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SOME THEOREMS CONCERNING EXTREMA
OF BROWNIAN MOTION WITH d-DIMENSIONAL TIME

Dedicated to Professor N. Ikeda on his 70th birthday

HIROSHI TANAKA

(Received October 16, 1999)

Introduction

Let $X = \{X(x), x \in \mathbb{R}^d\}$ be a Lévy’s Brownian motion with $d$-dimensional time ([2]) defined on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$; thus $X$ is a centered Gaussian system with continuous sample functions satisfying $X(0) = 0$ and $E\{X(x)X(y)\} = (|x| + |y| - |x - y|)/2$. For a nonempty subset $A$ of $\mathbb{R}^d$ we put

$$\underline{X}(A) = \inf\{X(x) : x \in A\}, \quad \overline{X}(A) = \sup\{X(x) : x \in A\}.$$ 

We often use the notation $X(A)$ to denote either $\underline{X}(A)$ or $\overline{X}(A)$. For example, $X(A) - X(B)$ denotes any one of $\underline{X}(A) - \underline{X}(B)$, $\underline{X}(A) - \overline{X}(B)$, $\overline{X}(A) - \underline{X}(B)$, and $\overline{X}(A) - \overline{X}(B)$. A point $x$ in $\mathbb{R}^d$ is called a point of local minimum (resp. local maximum) of a sample function $X$ if there exists a neighborhood $U$ of $x$ such that $X(x) = \underline{X}(U)$ (resp. $X(x) = \overline{X}(U)$). A point of either local minimum or local maximum is called an extreme-point. The following are typical of those problems and theorems we discuss in this paper.

(I) Under what condition on $A$ does the probability distribution of $X(A)$ admit a strictly positive $C^\infty$-density?

(II) Under what condition on $A$ and $B$ does the joint probability distribution of $X(A)$ and $X(B)$ admit a strictly positive $C^\infty$-density?

(III) Almost all sample functions $X$ have the following property: There are no distinct extreme-points $x$ and $y$ with $X(x) = X(y)$.

We give some sufficient conditions that will give positive answers to the problems (I) and (II) and then give a proof of (III). Formulating the problems somewhat generally we state our main results in the following theorems.

**Theorem 1.** Let $A_k$, $1 \leq k \leq n$, be nonempty bounded closed sets not containing the origin $0$. Then for any constants $c_k$, $1 \leq k \leq n$, such that $c_1 + c_2 + \cdots + c_n \neq 0$, the probability distribution of

$$c_1X(A_1) + c_2X(A_2) + \cdots + c_nX(A_n)$$

...
can be expressed as a convolution $\gamma \ast \mu$ where $\gamma$ is a nondegenerate Gaussian distribution with mean 0 and $\mu$ is some probability distribution in $\mathbb{R}$. In particular, the distribution of each of $X(A)$ and $\overline{X}(A)$ has a strictly positive $C^\infty$-density provided that $A$ is a nonempty bounded closed set not containing 0.

\textbf{Theorem 2.} Let $A_j, B_k, 1 \leq j \leq m, 1 \leq k \leq n$, be nonempty bounded closed sets such that $\cup_{j=1}^{m} A_j$ is separated from $\cup_{k=1}^{n} B_k$ by a certain $(d-1)$-dimensional hyperplane $\Pi$ passing through the origin 0. Then for any constants $c_j, c'_k, 1 \leq j \leq m, 1 \leq k \leq n$, such that $\sum_{j=1}^{m} c_j \neq 0$ and $\sum_{k=1}^{n} c'_k \neq 0$, the joint distribution of

\begin{equation}
(1)\quad f_1(X) = \sum_{j=1}^{m} c_j X(A_j), \quad f_2(X) = \sum_{k=1}^{n} c'_k X(B_k)
\end{equation}

has a form $(\gamma_1 \otimes \gamma_2) \ast \nu$ where each $\gamma_i$ is a nondegenerate Gaussian distribution with mean 0 and $\nu$ is some 2-dimensional probability distribution. In particular, the joint distribution of $X(A)$ and $X(B)$ has a strictly positive $C^\infty$-density provided that $A$ and $B$ are nonempty bounded closed sets separated from each other by a certain $(d-1)$-dimensional hyperplane passing through 0.

\textbf{Theorem 3.} Let $A_j, B_k, 1 \leq j \leq m, 1 \leq k \leq n$, be nonempty bounded closed sets such that $\cup_{j=1}^{m} A_j$ is separated from $\cup_{k=1}^{n} B_k$ by a certain $(d-1)$-dimensional hyperplane. Then for any constants $c_j, c'_k, 1 \leq j \leq m, 1 \leq k \leq n$, such that $\sum_{j=1}^{m} c_j = \sum_{k=1}^{n} c'_k \neq 0$, the probability distribution of $f_1(X) - f_2(X)$, with $f_1$ and $f_2$ given by (1), has a form $\gamma \ast \mu$ where $\gamma$ is a nondegenerate Gaussian distribution with mean 0 and $\mu$ is some distribution in $\mathbb{R}$. In particular, the distribution of each of $X(A) - \overline{X}(B), X(A) - \overline{X}(B)$ and $\overline{X}(A) - \overline{X}(B)$ has a strictly positive $C^\infty$-density provided that $A$ and $B$ are nonempty bounded closed sets separated from each other by a certain $(d-1)$-dimensional hyperplane.

\textbf{Theorem 4.} Almost all sample functions $X$ have the following property: There are no distinct extreme-points $x$ and $y$ of $X$ such that $X(x) = X(y)$.

An example of the applicability (or our motivation) of Theorem 4 will be given in the final section.

\section{A lemma}

Given a centered Gaussian system $\{X_\lambda, \lambda \in \Lambda\}$ defined on a certain probability space $(\Omega, \mathcal{P})$, we denote by $H$ the real Hilbert space spanned by $\{X_\lambda, \lambda \in \Lambda\}$ and by $H_0$ the closed linear span (abbreviation: c.l.s.) of $\{X_\lambda - X_\mu, \lambda, \mu \in \Lambda\}$. Clearly $H_0 \subset H \subset L^2(\Omega, \mathcal{P})$. We now introduce the following conditions.

Condition (A). There exists a nondegenerate Gaussian random variable $Y_0$ inde-
pendent of \( \{X_\lambda - Y_0, \ \lambda \in \Lambda\} \).

Condition (B). There exists \( \lambda \in \Lambda \) such that \( X_\lambda \not\in H_0 \).

It is easy to see that the condition (B) implies that \( X_\lambda \not\in H_0 \) for all \( \lambda \in \Lambda \). Denote by \( \mathbb{R}^\Lambda \) the space of real valued functions on \( \Lambda \); it has a Borel structure defined in a natural way. Then we can regard \( X_\Lambda = \{X_\lambda, \ \lambda \in \Lambda\} \) as a random variable taking values in \( \mathbb{R}^\Lambda \). The following lemma is rather trivial; nevertheless, it plays a fundamental role in this paper.

**Lemma 1.** (i) Let \( f \) be a Borel function from \( \mathbb{R}^\Lambda \) to \( \mathbb{R} \) such that

\[
(1.1) \quad f(w + t\mathbf{1}) = f(w) + ct
\]

for any \( w \in \mathbb{R}^\Lambda \) and \( t \in \mathbb{R} \) where \( c \) is some nonzero constant and \( \mathbf{1} \) denotes the function on \( \Lambda \) that identically equals 1. Then under the condition (A) we have \( f(X_\Lambda) = cY_0 + Y \) with a suitable random variable \( Y \) independent of \( Y_0 \); in particular, the probability distribution of \( f(X_\Lambda) \) has a strictly positive \( C^\infty \)-density.

(ii) Suppose \( \Lambda \) is a locally compact space with a countable open base and assume that \( X_\lambda \) is continuous in \( \lambda \) with probability 1. We regard \( X_\Lambda = \{X_\lambda, \ \lambda \in \Lambda\} \) as a random variable taking values in the space \( C(\Lambda) \) of continuous functions on \( \Lambda \), which is equipped with the compact uniform topology. Then, under the condition (A), the conclusion of (i) remains valid for any Borel function \( f \) from \( C(\Lambda) \) to \( \mathbb{R} \) satisfying (1.1) for \( w \in C(\Lambda) \) and \( t \in \mathbb{R} \).

(iii) The condition (B) implies the condition (A).

**Remark 1.** Let \( \Lambda_k, \ 1 \leq k \leq n, \) be subsets of \( \Lambda \) and let \( c_k, \ 1 \leq k \leq n, \) be constants such that \( c_1 + \cdots + c_n \neq 0 \). Let \( w(\Lambda_k) \) indicate either \( \inf\{w(\lambda) : \lambda \in \Lambda_k\} \) or \( \sup\{w(\lambda) : \lambda \in \Lambda_k\} \); the choice may depend on \( k \) but not on \( w \). Then

\[
(1.2) \quad f(w) = c_1w(\Lambda_1) + \cdots + c_nw(\Lambda_n)
\]

is a typical example of \( f \) satisfying (1.1) with \( c = c_1 + \cdots + c_n \) provided that \( f \) can be defined to be a Borel function.

**Remark 2.** Let \( F \) be a class of functions defined on \([0, 1]\) and taking values in \( \Lambda \) (an example of such an \( F \) is the space of continuous paths in \( \Lambda \) connecting two given points of \( \Lambda \)). Then the function \( f \) defined by \( f(w) = \inf\{g(w, u) : u \in F\} \) with \( g(w, u) = \sup\{w(u(t)) : 0 \leq t \leq 1\} \) satisfies (1.1).

**Remark 3.** If \( \{X_\lambda, \ \lambda \in \Lambda\} \) satisfies (A) (resp. (B)) and if \( \Lambda_1 \) is a nonempty subset of \( \Lambda \), then the sub-system \( \{X_\lambda, \ \lambda \in \Lambda_1\} \) also satisfies (A) (resp. (B)).

**Proof of Lemma 1.** (i) Under the condition (A) \( X_\Lambda - Y_0\mathbf{1} \) and \( Y_0 \) are independent so \( f(X_\Lambda) - cY_0 = f(X_\Lambda - Y_0\mathbf{1}) \) and \( Y_0 \) are independent. If we put \( Y = f(X_\Lambda) - cY_0 \),
then we have the expression \( f(X_\lambda) = cY_0 + Y \) in which \( Y_0 \) and \( Y \) are independent and \( Y_0 \) is a nondegenerate Gaussian random variable. The assertion (ii) follows from (i).

(iii) It is easy to see that \( X_\lambda + H_0 = \{X_\lambda + Y : Y \in H_0 \} \) does not depend on \( \lambda \). The condition (B) means that \( X_\lambda + H_0 \neq 0 \). Since \( X_\lambda + H_0 \) is a closed convex set, there exists a unique \( Y_0 \in X_\lambda + H_0 \) such that

\[
\sqrt{E\{Y_0^2\}} = \min \left\{ \sqrt{E\{|X_\lambda + Y|^2\}} : Y \in H_0 \right\} > 0.
\]

Then clearly \( Y_0 \perp H_0 \). Since \( X_\lambda - Y_0 \in H_0, X_\lambda - Y_0 \perp Y_0 \) for all \( \lambda \). This implies that \( Y_0 \) is independent of \( \{X_\lambda - Y_0, \lambda \in \Lambda\} \).

\[\Box\]

2. Proof of Theorem 1

As stated in Introduction let \( X = \{X(x), \ x \in \mathbb{R}^d\} \) be a Brownian motion with \( d \)-dimensional time. For any fixed pair of real numbers \( t_1 \) and \( t_2 \) such that \( 0 < t_1 < t_2 \) we put \( \Lambda = \{x \in \mathbb{R}^d : t_1 \leq |x| \leq t_2\} \), \( H = \text{c.l.s.}\{X(x), \ x \in \Lambda\} \) and \( H_0 = \text{c.l.s.}\{X(x) - X(y), x, y \in \Lambda\} \). First we prepare the following lemma.

**Lemma 2.** The condition (B) is satisfied for \( \{X(x), \ x \in \Lambda\} \), namely, there exists \( x \in \Lambda \) such that \( X(x) \notin H_0 \).

**Proof.** (i) We consider the case where the dimension \( d \) is odd and \( d \geq 3 \). Denoting by \( \hat{\theta} \) the uniform distribution on \( S^{d-1} = \{\theta \in \mathbb{R}^d : |\theta| = 1\} \), we put

\[
R(t) = \int_{S^{d-1}} X(t\theta)\hat{\theta} \text{d}\theta, \quad t \geq 0,
\]

\[
H_0 = \text{c.l.s.}\{R(t), \ t_1 \leq t \leq t_2\},
\]

\[
H_0^\perp = \text{the orthogonal complement of } H_0 \text{ in } H.
\]

Then we have

\[
(2.1) \quad X(x) - R(|x|) \in H_0^\perp \quad \text{for any } x \in \Lambda.
\]

In fact, it is easy to see that, for each fixed \( t \geq 0 \), \( E\{(X(x) - R(|x|))R(t)\} \) depends only on \( |x| \) and hence it must vanish, which implies (2.1). We are going to prove that \( X(t_1\theta) \notin H_0 \) for \( \theta \in S^{d-1} \). The relation (2.1) implies that \( X(t_1\theta) = R(t_1) + X' \) with \( X' \in H_0^\perp \) and that \( H_0 \subset H_{10} \oplus H_1^\perp \) where \( H_{10} = \text{c.l.s.}\{R(t) - R(s), t, s \in [t_1, t_2]\} \). Therefore, for the proof of \( X(t_1\theta) \notin H_0 \) it is enough to show that \( R(t_1) \notin H_{10} \). We now make use of the canonical representation of the Gaussian process \( \{R(t), \ t \geq 0\} \) due to McKean [5], which means that

\[
R(t) = \int_0^t f(t, r) \text{d}B(r), \quad t \geq 0,
\]
where \( \{B(r), \ r \geq 0\} \) is a one-dimensional standard Brownian motion and

\[
f(t, r) = k(d) \int_{r/t}^{1} (1 - u^2)^{(d-3)/2} du, \quad 0 \leq r \leq t,
\]

\( k(d) \) being a suitable constant depending only on \( d \). For any \( s \) and \( t \) with \( t_1 \leq s < t \leq t_2 \) we have

\[
R(t) - R(s) = \int_{0}^{t} f_{ts}(r) dB(r) + \int_{t_1}^{t} g_{ts}(r) dB(r),
\]

\[
R(t_1) = \int_{0}^{t_1} f(r) dB(r),
\]

where \( f_{ts}(r) = f(t, r) - f(s, r), f(r) = f(t_1, r) \) and \( g_{ts}(r) \) is a suitable function. Therefore, if we put

\[
\mathcal{H}_0 = c.l.s. \left\{ \int_{0}^{t_1} f_{ts}(r) dB(r), t, s \in [t_1, t_2] \right\},
\]

\[
\hat{\mathcal{H}} = c.l.s. \left\{ B(u) - B(r), r, u \in [t_1, t_2] \right\},
\]

then \( \mathcal{H}_0 \perp \hat{\mathcal{H}} \subseteq \mathcal{H}_0 \oplus \hat{\mathcal{H}} \) and \( R(t_1) \perp \hat{\mathcal{H}} \). From these observations we see that for the proof of \( R(t_1) \notin \mathcal{H}_1 \), it is enough to show

\[
\int_{0}^{t_1} f(r) dB(r) \notin \mathcal{H}_0.
\]

Let \( L_0^2 \) be the subspace of \( L^2[0, t_1] \) spanned by the functions \( f_{ts}(\cdot), t, s \in [t_1, t_2] \). Then the Hilbert space \( \mathcal{H}_0 \) is isomorphic to \( L_0^2 \) and (2.3) is equivalent to \( f \notin L_0^2 \). Now the assumption that \( d \) is an odd integer \( \geq 3 \) implies that \( f_{ts}(r), t, s \in [t_1, t_2] \), are polynomials of degree \( d - 2 \) vanishing at \( r = 0 \) (use (2.2)). Therefore all the functions in \( L_0^2 \) are also polynomials of degree at most \( d - 2 \) vanishing at \( r = 0 \). On the other hand it is easy to see that \( f \) is a polynomial of degree \( d - 2 \) with \( f(0) > 0 \). Therefore \( f \notin L_0^2 \), which finally implies \( X(t_1, 0) \notin \mathcal{H}_0 \). This completes the proof in the case where \( d \) is odd and \( d \geq 3 \).

(ii) The proof in the case where \( d \) is even can be obtained by the method of descent in which a Brownian motion with \( d \)-dimensional time is viewed as the restriction of a Brownian motion with \( (d+1) \)-dimensional time to \( \mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1} \) and also by using Remark 3. The proof in the case \( d = 1 \) is easy. The proof of Lemma 2 is finished.

We are now able to prove Theorem 1. From the assumption on \( A_k, 1 \leq k \leq n \), there exist \( t_1 \) and \( t_2 \) with \( 0 < t_1 < t_2 \) such that \( \Lambda = \{x \in \mathbb{R}^d : t_1 \leq |x| \leq t_2\} \) includes all \( A_k \). Then, by Lemma 2 the condition (B) is satisfied for \( X_\Lambda = \{X(x), x \in \Lambda\} \)
and by Remark 1 the condition (1.1) is satisfied for the function \( f(w) = c_1 w(A_1) + c_2 w(A_2) + \cdots + c_n w(A_n), w \in C(\Lambda), \) with \( c = c_1 + \cdots + c_n. \) Therefore by Lemma 1 the probability distribution of the random variable \( f(X_\Lambda) = c_1 X(A_1) + c_2 X(A_2) + \cdots + c_n X(A_n) \) has a form \( \gamma \ast \mu. \) This completes the proof of Theorem 1.

3. Proof of Theorem 2

Under the assumption on \( A_j \) and \( B_k \) in Theorem 2 we can take disjoint closed balls \( K \) and \( L \) with the following properties:

(3.1) \( K \supset \bigcup_{j=1}^m A_j, \quad L \supset \bigcup_{k=1}^n B_k. \)

(3.2) \( K \) is separated from \( L \) by the hyperplane \( \Pi. \)

(3.3) The center \( a \) of \( K \) and the center \( b \) of \( L \) are on the straight line that passes through the origin 0 and is perpendicular to \( \Pi. \)

We consider open balls \( U_1 \) and \( U_2 \) with a common radius \( \varepsilon \) and with centers \( \delta a \) and \( \delta b, \) respectively, where \( \delta > 0 \) is chosen so that \( \delta a \notin K \) and \( \delta b \notin L \) (see the figure).

We now make use of the Chentsov representation of \( X(\chi) \) ([11]), which asserts that

\[
X(\chi) = W(D_\chi),
\]

where \( D_\chi \) is the open ball with center \( \chi/2 \) and radius \(|\chi|/2, \) and \( \{W(d\xi)\} \) is a suitable white noise in \( \mathbb{R}^d \) associated with the measure \( c_d |\xi|^{-d+1} d\xi \) (\( c_d \) is a suitable constant), namely, a Gaussian random measure in \( \mathbb{R}^d \) such that \( E\{W(d\xi)\} = 0 \) and \( E\{W(d\xi)^2\} = c_d |\xi|^{-d+1} d\xi. \) By taking \( \varepsilon > 0 \) small enough, we can assume

\[
U_1 \subset \left\{ \bigcap_{x \in K} D_x \right\} \bigcap \left\{ \bigcup_{y \in L} D_y \right\}^c, \quad U_2 \subset \left\{ \bigcap_{y \in L} D_y \right\} \bigcap \left\{ \bigcup_{x \in K} D_x \right\}^c.
\]

If we write \( X(\chi) = W(D_\chi) = W(U_1) + \tilde{X}_x \) and \( X(\chi) = W(D_\chi) = W(U_2) + \tilde{X}_y, \) then (3.5) implies that the 2-dimensional random vector \((W(U_1), W(U_2))\) is independent of the Gaussian family \( \{\tilde{X}_x, \tilde{X}_y : x \in K, y \in L\}. \) Therefore we have

\[
f_1(X) = c W(U_1) + \tilde{f}_1, \quad f_2(X) = c' W(U_2) + \tilde{f}_2,
\]
with \( c = \sum_{j=1}^{\infty} c_j, c' = \sum_{k=1}^{\infty} c'_k \) and \((W(U_1), W(U_2))\) is independent of \((\tilde{f}_1, \tilde{f}_2)\). Since \(W(U_1)\) and \(W(U_2)\) are independent and each of them is a nondegenerate Gaussian random variable with mean 0, the joint distribution of \(f_1(X)\) and \(f_2(X)\) has a form \((\gamma_1 \otimes \gamma_2) * \nu\).

4. Proof of Theorem 3 and Theorem 4

By using the fact that \(\{X(\chi - X(\chi_0), \chi \in \mathbb{R}^d\}\) is identical in law to \(\{X(\chi - X_0), \chi \in \mathbb{R}^d\}\) for each \(X_0 \in \mathbb{R}^d\) and also by using the assumption \(\sum_{j=1}^{\infty} c_j = \sum_{k=1}^{\infty} c'_k\), we see that the probability distribution of \(f_1(X) - f_2(X)\) is invariant under any simultaneous shift of \(A_j\) and \(B_k\). Therefore, in proving Theorem 3 we may assume that \(A_j\) and \(B_k\) satisfy the same assumption as in Theorem 2. Then the joint distribution of \(f_1(X)\) and \(f_2(X)\) has a form \((\gamma_1 \otimes \gamma_2) * \nu\) by Theorem 2 and this implies the conclusion of Theorem 3.

Before going to the proof of Theorem 4 we introduce some notation. Denote by \(\mathcal{K}\) the set of all pairs \((K_1, K_2)\) of disjoint closed balls \(K_1\) and \(K_2\) with rational centers and rational radii. We put \(f(K_1, K_2; \sigma_1, \sigma_2) = X(K_1; \sigma_1) - X(K_2; \sigma_2)\) where each \(\sigma_i\) is either 0 or 1 and \(X(K_i; \sigma_i)\) denotes either \(X(K_i)\) or \(\bar{X}(K_i)\) according as \(\sigma_i = 0\) or 1. We also denote by \(\mathcal{E}(K_1, K_2; \sigma_1, \sigma_2)\) the event \(\{f(K_1, K_2; \sigma_1, \sigma_2) = 0\}\) and then put \(\mathcal{E}' = \cup \mathcal{E}(K_1, K_2; \sigma_1, \sigma_2)\) where the union is taken over all \((K_1, K_2) \in \mathcal{K}\) and all \((\sigma_1, \sigma_2) \in \{0, 1\}^2\). Finally let \(\mathcal{E}\) be the event such that there exist distinct extreme points \(x\) and \(y\) with \(X(x) = X(y)\). It is then easy to see that \(\mathcal{E} \subset \mathcal{E}'\). On the other hand Theorem 3 implies \(P\{\mathcal{E}(K_1, K_2; \sigma_1, \sigma_2)\} = 0\) and hence \(P\{\mathcal{E}'\} = 0\). This implies \(P\{\mathcal{E}\} = 0\) as was to be proved.

5. Remarks on a diffusion process in a \(d\)-dimensional Brownian environment

This section is to supply an example for the applicability of Theorem 4. We change the notation for a Brownian motion with a \(d\)-dimensional time since we want to use \(X(t)\) for a diffusion process. Let \(W\) be the space of continuous functions on \(\mathbb{R}^d\) vanishing at 0. In this section an element \(W\) of \(W\) is called an environment. We consider the probability measure \(P\) on \(W\) such that \(\{W(\chi), \chi \in \mathbb{R}^d\}\) is a Lévy’s Brownian motion with a \(d\)-dimensional time. Let \(\Omega\) be the space of continuous functions on \([0, \infty)\) taking values in \(\mathbb{R}^d\). The value of \(\omega(\in \Omega)\) at time \(t\) is denoted by \(X(t) = X(t, \omega) = \omega(t)\). For each fixed environment \(W\) we consider the probability measure \(P_W\) on \(\Omega\) such that \(\{X(t), t \geq 0, P_W\}\) is a diffusion process in \(\mathbb{R}^d\) with generator

\[
\frac{1}{2} (\Delta - \nabla W \cdot \nabla) = \frac{1}{2} e^W \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left( e^{-W} \frac{\partial}{\partial x_k} \right)
\]

and starting from 0. Let \(\mathcal{P}\) be the probability measure on \(W \times \Omega\) defined by \(\mathcal{P}(dWd\omega) = P(dW)P_W(d\omega)\). Then \(\{X(t), t \geq 0, \mathcal{P}\}\) can be regarded as a process defined on the probability space \((W \times \Omega, \mathcal{P})\), which we call a diffusion process in
a $d$-dimensional Brownian environment. When $d = 1$, this model is a diffusion analogue of well-known Sinai’s random walk in a random environment (1982) and much is known about the long-term behavior of $X(t)$ such as localization. When $d \geq 2$, a similar diffusion model appeared in [3]. Now our interest is the long-term behavior of $\{X(t), \ t \geq 0, \ P \}$ in the case $d \geq 2$. Tanaka [6] (see also [7]) proved that, for any dimension $d$, $\{X(t), \ t \geq 0, \ P_W \}$ is recurrent for almost all Brownian sample environments $W$. Mathieu [4] proved that localization takes place for $\{X(t), \ t \geq 0, \ P \}$, in the sense that

$$
\lim_{N \to \infty} \lim_{\lambda \to \infty} P(\lambda^{-2} \max\{|X(t)| : 0 \leq t \leq e^\lambda\} > N) = 0.
$$

However, in the case $d \geq 2$, it seems that the existence of the limiting distribution of $\{\lambda^{-2}X(e^\lambda), \ P \}$ as $\lambda \to \infty$ is still an open problem. We give a remark on this problem. We notice the scaling relation

$$\{X(t), \ t \geq 0, \ P_{\lambda W}\} \overset{d}{=} \{\lambda^{-2}X(\lambda^4 t), \ t \geq 0, \ P_W\},$$

where $\lambda > 0$ and $W \in W$ are fixed, $W_\lambda$ denotes an element of $W$ defined by $W_\lambda(x) = \lambda^{-1}W(\lambda x)$, $x \in \mathbb{R}^d$, and $\overset{d}{=} \text{means the equality in distribution.}$ This scaling relation combined with $W_\lambda \overset{d}{=} W$ imply the following: If we can prove that $\{X(e^{\lambda}), P_{\lambda W}\}$ has the limiting distribution as $\lambda \to \infty$ under the condition $r = r(\lambda) \to 1$, then so does $\{\lambda^{-2}X(e^\lambda), P\}$. From now on we are interested in $\{X(t), P_{\lambda W}\}$. For $W \in W$ we define the sub-level domain $D$ as the connected component of the open set $\{x \in \mathbb{R}^d : W(x) < 1\}$ containing 0. Then it is easy to see that $D$ is bounded, $P$-a.s. By making use of Theorem 4 we see that for $W$ not belonging to some $P$-negligible subset of $W$, there exists a point $\bar{b}$ of local (strict) minimum of $W$ with depth $> 1$ inside $D$. Such a point $\bar{b}$ is characterized by (i) $W(\bar{b}) < W(x)$ for $x \in U - \{\bar{b}\}$ and (ii) $U \subset D$, where $U$ denotes the connected component of the open set $\{x \in \mathbb{R}^d : W(x) - W(\bar{b}) < 1\}$ containing $\bar{b}$. It is obvious that the totality of such points $\bar{b}$ is a finite set, which is denoted by $\{b_k(W), 1 \leq k \leq l(W)\}$. Now suppose $l(W) = 1$ and put $b = b_1(W)$. Then from the argument of [4] we see that

$$X(e^{\lambda}) \to b \text{ (in probability with respect to } P_{\lambda W})$$

as $\lambda \to \infty$ provided $r = r(\lambda)(\text{non-random})$ tends to 1. If $l(W) \geq 2$, we do not know whether the limiting distribution of $X(e^{\lambda})$ exists. Hoping for the best, we think it might be possible to define $b$, in one way or another, as a single point among $b_k(W)$, $1 \leq k \leq l(W)$, and to prove (5.1) even in the case $l(W) \geq 2$, for almost all $w$. 
References


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