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ON THE DIFFERENTIABLE PINCHING PROBLEM

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Introduction

Let M be a compact, connected, simply connected Riemannian manifold with a metric d and denote by K the sectional curvature of M. Then it is known that if K satisfies the following inequality:

$$1/4 < K \leqq 1,$$

there exists a homeomorphism h of M onto S^n, the standard unit n-sphere ([1, 4, 6, 8]).

On the other hand, we also know that there is defined a positive $\mathfrak{l}(h)$ $(\geqq 1)$ for a homeomorphism h between two compact Riemannian manifolds, such that if $\mathfrak{l}(h)$ is sufficiently near to unity, that is, $(1 \leqq)$ $\mathfrak{l}(h) < 1+\varepsilon(n)$ ($\varepsilon(n)$ is a positive depending on n), then h is approximated arbitrarily by diffeomorphisms ([5]).

Our main aim in the note is to investigate a relation between $\mathfrak{l}(h)$ and the sectional curvature K to obtain an evaluation of $\mathfrak{l}(h)$ as in the following Proposition,

Proposition 1. *If K is δ-pinched, that is,*

$$\delta \leqq K \leqq 1$$

then with a constant c, $\mathfrak{l}$(h) *satisfies the following*:

$$(0 \leqq)\, \mathfrak{l}(h) - 1 \leqq c\sqrt{1-\delta}\,.$$

Therefore making $(1-\delta)$ so small as to satisfy

$$c\sqrt{1-\delta} < \varepsilon\,,$$

we get a diffeomorphism between M and (the standard) S^n.

Theorem 1. *If a compact, connected, simply connected Riemannian manifold M is δ-pinched with*

$$1-(\varepsilon/c)^2 < \delta$$

then M is diffeomorphic to the standard n-sphere.

Unfortunately, our evaluation itself is not as good as that of D. Gromoll [2], though our method might allow to generalize the pinching problem and make it possible to treat the problem from an interesting point of view.

1. Preliminary remarks

Lemma 1. *Let h be a homeomorphism between complete Riemannian manifolds $M_l(l=1, 2)$, with metrics $d_l(l=1, 2)$ and let $\{U_i\}$ be an open covering of M_1. Then if h satisfies on each open set U_i the following inequality;*

$$d_1(x, y)/k \leqq d_2(h(x), h(y)) \leqq k d_1(x, y) \quad (x, y \in U_i),$$

we have

$$\mathrm{I}(h) \leqq k .$$

Proof. For two points $p, q \in M_1$, take the minimizing geodesic $g(t)$ from p to q. It is possible to choose $t_j(j=0, \cdots, N)$ such that the geodesic segment $g([t_{j-1}, t_j])$ lies completely in one of open sets U_i.
Therefore we have,

$$\begin{aligned} d_2(h(p), h(q)) &\leqq \sum_j d_2(h(g(t_{j-1})), h(g(t_j))) \\ &\leqq \sum_j k d_1(g(t_{j-1}), g(t_j)) \\ &\leqq k d_1(p, q) . \end{aligned}$$

Also we have in quite a similar way (just replacing h by h^{-1}) that

$$d_2(h(p), h(q)) \geqq d_1(p, q)/k ,$$

finishing the proof.

The condition that U_i is open may be replaced by an assurance that the subdivision of a geodesic segment by U_i consist only of finite segments.
Therefore we get the following version of Lemma 1:

Corollary 1. *Let (K_1, f), (K_2, g) be differentiable triangulations of M_1, M_2, respectively, and assume that h satisfies the following* 1), 2).

1) $d_2(h(p), h(q)) \leqq k d_1(p, q)$, *for any p, q of each n-simplex Δ_1 of K_1.*

2) $d_1(h^{-1}(p), h^{-1}(q)) \leqq k d_2(p, q)$, *for any p, q of each n-simplex Δ_2 of K_2.*

Then we have

$$\mathrm{I}(h) \leqq k .$$

Lemma 2. *Suppose that there exist coordinate systems $\{U_i, f_i\}$, $\{U_i, g_i\}$ on M_1, M_2, having the same Euclidean open sets U_i as local parameter systems, and* *that the homeomorphism h is given by $g_i \cdot f_i^{-1}$ on each open set $f_i(U_1)$. Then if the line elements ds_1, ds_2 (written in the parameter system of U_i) satisfy that*

$$ds_1/k \leqq ds_2 \leqq k ds_1 ,$$

we also have

$$\mathrm{I}(h) \leqq k .$$

Corollary 2. *If h is piecewise differentiable on differentiable triangulations (K, f), (K, g) of M_1, M_2, then Lemma 2 holds when h is given by $g \cdot f^{-1}$ on each n-simplex Δ* of K *and the line elements ds_1, ds_2 (written in the coordinate of Δ) satisfy*

$$ds_1/k \leqq ds_2 \leqq k ds_1 \qquad \text{on each } \Delta \in K .$$

2. The computation of $\mathrm{I}(h)$

For a 1/4-pinched compact simply connected Riemannian manifold, the following facts i), ii) are known in [1, 4, 6, 8].

i) There are points p, $q \in M$ and a positive a, satisfying $\pi/2\sqrt{\delta} \leqq a \leqq \pi$ with $\delta = \min K$ such that

1) The open sets U, $V \subset M$ defined by

$$U = \{x \in M / d(x, p) < a\}, \quad V = \{y \in M / d(y, q) < a\}$$

cover M, that is, $U \cup V = M$.

2) The exponential maps defined at p, $q \in M$ send the open balls $\{X \in T_p(M) / |X| < a\}$, $\{Y \in T_q(M) / |Y| < a\}$ diffeomorphically onto U and V, respectively.

ii) Let N be a point set defined by

$$N = \{x \in M / d(x, p) = d(x, q)\} ,$$

then N possesses the following properties:

1) N is a differentiable submanifold of M and lies in $U \cap V$.

2) For every $x \in N$ there are a unique minimizing geodesic from p to x and a unique minimizing geodesic from q to x, we denote the initial directions of these geodesics by $g_+(x) \in T_p(M)$, $g_-(x) \in T_q(M)$, respectively.

3) Every geodesic segment of length a starting at p of initial direction X cuts N exactly at one point which we denote by $f_+(X)$. Also every geodesic segment of length a starting at q of initial direction Y cuts N exactly at one point which we denote by $f_-(Y)$.

Using the facts i), ii), a homeomorphism h of the standard unit n-sphere S^n onto M is constructed through following steps a)-e):

a) Let P, Q be the north pole and the south pole of S^n and express a point

x of the northern hemi-sphere E_+ by the standard polar coordinate system at P:

$$x=(G_+(x),\ R_+(x)),\ G_+(x)\in T_P(S^n),\ 0\leqq R_+(x)\leqq \pi/2\,.$$

Also write a point y in the southern hemi-sphere E_- by the polar coordinate system at Q:

$$y=(G_-(y),\ R_-(y)),\ G_-(y)\in T_Q(S^n),\ 0\leqq R_-(y)\leqq \pi/2\,.$$

b) For a direction $X\in T_p(S^n)$, denote by $F_+(X)$ the point in the equator E at which the geodesic segment of initial direction X crosses E:

$$F_+(X)=(X,\ \pi/2)\,.$$

Also define $F_-(Y)$ $(Y\in T_Q(S^n))$ to be the point in E at which the geodesic segment of initial direction y cuts E:

$$F_-(Y)=(Y,\ \pi/2)\,.$$

c) Take a linear isometry α of $T_P(S^n)$ onto $T_P(M)$ and define an one to one map β of $T_Q(S^n)$ onto $T_q(M)$ by

$$\beta(Y)=\begin{cases} g_-\circ f_+\circ\alpha\circ G_+\circ F_-(Y), & \textit{if} \quad |Y|=1 \\ |Y|\beta(Y/|Y|) & \textit{otherwise.}\end{cases}$$

d) Define an one to one map γ_+ of $T_p(M)$ onto itself by

$$\gamma_+(X)=2\ \mathrm{dist}\ (p,f_+(X/|X|))\mathrm{X}/\pi\,.$$

also define $\gamma_-(Y)$ on $T_q(M)$ by

$$\gamma_-(Y)=2\ \mathrm{dist}\ (p,f_-(Y/|Y|))\,Y/\pi\,.$$

e) Now the homeomorphism h of S^n onto M is given by

$$h(x)=\begin{cases} \exp\,(p)\circ\gamma_+\circ\alpha\circ\exp\,(P)^{-1}(x), & \textit{if} \quad x\in E_+ \\ \exp\,(q)\circ\gamma_-\circ\beta\circ\exp\,(Q)^{-1}(x), & \textit{if} \quad x\in E_-\,.\end{cases}$$

In order to prove that h is approximated by diffeomorphisms if the sectional curvature K of M is sufficiently pinched, we evaluate $\mathrm{I}(h)$ relative to the standard metric on S^n and the given Riemannian metric on M. The evaluation is done through the three steps: first we evaluate $\mathrm{I}(\exp\,(p)\circ\alpha\circ\exp\,(P)^{-1})$, next $\mathrm{I}(\exp\,(p)\circ\gamma_+\circ\exp\,(p)^{-1})$ and $\mathrm{I}(\exp\,(q)\circ\gamma_-\circ\exp\,(q)^{-1})$, and finally we evaluate $\mathrm{I}(\exp\,(q)\circ\beta\circ\exp\,(Q)^{-1})$. Since in general we know that

$$\mathrm{I}(A\circ B)\leqq\mathrm{I}(A)\ \mathrm{I}(B)$$

for any maps A, B, these three steps complete our evaluation.

2.1 First step, on $\mathfrak{l}(\exp(p)\circ\alpha\circ\exp(P)^{-1})$*.

Take orthogonal directions X, $Y\in T_p(M)$, then because of i) 2), we may apply Rauch's comparison theorem to the arc $c(\theta)=r(X\cos\theta+Y\sin\theta)$ $(t_1\leqq\theta\leqq t_2,\ 0\leqq r\leqq\pi/2)$ and to S^n, M, α. We get that

$$L(\exp(P)\circ\alpha^{-1}\circ c)\leqq L(\exp(p)\circ c)\,,$$

where $L(\varphi)$ denotes the length of the arc φ.

Let $S^n(\delta)$ be the sphere of constant curvature δ(δ is the positive pinching of the sectional curvature K of M from below; $\delta\leqq K\leqq 1$), then we also can apply Rauch's theorem to $c(\theta)$, M, $S^n(\delta)$ and a linear isometry α' of $T_P(S^n(\delta))$ onto $T_p(M)$, to get that

$$L(\exp(p)\circ c)\leqq L(\exp(P)\circ\alpha'^{-1}\circ c)$$

Since it is elementary to show that

$$L(\exp(P)\circ\alpha^{-1}\circ c)=(t_2-t_1)\sin r$$
$$L(\exp(P)\circ\alpha'^{-1}\circ c)=\frac{(t_2-t_1)}{\sqrt{\delta}}\sin\sqrt{\delta}\,r\,,$$

we deduce that

$$\sin r\leqq\frac{L(\exp(p)\circ c)}{t_2-t_1}\leqq\frac{1}{\sqrt{\delta}}\sin\sqrt{\delta}\,r\,.$$

In order to have an evaluation of the ratio of the line elements on S^n and M, consider the submanifold $M(X,\ Y)$ of M consisting of elements of the form $\exp(p)\,(rX\cos\theta+rY\sin\theta)$ and parametrize the plane of X, Y by (r,θ). The line element of M restricted on $M(X,\ Y)$, then, is written in the form $dr^2+\mu^2(r,\theta)\,d\theta^2$. Since the function $\mu(r,\theta)$ is nothing but the limit of $L(\exp(p)\circ c)/t_1-\theta$ when $t_1\to\theta$, the above inequality yields that

$$dr^2+\sin^2 r d\theta^2\leqq dr^2+\mu^2(r,\theta)\,d\theta^2\leqq dr^2+\frac{1}{\delta}\sin^2\sqrt{\delta}\,r d\theta^2\,.$$

Therefore we get that for any X, $Y\in T_p(M)$ it holds that

$$dr^2+\sin^2 r d\theta^2\leqq dr^2+\mu^2(r,\theta)\,d\theta^2\leqq\frac{1}{\delta}(dr^2+\sin^2 r d\theta^2)\,,$$

on $M(X,\ Y)$. Thus we may conclude that

$$\mathfrak{l}(\exp(p)\circ\alpha\circ\exp(P)^{-1})\leqq 1/\sqrt{\delta}\,,$$

by virtue of Lemma 2.

* The description in section 2.1, is due to professor Y. Tsukamoto and improves the author's original (less complete) one.

2.2 Second step, on $\mathfrak{l}(\exp(q)\circ\gamma_+\circ\exp(q)^{-1})$

The following fact iii) also is known for a compact connected δ-pinched simply connected manifold $M(\delta>1/4)$,

iii) 1) $\pi\leqq \operatorname{diam}(M)\leqq\pi/\sqrt{\delta}$

2) Let p, q be the points in i) 2), then for any $x\in M$,

$d(p, x)\leqq\pi/2\sqrt{\delta}$ or $d(q, x)\leqq\pi/2\sqrt{\delta}$.

In order to evaluate $\mathfrak{l}(\exp(q)\circ\gamma_+\circ\exp(q)^{-1})$, we first consider the differential in X of the function λ defined by

$$\lambda(X)=|\gamma_+(X)|/|X|.$$

Take $x, y\in N$ and let $\triangle PAB$, $\triangle QA'B'$ be triangles in euclidean space such that

$$d(p, x)=d(P, A),\quad d(p, y)=d(P, B),\quad d(x, y)=d(A, B)$$
$$d(q, x)=d(Q, A'),\quad d(q, y)=d(Q, B'),\quad d(x, y)=d(A', B').$$

Suppose $\angle A'\leqq\angle B'$ for instance, in $\triangle QA'B'$, we then have by Toponogov's comparison theorem that

$$\pi-\angle Q=\angle A'+\angle B'\leqq 2\angle B'\leqq 2\angle qyx.$$

Since, in general, it holds that

$$\angle qyx+\angle xyp+\angle pyq\leqq 2\pi,$$

we get that

$$\pi/2-\angle P/2\leqq\angle PBA\leqq\angle pyx\leqq\pi/2+\angle P/2+(\pi-\angle pyq),$$

hence we see that in $\triangle PBA$

$$\pi/2-3\angle P/2-(\pi-\angle pyq)\leqq\angle PAB=\angle QA'B'\leqq\pi/2-\angle P/2.$$

Therefore we have that

$$\begin{aligned}|d(p, x)-d(p, y)|&=|d(P, A)-d(P, B)|\\&\leqq d(P, A)\left|\frac{2\sin\angle P/2\sin(\angle PAB/2-\angle PBA/2)}{\sin\angle PBA}\right|\\&\leqq\pi|\sin\angle P/2\tan(\angle P+(\pi-\angle pyq))|/\sqrt{\delta}.\end{aligned}$$

Let now $x=f_+(X)$, $y=f_+(X+dX)$, then the inequality above yields that

$$|\gamma'(X)|\leqq\tan\angle pyq/\sqrt{\delta}$$

On the other hand, Toponogov's theorem applied to the geodesic triangle $\triangle pyq$, on which

$$\pi \leqq d(p, q), \quad d(p, y) = d(q, y) \leqq \pi/2\sqrt{\delta}\,,$$

yields that

$$\cos \angle pyq \leqq 1 - d^2(p, q)/2d^2(p, y) \leqq 1 - 2\delta\,.$$

Thus we get that

$$|\gamma'(X)| \leqq 4\sqrt{1-\delta}\,.$$

Since the homeomorphism $\exp(p)\circ\gamma_+\circ\exp(p)^{-1}$ leaves the submanifold $M(X, Y)$ of (2, 1) for orthogonal directions X, Y invariant, we may evaluate the effect of $(\exp(p)\circ\gamma_+\circ\exp(p)^{-1})^*$ on the line element ds of $M(X, Y)$, in order to get an evaluation of $I(\exp(p)\circ\gamma_+\circ\exp(p)^{-1})$. We compare two quadratic forms $I(x, y)$, $I_0(x, y)$ given by

$$I(x, y) = (\lambda(\theta)x)^2 + 2\lambda'(\theta)\lambda(\theta)rxy + \{\mu^2(\theta, \lambda(\theta)r) + (\lambda'(\theta)r)^2)\}y^2\,,$$
$$I_0(x, y) = x^2 + \mu^2(\theta, r)y\,,$$

to get the following: If a positive k satisfies that

1) $\lambda^2(\theta) \leqq k \leqq 4$
2) $4(\lambda'(\theta)r)^2 \leqq \mu^2(\theta, r)\left(k - \left(\dfrac{\mu(\theta, \lambda(\theta)r)}{\mu(\theta, r)}\right)^2\right)(k - \lambda^2)\,,$

then the quadratic form $kI_0(x, y)$ dominates $I(x, y)$, that is,

$$I(x, y) \leqq kI_0(x, y) \qquad \textit{for any } x, y\,.$$

Since we have that

$$\delta \sin r \leqq \mu(r, \theta) \leqq \frac{1}{\delta} \sin r, \quad 1 \leqq \lambda \leqq 1/\sqrt{\delta}$$

from (2. 1) and from iii) 1), 2), we see that the condition 2) above is fulfilled with k such that

$$k \geqq (1 + 4\pi\delta^2\sqrt{1-\delta})/\delta^3$$

Thus we have that, if $\delta \geqq 99/100$ e.g., then with $k_1 = (1 + 4\pi\delta^2\sqrt{1-\delta})/\delta^3$, it holds that

$$(\exp(p)\circ\gamma_+\circ\exp(p))^{-1})^*\, ds \leqq k_1 ds\,.$$

Quite similarly, we also have that with $k_2 = \delta(1 - 4\pi\sqrt{1-\delta})$, it holds that

$$(\exp(p)\circ\gamma_+\circ\exp(p)^{-1})^*\, ds \geqq k_2 ds\,.$$

Thus we may conclude that

$$I(\exp(p)\circ\gamma_+\circ\exp(p)^{-1})\leqq k_0\,,$$

where $k_0=\max(k_1, 1/k_2)$.

As in the same way above, we get that

$$I(\exp(q)\circ\gamma_-\circ\exp(q)^{-1})\leqq k_0\,.$$

2.3 Third step, on $I(\exp(q)\circ\beta\circ\exp(Q)^{-1})$.

We take two points x, y in E_- with polar coordinates (X, r) (Y, r) $(X, Y \in T_Q(S^n)$, $0\leqq r\leqq\pi/2)$. Apply the evaluation in 2. 1 and 2. 2 to points $F_-(X)$, $F_-(Y)\in E$, where two maps $h_+=\exp(p)\circ\gamma_+\circ\alpha\circ\exp(P)^{-1}$ and $h_-=\exp(q)\circ\gamma_-\circ\beta\circ\exp(Q)^{-1}$ coincide, to get that

$$\delta/k_0\leqq\frac{d(h_-\circ F_-(X),\ h_-\circ F_-(Y))}{d(F_-(X),\ F_-(Y))}\leqq k_0/\delta\,.$$

On the other hand, Rauch's theorem applied to a linear isometry $\tilde\beta$ of $T_Q(S^n)$ onto $T_q(M)$ and to S^n(or $S^n(\delta)$), M yields that

$$1\leqq\frac{d(\exp(q)\circ\tilde\beta\circ\exp(Q)^{-1}(a),\ \exp(q)\circ\tilde\beta\circ\exp(Q)^{-1}(b))}{d(a, b)}\leqq\frac{1}{\sqrt{\delta}}$$

for $a, b\in E_-$. Let $\xi=\tilde\beta^{-1}\beta(X)$, $\eta=\tilde\beta^{-1}\beta(Y)$, then we have that

$$1\leqq\frac{d(\tilde h\exp(Q)(sX),\ \exp\tilde h(Q)(sY))}{d(\exp(Q)(s\xi),\ \exp(Q)(s\eta))}\leqq\frac{1}{\sqrt{\delta}}\,,$$

where $\tilde h=\exp(q)\circ\beta\circ\exp(Q)^{-1}$. Substitute s by $\pi/2$ and by r in the inequality above to have that

$$\sqrt{\delta}\leqq\frac{d(\exp(Q)(r\xi),\ \exp(Q)(r\eta))}{d(F_-(\xi),\ F_-(\eta))}\cdot\frac{d(\tilde h\circ F_-(X),\ \tilde h\circ F_-(Y))}{d(\tilde h(x),\ \tilde h(y))}\leqq\frac{1}{\sqrt{\delta}}\,.$$

Since the ratio

$$\frac{d(\exp(Q)(r\xi),\ \exp(Q)(r(\xi+d\xi))}{d(F_-(\xi),\ F_-(\xi+d\xi))}=\sin r$$

depends only on r, we get that if Y is sufficiently near to X, then

$$\frac{\delta}{k_0}\leqq\frac{d(\tilde h(x),\ \tilde h(y))}{d(\tilde h\circ F_-(X),\ \tilde h\circ F_-(Y))}\cdot\frac{d(H\circ\tilde h\circ F_-(X),\ H\circ\tilde h\circ F_-(Y))}{d(x, y)}\leqq\frac{k_0}{\delta}\,,$$

where $H=\exp(q)\circ\gamma_-\circ\exp(q)^{-1}$. Combining this with the result of 2. 2, we have that

$$\frac{\delta}{k_0^2} \leqq \frac{d(\tilde{h}(x), \tilde{h}(y))}{d(x, y)} \leqq \frac{k_0^2}{\delta}$$

Thus we conclude that

$$\mathfrak{l}(\tilde{h}) \leqq k_0^2/\delta$$

because $\tilde{h}$ preserves length along longitude.

Consequently we get that

$$\mathfrak{l}(h_-) \leqq \mathfrak{l}(H) \cdot \mathfrak{l}(\tilde{h}) \leqq k_0^3/\delta$$

Therefore we finally have that

$$\mathfrak{l}(h) \leqq k_0^3/\delta ,$$

from Corollary 1, finishing the proof of Proposition 1 at the beginning.

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