On some doubly transitive permutation groups in which socle(Gα) is nonsolvable

Hiramine, Yutaka


VoR

https://doi.org/10.18910/8021

Osaka University Knowledge Archive

https://ir.library.osaka-u.ac.jp/

Osaka University
1. Introduction

Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ and $\alpha \in \Omega$. In [8], O'Nan has proved that $\text{socle}(G_{\alpha}) = A \times N$, where $A$ is an abelian group and $N$ is 1 or a nonabelian simple group. Here $\text{socle}(G_{\alpha})$ is the product of all minimal normal subgroups of $G_{\alpha}$.

In the previous paper [4], we have studied doubly transitive permutation groups in which $N$ is isomorphic to $PSL(2,q)$, $Sz(q)$ or $PSU(3,q)$ with $q$ even. In this paper we shall prove the following:

**Theorem.** Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ with $|\Omega|$ even and let $\alpha \in \Omega$. If $G_{\alpha}$ has a normal simple subgroup $N^*$ isomorphic to $PSL(2,q)$, where $q$ is odd, then one of the following holds.

(i) $G_{\Omega}$ has a regular normal subgroup.

(ii) $G_{\alpha} \cong A_6$ or $S_6$, $N^* \cong PSL(2,5)$ and $|\Omega| = 6$.

(iii) $G_{\alpha} \cong M_{11}$, $N^* \cong PSL(2,11)$ and $|\Omega| = 12$.

In the case that $G_{\alpha}$ has a regular normal subgroup, by a result of Hering [3] we have $(|\Omega|, q) = (16,9), (16,5)$ or $(8,7)$.

We introduce some notations:

- $F(X)$: the set of fixed points of a nonempty subset $X$ of $G$.
- $X(\Delta)$: the global stabilizer of a subset $\Delta \subseteq \Omega$ in $X$.
- $X_{\Delta}$: the pointwise stabilizer of $\Delta$ in $X$.
- $X^\Delta$: the restriction of $X$ on $\Delta$.
- $m|n$: an integer $m$ divides an integer $n$.
- $X^H$: the set of $H$-conjugates of $X$.
- $|X|_p$: maximal power of $p$ dividing the order of $X$.
- $I(X)$: the set of involutions in $X$.
- $D_m$: dihedral group of order $m$.

In this paper all sets and groups are finite.
2. Preliminaries

Lemma 2.1. Let $G$ be a transitive permutation group on $\Omega$, $\alpha \in \Omega$ and $N^\alpha$ a normal subgroup of $G_\alpha$ such that $F(N^\alpha) = \{\alpha\}$. Let the subgroup $X \leq N^\alpha$ be conjugate in $G_\alpha$ to every group $Y$ which lies in $N^\alpha$ and which is conjugate to $X$ in $G$. Then $N_\alpha(X)$ is transitive on $\Delta = \{\gamma \in \Omega | X \leq N^\alpha\}$.

Proof. Let $\beta \in \Delta$ and let $g \in G$ such that $\beta^g = \alpha$. Then, as $X \leq N^\beta$, $X^\beta \leq N_\alpha(X)$. By assumption, $(X^\beta)^h = X$ for some $h \in G_\alpha$. Hence $gh \in N_\alpha(X)$ and $\alpha^{(gh)^{-1}} = \alpha^{-1} = \beta$. Obviously $N_\alpha(X)$ stabilizes $\Delta$. Thus Lemma 2.1 holds.

Lemma 2.2. Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and $N^\alpha$ a nonabelian simple normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. If $C_G(N^\alpha) \neq 1$, then $N^\alpha = N^\beta \cap N^\gamma$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semiregular on $\Omega - \{\alpha\}$.

Proof. See Lemma 2.1 of [4].

Lemma 2.3. Let $G$ be a transitive permutation group on $\Omega$, $H$ a stabilizer of a point of $\Omega$ and $M$ a nonempty subset of $G$. Then

$$|F(M)| = |N_\alpha(M)| \times |M^G \cap H|/|H|.$$  

Here $M^G \cap H = \{g^{-1}Mg | g^{-1}Mg \subseteq H, g \in G\}$.

Proof. See Lemma 2.2 of [4].

Lemma 2.4. Let $G$ be a doubly transitive permutation group on $\Omega$ and $N^\alpha$ a normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. Assume that a subgroup $X$ of $N^\alpha$ satisfies $X^\alpha = X^\beta$. Then the following hold.

(i) $|F(X) \cap \beta^N| = |F(X) \cap \gamma^N|$ for $\beta, \gamma \in \Omega - \{\alpha\}$.

(ii) $|F(X)| = 1 + |F(X) \cap \beta^N| \times r$, where $r$ is the number of $N^\alpha$-orbits on $\Omega - \{\alpha\}$.

Proof. Let $\Gamma = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\alpha$-orbits on $\Omega - \{\alpha\}$. Since $G_\alpha$ is transitive on $\Omega - \{\alpha\}$ and $G_\alpha \supseteq N^\alpha$, we have $|\Delta_i| = |\Delta_j|$ for $1 \leq i, j \leq r$. By assumption, $G_\alpha = N_\alpha(X)N^\alpha$ and so $N_\alpha(X)$ is transitive on $\Gamma$. Hence for each $i$ with $1 \leq i \leq r$ there exists $g \in N_\alpha(X)$ such that $(\Delta_i)^g = \Delta_j$. Therefore $|F(X) \cap \Delta_i| = |F(X^g) \cap (\Delta_i)^g| = |F(X) \cap \Delta_j|$. Thus (i) holds and (ii) follows immediately from (i)

Lemma 2.5 (Huppert [5]). Let $G$ be a doubly transitive permutation group on $\Omega$. Suppose that $\theta_2(G) = 1$ and $G_\alpha$ is solvable. Then for any involution $z$ in $G_\alpha$, $|F(z)|^2 = |\Omega|$.

We list now some properties of $PSL(2, q)$ with $q$ odd which will be required
Lemma 2.6 ([2], [6], [10]). Set \( N = PSL(2, q) \) and \( G = Aut(N) \), where \( q = p^n \) and \( p \) is an odd prime. Let \( z \) be an involution in \( N \). Then the following hold.

(i) \( |N| = (q-1)q(q+1)/2 \), \( I(N) = z^N \) and \( C_N(z) = D_{q-\tau} \), where \( \tau \equiv 0 \mod 4 \).

(ii) If \( q \equiv 3 \mod 4 \), \( N \) is a nonabelian simple group and a Sylow \( r \)-subgroup of \( N \) is cyclic when \( r \neq 2, p \).

(iii) If \( X \) and \( Y \) are cyclic groups of \( N \) and \( |X| = |Y| = 2 \), \( p \), then \( X \) is conjugate to \( Y \) in \( \langle X, Y \rangle \) and \( N_\langle X \rangle = D_{q+\tau} \).

(iv) If \( X \leq N \) and \( X = \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( N_\langle X \rangle \) is isomorphic to \( A_4 \) or \( S_4 \).

(v) If \( |N| \geq 8 \), \( N \) has two conjugate classes of four-groups in \( N \).

(vi) There exist a field automorphism \( f \) of \( N \) of order \( n \) and a diagonal automorphism \( d \) of \( N \) of order 2 and if we identify \( N \) with its inner automorphism group, \( \langle d \rangle N = PGL(2, q) \), \( \langle f \rangle \langle d \rangle N = G \) and \( G \mid Z_2 \times Z_2 \).

(vii) \( C_N(d) = D_{q+\tau} \) and \( C_{\langle d \rangle N}(z) = D_{q-\tau} \).

(viii) Suppose \( n = mk \) for positive integers \( m, k \). Then \( C_N(f^m) = PSL(2, p^m) \) if \( k \) is odd and \( C_N(f^m) = PGL(2, p^m) \) if \( k \) is even.

(ix) Assume \( n \) is even and let \( u \) be a field automorphism of order 2. Then \( I(G) = I(N) \cup d^N \cup u^{\langle d \rangle N} \). If \( n \) is odd, \( I(G) = I(N) \cup d^N \).

Proof. By (viii) of Lemma 2.6, we can take an involution \( t \) satisfying \( \langle d \rangle H = \langle t \rangle W \) and \( [f, t] = 1 \). Since \( N_G(Y) = \langle f, d \rangle N_\langle Y \rangle = \langle f, d \rangle H \), \( C_G(Y) = C_{\langle f \rangle \langle d \rangle H}(Y) = W \cdot C_{\langle f \rangle \langle d \rangle H}(Y) \). Suppose \( ht \in C(Y) \) for some \( h \in \langle f \rangle \). Since \( t \) inverts \( Y \), \( h \) also inverts \( Y \) and so \( h^2 \) centralizes \( Y \). Hence some nontrivial 2-element \( g \in \langle h \rangle \) inverts \( Y \), so that \( C_H(g) \) contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, \( G^\O \) will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume \( G^\O \) has no regular normal subgroup.
Notation. $C^*=C_G(N^*)$, which is semi-regular on $\Omega - \{\alpha\}$ by Lemma 2.2. Let $r$ be the number of $N^*$-orbits on $\Omega - \{\alpha\}$.

Since $G_\beta \supseteq N^*$, $|\beta^N^*|=|\gamma^N^*|$ for $\beta, \gamma \in \Omega - \{\alpha\}$ and so $|\Omega|=1+r \times |\beta^N^*|$. Hence $r$ is odd and $N^*$ is a subgroup of $N^*$ of odd index. Therefore $N^*$ is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

**Lemma 2.8.** Let $Z$ be a cyclic subgroup of $N^*$ with $|Z| \neq 1, p$. Then

(i) $|Z|=2$, $|F(Z)|=1+(q-\varepsilon)|I(N^*)|/|N^*_\beta|$.

(ii) $|Z|=p$, $|F(Z)|=1+|N^*_\alpha(Z)|/|N^*_\beta(Z)|$.

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

**Lemma 2.9.** If $N^*_\beta \neq D_{q^\pm s}$ and $Z$ is a cyclic subgroup of $N^*$ with $|Z| \neq 1, p$ and $N_G(Z)^{F(Z)}$ is doubly transitive. Then $C^*=1$ and one of the following holds.

(i) $N_G(Z)^{F(Z)}=\text{AGL}(1, q_1)$ for some $q_1$.

(ii) $N_G(Z)^{F(Z)} \geq \text{PSL}(2, p_1)$, $r=1$ and $|F(Z)|=1+|N^*_\alpha(Z)|/|N^*_\beta(Z)|$.

(iii) $N_G(Z)^{F(Z)}=\text{R}(3)$, the smallest Ree group, $|F(Z)|=28$.

Proof. Set $N_G(Z)=L$ and $F(Z)=\Delta$. By Lemma 2.6(iii), $L \cap N^* \leq D_{q^\pm s}$ and $L \cap N^*=\langle t \rangle Y \geq Z$, where $0(t)=2$, $Y \cong Z_{(q^\pm s)/2}$.

If $(L \cap N^*)^\alpha=1$, then $L \cap N^*=N^*_\beta$ because $L \cap N^*$ is a maximal subgroup of $N^*$. Since $|N^*: N^*_\beta|$ is odd, $L \cap N^*=N^*_\beta=D_{q^\pm s}$, contrary to the assumption. Hence $(L \cap N^*)^\alpha=1$ and as $L_\alpha \geq L_\alpha \cap N^*$ and $L_\alpha \supseteq Y$, $(L_\alpha)^a$ has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:

(a) $L^\alpha$ has a regular normal subgroup

(b) $L^\alpha \geq \text{PSL}(2, p_1)$, $|\Delta|=p_1+1$, where $p_1(\geq 5)$ is a prime

(c) $L^\alpha \geq \text{PSL}(3, p_1)$, $p_1 \geq 3$, $|\Delta|=(p_1)^3+1$

(d) $L^\alpha=\text{R}(3)$, $|\Delta|=28$.

Suppose $C^*=1$. Then there exists a subgroup $D$ of $C^*$ of prime order such that $(L_\alpha)^a \supseteq\leq D^\alpha$. Since $[L_\alpha, D] \leq D \cdot L^\alpha \cap C^* = (L_\alpha \cap C^*) = D$, $D$ is a normal subgroup of $L_\alpha$. By (i) and (ii) of Lemma 2.6, $G_\alpha=L_\alpha \cdot N^*$ and so $D$ is a normal subgroup of $G_\alpha$. By Theorem 3 of [1], $G^\alpha$ has a regular normal subgroup, contrary to the hypothesis. Thus $C^*=1$.

If (a) occurs, $L^\alpha$ is solvable because $L_\alpha \cap N^* \leq L_\alpha N^*/N^* \leq \text{Out}(N^*)$ and $L \cap N^*=D_{q^\pm s}$. Hence by [5], (i) holds in this case.

If (b) occurs, we have $Y^\alpha=1$, for otherwise $(L \cap N^*)^\alpha=1$ and so $N^*_\beta=L \cap N^*=D_{q^\pm s}$, a contradiction. Hence $1 \neq C_G(Z)^{L^\alpha} \leq \text{PSL}(2, p_1)$ and $Y^\alpha \leq Z_{p_1}$. Therefore $|\Delta \cap \beta^{N^*}|=p_1$ and $r=1$ by Lemma 2.4 (ii). Since $|\beta^r|=p_1$, we have $|\beta^{L_N^*}|=p_1$, so that $L \cap N^*: L \cap N^*_\beta=p_1$. Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of $\text{PSU}(3, p_1)$,
some doubly transitive permutation groups 801

a Sylow $p_1$-subgroup of $(L_a)'$ is not cyclic, while $(L_a)' \leq L \cap N^a = D_{q_2}$, a contradiction.

3. Case (I)

In this section we assume that $N^a_\beta \leq D_{q_\Gamma}$, where $\beta \neq \alpha$, $q = p^a$.

(3.1) (i) If $N^a_\beta \neq Z_2 \times Z_2$, $N^a_\beta = N^a_\beta$ and $|F(N^\bullet_\beta)| = r + 1$.

(ii) If $N^a_\beta = Z_2 \times Z_2$, $N^a_\beta = A_4$ and $|F(N^\bullet_\beta)| = 3r + 1$.

Proof. Put $X = N^a_\beta(N^a_\beta)$. Let $S$ be a Sylow 2-subgroup of $N^a_\beta$ and $Y$ a cyclic subgroup of $N^a_\beta$ of index 2.

If $N^a_\beta \neq Z_2 \times Z_2$, then $|Y| > 2$ and so $Y$ is characteristic in $N^a_\beta$. Hence $X \leq N^a_\beta(Y) = D_{q_\Gamma}$. From this $[N^\bullet_\beta(S), S \cap Y] \leq S \cap Y$ and $0^\bullet(N_\beta(S))$ stabilizes a normal series $S \geq Y \geq S \cap Y$, so that $0^\bullet(N_\beta(S)) \leq C_{N^a_\beta}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^a_\beta}(S) \leq S$ and hence $N_\beta(S) = S$. On the other hand by a Frattini argument, $X = N_\beta(S)N^a_\beta$ and so $X = N^a_\beta$. By Lemma 2.6(i), $(N^a_\beta)^a = (N^a_\beta)^a$ and so by Lemmas 2.3 and 2.4(ii), $|F(N^a_\beta)| = 1 + |F(N^a_\beta) \cap \beta^a| \times r = 1 + |N^a_\beta| \times r = r + 1$. Thus (i) holds.

If $N^a_\beta = Z_2 \times Z_2$, $N^a_\beta = A_4$ by Lemma 2.6(iv). Similarly as in the case $N^a_\beta \neq Z_2 \times Z_2$, we have $|F(N^\bullet_\beta)| = 3r + 1$.

(3.2) $N^a_\beta \cap N^\beta \leq Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C_a = 1$. Suppose $C_a = 1$. Then $N^a_\beta/N^a_\alpha \cap N^\beta = N^a_\beta/N^a_\beta \leq \text{Out}(N^a_\bullet) = Z_2 \times Z_2$ by Lemma 2.6(vi) and hence $(N^a_\beta)^a = N^a_\beta \cap N^\beta$. Since $N^a_\beta$ is dihedral, $N^a_\beta(N^\bullet_\beta) = Z_2 \times Z_2$, so that $N^a_\beta/N^a_\alpha \cap N^\beta \leq Z_2 \times Z_2$.

(3.3) Suppose $N^a_\beta = N^a_\alpha \cap N^\beta$ and let $U$ be a subgroup of $N^a_\bullet$ isomorphic to $Z_2 \times Z_2$. Then $|F(U)| = 3r + 1$ and $N^a_\beta(U)^F(U)$ is doubly transitive.

Proof. Sex $X = N^a_\beta(N^a_\beta)$, $\Delta = F(N^a_\bullet)$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^a_\bullet$-orbits on $\Omega - \{\alpha\}$. If $g \in N^a_\beta \leq G^a_\beta$, then $g^{-1}N^a_\beta \leq N^a_\beta \cap N^\beta = N^a_\beta \cap N^\beta \leq N^a_\bullet$, where $\gamma = \alpha^a$. By a Witt's theorem, $X^a$ is doubly transitive.

If $U$ is a Sylow 2-subgroup of $N^a_\bullet$, by a Witt's theorem, $N^a_\beta(U)^F(U)$ is doubly transitive. Moreover $N_\beta(U) = A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)| = 1 + |A_4| \times |N^a_\beta| = |N^a_\beta(U)| \times r = 3r + 1$.

If $|N^a_\beta| > 2$, by Lemma 2.6(iv) and (v), $N^a_\beta(U) = S_4$ and $N^a_\bullet$ has two conjugate classes of four-groups, say $\pi = \{K_1, K_2\}$. Set $X_\pi = M$. Then $M \geq N^a_\bullet$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_i = \emptyset$ for each $i$ and so $|F(U) \cap \Delta_i| = 3$ by Lemma 2.3. Hence $|F(U)| = 3r + 1$. Since $N^a_\bullet(U) = S_4$, we may assume $r > 1$. Hence by (3.1) (i) $|\Delta| = r + 1 \geq 4$, so that $M^a$ is doubly transitive. Since $M = N^a_\beta N^a_\beta_\beta(U)$, $N^a_\beta_\beta(U)^a$ is also doubly transitive and so $N^a_\beta_\beta(U)$ is transitive on $\Delta$—
\{a\}. As \(|\Delta \cap \Delta_i|=1, \Delta \cap \Delta_i \subseteq F(U)\) and \(N_{\Delta}(U)\) is transitive on \(F(U) \cap \Delta_i\) for each \(i\), \(N_{\Delta}(U)^{F(U)}\) is doubly transitive.

\[(3.4)\]

(i) \(C^a=1\).

(ii) Let \(U\) be a subgroup of \(N_{\beta}^\sharp\) isomorphic to \(Z_2 \times Z_2\). If \(N_{\beta}^\sharp=N_{\beta} \cap N_{\beta}^\sharp\), then \(N_{\Delta}(U)^{F(U)}\) has a regular normal 2-subgroup. In particular \(|F(U)|=3r+1=2^b\) for positive integer \(b\).

Proof. Since \(N_{\Delta}(U)^{F(U)} \geq N_{\beta}^\sharp(U)^{F(U)}=S_3\) or \(Z_3\), by (3.3) and Theorem 3 of [1], \(N_{\Delta}(U)^{F(U)}\) has a regular normal subgroup, \(N_{\Delta}(U)^{F(U)} \geq \text{PSU}(3,3)\) or \(N_{\Delta}(U)^{F(U)}=R(3)\).

Suppose \(C^a \pm 1\). Let \(D\) be a minimal characteristic subgroup of \(C^a\). Clearly \(G_{\alpha} \supset D\). If \(N_{\Delta}(U)^{F(U)}+R(3), D\) is cyclic. By Theorem 3 of [1], \(C^a\) has a regular normal subgroup, contrary to the hypothesis. Hence \(N_{\Delta}(U)^{F(U)}=R(3)\). Therefore \((N_{\Delta}(U)^{F(U)})'\) contains an element of order 9. Since \(N_{\Delta}(U)^{C^a} N_{\beta}^\sharp(U) \simeq N_{\Delta}(U) C^a N_{\beta}^\sharp(U)\), by (vi) of Lemma 2.6 we have \((N_{\Delta}(U)^{C^a} N_{\beta}^\sharp(U))' \leq C^a \times N_{\beta}^\sharp(U)\). From this, \(C^a\) contains an element of order 9 and so \(C^a \simeq Z_9\) or \(M_3(3)\). In both cases, \(C^a\) contains a characteristic subgroup of order 3. Since \(G_{\alpha} \supset D\), by Theorem 3 of [1] \(G_{\alpha}\) has a regular normal subgroup, a contradiction. Thus \(C^a=1\).

Let \(R\) be a Sylow 3-subgroup of \(N_{\Delta}(U)\). Since \(N_{\Delta}(U)^{N_{\beta}^\sharp(U)} \geq N_{\Delta}(U)^{N_{\beta}^\sharp(U)} \geq \text{Out}(N_{\beta}^\sharp)\), \(R\) is cyclic. Clearly \(R \cap N_{\beta}^\sharp(U)=Z_3\). Therefore \(N_{\Delta}(U)^{F(U)} \simeq \text{PSU}(3,3), R(3)\). Thus (3.4) holds.

Since \(N_{\beta}^\sharp\) is dihedral, we set \(N_{\beta}^\sharp=\langle t \rangle W\) and \(Y=W \cap N_{\beta}^\sharp \cap N_{\beta}^\sharp\), where \(W\) is a cyclic subgroup of \(N_{\beta}^\sharp\) of index 2 and \(t\) is an involution in \(N_{\beta}^\sharp\) which inverts \(W\).

\[(3.5)\]

(i) If \(|Y| \geq 3, N_{\beta}^\sharp(Y)^{F(Y)}\) is doubly transitive.

(ii) If \(|Y|<3, N_{\beta}^\sharp=Z_2 \times Z_2\) or \(N_{\beta}^\sharp=D_4\) and \(N_{\beta}^\sharp \cap N_{\beta}^\sharp \leq Z_2 \times Z_2\).

Proof. Suppose \(|Y| \geq 3\). If \(Y^x \leq G_{\alpha^y}, Y^x \leq N^y \cap G_{\alpha^y} \leq N_{\beta}^\sharp\), where \(y=\alpha^x\).

If \(y=\alpha\), obviously \(Y^x \leq N^x\). If \(y=\alpha\), \(N_{\beta}^\sharp=Z_{\beta}\). Therefore, as \(|Y| \geq 3, N_{\beta}^\sharp\) has a unique cyclic subgroup of order \(|Y|\). Hence \(Y^x \leq N^y \cap N_{\beta}^\sharp \leq N_{\beta}^\sharp\), so that \(Y^x \leq N_{\beta}^\sharp\). Similarly \(Y^x \leq N_{\beta}^\sharp\). Thus \(Y^x \leq N_{\beta}^\sharp\) and so \(Y^x = Y\). By a Witt’s theorem, \(N_{\Delta}(Y)\) is doubly transitive on \(F(Y)\).

Suppose \(|Y|<3\). Since \(|N_{\beta}^\sharp \cap N_{\beta}^\sharp | Y | \leq 2\), we have \(N_{\beta}^\sharp \cap N_{\beta}^\sharp \leq Z_2 \times Z_2\). On the other hand, as \(N_{\beta}^\sharp\) is dihedral, \((N_{\beta}^\sharp)'\) is cyclic. Hence (ii) follows immediately from (3.2).

\[(3.6)\]

Set \(\Delta=F(N_{\beta}^\sharp), L=K_{\beta} \Delta, K=K_{\Delta} \) and suppose \(N_{\beta}^\sharp \neq Z_2 \times Z_2\). Then \(L \simeq N_{\beta}^\sharp, K' \leq N_{\beta}^\sharp, K' \leq N_{\beta}^\sharp \cap N_{\beta}^\sharp\) and \((L_{\alpha})^x=Z_r\). If \(r \neq 1\), \(L_{\alpha}\) is a doubly transitive Frobenius group of degree \(r+1\).

Proof. By Corollary B1 of [7] and (i) of (3.1), \(L_{\alpha}\) is doubly transitive and
Since $N^a \cap L \geq N^a \cap K = N^a$, by (i) of (3.1), we have $N^a \cap L = N^a$. Hence $L_a \geq N^a$. By (i) of (3.4), $L_a/N^a = L_a^\beta/N^a \leq \Out(N^a) = Z_2 \times Z_2$ and so $(L_a)^\beta \leq N^a$ and $(L_a)^\beta = Z_r$. If $r \neq 1$, then $(L_a)^\beta \neq 1$. On the other hand $(L_a)^\beta = 1$ as $(L_a)^\beta$ is abelian. Hence $L^\beta$ is a Frobenius group.

(3.7) Suppose $|Y| \geq 3$. Then there exists an involution $z$ in $N^a_\beta \cap Y$ such that $Z(N^a_\beta) = \langle z \rangle$.

Proof. Since $N^a_\beta \neq Z_2 \times Z_2$, $|N^a_\beta| \geq 2^2$ and $N^a_\beta$ is dihedral, we have $\langle I(W) \rangle = Z(N^a_\beta) = Z_2$ and $N^a_\beta/N^a_\gamma = Z_2 \times Z_2$. Let $Z(N^a_\beta) = \langle x \rangle$ and suppose that $z$ is not contained in $Y$. By (3.2), $(N^a_\beta)^\gamma \leq N^a \cap N^a \cap W = Y$ and so $|N^a_\beta|^\gamma$ is odd. Hence $|N^a_\beta| = 4$ and $q \equiv p^2 \equiv 3$ or $5$ (mod $8$), so that $n$ is odd. By (3.2) and (i) of (3.4), $N^a_\beta/N^a_\gamma \cap N^a_\beta = N^a_\beta/N^a_\beta = 1$ or $Z_2$. If $N^a_\beta = N^a \cap N^a$, then $W = Y$ and so $x \in Y$, contrary to the assumption. Therefore we have $N^a_\beta/N^a \cap N^a = Z_2$ and $N^a_\beta = \langle z \rangle \times (N^a \cap N^a)$. Since $n$ is odd and $z \leq N^a_\beta \cap N^a = N^a$, by Lemma 2.6 (vi), (vii) and (ix), $N^a_\beta/N^a \cap N^a = \langle z \rangle \times (N^a \cap N^a)$ and besides it is isomorphic to a subgroup of $D_{q^2}$. Hence $N^a \cap N^a = Z_2$ and $N^a_\beta = Z_2 \times Z_2$, a contradiction.

(3.8) Suppose $|Y| \geq 3$. Then $N^a_\beta = N^a \cap N^a$.

Proof. Suppose $N^a_\beta \neq N^a \cap N^a$ and let $\Delta, L, K$ be as defined in (3.6) and $x \in L_a$ such that its order is odd and $\langle x \rangle$ is transitive on $\Delta - \{a\}$. As $|Y| \geq 3$, $W$ is characteristic in $N^a_\beta$ and hence by (3.6), $x$ stabilizes a normal series $L_a \triangleright N^a_\beta \triangleright W \triangleright (N^a_\beta)^\gamma$. By Theorem 5.3.2 of [2], $[x, 0_\beta(L_a/(N^a_\beta)^\gamma)] = 1$. Since $L_a/(N^a_\beta)^\gamma$ has a normal Sylow 2-subgroup and $(N^a_\beta)^\gamma \leq K'$, we have $[x, 0_\beta(L_a/K')] = 1$, so that $[x, N^a_\beta] \leq K' \leq N^a \cap N^a$ by (3.6). If $r \neq 1$, then $\beta^a + \beta = \beta^a \in \Delta$, hence $N^a_\beta = x^{-1}N^a_\beta x = N^a_\beta$, where $\gamma = \beta^a$. Since $\gamma \in \Delta$ and $\Delta = F(N^a_\beta)$, $N^a_\beta \leq N^a \cap G_\gamma = N^a_\beta$ and so $N^a_\beta = N^a_\beta$. Similarly $N^a_\gamma = N^a_\beta$. Hence $N^a_\beta = N^a_\beta$, which implies $N^a_\beta = N^a \cap N^a$. By the doubly transitivity of $G$, we have $N^a_\beta = N^a \cap N^a$, contrary to the assumption. Therefore we obtain $r = 1$.

Let $z$ be as defined in (3.7) and put $k = (q - \varepsilon)/|N^a_\beta|$. By Lemma 2.8(i) we have $|F(z)| = 1 + (q - \varepsilon) \times (|N^a_\beta|/2 + 1)/|N^a_\beta| = (q - \varepsilon)/2 + k + 1$. Similarly $|F(Y)| = k + 1$. As $N^a_\beta \neq N^a \cap N^a$, there is an involution $t$ in $N^a_\beta$ which is not contained in $N^a_\beta$. By Lemma 2.6 (i), $t^\gamma = z$ for some $x \in N^a_\beta$. Set $\gamma = \beta^a$. Then $\gamma \in F(z)$ and $x \in N^a_\beta$. By Lemma 2.6 (vii), (viii) and (ix), $C_\Delta(x) = D_{q^2}$, or $PGL(2, q)$. Assume $C_\Delta(x) = D_{q^2}$, and let $R$ be a cyclic subgroup of $C_\Delta(x)$ of index 2. We note that $R$ is semi-regular on $\Omega - \{\alpha\}$. Set $X = C_\Delta(z)$. Since $2 \leq k + 1 \leq (q - \varepsilon)/|q - \varepsilon| + 1$, we have $(q + \varepsilon)/2 \geq k + 1$ and so $|\alpha^X| > k + 1$. By (i) of (3.5) and (3.7), $N^a_\gamma(L_\gamma) \subseteq C_\Delta(z) = X$ and $\alpha^X \subseteq F(Y)$. It follows from Lemma 2.1 that $\alpha^X = \{x | x \in N^a_\beta \} \neq \gamma$. Hence $|F(z)| > |\alpha^X| \geq |F(Y)| + (q + \varepsilon)/2 = k + 1 + (q - \varepsilon)/2 = |F(z)| + \varepsilon$. Therefore $\varepsilon = 1$ and $\gamma^X = \{\gamma\}$, so that $\gamma \in F(Y)$, a contradiction. Thus $C_\Delta(x) = PGL(2, \sqrt{q})$, $\varepsilon = 1$, $N^a_\beta/N^a \cap N^a = Z_2$ and $|\langle \alpha^X \cap G_\alpha \rangle|: N^a_\beta = 2$. 


Set $\Delta_1 = \alpha^x$ and $\Delta_2 = F(z) - \Delta_1$. Let $\delta \in \Delta_2$ and $g$ an element of $G$ satisfying $\delta^g = \gamma$. Then $x \in N_2 \cap N^x - N^x$ and so $x^g \in N_2 \cap N^x - N^x$, where $v = \alpha^x$. Since $\langle \alpha^x \cap G \rangle : N_2 = 2$ and $x \in G \cap N^x$, it follows from Lemma 2.6 (ix) that $(x^g)^h = x$ for some $h \in G$. Hence $gh \in X$ and $\delta^g = \gamma$. Let $\delta \in \Delta_2$. Then $z \in N_2$ and $g$ an element of $G$ satisfying $\delta^g = \gamma$. Let $\delta \in \Delta_2$. Then $z \leq N^x \cap N^x - N^x$ and so $z^g \in G \cap N^x \cap N^x - N^x$ where $\iota = \alpha^x$. Since $|O \cap G^\kappa| > 1$ and $\delta^g = \gamma$, it follows from Lemma 2.6 (ix) that $(x^g)^h = x$ for some $h \in G$. Hence $gh \in X$ and $\delta^g = \gamma$. Let $\delta \in \Delta_2$. Then $z \leq N^x \cap N^x - N^x$ and so $z^g \in G \cap N^x \cap N^x - N^x$. Hence $(\Delta_1, |\Delta_2|) = ((q-1)/4 + k + 1, (q-1)/4)$ or $(k+1, (q-1)/2)$. Let $P$ be a subgroup of $C_\S^\kappa(x)$ of order $\sqrt{q}$. Then $F(P) = \{\gamma\}$ and $P$ is semi-regular on $\Omega - \{\gamma\}$. If $|\Delta_2| = (q-1)/4$, then $\sqrt{q} | (q-1)/4 - 1 = (q-5)/4$ and $\sqrt{q} | (q-1)/4 + k + 1$. From this, $q = 5^4$, $k = 3$, $|\Delta_2| = 10$ and $|\Delta_2| = 6$. Since $(C_\S^\kappa(x))^2 \neq S_3$, $X^2 \simeq S_6$ and so $|X| \geq 3^2$. As $X$ acts on $\Delta_1$ and $|\Delta_1| = 1$ (mod 3), $|G_a| \geq |X_\alpha| \geq 3^2$, contrary to $N^\kappa \simeq PSL(2, 25)$. If $|\Delta_2| = (q-1)/2$, $\sqrt{q} | (q-1)/2 - 1 = (q-3)/2$, so $q = 3^3$, $k = 1$, $N_2 \simeq D_8$ and $\Delta_2 = \{\alpha, \beta\}$. Hence $C_\S^\kappa(x)$ fixes $\alpha$ and $\beta$, so that $\langle C_\S^\kappa(x) \rangle \cap \langle \alpha, \beta \rangle = \{e\}$. Hence $\langle C_\S^\kappa(x) \rangle \cap \langle \alpha, \beta \rangle = \{e\}$.

(3.9) Suppose $|Y| \geq 3$. Then $r = 1$.

Proof. By (3.6), $r+1 = 2^c$ for some integer $c \geq 0$. On the other hand $3r+1 = 2^b$ by (3.8) and (ii) of (3.4). Hence $2r = 2^c(2^k - c - 1)$ and so $c = 1$ as $r$ is odd. Thus $r = 1$.

(3.10) Put $k = (q-\varepsilon)/|N_2|$. If $N_2 = N^\kappa \cap N^\beta$ and $r = 1$, then

$$q - \varepsilon + 2k + 2 \leq ((2k + 2 - \varepsilon)(k + 1 - \varepsilon)k + 1)(2k + 2 - \varepsilon)(k + 1 - \varepsilon).$$

Proof. Set $S = \{(\gamma, u) \mid \gamma \in F(u), u \in \alpha^x\}$, where $z$ is an involution in $N_2$. We now count the number of elements of $S$ in two ways. Since $N_2 = N^\kappa \cap N^\beta$, $F(z) = \{\gamma \mid z \in N^\kappa\}$. If $|\Delta_2| = (q-1)/4$, then $\sqrt{q} \mid (q-1)/4 - 1 = (q-5)/4$ and $\sqrt{q} \mid (q-1)/4 + k + 1$. From this, $q = 5^4$, $k = 3$, $|\Delta_2| = 10$ and $|\Delta_2| = 6$. Since $(C_\S^\kappa(x))^2 \neq S_3$, $X^2 \simeq S_6$ and so $|X| \geq 3^2$. As $X$ acts on $\Delta_1$ and $|\Delta_1| = 1$ (mod 3), $|G_a| \geq |X_\alpha| \geq 3^2$, contrary to $N^\kappa \simeq PSL(2, 25)$. If $|\Delta_2| = (q-1)/2$, $\sqrt{q} \mid (q-1)/2 - 1 = (q-3)/2$, so $q = 3^3$, $k = 1$, $N_2 \simeq D_8$ and $\Delta_2 = \{\alpha, \beta\}$. Hence $C_\S^\kappa(x)$ fixes $\alpha$ and $\beta$, so that $\langle C_\S^\kappa(x) \rangle \cap \langle \alpha, \beta \rangle = \{e\}$. Hence $\langle C_\S^\kappa(x) \rangle \cap \langle \alpha, \beta \rangle = \{e\}$.

(3.11) Suppose $|Y| \geq 3$. Then one of the following holds.

(i) $N_2 = N^\kappa \cap N^\beta \simeq D_{q-4}$.

(ii) $N_2 = N^\kappa \cap N^\beta \simeq D_q$ and $N_6(Y)^{F(Y)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_6(Y)^{F(Y)} = R(3)$ or there exists a prime $p_1 \geq 5$ such that $C_\S^\kappa(Y)^{F(Y)} \simeq PSL(2, p_1)$ and $V/Y \simeq Z_{p_1}$, where $V = C_\S^\kappa(Y)$. By (i) of (3.1) and (3.9), $F(N_2) = \{\alpha, \beta\}$. On the other hand, $(N_2)^{F(Y)} \simeq N_2^\kappa \simeq Z_2$. Hence $N_6(Y)^{F(Y)} \simeq R(3)$ and $C_\S^\kappa(Y)^{F(Y)} \simeq R(3)$.
PSL(2, p).

By (i) of (3.4) and Lemma 2.7, we have $C_{G_{\alpha}}(Y) = V \langle f_{1} \rangle$, where $f_{1}$ is a field automorphism of $N^\alpha$. Let $t$ be the order of $f_{1}$, $n = tm$ and let $p^m \equiv \xi_1 \in \{ \pm 1 \} \pmod{4}$. Clearly $C_{G_{\alpha}}(Y)^{(Y)} \supseteq V^{(Y)} \cong Z_{n_{1}}$ and $|C_{G_{\alpha}}(Y)^{(Y)}| \mid t$, so that $(p_{1}-1)/2 \mid t$.

First we assume that $t$ is even and set $t = 2t_{1}$. Then $Y \leq C_{N^\alpha}(f_{1}) = PGL(2, p^m)$ by Lemma 2.6 (viii). As $|V/Y| = p_1$ and $p_1$ is a prime, $Y$ is a cyclic subgroup of $C_{N^\alpha}(f_{1})$ of order $p^m - \xi_1$ and $(p^m - 1)/2(p^m - \xi_1) = p_1$. Put $s = \sum_{i=1}^{t_1-1} (p_{2}^{2i})^i$. Then $(p^m + \xi_1)s/2 = p_1$, so that we have either (i) $t_1 = 1$ and $p_1 = (p^m + \xi_1)/2$ or (ii) $t_1 \geq 2$, $p^m = 3$ and $p_1 = s$. In the case (i), $2 \leq (p_{1}-1)/2 = (p^m + \xi_1 - 2)/4 | 2t_{1} = 2$. Hence $(p_{1}, q) = (5, 3)$ or $(4, 11^2)$. Let $s$ be as in (3.7). As mentioned in the proof of (3.10), $|F(z)| = (q-1)/2 + k^2$, $|\Omega| = kq(q+1)/2$ and $C_{G}(z)$ is transitive on $F(z)$. If $q = 3^2$, then $|F(z)| = 46$ and $|\Omega| = 2 \cdot 19^2$. Hence $|C_{G}(z)| = |F(z)|$ $|C_{G_{\alpha}}(z)| = |F(z)|$ $|C_{G_{\alpha}}(z)N^\alpha/N^\alpha| = q = 46 \cdot 2^{1} \cdot 80 = 2^5 \cdot 5 \cdot 13$ with $0 \leq i \leq 3$. Let $P$ be a Sylow 23-subgroup of $C_{G}(z)$ and $Q$ a Sylow 5-subgroup of $C_{G}(z)$. It follows from a Sylow's theorem that $P$ is a normal subgroup of $C_{G}(z)$ and so $[P, Q] = 1$. Theorefore $|F(Q)| \geq 23$, contrary to $5/\Omega | N^\alpha$. If $q = 11^2$, then $|F(z)| = 66$ and $|\Omega| = 2 \cdot 3 \cdot 6151$. Let $P$ be a Sylow 11-subgroup of $C_{G}(z)$. Since $11/\Omega | \Omega$, $P$ is a subgroup of $N^\alpha$ for some $\gamma \in \Omega$ and $F(P) = \{ \gamma \}$. Hence $\gamma \in F(z)$, so that $z \in N^\gamma$, contrary to $C_{N^\gamma}(z) = D_{120}$. In the case (ii), we have $(p_{1}-1)/2 = (t_1-1)q/2 | t = 2t_{1}$. From this, $9^{t_{1}-1} \leq 4t_1$, hence $t_1 = 1$, a contradiction.

Assume $t$ is odd. Then $Y \leq C_{N^\alpha}(f_{1}) = PGL(2, p^m)$ by Lemma 2.6 (viii). As $|V/Y| = p_1$ and $p_1$ is a prime, $Y \cong Z_{(p^m - \xi_1)/2}$ and $(q - \xi)(p^m - \xi_1) = p_1$. Hence $\sum_{i=0}^{t_1-1} (p^m)_{i}^{(s)} = p_1$ and $(p_{1}-1)/2 = (\sum_{i=1}^{t_1-1} (p_{2}^{2i})^{i}) - 1)/2 | t$. In particular $2t \geq (p^m)^{t_1-1} - (p^m)^{t_2-2} = (p^m)^{t_2-2} \geq 2(p^m)^{t_2-2}$. From this $t = 3$, $m = 1$, $p_1 = 7$ and $q = 3^2$, so that $N^\alpha = Z_{2} \times Z_{2}$, a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let $G^\alpha$ be a minimal counterexample to (3.12) and $M$ a minimal normal subgroup of $G$. By the hypothesis, $G$ has no regular normal subgroup and hence $M^\alpha = 1$. As $M^\alpha$ is a normal subgroup of $G^\alpha$, by (i) of (3.4), $M^\alpha$ contains $N^\alpha$. By (3.9), $r = 1$, hence $M$ is doubly transitive on $\Omega$. Therefore $G = M$ and $G$ is a nonabelian simple group.

Since $N^\alpha = D_{4 \cdot t}, k = 1$ and so $q - \xi + 4 | 2((4 - \xi)(2 - \xi) + 1)(4 - \xi)(2 - \xi)$ by (3.10). Hence we have $q = 7, 9, 11, 19, 27$ or 43.

Let $x$ be an element of $N^\alpha$. If $|x| > 2$, by Lemma 2.8, $|F(x)| = 1 + |N^\alpha| \times 1/|N^\alpha| = 2$ and if $|x| = 2$, similarly we have $|F(x)| = (q - \xi)/2 + 2$. Assume $q = 9$ and let $d$ be an involution in $G^\alpha - N^\alpha$ such that $\langle d \rangle N^\alpha$ is isomorphic to $PGL(2, p^m)$.
(2, q). We may assume \( d \in G_{a\beta} \). Since \( \langle d \rangle N^a \) is transitive on \( \Omega - \{ \alpha \} \), by Lemmas 2.3 and 2.6 (vii), (ix), \( |F(d)| = 2(q-1)(q+1)/2(q+1)+1 = (q+1)/2 \), while \( |F(x)| = (q+1)/2+2 \) for \( x \in I(N^a) \). Hence \( d \) is an odd permutation, contrary to the simplicity of \( G \). Thus \( G_a = N^a \) if \( q \neq 9, 27 \) and \( |G_a/N^a| = 3 \) if \( q = 27 \).

If \( q = 9 \), \( |\Omega| = 1+1 = |N^a/2| = 1+9-10/2 = 2 \), and \( |G_a| = 2^1|\text{PSL}(2, 9)| = 2^{3+1} \cdot 3 \cdot 5 \) with \( 0 \leq i \leq 2 \). Let \( P \) be a Sylow 23-subgroup of \( G \). Since Aut\((\mathbb{Z}_{23})\) \( \cong \mathbb{Z}_2 \times \mathbb{Z}_{11}, 3 | |N_G(P)| \), for otherwise \( P \) centralizes a nontrivial 3-element \( x \) and so \( F(P) \supseteq F(x) \) because \( |F(x)| = 1 \), contrary to \( |F(P)| = 0 \). Similarly \( 5 | |N_G(P)| \).

Hence \( |G : N_G(P)| = 2^4 \cdot 3 \cdot 5 \) for some \( a \) with \( 0 \leq a \leq 6 \). By a Sylow’s theorem, \( 2^4 \cdot 3 \cdot 5 \equiv -2^4 \equiv 1 \) (mod 23), a contradiction.

If \( q = 27 \), \( |\Omega| = 1+27+26/2 = 25 \cdot 11 \) and \( |G_a| = 2^2 \cdot 3^{4+i} \cdot 7 \cdot 13 \) with \( 0 \leq i \leq 1 \). Let \( P \) be a Sylow 11-subgroup of \( G \). Since \( P = \mathbb{Z}_{11} \) and Aut\((\mathbb{Z}_{11})\) \( \cong \mathbb{Z}_2 \times \mathbb{Z}_{11}, 3^i | |N_G(P)| \) by the similar argument as above. Hence \( |G : N_G(P)| = 2^2 \cdot 3^i \cdot 7 \cdot 13 \) with \( 0 \leq a \leq 7 \) and \( 3 \leq b \leq 3+i \). By a Sylow’s theorem, \( 2^2 \cdot 3^i \cdot 7 \cdot 13 = 2^2 \cdot 3^7 \cdot 3 \cdot 7 \cdot 13 = 2^2 \cdot 3^7 \cdot 5 \equiv 1 \) (mod 11). Hence \( a = 0, b = 4 \). Therefore \( N_G(P) \) contains a Sylow 2-subgroup \( S \) of \( G \). Let \( T \) be a Sylow 2-subgroup of \( N^a \) and \( g \) an element such that \( T^g \subseteq S \). Then \( T^g \cap C_G(P) = 1 \) as \( N_G(P) \cap C_G(P) \subseteq S \). Let \( u \) be an involution in \( T^g \cap C_G(P) \). Then \( |F(u)| = (27+1)/2 = 13 \), while \( |F(u)| = 0 \), a contradiction.

If \( q = 7, 11, 19 \) or \( 43 \), then \( G_a = N^a \) and \( \varepsilon = -1 \). Let \( \Gamma = \{ \gamma, \delta \} = \gamma \in \Omega, \gamma \neq \delta \}. We consider the action of \( G \) on \( \Gamma \). Since \( G^\sigma \) is doubly transitive, \( G^\sigma \) is transitive and \( G_r = 1 \). Let \( z \) be an involution of \( Z(N^a) \). There exists an involution \( t \) such that \( t \in \sigma^a \) and \( \sigma = \beta \). Since \( G_{a\beta} = N^a \) and \( F(N^a) = \{ \alpha, \beta \} \) we have \( G_{a\beta} = \langle t \rangle N^a \).

By Lemma 2.6, \( |F(x^t)| = |C_G(x)| = |t \rangle N^a \cap x^c = \langle t \rangle N^a \cap |x^c = |F(z)| = |C_G(z)| = |t \rangle N^a \cap |z^c = |F(z)| \) for 2. Moreover, \( |F(x^t)| = 2 = \langle t \rangle N^a \cap |z^c = |F(z)| \) for 2. In particular, \( |F(z)| = 1 \). Since \( |F(z)| = (q+1)/2+2 = (q+1)/2 \), we have \( q = 11 \) and \( |t \rangle N^a \cap |z^c = 13 \). Moreover, \( |\Omega| = 56 \), \( |G_a| = 1 |\text{PSL}(2, 11)| = 2^2 \cdot 3 \cdot 11 \) and \( |G| = 2^3 \cdot 3 \cdot 5 \cdot 11 \).

We now argue that \( \langle t \rangle N^a = D_{24} \). Let \( R \) be the Sylow 3-subgroup of \( N^a \). If \( t \) centralizes \( R \), \( R \) acts on \( F(t) \) and so \( F(R) \subseteq F(t) \) as \( |F(t)| = 3 \) and \( |F(R)| = 2 \). Hence \( \sigma = \alpha \), contrary to the choice of \( t \). Therefore \( t \) inverts \( R \) and \( \langle t \rangle N^a \) is isomorphic to \( Z_3 \times D_{12} \) or \( D_{24} \). Suppose \( \langle t \rangle N^a = Z_3 \times D_{12} \). Then \( \langle t \rangle N^a \) contains fifteen involutions and so we can take \( u \in I(\langle t \rangle N^a) \) satisfying \( |F(u)| = 0 \) and \( \langle t \rangle N^a = \langle u \rangle \times N^a \). As \( |F(u)| = 0 \), \( |F(u^t)| = |\Omega|/2 = 28 \). By Lemma 2.6, \( |C_G(u)| = |\langle u \rangle \times N^a \cap u^c| = 28 \) and hence \( |C_G(u)| = 2 \cdot 3 \cdot 7 \) or \( 2^2 \cdot 3 \cdot 7 \). Since \( \langle u \rangle N^a = N_G(R) \), we have \( |C_G(u)| = |C_G(u) \cap N_G(R)| = 2 \cdot 7 \) or \( 2 \cdot 7 \). By a Sylow’s theorem, \( |C_G(u)| = |C_G(u) \cap N_G(R)| = 2 \cdot 7 \), so that \( |C_G(u)| = 2^2 \cdot 3 \cdot 7 \). Let \( Q \) be a Sylow 7-subgroup of \( C_G(u) \). Then \( |C_G(u) \cap N_G(Q)| = 2 \cdot 3 \cdot 7 \) or \( 2 \cdot 3 \cdot 7 \) by a Sylow’s theorem. Hence \( 2^2 \cdot 3 \cdot 7 \) as \( N_G(Q) \). Since Aut\((Z_3) = Z_2 \times Z_3 \).
5|\langle N (Q) \rangle \text{ and } 11|\langle N (Q) \rangle \text{ by the similar argument as in the case } q=9. Therefore |G: N (Q)| = 2^a \cdot 5 \cdot 11 \text{ for some } a \text{ with } 0 \leq a \leq 3. Hence |G: N (Q)| \equiv 1 \pmod{7}, \text{ a contradiction. Thus } \langle \tau \rangle N = D_8.

Let U be a Sylow 2-subgroup of N and set L = N(U). It follows from (3.3) and Lemma 2.6 (iv) that L \cap N = A_4 and |L| = 2 \cdot 3. Let T, \langle x \rangle be Sylow 2- and 3-subgroup of L, respectively. Obviously L \triangleright T and C_T(x) = 1.

On the other hand T > L \triangleright \langle x \rangle N^* = D_8 and so T' = Z_2 \times Z_2 because C_T(x) = 1. By Theorem 5.4.5 of [2], T is dihedral or semi-dihedral. Hence N_G(T)/C_G(T) \leq \text{Aut}(T) \text{ is a 2-group, so that } C_T(x) = Y, a contradiction. Thus \langle \tau \rangle N = D_24.

Let U be a Sylow 2-subgroup of N with \tau \in U. Then F(x^{(q)}) = F(x^Y) = F(N^*) = \{x, \beta \} by (i) of (3.1) and (3.9). Since |F(Y)| = 1 + (q-\epsilon)/3, we have |N^*| = 1 + k \geq 4, x^{(q)} is an involution. By Lemma 2.5, 1 + k = 2^2 and so k = 3. By (3.11), q - \epsilon + 8 | 2((q - \epsilon)(4 - \epsilon) + 3 + 1)(8 - \epsilon)(4 - \epsilon). Hence q + 7 \mid 2^2 \cdot 3 \cdot 7 \text{ if } \epsilon = 1 \text{ and } q + 9 \mid 2^4 \cdot 3^2 \cdot 5 \cdot 17 \text{ if } \epsilon = -1. Since k = 3|q - \epsilon, 3|q - \epsilon + 8. From this q + 7 \mid 2^7 = 7 \text{ if } \epsilon = 1 \text{ and } q + 9 \mid 2^8 \cdot 5 \cdot 17 \text{ if } \epsilon = -1. Therefore q = 5^2, 7^2, 11^3, 59 \text{ or 71.}

Let p be an odd prime such that p \mid \Omega \text{ and } p \not{\mid} |G_a| \text{ and let } P be a Sylow p-subgroup of G. Clearly P is semi-regular on \Omega and so any element in C_G(P) has at least p fixed points. If x is an element of N^* and its order is at least three, |F(x)| = |F(Y)| = 4 by Lemma 2.8. Since |N^*| = (q-\epsilon)/3, we have |\Omega| = 1 + |N^*| = 1 + 3q(q + \epsilon)/2.

If q = 5^2, then |\Omega| = 2^4 \cdot 61 \text{ and } |G_a| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 13 \text{ (0 \leq i \leq 2)}. Let P be a Sylow 61-subgroup of G. Then P \cong Z_{61}. As mentioned above, 5, 13 \not{\mid} |C_G(P)| \text{ and so } 5^2, 13 \not{\mid} |N(G)|. Hence |G: N(G)| = 2^{a+1} \cdot 5 \cdot 13, \text{ where } 0 \leq a \leq 10 \text{ and } 0 \leq b, c \leq 1. But we can easily verify |G: N(G)| \equiv 1 \pmod{61}, contrary to a Sylow's theorem.

If q = 7^2, then |\Omega| = 2^3 \cdot 919 \text{ and } |G_a| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 7^2 \text{ (0 \leq i \leq 2)}. Let P be a Sylow 919-subgroup of G. By the similar argument as above, we obtain 5, 7 \not{\mid} |N(G)| \text{ and so } |G: N(G)| = 2^a \cdot 3^4 \cdot 5^2 \cdot 7^2 \equiv 2^a \cdot 306 \text{ or } 2^a \pmod{919}, \text{ where } 0 \leq a \leq 8 \text{ and } 0 \leq b \leq 1. Hence |G: N(G)| \equiv 1, a contradiction.

If q = 11^2, then |\Omega| = 2^7 \cdot 173 \text{ and } |G_a| = 2^{8+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \text{ (0 \leq i \leq 2)}. Let P be a Sylow 173-subgroup of G. Similarly we have 3, 5, 11, 61 \not{\mid} |N(G)| \text{ and so } |G: N(G)| = 2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -5 \cdot 2^a \pmod{173}, \text{ where } 0 \leq a \leq 12. Hence |G: N(G)| \equiv 1, a contradiction.

If q = 59, then |\Omega| = 2 \cdot 17 \cdot 151 \text{ and } |G_a| = 2^{3+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59 \text{ (0 \leq i \leq 1)}. Let P be a Sylow 17-subgroup of G. Similarly we have 3, 5, 29, 59 \not{\mid} |N(G)| \text{ and so } |G: N(G)| = 2^a \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151^4 \equiv 10 \cdot 2^a \text{ or } 12 \cdot 2^a \pmod{17}, \text{ where } 0 \leq a \leq 4 \text{ and } 0 \leq b \leq 1. From this, we have a contradiction.

If q = 71, then |\Omega| = 2^5 \cdot 233 \text{ and } |G_a| = 2^{3+i} \cdot 3 \cdot 5 \cdot 7 \cdot 71 \text{ (0 \leq i \leq 1)}. Let P be
a Sylow 233-subgroup of $G$. Since $3, 5, 7, 71 \not| N_{\alpha}(P)$, $|G|: N_{\alpha}(P) = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 71^{\pm 3 \cdot 2^e}$ (mod 233), where $0 \leq a \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N_{\beta}^a = Z_2 \times Z_2$ or $N_{\beta}^a = D_8$ and $N^a \cap N^B \leq Z_2 \times Z_2$.

(3.14) The case that $N_{\beta}^a = Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N_{\beta}^a)$. Then $|\Delta| = 3^{r+1}$ and $\Delta = F(N_{\beta}^a N_{\beta}^a)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^a|^2 = 4$, we have $q = p^e = 3, 5, 0 < p < 9$. Similarly we get a contradiction.

We now consider the case $|\gamma| < 3$. By (ii) of (3.5), $N^\gamma \times Z_2$ or $N^\gamma = D_8$ (3.14). The case that $N^\gamma = Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N_{\beta}^a)$. Then $|\Delta| = 3^{r+1}$ and $\Delta = F(N_{\beta}^a N_{\beta}^a)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^a|^2 = 4$, we have $q = p^e = 3, 5 \pmod{8}$ and so $n$ is odd. Hence $|C_{G^a} N^a|^2 \leq 2$ and $N^a \cap N^B = N^a N^B N^B = 1$ or $Z_2$ by (3.2). Suppose $N_{\beta}^a \cap N^a \cap N^B = Z_2$. Then $N^a \cap N^B$ is a Sylow 2-subgroup of $G$, hence $N_{\gamma}^a(N_{\beta}^a N_{\beta}^a)$ is doubly transitive by a Witt's theorem. Since $N^a \cap N^B = D_8$ and $|\Delta|$ is even, $C_{G^a}(N_{\beta}^a N_{\beta}^a)$ is also doubly transitive. Let $g$ be an element of $C_{G^a}(N_{\beta}^a N_{\beta}^a)$ such that $c^g = \beta$ and $d^g = \alpha$. Then $N^a \cap N^B = g N_{\beta}^a g = N_{\beta}^a$ and hence $N^a \cap N^B = N_{\beta}^a$, a contradiction. Thus $N_{\beta}^a = N^a \cap N^B = Z_2 \times Z_2$.

Let $z$ be an involution in $N_{\beta}^a$ and $t \in Z_2$ an involution such that $c^t = \beta$. Set $\Gamma = \{ (\gamma, \delta) | \gamma, \delta \in \Omega, \gamma \neq \delta \}$. We consider the action of the element $z$ on $\Gamma$. By the similar argument as in the proof of (3.12), $|F(z)(\{ (z, \delta) | z \in Z_2 \}| - |F(z)|)/2 + |\Omega| - |F(z)|) = 2 + |F(z)| = |C_{G^a}(z)| \times |\langle t \rangle |G_{\alpha^B}|| \langle t \rangle |G_{\beta^B}|$. Since $|N^a|^2 = N^a \cap N^B$, by Lemma 2.6 (i), $\langle t \rangle G_{\alpha^B} \times C_{G^a}(z)$ and hence $|G_{\alpha^B}| = |F(z)| \times |C_{G^a}(z)|$. Hence $|G_{\alpha^B}| = |F(z)| \times |C_{G^a}(z)| \times z_0 \times |\langle t \rangle |G_{\alpha^B}|$, so that $|G_{\alpha^B}| \times |\Omega| \equiv 0 \pmod{|F(z)|}$. Since $|G_{\alpha^B}| = N_{\beta}^a N_{\beta}^a |N^B| |2n|, we have $|G_{\alpha^B}| / 8n$. Clearly $|\Omega| = 1 + q(q - e) (q + e) r/8$ and by Lemma 2.8 (i), $|F(z)| = 1 + 3 (q - e) r/4$. Hence $1 + 3 (q - e) r/4 \leq |G_{\alpha^B}| / 8n$. Thus $3q - 3e + 4 | (4rs + 8q(q - e) (q + e) r^3 = 864 r^2 + 4s (3pq) (3pq - 3e) (3qr + 3e)$. Hence $3q - 3e + 4 | 864 r^2 + 4s (3q - 3e) (3qr + 3e) = 864 r^2 - 32s (3qr - 3e) - 2e (r - 2)$. (\*)

We argue that $r = 1$. Suppose false. Then $32s (3qr - 3e) (3q - 3e) > 0$ and so $3q - 3e < 864 r^2$. Therefore $288n + e > q = p^e \geq 3^e$ and so $288n > 3^e$. Hence $(n, r, p, e) = (5, 5, 3, -1), (3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (**) Thus $r = 1$.

Hence $3q - 3e + 4 | 64 (5 + 9e) n$ and $|F(z)| = 1 + 3 (q - e) r/4, |\Omega| = 1 + q(q - e) (q + e) r/8$. If $e = 1$, then $3^5 < 3q + 7 | 256 n$. Hence $n = 1$ or $(n, p) = (5, 3), (3, 3)$. Since $3^5 + 7 | 256$ and $3^3 + 7 | 256$, $n = 1$ and $3q + 7 | 256$. From this, $q = 19$ or 83. If $e = 1$, then $3^5 < 3q + 1 | 896 n$ and so $n = 1$ or $(n, p) = (3, 5)$. Since $3^5 + 1 | 896$, we have $n = 1$. From this, $q = 5, 37$ or 149. As $PSL(2, 5) \Rightarrow PSL(2, 4), q \equiv 5 \pmod{4}$. Thus $q = 19, 37, 83$ or 149.

Set $m = \{ z_0 \times \langle t \rangle | G_{\alpha^B} \}$. As we mentioned above, $|G_{\alpha^B}| = |G(z)| |F(z)| (|F(z)| - 1) + |\Omega| - |F(z)|) = |F(z)| \times |C_{G^a}(z)| m$. Since $|G_{\alpha^B}| = 1$ or 2, $|C_{G^a}(z)| \times |G_{\alpha^B}| = (q - e) r/4$. Therefore $m = (2q^2 + (2e + 9q - 9q) (3q - 3e + 4)$. It follows that $(q, m) = (19, 27/2), (37, 28), (83, 449/8)$ or $(149, 411/4)$. Since $m$ is an integer, we have $(q, m) = (37, 28)$. But $m \leq |\langle t \rangle |G_{\alpha^B}| \leq 16$, a contradiction. Thus (3.14)
holds.

\[(3.15)\quad \text{The case that } N^*_\beta = D_8 \text{ and } N^*_\beta \cap N^\beta \leq Z_2 \times Z_2 \text{ does not occur.}\]

Proof. Let \(\Delta, L\) and \(K\) be as defined in (3.6). By (3.6), there exists an element \(x\) in \(L\) such that its order is odd and \(\langle x^\alpha \rangle\) is regular on \(\Delta - \{\alpha\}\).

Since \((L\Lambda)_{\nu} \leq N^*_\beta\) by (3.6) and \(N^*_\beta = D_8\), \(x\) stabilizes a normal series \(N^*_\beta N^\mu \triangleright N^*_\beta \triangleright 1\). Hence \(x\) centralizes \(N^*_\beta N^\mu\) by Theorem 5.3.2 of [2] and so \(x^{-1}N^*_\beta x = N^*_\beta\). Put \(\gamma = \beta^x\). If \(r=\gamma\), then \(\beta = \gamma\), so that \(N^*_\beta = N^*_\beta\). From this, \(N^*_\beta = N^*_\beta\). By the doubly transitivity of \(G\), \(N^*_\beta = N^*_\beta\), hence \(N^*_\beta = N^*_\beta \cap N^\beta\), a contradiction. Therefore \(r=\gamma\) and \(\Delta = \{\alpha, \beta\}\).

Set \(\langle x \rangle = Z(\Lambda \Gamma Z), \Delta = \langle x \rangle \cap N^*_\beta\) and let \(\{\Delta_1, \Delta_2, \ldots, \Delta_k\}\) be the set of \(C_0(x)\)-orbits on \(F(x)\). Since \(L \triangleright N^* \cap N^\beta\) and by (3.2), \(N^* \cap N^\beta \not\leq 1\), \(x\) is contained in \(N^* \cap N^\beta\). Hence, by Lemma 2.1, \(\beta \in \Delta_1\) and \(k\) is at least two. By Lemma 2.8, \(|F(x)| = 1 + (q-\varepsilon)/8|N^*_\alpha| = 1 + (q-\varepsilon)/4\). Clearly \(|C_0(x)| = (q-\varepsilon)/8\) and so \(\Delta_1 \geq 1 + (q-\varepsilon)/4\). If \(\gamma \in F(x) - \Delta_1\), then \(C_0(x) = Z_2 \times Z_2\), for otherwise \(\langle x \rangle = Z(\Lambda \Gamma Z) \leq N^* \cap N^\gamma\) and by Lemma 2.1 \(\gamma \in \Delta_1\), a contradiction. Hence one of the following holds.

(i) \(k = 3\) and \(\Delta_1 = 1 + (q-\varepsilon)/4\), \(\Delta_2 = 1 + (q-\varepsilon)/2\), \(\Delta_3 = 1 + (q-\varepsilon)/4\).

(ii) \(k = 2\) and \(\Delta_1 = 1 + (q-\varepsilon)/8\), \(\Delta_2 = 1 + (q-\varepsilon)/2\).

(iii) \(k = 2\) and \(\Delta_1 = 1 + 3(q-\varepsilon)/4\), \(\Delta_2 = 1 + (q-\varepsilon)/4\).

Let \(\gamma \in F(x) - \Delta_1\). Then, \(z \in G_\gamma \cap N^\beta\) and so \(C_0(\gamma) = D_4 \times e\) or \(PGL(2, \sqrt{q})\) by Lemma 2.6 (vii), (viii), (ix). If \(C_0(\gamma) = D_4 \times e\), then \(\gamma^{+1}/2\mid |\Delta_1|\) and so \(q=7\) and (iii) occurs. But \((q+\varepsilon)/2 = 3\mid |\Delta_2| - 1 = 1\) is a contradiction. If \(C_0(\gamma) = PGL(2, \sqrt{q})\), then (i) does not occur because \(\sqrt{q} \not\equiv q - \varepsilon\). Hence \(\sqrt{q} \mid |\Delta_1|\) and \(\sqrt{q} \mid |\Delta_2|\) - 1. From this, \(q=25\) and (iii) occurs. In this case, we have \(|\Delta_1| = 10\), so that an element of \(C_0(\gamma)\) of order 3 is contained in \(N^*_\beta\) for some \(\gamma \in \Delta_1\), contrary to \(N^*_\beta = N^*_\beta\).

4. Case (II)

In this section we assume that \(N^*_\beta = PGL(2, p^m)\), where \(n = 2mk\) and \(k\) is odd. Since \(n\) is even, \(q = p^m \equiv 1\) (mod 4). We set \(p^m \equiv e \equiv \{\pm 1\}\) (mod 4). In section 7 we shall consider the case that \(N^*_\beta = S_4\). Therefore we assume \((p, m) \neq (3, 1)\) in this section.

(4.1) \(\text{The following hold.}\)

(i) \(N^*_\beta/N^*_\beta \cap N^\beta \leq 1\) or \(Z_2\) and \(N^*_\beta \cap N^\beta \geq (N^*_\beta) \cap PSL(2, p^m)\).

(ii) If \((p, m) \neq (5, 1)\), there exists a cyclic subgroup \(Y\) of \((N^*_\beta)\) such that \(N^*_\beta(Y) = D_4 \times e\) and \(N_0(Y)^{P(Y)}\) is doubly transitive.

Proof. As \(N^*_\beta \supseteq N^*_\beta \cap N^\beta\), either \(N^*_\beta \cap N^\beta \leq Z_2\) or \(N^*_\beta \cap N^\beta \equiv 1\). If \(N^*_\beta \cap N^\beta = 1\), by Lemma 2.2 and 2.6 (vi), \(N^*_\beta = N^*_\beta/N^*_\beta \cap N^\beta = N^*_\beta/N^*_\beta \cap N^\beta = Z_2 \times Z_2\), a
contradiction. Therefore \( N^\alpha_\beta/N^\alpha \cap N^\beta = 1 \) or \( N^\alpha \cap N^\beta \supseteq (N^\alpha_\beta)' = \text{PSL}(2,p^m) \).

Now we assume that \((p,m) = (3,1)\) and let \( z \) be an involution in \((N^\alpha_\beta)'\). Then \( C_{N^\alpha_\beta}(z) = D_{2p^m-2} \) by Lemma 2.6 (vii). Suppose \( C_{N^\alpha_\beta}(z) \) is not a 2-subgroup and put \( Y = 0(C_{N^\alpha_\beta}(z)) \). Then, if \( Y^z \leq G_{ab} \) for some \( g \in G \), we have \( Y^z \leq N^\alpha \) and \( Y^z \leq N^\beta \), where \( \gamma = \alpha^\delta \) and \( \delta = \beta^\delta \). By (i) \( Y^z \leq N^\alpha \cap N^\beta \) and so \( Y^z = Y^\delta \) for some \( h \in N^\alpha \cap N^\beta \). Thus \( N_G(Y)^F \) is doubly transitive. Assume that \( C_{N^\alpha_\beta}(z) \) is a 2-subgroup and set \( C_{N^\alpha_\beta}(z) = \langle u, v | u^2 = u^{-1}, v^2 = 1 \rangle \). We may assume that \( v \in (N^\alpha_\beta)' \) and \( \langle u^2, v \rangle \) is a Sylow 2-subgroup of \((N^\alpha_\beta)'\). Since \( p^m + 3,5 \), the order of \( u^2 \) is at least four. On the other hand there is no element of order \( |u^2| \) in \( \langle u, v \rangle = \langle u^2, v \rangle \). Hence any element of order \( |u^2| \) which is contained in \( N^\alpha \) is necessarily an element of \( N^\alpha \cap N^\beta \). By the similar argument as above, \( N_G(Y)^F \) is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose \((p,m) \neq (3,1), (5,1)\) and set \( \Delta = F(Y) \) and \( X = N_G(Y) \). Then \( |\Delta| = rs(p^m + \varepsilon)/2 + 1 \), where \( s = \sum_{i=0}^{\ell} p^{2mi} \), \( C_G(N^\alpha) = 1 \) and one of the following holds.

(i) \( X^\Delta \leq \text{Alt}(1,2)^c \) for some integer \( c \).

(ii) \( X^\Delta = \text{PSL}(2,p_1) \) or \( \text{PGL}(2,p_2) \), \( r = 1 \) and \( 2p_1 = p^m + \varepsilon \).

Proof. By Lemma 2.8 (ii), \( |\Delta| = 1 + |N^\alpha \cap X| |r| |N^\alpha_\beta \cap X| = 1 + (p^{2mk} - 1) \) \( r/2(p^m - \varepsilon) = rs(p^m + \varepsilon)/2 + 1 \). By (4.1) and Lemma 2.9, we have (i), (ii) or \( X^\Delta = R(3) \).

Assume that \( X^\Delta = R(3) \). Then \( rs(p^m + \varepsilon)/2 + 1 = 28 \), hence \( k = 1 \) and \( r(p^m + \varepsilon)/2 = 27 \). Since \( r \) is odd and \( r | 2m = n \), we have \( r = m = 1 \) and \( q = 53 \). But a Sylow 3-subgroup of \( X^\alpha \) is cyclic because \( N^\alpha \cap X = D_{2q-4} \) and \( X^\alpha/X \cap N^\alpha \cong X_{A^*}N^\alpha/N^\alpha \leq \mathbb{Z}_2 \times \mathbb{Z}_2 \), a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose \( X^\Delta = \text{Alt}(1,2^2) \) and put \( W = N^\alpha_\beta(Y) \). Then \( Y \leq W = Z_{p^m - 2} \). Since \( C_N(Y) \) is cyclic, \( W \) is a characteristic subgroup of \( C_N(Y) \) and so \( W \) is a normal subgroup of \( X_\alpha \). Hence \( W \leq X^\Delta \) and \( (X \cap N^\alpha_\beta) \cong 1 \) or \( Z_2 \). By Lemmas 2.4 and 2.6, \( F(X \cap N^\alpha_\beta) = 1 + |X \cap N^\alpha_\beta| + \sum_{r \leq |\Delta|} |X \cap N^\alpha_\beta| \times r + N^\alpha_\beta| = 1 + r \). Since \( r < |\Delta| \), \( (X \cap N^\alpha_\beta) \cong \mathbb{Z}_2 \) and hence \( 1 + r = rs(p^m + \varepsilon)/2 + 1 \) by Lemma 2.5. From this, \( r = s(p^m + \varepsilon)/2 - 2 |mk \) and so \( p^{2m(k-1)} + mk \leq 2 \). Hence \( m = k = r = 1 \) and \( q = 7^2 \).

Let \( R \) be a Sylow 3-subgroup of \( N^\alpha_\beta \). Since \( N^\alpha_\beta = \text{PGL}(2,7) \), we have \( R = Z_3 \). By Lemmas 2.4 and 2.6, \( |F(R)| = 1 + (7^2 - 1) |N^\alpha_\beta| \). Hence \( N_G(R)^F = A_4 \) or \( S_4 \). But is a Sylow 3-subgroup of \( N_G(R) \) because \( N^\alpha = \text{PSL}(2,7) \), contrary to \( N_G(R)^F = A_4 \) or \( S_4 \).

(4.4) (ii) of (4.2) does not occur.
Proof. Let notations be as in (4.2). Suppose \( X^a \geq PSL(2, p_1) \). By the similar argument as in (4.3), \( C_{N^a}(Y) \leq X_\Delta \) and so \( C_{N^a}(Y) \cong Z_{p_1} \), and \( N_{N^a}(Y)^a \cong D_{2p_1} \). Hence \(|X^a| = |2p_1-2n|\). Since \( X^a \geq PSL(2, p_1) \), \( p_1(p_1-1) - 2| |(X^a)| \), hence \( p_1 - 1 | 8n \). As \( k = 1 \) and \( 2p_1 = p^m + \varepsilon \), we have \( p^m + \varepsilon - 2 | 32m \). From this, \( (p, m, p_1) = (11, 1, 5), (3, 2, 5) \) or \( (3, 3, 13) \).

Let \( R \) be a cyclic subgroup of \( N^a_5 \) such that \( R = Z_{(p^m + \varepsilon)/2} \). By Lemma 2.6, \( N_G(R)^{F(R)} \) is doubly transitive and by Lemma 2.8 (ii), \(|F(R)| = 1 + |N_{N^a}(R)|/(|N_{N^a}(R)| - 1) + (p^m - \varepsilon)/2 + 1\).

If \( (p, m, p_1) = (11, 1, 5) \), \(|F(R)| = 7\) and so by [9] \(|N_G(R)^{F(R)}| = 42\) and \( N_{G_5}(R)^{F(R)} \cong Z_6 \). Since \(|N_{N^a}(R)| = 6\), \( N_{N^a}(R)^{F(R)} = N_{G_5}(R)^{F(R)} \). Hence \( N_{N^a}(R)(N_{N^a}(R))^\Delta \cong Z_2 \times Z_2 \), a contradiction.

If \( (p, m, p_1) = (3, 2, 5) \), \(|F(R)| = 5\) and so by [9], \(|N_G(R)^{F(R)}| = 10\) and \( N_{G_5}(R)^{F(R)} \cong Z_6 \). Since \(|N_{N^a}(R)| = 4\), \( N_{N^a}(R)^a = Z_4 \), contrary to \( N_{N^a}(R)/\langle N_{N^a}(R) \rangle \cong Z_2 \times Z_2 \).

If \( (p, m, p_1) = (3, 3, 13) \), \(|F(R)| = 15\). By [9], \( N_{G_5}(R)^{F(R)} \) is not solvable, a contradiction.

(4.5) \( p^m = 5 \).

Proof. Assume that \( p^m = 5 \). Then \( n = 2k \) with \( k \) odd and \( N^a_5 = PGL(2, 5) \) \( \cong S_5 \). First we argue that \( N^a_5 = N^a \cap N^a \) \( \cong S_5 \) and the outer automorphism group of \( S_5 \) is trivial, we have \( Z(N^a_5 N^a) \cong Z_2 \).

Let \( \omega \) be the involution of \( Z(N^a_5 N^a) \) and let \( \omega \in \mathcal{I}(N^a_5) \). Since \( C_{N^a}(\omega) \cong PGL(2, 5) \), \( |F(\omega)| = 1 + q(\omega - \varepsilon)/24 \).

By Lemma 2.8 \(|N^a_5| = 1 + 5r(25^2 - 1)/24\). Let \( P \) be a Sylow 5-subgroup of \( C_{N^a}(\omega) \). Then \(|P| = 5^k\) and \(|\gamma^P| = 5^{k_1}\) for each \( \gamma \in \Omega - \{\alpha\} \).

By Lemma 2.9, \( C_{G_5}(N^a_5) = 1 \) and \( N^a_5 = N^a \cap N^a \).

Let \( V \) be a cyclic subgroup of \( N^a_5 \) of order 4. Since \( N^a_5 = N^a \cap N^a \), \( N_G(V)^{F(V)} = S_5 \), \( N_G(V)^{F(V)} \) is doubly transitive and by Lemma 2.8, \(|F(V)| = 1 + |N_{N^a}(V)| = 3s^2 + 1 \) and \( s^2 = 25^t \) for each \( \gamma \in \Omega\). By Lemma 2.9, \( C_{G_5}(N^a) = 1 \) and \( a) N_G(V)^{F(V)} \cong SL(2, 3) \) or \( b) N_G(V)^{F(V)} \cong \Gamma L(1, 2) \).

Put \( P = N_{N^a_5}(V) \). Then \( P = D_5 \) and \(|F(P)| = 1 + |N_{N^a}(P)| = 120 \) and \( P^{F(V)} \cong Z_2 \times Z_2 \).

If \( (a) \) occurs, \( k = 1 \) and \( r = 9 \), hence \(|F(P)| = 10\), a contradiction. Therefore \( (a) \) holds.

By Lemma 2.5, \(|r + 1|^2 = 3s^2 + 1\) and so \( r = 3s - 2 | k \). Hence \( k = r = 1 \) and \( G_5|N^a \leq Z_2 \times Z_2 \). Let \( \zeta \) be an involution in \( N^a_5 \). Then \(|F(\zeta)| = 1 + 24 \cdot 25/120 = 6 \).
by Lemma 2.8 and $|\Omega|=1+|N^a|: N^a\beta|=66$ as $r=1$. By the similar argument as in the proof of (3.12), $|F(\alpha)|/(|F(\alpha)|-1)/2+(|\Omega|-|F(\alpha)|)/2=|C_G(\alpha)|/|z^e\cap \langle t\rangle G_{ab}\rangle/|\langle t\rangle G_{ab}\rangle|$, where $t$ is an involution such that $\alpha^t=\beta$. Hence $|z^e\cap \langle t\rangle G_{ab}\rangle|=15|G_{ab}\rangle/|C_G(\alpha)\rangle$. Set $H=\langle t\rangle G_{ab}$ and let $R$ be a Sylow 3-subgroup of $N^a\beta$. By Lemma 2.8, $|F(R)|=1+24\cdot 10/20=3$. Set $F(R)=\{\alpha,\beta,\gamma\}$. On the other hand, as $N^{ab}=S_5$ and $Out(S_5)=1$, we have $H=Z(H)\times N^{ab}$ and $|Z(H)|=2,4$ or $H=C_H(N^a\beta)\times N^a\beta$ and $Z(G_{ab})=Z_2\times Z_2$, contrary to Lemma 2.6 (ix). In the former case, we have $|Z(H)|=2$. For otherwise $Z(H)<G_{ab}$ and $Z(H)\cap G_{ab}$ and so letting $u^Z(H)$, we have $|2|=3|F(\kappa)|-1=5$, a contradiction. Therefore $Z(H)=Z_2$ and so $|G_{ab}|=25+25=50$, while $|G_{ab}|=15|G_{ab}|/|C_G(\alpha)|=15\cdot 120/24=75$, a contradiction.

5. Case (III)

In this section we assume that $N^a\beta=PSL(2,p^{\alpha})$, where $n=mk$ and $k$ is odd. Set $p^m=\pm 1 \pmod 4$. Then $q^m=\pm 1 \pmod 4$ as $k$ is odd. In section 6 we shall consider the case that $N^a\beta=A_4$, so we assume $(p,m)=(3,1)$ in this section. From this $N^a\beta$ is a nonabelian simple group and so $N^a\beta=N^a\cap N^b$ or $N^a\cap N=1$. If $N^a\cap N^b=1$, then $C_G(N^a)a=1$ by Lemma 2.2 and $N^a\beta=N^a\cap N^b=N^a\cap N^b/N^a\approx Z_2\times Z_2$, a contradiction. Hence $N^a\beta=N^a\cap N^b$.

Let $z$ be an involution of $N^a\beta$. Suppose $z^e\in G_{ab}$ for some $g\in G$ and set $\gamma=\alpha^z, \delta=\beta^z$. Then $z^e\in N_3\cap G_{ab} \leq N_3^\alpha \cap N_3^a \leq N^a \cap N^b$ and so $z^e\in z^e\beta^e$. Hence $C_G(z)^{F(z)}$ is doubly transitive and by Lemma 2.8 (i), $|F(z)|=(q-\delta)r/(p^m-\epsilon)+1$.

In particular $|F(z)|>3r+1$ as $(p^m-\epsilon)/(p^m-\epsilon)\geq p^m+1/p^m+1>3$.

By Lemma 2.9, $C_G(N^a)=1$ and one of the following holds.

(a) $C_G(z)^{F(z)}\leq ATL(1,2')$.

(b) $C_G(z)^{F(z)}\geq PSL(2,p_1)$.

(c) $C_G(z)^{F(z)}=R(3)$.

Let $Y$ be a cyclic subgroup of $C_{N^a\beta}(z)\approx D_{p^m-\epsilon}$ of index 2. Since $C_{Ga}(z)\geq Y, z\in Y$ and $C_G(z)^{F(z)}$ is doubly transitive, we have $F(Y)=F(z)$. By the similar argument as in (3.1), $N^a\cap N(C_{Na}(z))=C_{Na}(z)$ or $N^a\cap N(C_{Na}(z))=A_4$. Hence by Lemmas 2.3 and 2.4, $|F(C_{Na}(z))|=1+|C_{Na}(z)|/|N^a|: N^a\beta\approx |N^a\beta|/|N^a\beta|$ or $1+|A_4|/|N^a\beta|: N^a\beta\approx |N^a\beta|/|N^a\beta|$. Therefore $|F(C_{Na}(z))|=r+1$ or $3r+1$. From this $C_{Na}(z)^{F(z)}\approx Z_2$.

In the case (a), $(r+1)^2=1+(p^m-\epsilon)r/(p^m-\epsilon)$ by Lemma 2.5 and hence $r=(p^m-\epsilon)/((p^m-\epsilon)+1)/(p^m+1)=\sum_{i=0}^{k-1}(-p^m)^i/k$, where $k=3, m=1$ and $p=3$, contrary to $(p,m)=(3,1)$.

In the case (b), $r=1, p_1=(p^m-\epsilon)/(p^m-\epsilon), p\cap (p-1)/2$ and $s<s^4mkp_1$, where $s$ is the order of $C_{Ga}(z)^{F(z)}$. Hence $p_1-1=8mk$. Since $p_1-1=(p^m-\epsilon)/(p^m-\epsilon)-1$
(p^m+1)(p^m+1) - 1 = \sum_{k=0}^{\infty} (-1)^k \geq p^{m(k-2)}(p^m-1), we have \( p^m(k-2)/2k \leq 4m(p^m-1) \geq 1 \) because \( p^m \neq 3 \). Hence \( k=3 \) and \( p^m = 5 \), so that \( p_1-1 = 30 \times 8m = 24 \), a contradiction.

In the case (c), \( r+1 = 4 \) and \( 1 + (p^m-\varepsilon)r/(p^m-\varepsilon) = 28 \) and so \( r = 3 \) and \( (p^m-\varepsilon)/(p^m-\varepsilon) = 9 \). Hence \( 9 \geq (p^m+1)/(p^m+1) \geq p^{m-1} - p^m + 1 \), so that \( p^m = 3 \), a contradiction.

6. Case (IV)

In this section we assume that \( N_\phi = A_4 \) and \( q = 3, 5 \pmod{8} \). If \( N_\phi \cap N_\phi = 1 \), by Lemma 2.2, \( C_G(N_\phi) = 1 \) and so \( N_\phi/N_\phi \cap N_\phi = N_\phi/N_\phi \leq Z_2 \times Z_2 \). Hence \( N_\phi/N_\phi \cap N_\phi = 1 \) or \( Z_3 \), so that \( z^G \cap G_{ab} = z^G \cap N_\phi = z^N_\phi \) for an involution \( z \in N_\phi \).

Therefore \( C_G(z)^F(z) \) is doubly transitive. By Lemma 2.9, \( C_G(N_\phi) = 1 \) and one of the following holds.

(a) \( C_G(z)^F(z) \leq AGL(1, 2^c) \) for some integer \( c \geq 1 \).
(b) \( C_G(z)^F(z) \geq PSL(2, p_1) (p_1 > 5) \), \( r = 1 \) and \( |C_{N_\phi}(z_\phi)| \leq p_1 \).
(c) \( C_G(z)^F(z) = R(3) \).

Let \( T \) be a Sylow 2-subgroup of \( N_\phi \). Then \( z \in T \) and by Lemmas 2.3 and 2.4, \( |F(T)| = 1 + |N_{N_\phi}(T)| |N_\phi| = r+1 \). By Lemma 2.8 (i), \( |F(z)| = (q-\varepsilon)r/4 + 1 \). Hence \( T^F(z) = Z_2 \) if \( q = 5 \). If \( q = 5 \), as \( PSL(2, 5) \leq PSL(2, 4) \), (ii) of our theorem holds by [4]. Therefore we may assume \( q = 5 \).

In the case (a), \( (r+1)^2 = 1 + (q-\varepsilon)r/4 \) by Lemma 2.5. Hence \( r = (q-\varepsilon-8)/4 \) and \( r \mid n \), so that \( q = 11 \) or \( q = 33 \). Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \). Then \( R \leq N_\phi \) because \( G_{ab}/N_\phi = G_{ab}/N_\phi \leq 1 \) or \( Z_3 \) and \( N_\phi = A_4 \).

By Lemma 2.8 (ii), \( |F(R)| = 1 + 12/3 = 5 \) and \( N_G(R)^F(R) \) is doubly transitive. Since \( N_G(R) = D_{12} \) or \( D_{24} \) and \( |F(R)| = 5 \), we have \( |N_G(R)| = 5 \). Let \( S \) be a Sylow 5-subgroup of \( N_G(R) \). Then \( [S, R] = 1 \) as \( N_G(R) \leq Z_3 \). Since \( 5 \mid |G_{ab}| \), \( |F(S)| = 0 \) or 1. If \( |F(S)| = 1 \), then \( F(S) \leq F(R) \) and so \( 5 \mid |F(R)| = 1 = 4 \), a contradiction.

Therefore \( S \) is semi-regular on \( \Omega \). But \( |\Omega| = 1 + |N_\phi| = 1 + 56 \) or \( 92 \). This is a contradiction.

In the case (b), \( p_1(p_1-1)/2 \mid s \) and \( s/2n(q-\varepsilon)/2 = 4n \), where \( s \) is the order of \( C_{G_{ab}}(z) \). Hence \( p_1 = 1 \mid 8n \). Since \( p_1 = (q-\varepsilon)/4 \), \( p^m = 4 \mid 32n \) and so we have \( q = 11, 13, 19, 27 \) or 37. If \( q = 27 \), by Lemma 2.6, \( C_{G_{ab}}(z) = D_{12} \) or \( D_{24} \) and so \( C_{G_{ab}}(z)^F(z) = Z_2 \). Hence \( (p_1-1)/2 = -2 \). From this \( q = 19 \).

Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \). By the simmilar argument as in the case (a), \( N_G(R)^F(R) \) is doubly transitive and \( |F(R)| = 1 + 18/3 = 7 \). Hence \( |G| = 7 \mid |G| \). On the other hand \( |G| = |\Omega| \mid G_{ab} = (1 + |N_\phi| \mid N_\phi|)|G_{ab}| = (1 + 18 \cdot 19 \cdot 20/2 \cdot 12 \cdot 2^4 \cdot 18 \cdot 19 \cdot 20/2 = 2^3 \cdot 3^5 \cdot 11 \cdot 13 \cdot 19 \) with \( 0 \leq \lambda \leq 1 \), a contradiction.

If \( q = 27 \), then \( |C_G(z)| = |F(z)| \cdot |C_{G_{ab}}(z)| = 8 \cdot |G_{ab}| = 2 \cdot 4 \mid |G_{ab}| \), while \( |\Omega| = 1 + |N_\phi| \mid N_\phi| = 1 + 26 \cdot 27 \cdot 28/2 \cdot 12 = 820 = 2^3 \cdot 5 \cdot 41 \) and so \( |G_{ab}| = 4 \mid |G_{ab}| \). Therefore \( |C_G(z)| \mid |G| \), a contradiction.

In the case (c), \( r+1 = 4 \) and \( 1 + (q-\varepsilon)r/4 = 28 \). Hence \( r = 3 \) and \( q = 37 \),
contrary to \( r \mid n \).

7. Case (V)

In this section we assume that \( N^* = S_4 \) and \( q \equiv 7,9 \pmod{16} \). We note that \( 4 \not\mid n \).

First we argue that \( N^* = N^* \cap N^\beta \). Suppose \( N^* = N^* \cap N^\beta \). Then \( C_\omega(N^\beta) = 1 \) by Lemma 2.2. Since \( N^\beta_\omega/N^\omega \cap N^\beta = N^\omega_\beta N^\beta_\omega/N^\beta \leq Z_2 \times Z_2 \), we have \( N^\omega \cap N^\beta = A_4 \) and \( N^\beta_\omega/N^\omega \cap N^\beta = Z_2 \), so that \( N^\omega_\beta N^\beta_\omega/N^\beta = N^\omega_\beta N^\beta_\omega/N^\beta \cap N^\beta = Z_2 \). Hence as \( \text{Out}(S_4) = 1 \), \( Z(N^\beta_\omega) = Z \) and \( t \in I(N^\beta_\omega) - I(N^\beta) \). Since \( C_{N^\beta}(t) \geq N^\beta = S_4 \) and \( C_{N^\beta}(t) = N^\beta \), by Lemma 2.6, we have \( C_{N^\beta}(t) = PGL(2, \sqrt{q}) \) and \( |G(t)| = 1 + 3(q - \ell)/8 \) by Lemma 2.8.

Let \( P \) be a Sylow \( p \)-subgroup of \( C_{N^\beta}(t) \). Then \( |P| = \sqrt{q} \) if \( p \neq 3 \), \( P \) acts semi-regularly on \( F(t) \) and so \( N^\beta = N^\omega \cap N^\beta \). Hence \( \sqrt{q} \leq n^2 \) for any positive integer \( n \). This is a contradiction. If \( p = 3 \), \( |P| = \sqrt{q} \) for each \( \gamma \in \Omega - \{\alpha\} \). Hence \( \sqrt{q}/3 \mid |\gamma - 3\ell|/8 \) and so \( q > 81r^2 \). In particular, \( 3 \leq q \leq 81n^2 \). From this, \( n \leq 7 \). Since \( q = 3^{\ell} \equiv 7 \) or 9 (mod 16), we have \( q = 3^2 \) or \( 3^6 \). If \( q = 3^6 \), \( |\Omega| = 1 + |N^\beta| = 1 + 8 \cdot 9 \cdot 10 \cdot 24 = 16 \), a contradiction by [9]. If \( q = 3^6 \), \( |F(t)| = 1 + 273r \) and \( |F(t) - \{\alpha\}| \geq |C_{N^\beta}(t)| \geq |PGL(2, 3^6)|/8 = 2457 \) contrary to \( r \mid 3 \).

Let \( V \) be a cyclic subgroup of \( N^\beta_\omega \) of order 4 and let \( U \) be a Sylow 2-subgroup of \( N^\beta_\omega \). Then \( U = N^\omega_\beta(V) \). \( |F(V)| = 1 + (q - 6)/8 \) by Lemma 2.8 and \( |F(U)| = 1 + 8 \cdot 3r/24 = r + 1 \) by Lemmas 2.3 and 2.4. If \( q \neq 7,9 \), then \( |F(U)| < |F(V)| \) and hence \( U^{F(V)} = Z_2 \). Suppose \( q = 7 \) or 9. Then \( r = 1 \) as \( r \mid n \). Hence \( |\Omega| = 1 + |N^\beta| = 8 \) or 16. By [10], we have a contradiction. Therefore \( U^{F(V)} = Z_2 \).

Suppose \( V^\ell \leq G_{ab} \) for some \( g \in G \) and set \( \gamma = \gamma^\ell \). Then \( V^\ell \leq g^{-1} N^\omega_\omega g \cap G_{ab} \leq N^\gamma \cap G_{ab} \leq N^\beta \cap N^\beta = N^\beta_\omega \). As \( N^\beta_\omega = S_4 \), \( V^\ell = V^h \) for some \( h \in N^\beta_\omega \). Hence \( N_\omega(V)^{F(V)} \) is doubly transitive. By Lemma 2.9, \( C_\omega(N^\omega) = 1 \) and one of the following holds.

(a) \( N_\omega(V)^{F(V)} \leq AGL(1,2^2) \).
(b) \( N_\omega(V)^{F(V)} \geq PSL(2, p_1), p_1 = (q - \ell)/8 \geq 5 \).
(c) \( N_\omega(V)^{F(V)} = R(3) \).

In the case (a), \( (r + 1)/2 = 1 + (q - \ell)/8 \) by Lemma 2.5 and so \( r = (q - \ell - 16)/8 \) and \( r \mid n \). From this \( q = 23 \) or 25 and \( r = 1 \). Since \( |\Omega| = 1 + |N^\omega| = 2 \cdot 127 \) or \( 2 \cdot 163 \), we have \( |G| = 2 \cdot |G_{ab}| \) while \( |N_\omega(V)| = |F(V)| = |\omega_\omega(V)| = 4 \cdot |G_{ab}| \), contrary to \( |\omega_\omega(V)| = |G| \).

In the case (b), \( p_1 = (p_1 - 1)/2 \) and \( 2n(q - \ell)/4 = 4np_1 \), where \( s \) is the order of \( N_{ab}(V)^{F(V)} \). Hence \( p_1 = 1 \). From this, \( p^\ell - \ell - 8 \mid 64n \) and so \( q = 23, 41, 71 \) or 73. Since \( p_1 \) is a prime and \( p_1 = (q - \ell)/8 \geq 5 \), \( q = 23, 41, 71, 73 \). Therefore \( q = 41 \) and \( |\Omega| = 1 + |N^\omega| = 1 + 40 \cdot 41 \cdot 42 \cdot 24 = 2^2 \cdot 359 \), so that \( |G| = 4 \cdot |G_{ab}| \).
Since $N^\alpha_\beta=N^\alpha \cap N^\beta$, $C_\alpha(z)^{F(z)}$ is transitive by Lemma 2.1. On the other hand $|F(z)|=1+\frac{40 \cdot 9}{24}=16$ by Lemma 2.8 (i) and so $|C_\alpha(z)| = 16|C_\alpha(z)|_{2^3}=16|G|$, contrary to $|C_\alpha(z)| \neq |G|$.

In the case (c), $r+1=4$ and $1+(q-\varepsilon)r/8=28$. Hence $r=3$ and $q=71$ or 73, contrary to $r \mid n$.

8. Case (VI)

In this section we assume that $N^\alpha_\beta=A_5$ and $q \equiv 3,5 \pmod{8}$. In particular, $n$ is odd. If $N^\alpha_\beta \neq N^\alpha \cap N^\beta$, then $N^\alpha \cap N^\beta=1$, $C_\alpha(N^\alpha)=1$ and so $N^\alpha_\beta=N^\alpha_\beta N^\beta/N^\beta \leq \text{Out}(N^\beta) \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$, a contradiction. Hence $N^\alpha_\beta=N^\alpha \cap N^\beta$. Let $z$ be an involution in $N^\alpha_\beta$ and $T$ a Sylow 2-subgroup of $N^\alpha_\beta$ containing $z$. Then, by Lemma 2.8, $|F(z)| = 1+(q-\varepsilon)15r/60=1+(q-\varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)| = 1+12 \cdot 5r/60=1+r$. Since $N^\alpha_\beta=N^\alpha \cap N^\beta$, $z^G \cap N^\alpha_\beta = z^G \cap N^\alpha_\beta$ and so $C_\alpha(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_\alpha(N^\alpha)=1$ and one of the following holds.

(a) $C_\alpha(z)^{F(z)} \leq \text{AGL}(1,2^3)$.

(b) $C_\alpha(z)^{F(z)} \cong \text{PSL}(2,p_1), p_1=(q-\varepsilon)/4 \geq 5$.

(c) $C_\alpha(z)^{F(z)} = R(3)$.

In the case (a), by Lemma 2.5, $(q-\varepsilon)/4=1$ or $(r+1)^2=1+(q-\varepsilon)r/4$. Hence $q=5$ or $r=(q-\varepsilon-8)/4 \mid n$. If $q=5$, then $N^\alpha_\beta=N^\alpha$, a contradiction. Therefore $p^\alpha-\varepsilon-8 \mid 4n$ and so $n=1$ and $q=11$ or 13. If $q=13$, we have $5 \nmid |G_\alpha|$, a contradiction. Hence $q=11$ and $|\Omega|=1+|N^\alpha|: N^\alpha_\beta = 1+10 \cdot 11 \cdot 12/2 \cdot 60=12$. By [9], $G_\alpha \simeq M_{11}, |\Omega|=12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/2=4np_1$, where $s$ is the order of $C_\alpha(a)^{F(a)}$. Hence $p_1 \mid 18n$ and so $p^\alpha-\varepsilon-4 \mid 32n$. From this $q=19, 27$ or 37. Since $5 \mid |G_\alpha|$, $q \equiv 27, 37$. Hence $q=19$ and $|\Omega|=1+|N^\alpha|: N^\alpha_\beta = 1+18 \cdot 19 \cdot 20/2 \cdot 60=2 \cdot 29$. Since $G_\alpha=\text{PSL}(2,19)$ or $\text{PGL}(2,19)$, $|G|=|\Omega|: |G_\alpha|=2 \cdot 29 \cdot 2^i \cdot 18 \cdot 19 \cdot 20/2=2^i \cdot 3^2 \cdot 5 \cdot 19 \cdot 29$ with $0 \leq i \leq 1$. Let $P$ be a Sylow 29-subgroup of $G$. Then $P$ is semi-regular on $\Omega$ and 3, 5, 19 \nmid |G_\alpha|$ because $N_\alpha(P)/C_\alpha(P) \leq \mathbb{Z}_4 \times \mathbb{Z}_2$. Hence $|G: N_\alpha(P)|=2^i \cdot 3^2 \cdot 5 \cdot 19$ with $0 \leq j \leq 4$, while $2^i \cdot 3^2 \cdot 5 \cdot 19 \equiv 1 \pmod{29}$ for any $j$ with $0 \leq j \leq 4$, contrary to a Sylow's theorem.

If $C_\alpha(z)^{F(z)} = R(3), r+1=4$ and $1+(q-\varepsilon)r/4=28$ and hence $r=3, q=37$, contrary to $r \mid n$.

**References**


[3] C. Hering: *Transitive linear groups and linear groups which contain irreducible


