = 0$. Then $I'x \subseteq I^c \cap I^c = 0$. Hence $(I/x)^R \cap I' \neq 0$. Next let $(I^c + I'x) \cap I \ni z \neq 0$ and $z = a + b$, $a \in I^c$, $b \in I'x$. Then $a = z - b \in I^c \cap (I + I'x) \subseteq I^c \cap I^c = 0$. Thus $0 \neq z = b \in I \cap I'x$ so that $(I/x)^R \cap I' \neq 0$. Therefore we see that $(I/x)^R \in R^\Delta$.

Theorem 3. *If $R \leq R$, the following conditions are equivalent:*

(1) *R is a C -ring.*

(2) If $I \in R^\Delta$ and $Ix=0$, then $x=0$.

(3) $R^\Delta = R^\Delta$.

In this case, I^{cc} is uniquely determined for every left ideal I , and is in fact the smallest M -ideal $\Delta_R^R I$ containing I .

Proof. (1) \Rightarrow (2): If $x \neq 0$, then $l_R(x)$ is an M -ideal by (3.10), hence it is a complemented left ideal. Clearly $l_R(x) \neq R$. Hence $l_R(x) \notin R^\Delta$. (2) \Rightarrow (3): It follows immediately from the definition that $R^\Delta \subseteq R^\Delta$. Let $R^\Delta \ni I$ and let m be a nonzero I -left submodule of R . Then Im is a nonzero left ideal by the assumption. Hence $I \cap m \supseteq I \cap Im \neq 0$, which shows that the assumption of (4.2) is satisfied by R and I . Thus $(I/x)^I \in I^\Delta$ for every $x \in R$. It follows easily from this that $(I/x)^I \in R^\Delta$. If $0 \neq y \in R$, then $(I/x)^I y \neq 0$. This shows that there exists $a \in I$ such that $ay \neq 0$, $ax \in I$. Therefore $I \in R^\Delta$ and hence $R^\Delta \subseteq R^\Delta$. Thus $R^\Delta = R^\Delta$. (3) \Rightarrow (1): Let I be a left ideal of R and let $x \in I^{cc}$. By (4.3) we see that $(I/x)^R \in R^\Delta = R^\Delta$ or $x \in \Delta_R^R I$. This implies $I^{cc} \subseteq \Delta_R^R I$. Since I^{cc} is an M -ideal by (3.13), $\Delta_R^R I \subseteq \Delta_R^R I^{cc} = I^{cc} \subseteq \Delta_R^R I$ and whence $I^{cc} = \Delta_R^R I$. In particular, if I itself is an M -ideal, then $I^{cc} = I$ and I is a complemented left ideal. Therefore R is a C -ring as required.

Here we note that (1) the assumption $R \leq R$ follows directly from the condition (2), and (2) means that R is a ring with zero singular ideal by the terminology of R. E. Johnson [8].

(4.4) Let R be a C -ring. Then $S(\geq) R$ if and only if $S \geq R$.

The "if" part is trivial. To see the "only if" part let $S \ni x (\neq 0)$, y . Then $0 \neq ax \in R$ for some $a \in R$. By (4.2), $(R/ay)^R \in R^\Delta$. Since R is a C -ring, $(R/ay)^R ax \neq 0$ by (2) of Theorem 3. It follows from this that there is $c \in R$ such that $ca, cay \in R$ and $cax \neq 0$. Therefore $S \geq R$.

A unitary left module over a ring with a unit element is *injective* if it is a direct summand of every unitary extension module.²⁾ A necessary and sufficient condition for a unitary left R -module M to be injective is that any R -left homomorphism defined on a left ideal of R and having the values in M is obtained by the right multiplication of some element of M .³⁾ When a ring R is injective as an R -left module, we call it a *(left) injective ring*.

(4.5) (See R. E. Johnson [8]) If I is a left ideal of R , then $I + I^c \in R^\Delta$.

(4.6) Let R be a C -ring with a unit element. If I is an M -ideal of \overline{R} , then the R -left module I is injective.

2) See [2] Proposition 3.4.

3) See [2] Theorem 3.2.

Let I' be a left ideal of R and θ an R -left homomorphism of I' into I . We extend θ to an R -left homomorphism of $I' + I'^c$ into I by making it vanish on I'^c . Then the extended θ belongs to \mathfrak{A}_R since $I + I^c \in R^\Delta = R^\Delta$. By (1.12) there is $a \in R$ such that $x\theta = xa$ for every $x \in I + I^c$. From $(I + I^c)a \subseteq I$ we see that $a \in I$ since I is an M -ideal.

Theorem 4. *If R is a C -ring the following conditions are equivalent :*

- (1) $\bar{R} = R$.
- (2) R is an injective ring.
- (3) R is a regular ring⁴⁾ with unit element and has the property that if a family $\{x_\alpha + e_\alpha R\}$ of cosets of principal right ideals has the finite intersection property then the total intersection is non-void.

Proof. (1) \Rightarrow (2) is a special case of (4.6). (2) \Rightarrow (3) The regularity of R is a result of R. E. Johnson.⁵⁾ This is easily shown by (4.5) and Theorem 3. Next, we assume that a family $\{x_\alpha + e_\alpha R\}$ has the finite intersection property. We set $\alpha = \sum R(1 - e_\alpha)$ and consider the correspondence $\theta : \sum u_{\alpha_i}(1 - e_{\alpha_i}) (\in \alpha) \rightarrow \sum u_{\alpha_i}(1 - e_{\alpha_i}) x_{\alpha_i} = \sum u_{\alpha_i}(1 - e_{\alpha_i}) A_{\alpha_i}$. If $\sum u_{\alpha_i}(1 - e_{\alpha_i}) = 0$, then $\sum u_{\alpha_i}(1 - e_{\alpha_i}) A_{\alpha_i} = \sum u_{\alpha_i}(1 - e_{\alpha_i}) x = 0$ where x is an element in $\bigcap A_{\alpha_i}$. It is easy to see that θ is an R -left homomorphism. By (2) there is an element u such that $z\theta = zu$ for every $z \in \alpha$. Since $(1 - e_\alpha)x_\alpha = (1 - e_\alpha)u$ we know that $u \in x_\alpha + e_\alpha R$ or $u \in \bigcap A_\alpha$. (3) \Rightarrow (1) Let α be a left ideal of R and θ an R -left homomorphism of α into R . We set $A_\alpha = e_\alpha \theta + (1 - e_\alpha)R$ for every idempotent $e_\alpha \in \alpha$. For each finite subfamily $\{A_{\alpha_i}\}$ of the family $\{A_\alpha\}$ there exists an idempotent e_β such that $\sum R_{\alpha_i} = Re_\beta$. $e_\beta \theta - e_{\alpha_i} \theta = (1 - e_{\alpha_i})e_\beta \theta \in (1 - e_{\alpha_i})R$ and hence $e_\beta \theta \in A_{\alpha_i}$ for every $A_{\alpha_i} \in \{A_{\alpha_i}\}$. Thus $\{A_\alpha\}$ has the finite intersection property. Therefore there is $x \in \bigcap A_\alpha$ by our assumption. $e_\alpha \theta \in x + (1 - e_\alpha)R$ and $e_\alpha \theta = e_\alpha x$. From $\alpha = \sum Re_\alpha$ we see that $y\theta = yx$ for any $y \in \alpha$. This implies $R = \bar{R}$ by Theorem 1.

The following (4.7)–(4.9) are corollaries of this Theorem.

(4.7) *Let R be a C -ring such that $R = \bar{R}$. Then a left ideal of R is a complemented left ideal if and only if it is a principal left ideal.*

The “only if” part is evident by (4.6) Since R is regular, every principal left ideal is a direct summand and hence it is a complemented left ideal.

(4.8) *If R is a C -ring, then the set of all complemented left ideals*

4) See [13].

5) See [8] Theorem 2.

of R forms a complete complemented modular lattice.⁶⁾

In fact, by Theorem 2 and (4.7) the set of complemented left ideals of R forms a lattice isomorphic to that of principal left ideals of a regular ring with unit element. The completeness follows from (3.7).

In an obvious way, we may also define the notions of *right C-ring* and *right maximal quotient ring*.

(4.9) *Let R be a left and right C-ring and the left maximal quotient ring \bar{R} be simultaneously the right maximal quotient ring.⁷⁾ Then a left ideal of R is a complemented left ideal if and only if it is a left annihilator ideal. The set of all left annihilator ideals and the set of all right annihilator ideals form the mutually dual isomorphic lattices.*

This follows easily from Theorem 2, (3.10)–(3.15) and (4.7).

An example of C-rings. Levitzki [10] called a ring to be a *semisimple I-ring* if every nonzero right ideal contains a nonzero idempotent. It is well known that this concept is right-left symmetric.

(4.10) *Every semisimple I-ring is a C-ring.*

Let $x \in R$ and $l_R(x) \in R^\perp$. If e is an idempotent in xR , then $0 = l_R(x)e \supseteq l_R(x) \cap Re$. Hence $Re = 0$ and $e = 0$. This shows $x = 0$.

5. The left maximal quotient ring \bar{R} of a ring R is not always the right maximal quotient ring even if R is a both right and left C-ring. In the following we shall show this by treating a primitive with nonzero socle.

Let R be a primitive ring with nonzero socle and eR be its minimal right ideal. Then R may be regarded as a dense ring of linear transformations of the eRe -left module eR . We denote by L the ring of all linear transformations of eR .

(5.1) *L is the left maximal quotient ring of R .*

Indeed, since eR is a faithful R -right module, we see easily that $eR \leq R$. Hence \bar{eR} is the (left) maximal quotient ring of R by (1.14). In eR every eRe -left submodule is a left ideal. Since eRe is a division ring we see that eR is completely reducible for left ideals. Hence $(eR)^\wedge$ consists of eR alone. Thus eR satisfies the condition (2) of Theorem 3 and this implies that eR is a C-ring. Therefore $(eR)^\wedge = (eR)^\Delta$. It follows

6) This lattice is meet-homomorphic to that of all left ideals of R by (3.2) and Theorem 3. See [14].

7) On account of (1.5) and Theorem 1, this second assumption is, of course, equivalent to the condition that every left quotient ring of R is a right quotient ring of R and vice versa.

from this that \overline{eR} is the ring of all endomorphisms of the eR - (or eRe -) left module eR and hence equal to L .

As an immediate corollary of (5.1) we obtain the following

(5.2) *Let R be a primitive ring with nonzero socle. Then R is also the right maximal quotient ring if and only if R is a simple ring with minimum condition.*

Next, we regard the minimal right ideal eR of R as a topological vector space over eRe of which open base is the set of left annihilators $l_{eR}(x)$ for all x in the socle $S(R)$ of R .⁸⁾ Then the right multiplication of any element in R is a continuous linear transformation of the space eR . We denote by \tilde{R} the ring of all continuous transformations of eR . Then \tilde{R} is also a primitive ring with nonzero socle and has the property that the socle of $\tilde{S}=S(R)\subseteq R\subseteq\tilde{R}\subseteq L$. This shows the part (3) of the following proposition.

(5.3) (1) *\tilde{R} is the greatest one among the right quotient ring of R which is a subring of L .*

(2) *$(S(R)/S(R))^L=\tilde{R}$. In other words, \tilde{R} is the left idealizer of $S(R)$ in L .*

(3) *\tilde{R} is the greatest subring of L such that its intersection with the socle of L is $S(R)$.*

In fact, if $(S(R)/S(R))^L\ni x\neq 0$, then $0\neq xS(R)\subseteq S(R)$. Since $S(R)$ is a C -ring, it follows from this by (4.3) that $(S(R)/S(R))^L$ is a right quotient ring of $S(R)$. Clearly $R\subseteq(S(R)/S(R))^L$. Hence $(S(R)/S(R))^L$ is a right quotient ring of R . Now let A be any right quotient ring of R contained in L . Then A is, of course, that of $S(R)$. The right ideal of $S(R)$, which has $S(R)$ as its right quotient ring, is $S(R)$ itself alone since $S(R)$ is completely reducible for right ideals. Hence $A\subseteq(S(R)/S(R))^L$ by (1.10). Therefore $(S(R)/S(R))^L$ is the greatest right quotient ring of R contained in L . Next, let $x\in(S(R)/S(R))^L$ and $y\in S(R)$. Then $l_{eR}(xy)xy=0$; hence $l_{eR}(xy)x\in l_{eR}(y)$. This shows that $x\in\tilde{R}$. Thus $(S(R)/S(R))^L\subseteq\tilde{R}$. The converse inclusion is evident since $S(R)$ is the socle of \tilde{R} . This completes the proof.

6. First we prepare a certain number of terms we need. If the nilpotency indices of nilpotent elements in a ring is bounded, the ring is called to be *of bounded index* and its least upper bound is called the *index* of the ring.⁹⁾ A (semisimple) I -ring is said to be *plain* if it is of

8) See [3], [7]. This topology is the *weak* topology.

9) See [6].

index 1.¹⁰⁾ It is well known that every idempotent in a ring of index 1 is central.¹¹⁾ Thus,

(6.1) *A ring is plain if and only if every nonzero right ideal of R contains a nonzero central idempotent.*

The “only if” part follows immediately from the definition. If a ring R satisfies the condition, then R is evidently a semisimple I -ring. Let $0 \neq x \in R$. Then there is a nonzero central idempotent $e = xy$. Now $x^n y^n e = x^{n-1} e y^{n-1} e = x^{n-1} y^{n-1} e = \dots = x y e = e \neq 0$. This shows $x^n \neq 0$ and that R is plain.

If a two-sided ideal of a ring R is the total matrix ring, of finite degree, over a plain ring with unit element, then it is called a *matrix ideal* of R .¹²⁾ Of course, the unit element of any matrix ideal is central in R and hence every matrix ideal is a direct summand of R . A ring is called *semisimple weakly reducible* if every nonzero two-sided ideal contains a nonzero matrix ideal.¹³⁾ Levitzki [12] has proved the following facts:

(1) Every semisimple weakly reducible ring is a semisimple I -ring [12, Theorem 3.1];

(2) Every semisimple I -ring of bounded index is semisimple weakly reducible [12, Theorem 3.3];

(3) Every semisimple I -ring, of which each primitive image is a simple ring with minimum condition, is semisimple weakly reducible [12, Theorem 3.4]. We note that this assumption is satisfied by every semisimple I -ring with a polynomial identity.¹⁴⁾

To investigate the maximal quotient ring of a semisimple weakly reducible ring it seems pertinent to re-construct it by a special manner.

A family B of central idempotents in a ring R is called a B -family if the following conditions are satisfied:

(B1) Let f be a central idempotent in R . If $ef = f$ for some $e \in B$, then $f \in B$.

(B2) For every nonzero central idempotent f in R there exists a nonzero idempotent e in B such that $ef = f$.

We say that a mapping θ of a B -family B into the ring R is an H -mapping if θ satisfies the condition (H) that if $e, f \in B$ and $ef = f$ then $(e\theta)f = f\theta$.

The totality of H -mappings defined on a B -family B forms a ring H_B by the operations $e(\theta + \varphi) = e\theta + e\varphi$ and $e(\theta\varphi) = (e\theta)(e\varphi)$. It is easy to

10) See [12].

11) See [4], Lemma 1.

12), 13) See [12].

14) See [10] and [11].

see that the intersection of any pair of B -families is also a B -family. Now we say that two H -mapping are equivalent if their restrictions to some B -family coincide. Then this relation is reflexive, symmetric and transitive, and the set of equivalence classes forms evidently a ring R° . We note that for every $x \in R$ and every B -family B the mapping $x_B: e \rightarrow ex$ ($e \in B$) belongs to H_B .

(6.2) *Let R be a semisimple weakly reducible ring. Identifying each $x \in R$ with the class $\bar{x}_B \in R^\circ$ containing x_B we can regard R° as an extension ring of R . Then $R^\circ \simeq R$ over R .*

(1) Let B be a B -family. If $x \in R$ is nonzero, then $Bx \neq 0$. In fact, we assume $\bigcap_{e \in B} (1-e)R \neq 0$. Then $\bigcap (1-e)R$ would contain a nonzero matrix ideal and hence a nonzero central idempotent. By (B2) some nonzero $g \in B$ would be contained in $\bigcap (1-e)R$. Then $gR \subseteq \bigcap (1-e)R \subseteq (1-g)R$ and $g=0$, which is a contradiction. This shows that $\bigcap (1-e)R = 0$. If $x \neq 0$, then $x \notin (1-e)R$ or $ex \neq 0$ for some $e \in B$.

From (1) it is easy to see that the identification in (6.2) is allowable.

(2) Let $\mathfrak{m} \in R^\Delta$. Then the set $B_{\mathfrak{m}}$ of central idempotents contained in \mathfrak{m} forms a B -family. In fact, $B_{\mathfrak{m}}$ satisfies evidently (B1). Let e be a nonzero central idempotent. The Re contains a nonzero matrix ideal T_n over a plain ring T . Since T_n is a direct summand of R it follows from (1) of the proof of (2.1) that $T_n \cap \mathfrak{m} \in T_n^\Delta$. By (2) of the proof of (2.3) there is $\mathfrak{m}' \in T^\Delta$ such that $\mathfrak{m}' \subseteq T_n \cap \mathfrak{m}$. By (6.1) \mathfrak{m}' contains a nonzero central idempotent f . By (1.3) f is central in T and hence in R . Now $f \in \mathfrak{m}' \subseteq T_n \cap \mathfrak{m} \subseteq Re \cap \mathfrak{m}$. This implies $fe = f$ and $f \in \mathfrak{m}$. Therefore $B_{\mathfrak{m}}$ satisfies (B2) and it is a B -family.

(3) Let $e \in B$ and $\theta \in H_B$. Then $e\theta = e\bar{\theta}$ where $\bar{\theta}$ is the class $\in R^\circ$ containing θ . In fact, if $e, f \in B$, then $e(e\theta) = e\theta$ and $(fe)(e\theta) = (fe)\theta$ by (H). Hence $f(e\theta) = (fe)(e\theta) = (fe)\theta = (ef)(f\theta) = (fe_B)(f\theta)$.

(4) The extension R° of R satisfies the condition (1) of Theorem 1. In fact, we let $\mathfrak{m} \in R^\Delta$ and let φ be an R -left homomorphism of \mathfrak{m} into R . Then the restriction θ of φ to $B_{\mathfrak{m}}$ is clearly an H -mapping. On the other hand, R is a C -ring since it is a semisimple I -ring. Hence $R^\Delta = R^\Delta$ by Theorem 3. From (1), (2) it is easy to see that $\sum_{B_{\mathfrak{m}} \ni e} Re \in R^\Delta = R^\Delta$. For any element $\sum x_i e_i$ in $\sum Re$, $(\sum x_i e_i)\varphi = \sum x_i (e_i \varphi) = \sum x_i (e_i \theta) = \sum x_i (e_i \bar{\theta}) = (\sum x_i e_i) \bar{\theta}$.

(5) Let $0 \neq \bar{\theta} \in R^\circ$ and let $\theta \in H_B$ be a representative of $\bar{\theta}$. Since $B\theta \neq 0$, we see that $e\bar{\theta} = e\theta \neq 0$ for some $e \in B$. This shows $R \leq R^\circ$ by (4.4). Therefore $R^\circ \sim \bar{R}$ over R by (4) and Theorem 1.

Theorem 5. *Let R be a semisimple weakly reducible ring.*

- (1) The left maximal quotient ring \bar{R} of R is also the right maximal quotient ring of R ;
 (2) If R is of index n , then so is \bar{R} ;
 (3) If R satisfies a polynomial identity, then \bar{R} satisfies the same polynomial identity;
 (4) \bar{R} is also semisimple weakly reducible.

Proof. By the left-right symmetry of our method in (5.2) we see that R° is also the right maximal quotient ring. (2) Let $\bar{\theta} \in R^\circ$ be nilpotent and $\theta \in H_B$ be its representative. Then θ is nilpotent and hence so is $e\theta$ for every $e \in B$. $e\theta^n = (e\theta)^n = 0$. Thus $\bar{\theta}^n = 0$ and $\bar{\theta}^n = 0$. This shows that the index of R° (or \bar{R}) is at most n and hence is equal to n . (3) H_B may be regarded as a subdirect sum of Re for all $e \in B$. The identity holds in each Re . Hence it holds in H_B and in its limit R° . (4) Let α be a nonzero two-sided ideal of \bar{R} . Then $\alpha \cap R$ is nonzero and contains a nonzero matrix ideal $Re = T_n$ over a plain ring T . Since e is central in R it follows by (1.3) that e is central also in \bar{R} . $\bar{R} = e\bar{R} \oplus (1-e)\bar{R}$. By (2.2), $e\bar{R} \simeq e\bar{R} \cap \bar{R} = \overline{eR} = \overline{(T_n)}$. Hence $e\bar{R} \simeq (\bar{T})_n$ by (2.3). Now T is regular (Theorem 4) and of index 1 ((2) of this Theorem), and hence plain. Thus $e\bar{R}$ is a nonzero matrix ideal of \bar{R} and is contained in α . This shows that \bar{R} is semisimple weakly reducible.

7. In this section we consider some matrix rings as an application of Theorem 5.

A ring is called *strongly regular* if for any element x there is an element y such that $x^2y = x$. A necessary and sufficient condition for a ring to be strongly regular is that it is regular ring of index 1.¹⁵⁾

(7.1) Every plain ring R is embedded isomorphically into a strongly regular ring.

In fact, the regular ring \bar{R} is of index 1 by Theorem 5.

(7.2) If R is a nonzero plain ring, then R_n is of index n .

Every strongly regular ring is a subdirect sum of division rings.¹⁶⁾ Thus $R \leq \Sigma_{\oplus}^e P^{(\alpha)}$, $P^{(\alpha)}$ division rings. Then $R_n \leq (\Sigma_{\oplus}^e P^{(\alpha)})_n \sim \Sigma_{\oplus}^e P_n^{(\alpha)}$. Since $P_n^{(\alpha)}$ is of index n the index of R_n is at most n . On the other hand, for any nonzero idempotent $e \in R$, $\Sigma_{i=1}^{n-1} ee_{ii+1}$ is of index n . Therefore R_n is of index n .

(7.3) Let R be a semisimple I -ring. Then R is of index n if and

14) See [10], and [11].

15) See [4], Lemma 3.

16) See [4], Theorem 3.

only if R is a subdirect sum of its matrix ideals $T_{n_\alpha}^{(\alpha)}$ over plain rings $T^{(\alpha)}$ and $\text{Max } n_\alpha = n$.

“If” part: The index of R is evidently at most n . And some $T_{n_\alpha}^{(\alpha)}$ contains a nilpotent element of index n by (7.2). “Only if” part: From the assumption we see that R is a semisimple weakly reducible ring. Hence it follows from a result of Levitski [12, Theorem 3.1] that R is a subdirect sum of its matrix ideals $T_{n_\alpha}^{(\alpha)}$. By (7.2), $\text{Max } n_\alpha = n$.

Theorem 6. *A ring R is semisimple I -ring if and only if so is the total matrix ring R_n . In this case, R is of index m if and only if R_n is of index mn .*

Proof. (1) Let R be a semisimple I -ring. We assume that AR_n contains no nonzero idempotent where $A = \sum a_{ij}e_{ij} \in R_n$. Then $(xa_{ij}ye_{11})R_n = (xe_{11})A(ye_{11})R_n$ contains no nonzero idempotent for any $x, y \in R$. Let $e = xa_{ij}yz$ be an idempotent. Then $ee_{11} = (xa_{ij}ye_{11})(ze_{11})$ is also an idempotent. Hence $ee_{11} = 0$ and $e = 0$. This implies $xa_{ij}y = 0$. Therefore $a_{ij} = 0$ and $A = 0$. It follows from this that S_n is a semisimple I -ring.

(2) Let R_n be a semisimple I -ring and I a nonzero left ideal of R . The $\sum Ie_{11}$ is a nonzero left ideal of R_n . Hence it contains a nonzero idempotent $\sum x_{i1}e_{i1}$. $\sum x_{1i}e_{i1} = (\sum x_{i1}e_{i1})^2 = \sum x_{i1}x_{11}e_{i1}$. Therefore x_{11} is a nonzero idempotent in I which shows that R is a semisimple I -ring.

(3) Let R be a semisimple I -ring of index m . Then by (7.3) R is a subdirect sum of its matrix ideals $T_{n_\alpha}^{(\alpha)}$ and $\text{Max } n_\alpha = m$. Hence R_n is a subdirect sum of its matrix ideal $T_{n_\alpha n}^{(\alpha)}$ and $\text{Max } n_\alpha n = mn$. By (7.3) this means that R_n is a semisimple I -ring of index mn .

(4) Let R_n be a semisimple I -ring of index mn . Then R is also a semisimple I -ring by (2). Since R_n contains a subring isomorphic to R , we see that R is of bounded index. Hence the index of R is m by (3).

As a corollary of this Theorem, we have

(7.4) *Let R be a ring with a unit element and assume that some homomorphic image of some two-sided ideal of R is a nonzero semisimple I -ring of bounded index. Then $R_n \simeq R_m$ implies $n = m$.*

The minimum of the indeces of those rings, each of which is a nonzero semisimple I -ring of bounded index and is a homomorphic image of some two-sided ideal of R , is denoted by $\rho(R)$, $\rho(R_n)$ and $\rho(R_m)$ are similarly defined. Then $\rho(R_n) = n\rho(R)$ and $\rho(R_m) = m\rho(R)$. Therefore $n = m$ if $R_n \simeq R_m$.

(7.5) *Assume that a ring R satisfies the condition of (7.4). Let M be a unitary R -module with a basis consisting of k elements. Then any other basis is also finite and consists of k elements.*

This is evident from (7.4) since the R -endomorphism ring of M is R_k .

We note that every ring with a unit element, which is semisimple weakly reducible modulo its radical, satisfies the assumption in (7.4) and (7.5).

8. In this supplementary section we take a glance at the extended centralizer defined in [8]. We denote the extended centralizer over a module M as $E(M)$ and the family of submodules of M each of which has a nonzero intersection with every nonzero submodule of M as M^Δ .

Theorem 7. $E(N) \subseteq E(M)$ for every submodule N of M .

We omit the detailed proof. It is easy to see that (1) $E(N) = E(M)$ if $N \in M^\Delta$ and (2) $E(N) \subseteq E(M)$ if N is a direct summand of M . Now let N be an arbitrary submodule of M . Then $N + N^c \in M^\Delta$, where N^c is a maximal one among submodules having zero intersections with N . Hence $E(M) = E(N \oplus N^c) \supseteq E(N)$.

(8.1) Let K be a module and M the direct sum of n isomorphic copies $\{K_i\}$ of K . Then $E(M) \simeq (E(K))_n$.

Let θ_i be an isomorphism of K onto K_i . For any submodule H of K we denote the sum $\sum H\theta_i$ as H^* . Then we know that $H^* \in M^\Delta$ and that for any $N \in M^\Delta$ there is a submodule H of K such that $H^* \subseteq N$. From these facts we can prove the Theorem by the usual method.

Finally we add a simple application:

(8.2) Let R be a semisimple I -ring of bounded index and have a unit element. We assume that a unitary R -module M has a basis consisting of n elements. Then any basis of any free submodule N of M consists of at most n elements.

Owing to (8.1) we have $E(M) \simeq (E(R))_n$. Moreover, $E(R) = \bar{R}$ since R is a C -ring by (4.10). Let r be the index of R . Then the index of $E(R)$ is also r by Theorem 5. Hence that of $E(M)$ is rn by Theorem 6. Let t be a natural number which is not greater than the cardinal number of the given basis elements of N . Then N contains a submodule L which has a basis consisting of t elements. Now Theorem 7 assures that $E(L) \subseteq E(M)$. Since $E(L) \simeq (E(R))_t = (\bar{R})_t$, we know that the index of $E(L)$ is rt . Therefore $rt \leq rn$ whence $t \leq n$. This proves the proposition.

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