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# SOME THEOREMS CONCERNING EXTREMA OF BROWNIAN MOTION WITH $d$-DIMENTIONAL TIME

Dedicated to Professor N. Ikeda on his 70th birthday

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## Introduction

Let $X = \{X(x), x \in \mathbf{R}^d\}$ be a Lévy's Brownian motion with $d$-dimensional time ([2]) defined on a certain probability space $(\Omega, P)$; thus $X$ is a centered Gaussian system with continuous sample functions satisfying $X(0) = 0$ and $E\{X(x)X(y)\} = (|x| + |y| - |x - y|)/2$. For a nonempty subset $A$ of $\mathbf{R}^d$ we put

$$\underline{X}(A) = \inf\{X(x) : x \in A\}, \quad \overline{X}(A) = \sup\{X(x) : x \in A\}.$$

We often use the notation $X(A)$ to denote either $\underline{X}(A)$ or $\overline{X}(A)$. For example, $X(A) - X(B)$ denotes any one of $\underline{X}(A)-\underline{X}(B)$, $\underline{X}(A)-\overline{X}(B)$, $\overline{X}(A)-\underline{X}(B)$ and $\overline{X}(A)-\overline{X}(B)$. A point $x$ in $\mathbf{R}^d$ is called a point of local minimum (resp. local maximum) of a sample function $X$ if there exists a neighborhood $U$ of $x$ such that $X(x) = \underline{X}(U)$ (resp. $X(x) = \overline{X}(U)$). A point of either local minimum or local maximum is called an extreme-point.

The following are typical of those problems and theorems we discuss in this paper.

(I) Under what condition on $A$ does the probability distribution of $X(A)$ admit a strictly positive $C^\infty$-density?

(II) Under what condition on $A$ and $B$ does the joint probability distribution of $X(A)$ and $X(B)$ admit a strictly positive $C^\infty$-density?

(III) Almost all sample functions $X$ have the following property: There are no distinct extreme-points $x$ and $y$ with $X(x) = X(y)$.

We give some sufficient conditions that will give positive answers to the problems (I) and (II) and then give a proof of (III). Formulating the problems somewhat generally we state our main results in the following theorems.

**Theorem 1.** *Let $A_k$, $1 \le k \le n$, be nonempty bounded closed sets not containing the origin* 0. *Then for any constants $c_k$, $1 \le k \le n$, such that $c_1 + c_2 + \cdots + c_n \ne 0$, the probability distribution of*

$$c_1 X(A_1) + c_2 X(A_2) + \cdots + c_n X(A_n)$$

*can be expressed as a convolution* $\gamma * \mu$ *where* $\gamma$ *is a nondegenerate Gaussian distribution with mean* 0 *and* $\mu$ *is some probability distribution in* $\boldsymbol{R}$. *In particular, the distribution of each of* $\underline{X}(A)$ *and* $\overline{X}(A)$ *has a strictly positive* $C^\infty$*-density provided that* $A$ *is a nonempty bounded closed set not containing* 0.

**Theorem 2.** *Let* $A_j, B_k,\ 1 \le j \le m,\ 1 \le k \le n$, *be nonempty bounded closed sets such that* $\cup_{j=1}^m A_j$ *is separated from* $\cup_{k=1}^n B_k$ *by a certain* $(d-1)$*-dimensional hyperplane* $\Pi$ *passing through the origin* 0. *Then for any constants* $c_j,\ c'_k,\ 1 \le j \le m,\ 1 \le k \le n$, *such that* $\sum_{j=1}^m c_j \neq 0$ *and* $\sum_{k=1}^n c'_k \neq 0$, *the joint distribution of*

$$f_1(X) = \sum_{j=1}^m c_j X(A_j), \qquad f_2(X) = \sum_{k=1}^n c'_k X(B_k) \tag{1}$$

*has a form* $(\gamma_1 \otimes \gamma_2) * \nu$ *where each* $\gamma_i$ *is a nondegenerate Gaussian distribution with mean* 0 *and* $\nu$ *is some* 2*-dimensional probability distribution. In particular, the joint distribution of* $X(A)$ *and* $X(B)$ *has a strictly positive* $C^\infty$*-density provided that* $A$ *and* $B$ *are nonempty bounded closed sets separated from each other by a certain* $(d-1)$*-dimensional hyperplane passing through* 0.

**Theorem 3.** *Let* $A_j,\ B_k,\ 1 \le j \le m,\ 1 \le k \le n$, *be nonempty bounded closed sets such that* $\cup_{j=1}^m A_j$ *is separated from* $\cup_{k=1}^n B_k$ *by a certain* $(d-1)$*-dimensional hyperplane. Then for any constants* $c_j,\ c'_k,\ 1 \le j \le m,\ 1 \le k \le n$, *such that* $\sum_{j=1}^m c_j = \sum_{k=1}^n c'_k \neq 0$, *the probability distribution of* $f_1(X) - f_2(X)$, *with* $f_1$ *and* $f_2$ *given by* (1), *has a form* $\gamma * \mu$ *where* $\gamma$ *is a nondegenerate Gaussian distribution with mean* 0 *and* $\mu$ *is some distribution in* $R$. *In particular, the distribution of each of* $\underline{X}(A) - \underline{X}(B)$, $\underline{X}(A) - \overline{X}(B)$ *and* $\overline{X}(A) - \overline{X}(B)$ *has a strictly positive* $C^\infty$*-density provided that* $A$ *and* $B$ *are nonempty bounded closed sets separated from each other by a certain* $(d-1)$*-dimensional hyperplane.*

**Theorem 4.** *Almost all sample functions* $X$ *have the following property*: *There. are no distinct extreme-points* $x$ *and* $y$ *of* $X$ *such that* $X(x) = X(y)$.

An example of the applicability (or our motivation) of Theorem 4 will be given in the final section.

## 1. A lemma

Given a centered Gaussian system $\{X_\lambda,\ \lambda \in \Lambda\}$ defined on a certain probability space $(\Omega, P)$, we denote by $H$ the real Hilbert space spanned by $\{X_\lambda,\ \lambda \in \Lambda\}$ and by $H_0$ the closed linear span (abbreviation: c.l.s.) of $\{X_\lambda - X_\mu,\ \lambda,\ \mu \in \Lambda\}$. Clearly $H_0 \subset H \subset L^2(\Omega, P)$. We now introduce the following conditions.

Condition (A). There exists a nondegenerate Gaussian random variable $Y_0$ independent of $\{X_\lambda - Y_0,\ \lambda \in \Lambda\}$.

Condition (B). There exists $\lambda \in \Lambda$ such that $X_\lambda \notin H_0$.
It is easy to see that the condition (B) implies that $X_\lambda \notin H_0$ for all $\lambda \in \Lambda$. Denote by $\mathbf{R}^\Lambda$ the space of real valued functions on $\Lambda$; it has a Borel structure defined in a natural way. Then we can regard $X_\Lambda = \{X_\lambda,\ \lambda \in \Lambda\}$ as a random variable taking values in $\mathbf{R}^\Lambda$. The following lemma is rather trivial; nevertheless, it plays a fundamental role in this paper.

**Lemma 1.** (i) *Let $f$ be a Borel function from $\mathbf{R}^\Lambda$ to* R *such that*

$$f(w + t\mathbf{1}) = f(w) + ct \tag{1.1}$$

*for any $w \in \mathbf{R}^\Lambda$ and $t \in$ R where $c$ is some nonzero constant and* **1** *denotes the function on $\Lambda$ that identically equals* 1*. Then under the condition* (A) *we have $f(X_\Lambda) = cY_0 + Y$ with a suitable random variable $Y$ independent of $Y_0$; in particular, the probability distribution of $f(X_\Lambda)$ has a strictly positive $C^\infty$-density.*
(ii) *Suppose $\Lambda$ is a locally compact space with a countable open base and assume that $X_\lambda$ is continuous in $\lambda$ with probability* 1*. We regard $X_\Lambda = \{X_\lambda,\ \lambda \in \Lambda\}$ as a random variable taking values in the space $C(\Lambda)$ of continuous functions on $\Lambda$, which is equipped with the compact uniform topology. Then, under the condition* (A)*, the conclusion of* (i) *remains valid for any Borel function $f$ from $C(\Lambda)$ to* R *satisfying* (1.1) *for $w \in C(\Lambda)$ and $t \in$* R.
(iii) *The condition* (B) *implies the condition* (A).

REMARK 1. Let $\Lambda_k,\ 1 \le k \le n$, be subsets of $\Lambda$ and let $c_k,\ 1 \le k \le n$, be constants such that $c_1 + \cdots + c_n \neq 0$. Let $w(\Lambda_k)$ indicate either $\inf\{w(\lambda) : \lambda \in \Lambda_k\}$ or $\sup\{w(\lambda) : \lambda \in \Lambda_k\}$; the choice may depend on $k$ but not on $w$. Then

$$f(w) = c_1 w(\Lambda_1) + \cdots + c_n w(\Lambda_n) \tag{1.2}$$

is a typical example of $f$ satisfying (1.1) with $c = c_1 + \cdots + c_n$ provided that $f$ can be defined to be a Borel function.

REMARK 2. Let $F$ be a class of functions defined on $[0, 1]$ and taking values in $\Lambda$ (an example of such an $F$ is the space of continuous paths in $\Lambda$ connecting two given points of $\Lambda$). Then the function $f$ defined by $f(w) = \inf\{g(w, u) : u \in F\}$ with $g(w, u) = \sup\{w(u(t)) : 0 \le t \le 1\}$ satisfies (1.1).

REMARK 3. If $\{X_\lambda,\ \lambda \in \Lambda\}$ satisfies (A) (resp. (B)) and if $\Lambda_1$ is a nonempty subset of $\Lambda$, then the sub-system $\{X_\lambda,\ \lambda \in \Lambda_1\}$ also satisfies (A) (resp. (B)).

Proof of Lemma 1. (i) Under the condition (A) $X_\Lambda - Y_0\mathbf{1}$ and $Y_0$ are independent so $f(X_\Lambda) - cY_0 = f(X_\Lambda - Y_0\mathbf{1})$ and $Y_0$ are independent. If we put $Y = f(X_\Lambda) - cY_0$,

then we have the expression $f(X_\Lambda) = cY_0 + Y$ in which $Y_0$ and $Y$ are independent and $Y_0$ is a nondegenerate Gaussian random variable. The assertion (ii) follows from (i).
(iii) It is easy to see that $X_\lambda + H_0 = \{X_\lambda + Y : Y \in H_0\}$ does not depend on $\lambda$. The condition (B) means that $X_\lambda + H_0 \not\ni 0$. Since $X_\lambda + H_0$ is a closed convex set, there exists a unique $Y_0 \in X_\lambda + H_0$ such that

$$\sqrt{E\{Y_0^2\}} = \min\left\{\sqrt{E\{|X_\lambda + Y|^2\}} : Y \in H_0\right\} > 0.$$

Then clearly $Y_0 \perp H_0$. Since $X_\lambda - Y_0 \in H_0$, $X_\lambda - Y_0 \perp Y_0$ for all $\lambda$. This implies that $Y_0$ is independent of $\{X_\lambda - Y_0, \lambda \in \Lambda\}$. □

## 2. Proof of Theorem 1

As stated in Introduction let $X = \{X(x),\ x \in \mathrm{R}^d\}$ be a Brownian motion with $d$-dimensional time. For any fixed pair of real numbers $t_1$ and $t_2$ such that $0 < t_1 < t_2$ we put $\Lambda = \{x \in \mathrm{R}^d : t_1 \le |x| \le t_2\}$, $H = \text{c.l.s.}\{X(x),\ x \in \Lambda\}$ and $H_0 = \text{c.l.s.}\{X(x) - X(y), x, y \in \Lambda\}$. First we prepare the following lemma.

**Lemma 2.** *The condition* (B) *is satisfied for* $\{X(x),\ x \in \Lambda\}$, *namely, there exists* $x \in \Lambda$ *such that* $X(x) \notin H_0$.

Proof. (i) We consider the case where the dimension $d$ is odd and $d \ge 3$. Denoting by $\hat{d}\theta$ the uniform distribution on $S^{d-1} = \{\theta \in \mathrm{R}^d : |\theta| = 1\}$, we put

$$\begin{aligned} R(t) &= \int_{S^{d-1}} X(t\theta)\hat{d}\theta, \qquad t \ge 0, \\ H_1 &= \text{c.l.s.}\{R(t), \quad t_1 \le t \le t_2\}, \\ H_1^\perp &= \text{the orthogonal complement of } H_1 \text{ in } H. \end{aligned}$$

Then we have

$$X(x) - R(|x|) \in H_1^\perp \qquad \text{for any } x \in \Lambda. \tag{2.1}$$

In fact, it is easy to see that, for each fixed $t \ge 0$, $E\{(X(x) - R(|x|))R(t)\}$ depends only on $|x|$ and hence it must vanish, which implies (2.1). We are going to prove that $X(t_1\theta) \notin H_0$ for $\theta \in S^{d-1}$. The relation (2.1) implies that $X(t_1\theta) = R(t_1) + X'$ with $X' \in H_1^\perp$ and that $H_0 \subset H_{10} \oplus H_1^\perp$ where $H_{10} = \text{c.l.s.}\{R(t) - R(s), t, s \in [t_1, t_2]\}$. Therefore, for the proof of $X(t_1\theta) \notin H_0$ it is enough to show that $R(t_1) \notin H_{10}$. We now make use of the canonical representation of the Gaussian process $\{R(t),\ t \ge 0\}$ due to McKean [5], which means that

$$R(t) = \int_0^t f(t, r)\, dB(r), \quad t \ge 0,$$

where $\{B(r),\ r \geq 0\}$ is a one-dimensional standard Brownian motion and

$$f(t,r) = k(d)\int_{r/t}^{1}(1-u^2)^{(d-3)/2}\,du, \qquad 0 \leq r \leq t, \tag{2.2}$$

$k(d)$ being a suitable constant depending only on $d$. For any $s$ and $t$ with $t_1 \leq s < t \leq t_2$ we have

$$R(t) - R(s) = \int_0^{t_1} f_{ts}(r)\,dB(r) + \int_{t_1}^{t} g_{ts}(r)dB(r),$$
$$R(t_1) = \int_0^{t_1} f(r)\,dB(r),$$

where $f_{ts}(r) = f(t,r) - f(s,r)$, $f(r) = f(t_1, r)$ and $g_{ts}(r)$ is a suitable function. Therefore, if we put

$$\tilde{H}_0 = \text{c.l.s.}\left\{\int_0^{t_1} f_{ts}(r)\,dB(r), t, s \in [t_1, t_2]\right\},$$
$$\tilde{H}_+ = \text{c.l.s.}\left\{B(u) - B(r), r, u \in [t_1, t_2]\right\},$$

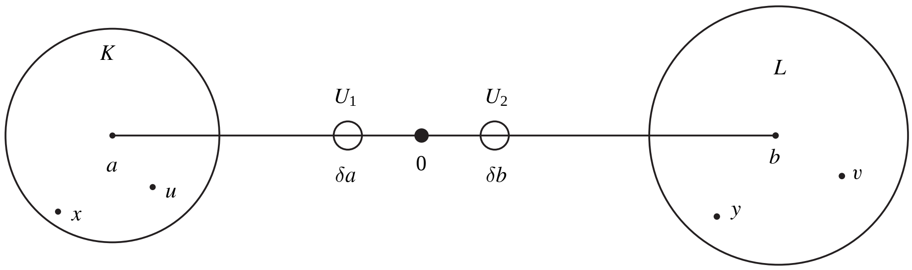
then $\tilde{H}_0 \perp \tilde{H}_+$, $H_{10} \subset \tilde{H}_0 \oplus \tilde{H}_+$ and $R(t_1) \perp \tilde{H}_+$. From these observations we see that for the proof of $R(t_1) \notin H_{10}$, it is enough to show

$$\int_0^{t_1} f(r)dB(r) \notin \tilde{H}_0. \tag{2.3}$$

Let $L_0^2$ be the subspace of $L^2[0, t_1]$ spanned by the functions $f_{ts}(\cdot), t, s \in [t_1, t_2]$. Then the Hilbert space $\tilde{H}_0$ is isomorphic to $L_0^2$ and (2.3) is equivalent to $f \notin L_0^2$. Now the assumption that $d$ is an odd integer $\geq 3$ implies that $f_{ts}(r), t, s, \in [t_1, t_2]$, are polynomials of degree $d-2$ vanishing at $r = 0$ (use (2.2)). Therefore all the functions in $L_0^2$ are also polynomials of degree at most $d-2$ vanishing at $r = 0$. On the other hand it is easy to see that $f$ is a polynomial of degree $d-2$ with $f(0) > 0$. Therefore $f \notin L_0^2$, which finally implies $X(t_1\theta) \notin H_0$. This completes the proof in the case where $d$ is odd and $d \geq 3$.

(ii) The proof in the case where $d$ is even can be obtained by the method of descent in which a Brownian motion with $d$-dimensional time is viewed as the restriction of a Brownian motion with $(d+1)$-dimensional time to $\mathrm{R}^d \times \{0\} \subset \mathrm{R}^{d+1}$ and also by using Remark 3. The proof in the case $d = 1$ is easy. The proof of Lemma 2 is finished. □

We are now able to prove Theorem 1. From the assumption on $A_k$, $1 \leq k \leq n$, there exist $t_1$ and $t_2$ with $0 < t_1 < t_2$ such that $\Lambda = \{x \in \mathrm{R}^d : t_1 \leq |x| \leq t_2\}$ includes all $A_k$. Then, by Lemma 2 the condition (B) is satisfied for $X_\Lambda = \{X(x), x \in \Lambda\}$




and by Remark 1 the condition (1.1) is satisfied for the function $f(w) = c_1 w(A_1) + c_2 w(A_1) + \cdots + c_n W(A_n)$, $w \in C(\Lambda)$, with $c = c_1 + \cdots + c_n$. Therefore by Lemma 1 the probability distribution of the random variable $f(X_\Lambda) = c_1 X(A_1) + c_2 X(A_2) + \cdots + c_n X(A_n)$ has a form $\gamma * \mu$. This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

Under the assumption on $A_j$ and $B_k$ in Theorem 2 we can take disjoint closed balls $K$ and $L$ with the following properties:

(3.1) $K \supset \cup_{j=1}^m A_j, \qquad L \supset \cup_{k=1}^n B_k$.

(3.2) $K$ is separated from $L$ by the hyperplane $\Pi$.

(3.3) The center $a$ of $K$ and the center $b$ of $L$ are on the straight line that passes through the origin 0 and is perpendicular to $\Pi$.

We consider open balls $U_1$ and $U_2$ with a common radius $\varepsilon$ and with centers $\delta a$ and $\delta b$, respectively, where $\delta > 0$ is chosen so that $\delta a \notin K$ and $\delta b \notin L$ (see the figure). We now make use of the Chentsov representation of $X(x)$ ([1]), which asserts that

$$X(x) = W(D_x), \tag{3.4}$$

where $D_x$ is the open ball with center $x/2$ and radius $|x|/2$, and $\{W(d\xi)\}$ is a suitable white noise in $\mathrm{R}^d$ associated with the measure $c_d|\xi|^{-d+1}d\xi$ ($c_d$ is a suitable constant), namely, a Gaussian random measure in $\mathrm{R}^d$ such that $E\{W(d\xi)\} = 0$ and $E\{W(d\xi)^2\} = c_d|\xi|^{-d+1}d\xi$. By taking $\varepsilon > 0$ small enough, we can assume

$$U_1 \subset \left\{\bigcap_{x\in K} D_x\right\} \bigcap \left\{\bigcup_{y\in L} D_y\right\}^c, \qquad U_2 \subset \left\{\bigcap_{y\in L} D_y\right\} \bigcap \left\{\bigcup_{x\in K} D_x\right\}^c. \tag{3.5}$$

If we write $X(x) = W(D_x) = W(U_1) + \tilde{X}_x$ and $X(y) = W(D_y) = W(U_2) + \tilde{X}_y$, then (3.5) implies that the 2-dimensional random vector $(W(U_1), W(U_2))$ is independent of the Gaussian family $\{(\tilde{X}_x, \tilde{X}_y) : x \in K, y \in L\}$. Therefore we have

$$f_1(X) = cW(U_1) + \tilde{f}_1, \qquad f_2(X) = c'W(U_2) + \tilde{f}_2,$$

with $c = \sum_{j=1}^m c_j$, $c' = \sum_{k=1}^n c'_k$ and $(W(U_1), W(U_2))$ is independent of $(\tilde{f}_1, \tilde{f}_2)$. Since $W(U_1)$ and $W(U_2)$ are independent and each of them is a nondegenerate Gaussian random variable with mean 0, the joint distribution of $f_1(X)$ and $f_2(X)$ has a form $(\gamma_1 \otimes \gamma_2) * \nu$.

## 4. Proof of Theorem 3 and Theorem 4

By using the fact that $\{X(x)-X(x_0), x \in \mathrm{R}^d\}$ is identical in law to $\{X(x-x_0), x \in \mathrm{R}^d\}$ for each $x_0 \in \mathrm{R}^d$ and also by using the assumption $\sum_{j=1}^m c_j = \sum_{k=1}^n c'_k$, we see that the probability distribution of $f_1(X) - f_2(X)$ is invariant under any simultaneous shift of $A_j$ and $B_k$. Therefore, in proving Theorem 3 we may assume that $A_j$. and $B_k$ satisfy the same assumption as in Theorem 2. Then the joint distribution of $f_1(X)$ and $f_2(X)$ has a form $(\gamma_1 \otimes \gamma_2) * \nu$ by Theorem 2 and this implies the conclusion of Theorem 3.

Before going to the proof of Theorem 4 we introduce some notation. Denote by $\mathcal{K}$ the set of all pairs $(K_1, K_2)$ of disjoint closed balls $K_1$ and $K_2$ with rational centers and rational radii. We put $f(K_1, K_2; \sigma_1, \sigma_2) = X(K_1; \sigma_1) - X(K_2; \sigma_2)$ where each $\sigma_i$ is either 0 or 1 and $X(K_i; \sigma_i)$ denotes either $\underline{X}(K_i)$ or $\overline{X}(K_i)$ according as $\sigma_i = 0$ or 1. We also denote by $\mathcal{E}(K_1, K_2; \sigma_1, \sigma_2)$ the event $\{f(K_1, K_2; \sigma_1, \sigma_2) = 0\}$ and then put $\mathcal{E}' = \cup \mathcal{E}(K_1, K_2; \sigma_1, \sigma_2)$ where the union is taken over all $(K_1, K_2) \in \mathcal{K}$ and all $(\sigma_1, \sigma_2) \in \{0, 1\}^2$. Finally let $\mathcal{E}$ be the event such that there exist distinct extreme-points $x$ and $y$ with $X(x) = X(y)$. It is then easy to see that $\mathcal{E} \subset \mathcal{E}'$. On the other hand Theorem 3 implies $P\{\mathcal{E}(K_1, K_2; \sigma_1, \sigma_2)\} = 0$ and hence $P\{\mathcal{E}'\} = 0$. This implies $P\{\mathcal{E}\} = 0$ as was to be proved.

## 5. Remarks on a diffusion process in a *d*-dimensional Brownian environment

This section is to supply an example for the applicability of Theorem 4. We change the notation for a Brownian motion with a $d$-dimensional time since we want to use $X(t)$ for a diffusion process. Let **W** be the space of continuous functions on $\mathrm{R}^d$ vanishing at 0. In this section an element $W$ of **W** is called an environment. We consider the probability measure $P$ on $W$ such that $\{W(x),\ x \in \mathrm{R}^d, P\}$ is a Lévy's Brownian motion with a $d$-dimensional time. Let $\Omega$ be the space of continuous functions on $[0, \infty)$ taking values in $\mathrm{R}^d$. The value of $\omega (\in \Omega)$ at time $t$ is denoted by $X(t) = X(t, \omega) = \omega(t)$. For each fixed environment $W$ we consider the probability measure $P_W$ on $\Omega$ such that $\{X(t), t \geq 0, P_W\}$ is a diffusion process in $\mathrm{R}^d$ with generator

$$\frac{1}{2}(\Delta - \nabla W \cdot \nabla) = \frac{1}{2} e^W \sum_{k=1}^d \frac{\partial}{\partial x_k} \left( e^{-W} \frac{\partial}{\partial x_k} \right)$$

and starting from 0. Let $\mathcal{P}$ be the probability measure on $W \times \Omega$ defined by $\mathcal{P}(dW d\omega) = P(dW) P_W(d\omega)$. Then $\{X(t), t \geq 0, \mathcal{P}\}$ can be regarded as a process defined on the probability space $(W \times \Omega, \mathcal{P})$, which we call a diffusion process in a $d$-dimensional Brownian environment. When $d = 1$, this model is a diffusion analogue of well-known Sinai's random walk in a random environment(1982) and much is known about the long-term behavior of X(t) such as localization. When $d \geq 2$, a similar diffusion model appeared in [3]. Now our interest is the long-term behavior of $\{X(t), t \geq 0, \mathcal{P}\}$ in the case $d \geq 2$. Tanaka [6](see also [7]) proved that, for any dimension $d$, $\{X(t), t \geq 0, P_W\}$ is *recurrent* for almost all Brownian sample environments $W$. Mathieu[4] proved that *localization* takes place for $\{X(t), t \geq 0, \mathcal{P}\}$, in the sense that

$$\lim_{N\to\infty} \overline{\lim_{\lambda\to\infty}} \mathcal{P}\left(\lambda^{-2}\max\{|X(t)| : 0 \leq t \leq e^{\lambda}\} > N\right) = 0.$$

However, in the case $d \geq 2$, it seems that the existence of the limiting distribution of $\{\lambda^{-2}X(e^{\lambda}), \mathcal{P}\}$ as $\lambda \to \infty$ is still an open problem. We give a remark on this problem. We notice the scaling relation

$$\{X(t),\ t \geq 0,\ P_{\lambda W_{\lambda}}\} \overset{d}{=} \{\lambda^{-2}X(\lambda^4 t),\ t \geq 0,\ P_W\},$$

where $\lambda > 0$ and $W \in \mathbf{W}$ are fixed, $W_{\lambda}$ denotes an element of $W$ defined by $W_{\lambda}(x) = \lambda^{-1}W(\lambda^2 x),\ x \in \mathrm{R}^d$, and $\overset{d}{=}$ means the equality in distribution. This scaling relation combined with $W_{\lambda} \overset{d}{=} W$ imply the following: If we can prove that $\{X(e^{r\lambda}),\ P_{\lambda W}\}$ has the limiting distribution as $\lambda \to \infty$ under the condition $r = r(\lambda) \to 1$, then so does $\{\lambda^{-2}X(e^{\lambda}), \mathcal{P}\}$. From now on we are interested in $\{X(t), P_{\lambda W}\}$. For $W \in \mathbf{W}$ we define the sub-level domain $D$ as the connected component of the open set $\{x \in \mathrm{R}^d : W(x) < 1\}$ containing 0. Then it is easy to see that $D$ is bounded, $P$-a.s. *By making use of* Theorem 4 we see that for $W$ not belonging to some $P$-negligible subset of $\mathbf{W}$, there exists a point $\tilde{b}$ of local (strict) minimum of $W$ with depth $> 1$ inside $D$. Such a point $\tilde{b}$ is characterized by (i) $W(\tilde{b}) < W(x)$ for $x \in U - \{\tilde{b}\}$ and (ii) $U \subset D$, where $U$ denotes the connected component of the open set $\{x \in \mathrm{R}^d : W(x) - W(\tilde{b}) < 1\}$ containing $\tilde{b}$. It is obvious that the totality of such points $\tilde{b}$ is a finite set, which is denoted by $\{b_k(W),\ 1 \leq k \leq l(W)\}$. Now suppose $l(W) = 1$ and put $b = b_1(W)$. Then from the argument of [4] we see that

$$X(e^{r\lambda}) \to b \text{ (in probability with respect to } P_{\lambda W}) \tag{5.1}$$

as $\lambda \to \infty$ provided $r = r(\lambda)$(non-random) tends to 1. If $l(W) \geq 2$, we do not know whether the limiting distribution of $X(e^{r\lambda})$ exists. Hoping for the best, we think it might be possible to define $b$, in one way or another, as *a single point* among $b_k(W),\ 1 \leq k \leq l(W)$, and to prove (5.1) even in the case $l(W) \geq 2$, for almost all $w$.

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