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ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS IN WHICH \text{socle}(G_\alpha) IS NONSOLVABLE

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1. Introduction

Let \( G \) be a doubly transitive permutation group on a finite set \( \Omega \) and \( \alpha \in \Omega \). In \cite{O'Nan}, O'Nan has proved that \( \text{socle}(G_\lambda) = A \times N \), where \( A \) is an abelian group and \( N \) is 1 or a nonabelian simple group. Here \( \text{socle}(G_\lambda) \) is the product of all minimal normal subgroups of \( G_\lambda \).

In the previous paper \cite{previous}, we have studied doubly transitive permutation groups in which \( N \) is isomorphic to \( \text{PSL}(2,q) \), \( \text{Sz}(q) \) or \( \text{PSU}(3,q) \) with \( q \) even.

In this paper we shall prove the following:

\textbf{Theorem.} Let \( G \) be a doubly transitive permutation group on a finite set \( \Omega \) with \( |\Omega| \) even and let \( \alpha \in \Omega \). If \( G_\alpha \) has a normal simple subgroup \( N^* \) isomorphic to \( \text{PSL}(2,q) \), where \( q \) is odd, then one of the following holds.

(i) \( G_\Omega \) has a regular normal subgroup.
(ii) \( G_\Omega \cong A_6 \) or \( S_6 \), \( N^* \cong \text{PSL}(2,5) \) and \( |\Omega| = 6 \).
(iii) \( G_\Omega \cong M_{11}, N^* \cong \text{PSL}(2,11) \) and \( |\Omega| = 12 \).

In the case that \( G_\alpha \) has a regular normal subgroup, by a result of Hering \cite{Hering} we have \(|\Omega|, q|=(16,9), (16,5) \) or \((8,7)\).

We introduce some notations:

- \( F(X) \): the set of fixed points of a nonempty subset \( X \) of \( G \)
- \( X(\Delta) \): the global stabilizer of a subset \( \Delta \subseteq \Omega \) in \( X \)
- \( X_\Delta \): the pointwise stabilizer of \( \Delta \) in \( X \)
- \( X^A \): the restriction of \( X \) on \( \Delta \)
- \( m|n \): an integer \( m \) divides an integer \( n \)
- \( X^H \): the set of \( H \)-conjugates of \( X \)
- \( |X|_p \): maximal power of \( p \) dividing the order of \( X \)
- \( I(X) \): the set of involutions in \( X \)
- \( D_m \): dihedral group of order \( m \)

In this paper all sets and groups are finite.
2. Preliminaries

**Lemma 2.1.** Let $G$ be a transitive permutation group on $\Omega$, $\alpha \in \Omega$ and $N^\alpha$ a normal subgroup of $G_\alpha$ such that $F(N^\alpha) = \{\alpha\}$. Let the subgroup $X \leq N^\alpha$ be conjugate in $G_\alpha$ to every group $Y$ which lies in $N^\alpha$ and which is conjugate to $X$ in $G$. Then $N_G(X)$ is transitive on $\Delta = \{\gamma \in \Omega \mid X \leq N^\gamma\}$.

Proof. Let $\beta \in \Delta$ and let $g \in G$ such that $\beta^g = \alpha$. Then, as $X \leq N^\beta$, $X^g \leq N^\beta = N^\alpha$. By assumption, $(X^g)^h = X$ for some $h \in G_\alpha$. Hence $gh \in N_G(X)$ and $\alpha^{(gh)^{-1}} = \alpha^g = \beta$. Obviously $N_G(X)$ stabilizes $\Delta$. Thus Lemma 2.1 holds.

**Lemma 2.2.** Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and $N^\alpha$ a nonabelian simple normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. If $C_G(N^\alpha) = 1$, then $N^\alpha = N^\alpha \cap N^\gamma$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semiregular on $\Omega - \{\alpha\}$.

Proof. See Lemma 2.1 of [4].

**Lemma 2.3.** Let $G$ be a transitive permutation group on $\Omega$, $H$ a stabilizer of a point of $\Omega$ and $M$ a nonempty subset of $G$. Then

$$|F(M)| = |N_G(M)| \times |M^G \cap H|/|H| .$$

Here $M^G \cap H = \{g^{-1}Mg \mid g \in G\}$. 

Proof. See Lemma 2.2 of [4].

**Lemma 2.4.** Let $G$ be a doubly transitive permutation group on $\Omega$ and $N^\alpha$ a normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. Assume that a subgroup $X$ of $N^\alpha$ satisfies $X^G = X^N$. Then the following hold.

(i) $|F(X) \cap N^\beta| = |F(X) \cap \gamma N^\beta|$ for $\beta, \gamma \in \Omega - \{\alpha\}$.

(ii) $|F(X)| = 1 + |F(X) \cap N^\beta| \times r$, where $r$ is the number of $N^\alpha$-orbits on $\Omega - \{\alpha\}$.

Proof. Let $\Gamma = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\alpha$-orbits on $\Omega - \{\alpha\}$. Since $G_\alpha$ is transitive on $\Omega - \{\alpha\}$ and $G_\alpha \geq N^\alpha$, we have $|\Delta_i| = |\Delta_j|$ for $1 \leq i, j \leq r$. By assumption, $G_\alpha = N_G(X)N^\alpha$ and so $N_G(X)$ is transitive on $\Gamma$. Hence for each $i$ with $1 \leq i \leq r$ there exists $g \in N_G(X)$ such that $(\Delta_i)^g = \Delta_i$. Therefore $|F(X) \cap \Delta_i| = |F(X^g) \cap (\Delta_i)^g| = |F(X) \cap \Delta_i|$. Thus (i) holds and (ii) follows immediately from (i).

**Lemma 2.5** (Huppert [5]). Let $G$ be a doubly transitive permutation group on $\Omega$. Suppose that $\vartheta(G) = 1$ and $G_\alpha$ is solvable. Then for any involution $z$ in $G_\alpha$, $|F(z)|^2 = |\Omega|$. 

We list now some properties of $PSL(2, q)$ with $q$ odd which will be required
in the proof of our theorem.

**Lemma 2.6** ([2], [6], [10]). Set \( N = PSL(2, q) \) and \( G = Aut(N) \), where \( q = p^n \) and \( p \) is an odd prime. Let \( z \) be an involution in \( N \). Then the following hold.

(i) \(|N| = (q-1)q(q+1)/2\), \( I(N) = \langle z^N \rangle \) and \( C_N(z) = D_{q-1} \), where \( q \equiv \varepsilon \in \{ \pm 1 \} \) \((\text{mod} \ 4)\).

(ii) If \( q \equiv 3 \), \( N \) is a nonabelian simple group and a Sylow \( r \)-subgroup of \( N \) is cyclic when \( r \equiv \pm 2, p \).

(iii) If \( X \) and \( Y \) are cyclic groups of \( N \) and \( |X| = |Y| \equiv 2, p \), then \( X \) is conjugate to \( Y \) in \( \langle X, Y \rangle \) and \( N_g(X) = D_{q-1} \).

(iv) If \( X \leq N \) and \( X = Z_2 \times Z_2 \), \( N_N(X) \) is isomorphic to \( A_4 \) or \( S_4 \).

(v) If \( |N| \geq 8 \), \( N \) has two conjugate classes of four-groups in \( N \).

(vi) There exist a field automorphism \( f \) of \( N \) of order \( n \) and a diagonal automorphism \( d \) of \( N \) of order 2 and if we identify \( N \) with its inner automorphism group, \( \langle d, N = PGL(2, q) \rangle, \langle f, d \rangle N = G \) and \( G|N = Z_2 \times Z_n \).

(vii) \( C_N(d) = D_{q+1} \) and \( C_N(d)(z) = D_{q+1} \).

(viii) Suppose \( n = mk \) for positive integers \( m, k \). Then \( C_N(f^m) = PSL(2, p^m) \) if \( k \) is odd and \( C_N(f^m) = PGL(2, p^m) \) if \( k \) is even.

(ix) Assume \( n \) is even and let \( u \) be a field automorphism of order 2. Then \( I(G) = I(N) \cup d^N \cup u(d)^N \). If \( n \) is odd, \( I(G) = I(N) \cup d^N \).

(x) If \( H \) is a subgroup of \( N \) of odd index, then one of the following holds:

1. \( H \) is a subgroup of \( C_N(z) \) of odd index for some involution \( z \in N \).
2. \( H = PGL(2, p^m) \), where \( n = 2mk \) and \( k \) is odd.
3. \( H = PSL(2, p^m) \), where \( n = mk \) and \( k \) is odd.
4. \( H = A_4 \) and \( q \equiv 3, 5 \mod 8 \).
5. \( H = S_4 \) and \( q \equiv 7, 9 \mod 16 \).
6. \( H = A_5 \), \( q \equiv 3, 5 \mod 8 \) and \( 5 | (q-1)q(q+1) \).

**Lemma 2.7.** Let \( G, N, d, f \) be as defined in Lemma 2.6 and \( H \) an \( \langle d, N \rangle \)-invariant subgroup of \( N \) isomorphic to \( D_{q-1} \). Let \( W \) be a cyclic subgroup of \( \langle d \rangle \) of index 2 \((\text{cf. (vii) of Lemma 2.6}) \) and set \( Y = \langle 0_d(W \cap H) \rangle \). Then \( C_H(Y) = W \cdot C_N(Y) \).

Proof. By (viii) of Lemma 2.6, we can take an involution \( t \) satisfying \( \langle d \rangle H = \langle t \rangle W \) and \([f, t] = 1\). Since \( N_N(Y) = \langle f, d \rangle N_N(Y) = \langle f, d \rangle H, C_N(Y) = C_{\langle f, d \rangle H}(Y) = W \cdot C_N(Y) \). Suppose \( ht \in C(Y) \) for some \( h \in \langle f \rangle \). Since \( t \) inverts \( Y, h \) also inverts \( Y \) and so \( h^2 \) centralizes \( Y \). Hence some nontrivial 2-element \( g \in \langle h \rangle \) inverts \( Y \), so that \( C_H(g) \) contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, \( G^2 \) will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume \( G^2 \) has no regular normal subgroup.
Notation. \( C^* = C_G(N^*) \), which is semi-regular on \( \Omega - \{ \alpha \} \) by Lemma 2.2. Let \( r \) be the number of \( N^* \)-orbits on \( \Omega - \{ \alpha \} \).

Since \( G_\beta \supseteq N^* \), \( |\beta^N| = |\gamma^N| \) for \( \beta, \gamma \in \Omega - \{ \alpha \} \) and so \( |\Omega| = 1 + r \times |\beta^N| \).

Hence \( r \) is odd and \( N^*_\beta \) is a subgroup of \( N^* \) of odd index. Therefore \( N^*_\beta \) is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

**Lemma 2.8.** Let \( Z \) be a cyclic subgroup of \( N^*_\beta \) with \( |Z| \neq 1, p \). Then

(i) \( |Z| = 2, |F(Z)| = 1 + (q - \varepsilon) |I(N^*_\beta)| / |N^*_\beta| \).

(ii) \( |Z| = 2, |F(Z)| = 1 + |N^*_\beta(Z)| / |N^*_\beta(Z)| \).

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

**Lemma 2.9.** If \( N^*_\beta \neq D_{q-1} \), and \( Z \) is a cyclic subgroup of \( N^*_\beta \) with \( |Z| \neq 1, p \) and \( N_G(Z) \) is doubly transitive. Then \( C^* = 1 \) and one of the following holds.

(i) \( N_G(Z) \leq AGL(1, q) \) for some \( q \).

(ii) \( C_G(Z) \supseteq PSL(2, p_1), r = 1 \) and \( |F(Z)| = 1 + |N^*_\beta(Z)| / |N^*_\beta(Z)| \).

(iii) \( |N_G(Z)| = 28 \).

Proof. Set \( N_G(Z) = L \) and \( F(Z) = \Delta \). By Lemma 2.6(iii), \( L \cap N^* = D_{q-1} \) and \( L \cap N^* = \langle t \rangle \subseteq \Delta \geq Y \geq \Omega \), where \( 0(t) = 2, Y \subseteq Z(q^2 + 1) \).

If \( (L \cap N^*)^a = 1, \) then \( L \cap N^* = \Delta \) because \( L \cap N^* \) is a maximal subgroup of \( N^* \). Since \( |N^*| : \Delta | \) is odd, \( L \cap N^* = N^*_\beta = D_{q-1} \), contrary to the assumption. Hence \( (L \cap N^*)^a \neq 1 \) and as \( L_a \supseteq L_a \cap N^* \) and \( L_a \supseteq Y \), \( (L_a)^a \) has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:

(a) \( L^a \) has a regular normal subgroup
(b) \( L^a \supseteq PSL(2, p_1), |\Delta| = p_1 + 1 \), where \( p_1 \geq 5 \) is a prime
(c) \( L^a \supseteq PSL(3, p_1), p_1 \geq 3, |\Delta| = (p_1)^2 + 1 \)
(d) \( L^a = R(3), |\Delta| = 28 \).

Suppose \( C^* = 1. \) Then there exists a subgroup \( D \) of \( C^* \) of prime order such that \( (L_a)^D = D^a \). Since \( [L_a, D] \leq D \cdot L_a \cap C^* = D(L_a \cap C^*) = D \), \( D \) is a normal subgroup of \( L_a \). By (i) and (iii) of Lemma 2.6, \( G_a = L_a \cdot N^* \) and so \( D \) is a normal subgroup of \( G_a \). By Theorem 3 of [1], \( G^a \) has a regular normal subgroup, contrary to the hypothesis. Thus \( C^* = 1 \).

If (a) occurs, \( L^a \) is solvable because \( L_a / L \cap N^* \simeq L_a N^*/N^* \leq \text{Out}(N^*) \) and \( L \cap N^* = D_{q-1} \). Hence by [5], (i) holds in this case.

If (b) occurs, we have \( Y^a = 1 \), for otherwise \( (L \cap N^*)^a = 1 \) and so \( N^*_\beta = L \cap N^* = D_{q-1} \), a contradiction. Hence \( 1 \neq C_G(Z)^a \leq L^a \) and so \( C_G(Z)^a \supseteq PSL(2, p_1) \) and \( Y^a \simeq Z_{p_1} \). Therefore \( |\Delta \cap \beta^N| = p_1 \) and \( r = 1 \) by Lemma 2.4 (ii). Since \( |\beta^N| = p_1 \), we have \( |\beta^L \cap N^*| = p_1 \), so that \( L \cap N^*: L \cap N^* = p_1 \). Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of \( PSU(3, p_1) \),
a Sylow $p_1$-subgroup of $(L_\alpha)^\prime$ is not cyclic, while $(L_\alpha)^\prime \leq L \cap N^\sigma=D_{q_{zz}}$, a contradiction.

3. Case (I)

In this section we assume that $N^\sigma_\beta \leq D_{q-1}$, where $\beta \neq \alpha$, $q=p^\sigma$.

(3.1) (i) If $N^\sigma_\beta \neq Z_2 \times Z_2$, $N^\sigma_\beta(N^\sigma_\beta)=N^\sigma_\beta$ and $|F(N^\sigma_\beta)|=r+1$.

(ii) If $N^\sigma_\beta=Z_2 \times Z_2$, $N^\sigma_\beta(N^\sigma_\beta)=A_4$ and $|F(N^\sigma_\beta)|=3r+1$.

Proof. Put $X=N^\sigma_\beta(N^\sigma_\beta)$. Let $S$ be a Sylow 2-subgroup of $N^\sigma_\beta$ and $Y$ a cyclic subgroup of $N^\sigma_\beta$ of index 2.

If $N^\sigma_\beta \neq Z_2 \times Z_2$, then $|Y|>2$ and so $Y$ is characteristic in $N^\sigma_\beta$. Hence $X \leq N^\sigma_\beta(Y)=D_{q-1}$. From this $[N^\sigma_\beta(S), S \cap Y] \leq S \cap Y$ and $\theta(N^\sigma_\beta(S))$ stabilizes a normal series $S \supset S \cap Y \supset 1$, so that $\theta(N^\sigma_\beta(S)) \leq C_{N^\sigma_\beta}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^\sigma_\beta}(S) \leq S$ and hence $N^\sigma_\beta(S)=S$. On the other hand by a Frattini argument, $X=N^\sigma_\beta(S)N^\sigma_\beta$ and so $X=N^\sigma_\beta$. By Lemma 2.6(i), $(N^\sigma_\beta)^{G^\sigma}=(N^\sigma_\beta)^{N^\sigma_\beta}$ and so by Lemmas 2.3 and 2.4 (ii), $|F(N^\sigma_\beta)|=1+|F(N^\sigma_\beta) \cap \beta^\sigma|^r=1+|N^\sigma_\beta||r|\ |N^\sigma_\beta|=r+1$. Thus (i) holds.

If $N^\sigma_\beta=Z_2 \times Z_2$, $N^\sigma_\beta(N^\sigma_\beta)=A_4$ by Lemma 2.6 (iv). Similarly as in the case $N^\sigma_\beta \neq Z_2 \times Z_2$, we have $|F(N^\sigma_\beta)|=3r+1$.

(3.2) $N^\sigma_\beta/N^\sigma_\beta \cap N^\sigma \leq Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^\sigma=1$. Suppose $C^\sigma=1$. Then $N^\sigma_\beta/N^\sigma \cap N^\sigma \leq N^\sigma_\beta/N^\sigma \cap N^\sigma \leq \text{Out}(N^\sigma)=Z_2 \times Z_2$ by Lemma 2.6 (vi) and hence $(N^\sigma_\beta)^{G^\sigma} \leq N^\sigma_\beta \cap N^\sigma$. Since $N^\sigma_\beta$ is dihedral, $N^\sigma_\beta/(N^\sigma_\beta)^{G^\sigma} \cong Z_2 \times Z_2$, so that $N^\sigma_\beta/N^\sigma \cap N^\sigma \leq Z_2 \times Z_2$.

(3.3) Suppose $N^\sigma_\beta=N^\sigma \cap N^\sigma$ and let $U$ be a subgroup of $N^\sigma_\beta$ isomorphic to $Z_2 \times Z_2$. Then $|F(U)|=3r+1$ and $N^\sigma_\beta(U)^{F(U)}$ is doubly transitive.

Proof. Sex $X=N^\sigma_\beta(N^\sigma_\beta)$, $\Delta=F(N^\sigma_\beta)$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\sigma$-orbits on $\Omega-\{\alpha\}$. If $g^{-1}N^\sigma_\beta g \leq G_{\alpha\beta}$, then $g^{-1}N^\sigma_\beta g \leq N^\sigma_\gamma \cap N^\sigma_\delta=N^\sigma_\gamma \cap N^\sigma_\delta \leq N^\sigma_\beta$, where $\gamma=\alpha^\sigma$. By a Witt’s theorem, $X^\sigma$ is doubly transitive.

If $U$ is a Sylow 2-subgroup of $N^\sigma_\beta$, by a Witt’s theorem, $N^\sigma_\beta(U)^{F(U)}$ is doubly transitive. Moreover $N^\sigma_\beta(U)=N^\sigma_\beta$ and so by Lemmas 2.3 and 2.4 (ii), $|F(U)|=1+|A_4| \times |N^\sigma_\beta| \times r/|N^\sigma_\beta|=3r+1$.

If $|N^\sigma_\beta|>4$, by Lemma 2.6 (iv) and (v), $N^\sigma_\beta(U)=S_4$ and $N^\sigma_\beta$ has two conjugate classes of four-groups, say $\pi=\{K_1, K_2\}$. Set $X_\pi=M$. Then $M \geq N^\sigma_\beta$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_i \neq \phi$ for each $i$ and so $|F(U) \cap \Delta_i|=3$ by Lemma 2.3. Hence $|F(U)|=3r+1$. Since $N^\sigma_\beta(U)=S_4$, we may assume $r>1$. Hence by (3.1) (i), $|\Delta|=r+1 \geq 4$, so that $M^\sigma$ is doubly transitive. Since $M=N^\sigma_\beta N^\sigma_\beta(U)$, $M^\sigma(U)^{F(U)}$ is also doubly transitive and so $N^\sigma_\beta(U)$ is transitive on $\Delta$—
\{α\}. As \(|Δ \cap Δ_i| = 1\), \(Δ \cap Δ_i \subseteq F(U)\) and \(N_\ast(U)\) is transitive on \(F(U) \cap Δ_i\) for each \(i\), \(N_\ast(U)^{F(U)}\) is doubly transitive.

\[(3.4)\] (i) \(C^\ast = 1\).
(ii) Let \(U\) be a subgroup of \(N^\ast_\beta\) isomorphic to \(Z_2 \times Z_2\). If \(N^\ast_\beta = N^\ast \cap N^\beta\), then \(N_\ast(U)^{F(U)}\) has a regular normal 2-subgroup. In particular \(|F(U)| = 3r + 1 = 2^b\) for positive integer \(b\).

Proof. Since \(N_\ast(U)^{F(U)} \supseteq N^\ast_\beta(U)^{F(U)} \supseteq S_3\) or \(Z_3\), by (3.3) and Theorem 3 of [1], \(N_\ast(U)^{F(U)}\) has a regular normal subgroup, \(N_\ast(U)^{F(U)} \supseteq \text{PSU}(3,3)\) or \(N_\ast(U)^{F(U)} = R(3)\).

Suppose \(C^\ast = 1\). Let \(D\) be a minimal characteristic subgroup of \(C^\ast\). Clearly \(G_\ast D\). If \(N_\ast(U)^{F(U)} \supseteq R(3)\), \(D\) is cyclic. By Theorem 3 of [1], \(G^\ast\) has a regular normal subgroup, contrary to the hypothesis. Hence \(N_\ast(U)^{F(U)} = R(3)\).

Thus (3.4) holds.

(3.5) (i) If \(|Y| \geq 3\), \(N_\ast(Y)^{F(Y)}\) is doubly transitive.
(ii) If \(|Y| < 3\), \(N^\ast_\beta = Z_2 \times Z_2\) or \(N^\ast_\beta = D_4\) and \(N^\ast \cap N^\beta \leq Z_2 \times Z_2\).

Proof. Suppose \(|Y| \geq 3\). If \(Y^g \leq G_\ast\beta\), \(Y^g \leq N^\ast \cap G_\ast\beta \leq N^\ast_\beta\), where \(γ = α^g\).
If \(γ = α\), obviously \(Y^g \leq N^\ast\). If \(γ \neq α\), \(N^\ast_\alpha = N^\ast_\beta\). Therefore, as \(|Y| \geq 3\), \(N^\ast_\alpha\) has a unique cyclic subgroup of order \(|Y|\). Hence \(Y^g \leq N^\ast \cap N^\beta \leq N^\ast\), so that \(Y^g \leq N^\ast\). Similarly \(Y^g \leq N^\beta\). Thus \(Y^g \leq N^\ast \cap N^\beta\) and so \(Y^g = Y\). By a Witt’s theorem, \(N_\ast(Y)\) is doubly transitive on \(F(Y)\).

Suppose \(|Y| < 3\). Since \(|N^\ast \cap N^\beta| \leq 2\), we have \(N^\ast \cap N^\beta \leq Z_2 \times Z_2\).
On the other hand, as \(N^\ast_\beta\) is dihedral, \((N^\ast_\beta)^\prime\) is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set \(Δ = F(N^\ast_\beta)\), \(L = G(Δ)\), \(K = G_Δ\) and suppose \(N^\ast_\beta \neq Z_2 \times Z_2\). Then \(L_\ast \supseteq N^\ast_\beta, (L_\ast)^\prime \leq N^\ast_\beta, K^\prime \leq N^\ast \cap N^\beta\) and \((L_\ast)^\prime = Z_r\). If \(r \neq 1\), \(L^\ast\) is a doubly transitive Frobenius group of degree \(r + 1\).

Proof. By Corollary B1 of [7] and (i) of (3.1), \(L^\ast\) is doubly transitive and
$|\Delta|=r+1$. Since $N^* \cap L \geq N^* \cap K = N^*_a$, by (i) of (3.1), we have $N^* \cap L = N^*_a$. Hence $L_a \geq N^*_a$. By (i) of (3.4), $L_a/N^*_a = L_a/N^* \leq \text{Out}(N^*) = Z_2 \times Z_4$ and so $(L_a)_{\Delta} = Z_r$. If $r \neq 1$, then $(L_a)_{\Delta} = 1$. On the other hand $(L_a)^3 = 1$. As $L_a$ is abelian. Hence $L_a$ is a Frobenius group.

(3.7) Suppose $|Y| \geq 3$. Then there exists an involution $y$ in $N^*_a \cap Y$ such that $Z(N^*_a) = \langle y \rangle$.

Proof. Suppose $N^*_a \neq Z_2 \times Z_2$, $|N^*_a| \geq 2^2$ and $N^*_a$ is dihedral, we have $\langle I(W) \rangle = Z(N^*_a) = Z_2$ and $N^*_a/(N^*_a)^\gamma = Z_2 \times Z_2$. Let $Z(N^*_a) = \langle y \rangle$ and suppose that $z$ is not contained in $Y$. By (3.2), $(N^*_a)^\gamma \leq N^* \cap N^\beta \leq W = Y$ and so $(N^*_a)^\gamma$ is odd. Hence $|N^*_a|^2 = 4$ and $q \equiv \beta^2 \equiv 3 \text{ or } 5 \pmod 8$, so that $n$ is odd. By (3.2) and (i) of (3.4), $N^*_a/N^* \cap N^\beta = N^*_a/N^\beta = 1$ or $Z_2$. If $N^*_a = N^* \cap N^\beta$, then $W = Y$ and so $z \in Y$, contrary to the assumption. Therefore we have $N^*_a \cap N^\beta = Z_2$ and $N^*_a = \langle z \rangle \times (N^* \cap N^\beta)$. Since $n$ is odd and $z \in N^* \cap N^\beta = N^\beta$, by Lemma 2.6 (vi), (vii) and (ix), $N^*_a/N^\beta = \text{PGL}(2, q)$ and $C_{N^a}(z) = D_{q+1}$. But $N^* \cap N^\beta \leq C_{N^a}(z)$ and besides it is isomorphic to a subgroup of $D_{q-v}$. Hence $N^* \cap N^\beta = Z_2$ and $N^*_a = Z_2 \times Z_2$, a contradiction.

(3.8) Suppose $|Y| \geq 3$. Then $N^*_a = N^* \cap N^\beta$.

Proof. Suppose $N^*_a \neq N^* \cap N^\beta$ and let $\Delta, L, K$ be as defined in (3.6) and $x \in L_a$ such that its order is odd and $\langle x \rangle$ is transitive on $\Delta - \{\alpha\}$. As $|Y| \geq 3$, $W$ is characteristic in $N^*_a$ and hence by (3.6), $x$ stabilizes a normal series $L_a \geq N^*_a \geq W \geq (N^*_a)^\gamma$. By Theorem 5.3.2 of [2], $[x, 0, (L_a/(N^*_a)^\gamma)] = 1$. Since $L_a/(N^*_a)^\gamma$ has a normal Sylow $2$-subgroup and $(N^*_a)^\gamma \leq K'$, we have $[x, 0, (L_a/K')] = 1$, so that $[x, N^*_a] \leq K \leq N^* \cap N^\beta$ by (3.6). If $r \neq 1$, then $\beta^2 = \beta$ and $\beta^2 \in \Delta$, hence $N^*_a = x^{-1}N^*_a x = N^*_a$, where $\gamma = \beta^2$. Since $\gamma \in \Delta$ and $\Delta = F(N^*_a)$, $N^*_a \leq N^\beta \cap G_\gamma = N^\gamma$ and so $N^*_a = N^\gamma$. Similarly $N^\gamma = N^*_a$. Hence $N^*_a = N^\gamma$, which implies $N^*_a = N^* \cap N^\beta$. By the doubly transitivity of $G$, we have $N^*_a = N^* \cap N^\beta$, contrary to the assumption. Therefore we obtain $r = 1$.

Let $z$ be as defined in (3.7) and put $k = (q - \xi)/|N^*_a|$. By Lemma 2.8(i) we have $|F(z)| = 1 + (q - \xi)(|N^*_a|/2 + 1)/|N^*_a| = (q - \xi)/2 + k + 1$. Similarly $|F(Y)| = k + 1$. As $N^*_a \neq N^* \cap N^\beta$, there is an involution $t$ in $N^*_a$ which is not contained in $N^\beta$. By Lemma 2.6 (i), $t^z = z$ for some $z \in N^*$. Set $\gamma = \beta^2$. Then $\gamma \in F(z)$ and $z \in N^\gamma$. By Lemma 2.6 (vii), (viii) and (ix), $C_{N^a}(z) = D_{q+1}$ or $\text{PGL}(2, \sqrt{q})$. Assume $C_{N^a}(z) = D_{q+1}$ and let $R$ be a cyclic subgroup of $C_{N^a}(z)$ of index 2. We note that $R$ is semi-regular on $\Omega - \{\alpha\}$. Set $X = C_0(z)$. Since $2 \leq k + 1 \leq (q - \xi)/|q - \xi| + 1$, we have $(q - \xi)/2 + k + 1 = (q - \xi)/2 + k + 1$. By (i) of (3.5) and (3.7), $N_0(Y) \leq C_0(z) = X$ and $\alpha^X \in F(Y)$. It follows from Lemma 2.1 that $\alpha^X = \{z \in N^* \cup \gamma \neq \gamma \}$. Hence $|F(z)| = |\alpha^X| \geq |F(Y)| + (q + \xi)/2 = k + 1 + (q - \xi)/2 + \xi = |F(z)| + \xi$. Therefore $\xi = -1$ and $\gamma^X = \{\gamma \}$, so that $\gamma \in F(Y)$, a contradiction. Thus $C_{N^a}(z) = \text{PGL}(2, \sqrt{q})$, $\xi = 1$, $N^*_a/N^* \cap N^\beta = Z_2$ and $|\langle z^\rho \cap G_a \rangle| = N^* = 2$. 

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Set $\Delta_1 = \alpha^x$ and $\Delta_2 = F(z) - \Delta_1$. Let $\delta \in \Delta_2$ and $g$ an element of $G$ satisfying $\delta^g = \gamma$. Then $z \in N_\gamma N^g - N^\gamma$ and so $z^2 \in N_\gamma N^g - N^\gamma$, where $v = \alpha^x$. Since $\langle \delta^g \cap G_\gamma \rangle = N^g| = 2$ and $x \in G_\gamma - N^\gamma$, it follows from Lemma 2.6 (ix) that $(x)^g = x$ for some $h \in G_\gamma$. Hence $gh \in X$ and $\delta^h = \gamma$. Thus $\Delta_2 = \gamma^x$. Let $\delta \in \Delta_2$. Then $\delta \in N^z$ and so $z \in N^\gamma \cap \Delta_2 = \{\alpha, \beta\}$. Hence $C_{N^\gamma}(z)$ fixes $\alpha$ and $\beta$, so that $PGL(2,3) = \langle C_{N^\gamma}(z) \rangle \leq N_{\gamma}^\gamma = N^\gamma / \Delta_2$, a contradiction.

(3.9) **Suppose** $|Y| \geq 3$. Then $r = 1$.

**Proof.** By (3.6), $r + 1 = 2^c$ for some integer $c \geq 0$. On the other hand $3r + 1 = 2^k$ by (3.8) and (ii) of (3.4). Hence $2r = 2^c(2^k - c - 1)$ and so $c = 1$ as $r$ is odd. Thus $r = 1$.

(3.10) **Put** $k = (q - \varepsilon)/|N^\gamma|$. If $N^\gamma_\beta = N^\gamma \cap N^\beta$ and $r = 1$, then

$$q - \varepsilon + 2k + 2|2((2k + 2 - \varepsilon)(2k + 2 - \varepsilon)(k + 1 - \varepsilon)(2k + 2 - \varepsilon)(k + 1 - \varepsilon)).$$

**Proof.** Set $S = \{(z, u) | z \in F(u), u \in z^g\}$, where $z$ is an involution in $N^g$. We now count the number of elements of $S$ in two ways. Since $N^\gamma_\beta = N^\gamma \cap N^\beta$, $F(z) = \{\gamma \in F(u), u \in z^g\}$ and hence $C_G(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S| = |\Omega| |z^g| = |z^g| |F(z)|$. Since $r = 1$, $|\Omega| = 1 + |N^\gamma_\beta| = kq(q + \varepsilon)2 + 1$ and by Lemma 2.8 $|F(z)| = (q - \varepsilon)/2 + k + 1$. Since $G_\gamma \geq N^\gamma$, $z^g$ is contained in $N^\gamma$ and so $|G_\alpha| = |C_G(z)| = q(q + \varepsilon)/2$. Hence $|G_\alpha| = (q - \varepsilon)/2 + k + 1$. On the other hand $|F(z)| = |C_G(z)| |G_\alpha(z)| = (q - \varepsilon)/2 + k + 1$. Because $|G_\alpha(z)| = q(q + \varepsilon)/2 \geq 1$ (mod 2). Hence $q - \varepsilon + 2k + 2|2(k + 2 - \varepsilon)(k + 1 - \varepsilon)(2k + 2 - \varepsilon)(k + 1 - \varepsilon)$, we have (3.10).

(3.11) **Suppose** $|Y| \geq 3$. Then one of the following holds.

(i) $N^\gamma_\beta = N^\gamma \cap N^\beta = D_{q - r}$.

(ii) $N^\gamma_\beta = N^\gamma \cap N^\beta \neq D_{q - r}$ and $N_G(Y)^{F(Y)}$ has a regular normal subgroup.

**Proof.** Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(Y)} = R(3)$ or there exists a prime $p_1 \geq 5$ such that $C_G(Y)^{F(Y)} \geq PSL(2, p_1)$ and $V/Y = Z_{p_2}$ where $V = C_N(Y)$. By (i) of (3.1) and (3.9), $F(N^\gamma_\beta) = \{\alpha, \beta\}$. On the other hand, $(N^\gamma_\beta)^{F(Y)} \geq N^\gamma_\beta / Y = Z_2$. Hence $N_G(Y)^{F(Y)} \geq R(3)$ and $C_G(Y)^{F(Y)} \geq$
By (i) of (3.4) and Lemma 2.7, we have \( C_{G_a}(Y) = V \langle f_1 \rangle \), where \( f_1 \) is a field automorphism of \( N^* \). Let \( t \) be the order of \( f_1 \), \( n = tm \) and let \( p^m \equiv \varepsilon_1 \equiv \{ \pm 1 \} \) (mod 4). Clearly \( C_{G_a}(Y)^{F(Y)} \geq V \langle f_1 \rangle = Z_2 \) and \( |C_{G_a}(Y)^{F(Y)}| \mid |t| \), so that \( (p_1-1)/2 \mid |t| \).

First we assume that \( t \) is even and set \( t = 2t_1 \). Then \( Y \leq C_{N^*}(f_1) = PGL(2, p^m) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \) is a cyclic subgroup of \( C_{N^*}(f_1) \) of order \( p^m-\varepsilon_1 \) and \( (p^m-1)/2(p^m-\varepsilon_1) = p_1 \). Put \( s = \sum_{i=1}^{t_1} (p^m)^i \). Then \( (p^m+\varepsilon_1)s/2 = p_1 \), so that we have either (i) \( t_1 = 1 \) and \( p_1 = (p^m+\varepsilon_1)/2 \) or (ii) \( t_1 \geq 2 \), \( p^m = 3 \) and \( p_1 = s \). In the case (i), \( 2 \leq (p_1-1)/2 = (p^m+\varepsilon_1-2)/4 \mid 2t_1 = 2 \). Hence \( (p_1, q) = (5, 3^4) \) or \( (4, 11^2) \). Let \( z \) be as in (3.7). As mentioned in the proof of (3.10), \( |F(z)| = (q-1)/2+k+1 \), \( |\Omega| = kq(q+1)/2+1 \) and \( C_\Omega(z) \) is transitive on \( F(z) \). If \( q = 3^4 \), then \( |F(z)| = 46 \) and \( |\Omega| = 2 \cdot 19 \cdot 23 \). Hence \( |C_\Omega(z)| = |F(z)| \mid C_{G_a}(z) = |F(z)| \mid C_{G_a}(z) = 46 \cdot 2^t \cdot 31 = 2^{11} \cdot 5 \cdot 23 \) with \( 0 \leq i \leq 3 \). Let \( P \) be a Sylow 23-subgroup of \( C_\Omega(z) \) and \( Q \) a Sylow 5-subgroup of \( C_\Omega(z) \). Since \( 11 \mid |\Omega| \), \( P \) is a subgroup of \( N_\Omega \) for some \( \gamma \in \Omega \) and \( F(P) = \{ \gamma \} \). Hence \( \gamma \in F(z) \), so that \( z \in N_\Omega \), contrariwise to \( C_{N^*}(z) = D_{160} \). In the case (ii), we have \( (p_1-1)/2 = (\sum_{i=1}^{t_1-1} 9^i)/2 = 2t_1 \). From this, \( 9^{t_1-1} \leq 4t_1 \), hence \( t_1 = 1 \), a contradiction.

Assume \( t \) is odd. Then \( Y \leq C_{N^*}(f_1) = PSL(2, p^m) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \) is a cyclic subgroup of \( C_{N^*}(f_1) \) of order \( p^m-\varepsilon_1 \) and \( (p^m-1)/2(p^m-\varepsilon_1) = p_1 \). Hence \( \sum_{i=0}^{t_1} (p^m)^i (\varepsilon_i)^{t_1-i} = p_1 \) and \( (p_1-1)/2 = (\sum_{i=1}^{t_1} (p^m)^i (\varepsilon_i)^{t_1-i})/2 \mid t_1 \). In particular \( 2t \geq (p^m)^{t_1-1}-(p^m)^{t_2} = (p^m)^{t_1-2} \geq 2(p^m)^{t_1-2} \). From this \( t = 3 \), \( m = 1 \), \( p_1 = 7 \) and \( q = 3^3 \), so that \( N_\Omega = D_{160} \times D_{160} \), a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let \( G^a \) be a minimal counterexample to (3.12) and \( M \) a minimal normal subgroup of \( G \). By the hypothesis, \( G \) has no regular normal subgroup and hence \( M \neq \pm 1 \). As \( M_a \) is a normal subgroup of \( G_a \), by (i) of (3.4), \( M_a \) contains \( N^* \). By (3.9), \( r = 1 \), hence \( M \) is doubly transitive on \( \Omega \). Therefore \( G = M \) and \( G \) is a nonabelian simple group.

Since \( N_\Omega = D_{160} \), \( k = 1 \) and so \( q-\varepsilon+4 \mid 2((4-\varepsilon)(2-\varepsilon)+1)(4-\varepsilon)(2-\varepsilon) \) by (3.10). Hence we have \( q = 7, 9, 11, 19, 27 \) or 43.

Let \( x \) be an element of \( N_\Omega \). If \( |x| > 2 \), by Lemma 2.8, \( |F(x)| = 1+|N_\Omega| \times 1/|N^*_\Omega| = 2 \) and if \( |x| = 2 \), similarly we have \( |F(x)| = (q-\varepsilon)/2+2 \). Assume \( q = 9 \) and let \( d \) be an involution in \( G_a - N^* \) such that \( \langle d \rangle N^* \) is isomorphic to \( PGL(2, p^m) \).
We may assume \(d \in G_{ab}\). Since \(\langle d \rangle N^{a}\) is transitive on \(\Omega - \{\alpha\}\), by Lemmas 2.3 and 2.6 (vii), (ix), \(|F(d)| = 2(q-1)(q+1)2/(q+1) + 1 = (q+1)/2\), while \(|F(x)| = (q+1)/2 + 2\) for \(x \in I(N^{a})\). Hence \(d\) is an odd permutation, contrary to the simplicity of \(G\). Thus \(G_{a} = N^{a}\) if \(q \neq 9, 27\) and \(|G_{a}/N^{a}| = 1, 3\) if \(q = 27\).

If \(q = 9\), \(|\Omega| = 1 + |N^{a}|: N_{a}^{a}| = 1 + 9 - 10/2 = 2.3.2\) and \(|G_{a}| = 2^{4} |PSL(2,9)| = 2^{3} \cdot 3 \cdot 5\) with \(0 \leq i \leq 2\). Let \(P\) be a Sylow 23-subgroup of \(G\). Since \(\text{Aut}(Z_{23}) \cong Z_{2} \times Z_{11}, 3 \cdot X |N_{G}(P)|, \) for otherwise \(P\) centralizes a nontrivial 3-element \(x\) and so \(F(P) \supseteq F(x)\) because \(|F(x)| = 1\), contrary to \(|F(P)| = 0\). Similarly \(5 \cdot X |N_{G}(P)|\). Hence \(|G: N_{G}(P)| = 2 \cdot 3 \cdot 5\) for some \(a\) with \(0 \leq a \leq 6\). By a Sylow’s theorem, \(2 \cdot 3 \cdot 5 \equiv -2^{2} \equiv 1 \) (mod 23), a contradiction.

If \(q = 27\), \(|\Omega| = 1 + 27 \cdot 2.6/2 = 25 \cdot 11\) and \(|G_{a}| = 2^{2} \cdot 3 \cdot 5 \cdot 7 \cdot 13\) with \(0 \leq i \leq 1\). Let \(P\) be a Sylow 11-subgroup of \(G\). Since \(P \cong Z_{11}\) and \(\text{Aut}(Z_{11}) \cong Z_{2} \times Z_{5}, 3 \cdot 7 \cdot 13\), \(13 \cdot X |N_{G}(P)|\) by the similar argument as above. Hence \(|G: N_{G}(P)| = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13\) with \(0 \leq a \leq 7\) and \(3 \leq b \leq 3 + i\). By a Sylow’s theorem, \(2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13\). \(7 \cdot 13 \equiv 2 \cdot 3 \cdot 5 \cdot 4 \equiv 1 \) (mod 11). Hence \(a = 0, b = 4\). Therefore \(N_{G}(P)\) contains a Sylow 2-subgroup \(S\) of \(G\). Let \(T\) be a Sylow 2-subgroup of \(N_{a}\) and \(g\) an element such that \(T^{g} \subseteq S\). Then \(T^{g} \cap C_{G}(P) \neq 1\) as \(N_{G}(P)/C_{S}(P) \subseteq Z_{2}\). Let \(u\) be an involution in \(T^{g} \cap C_{G}(P)\). Then \(|F(u)| = (27 + 1)/2 = 13\), while \(11 \mid |F(u)|\) because \([P, u] = 1\) and \(|F(P)| = 0\), a contradiction.

If \(q = 7, 11, 19\) or 43, then \(G_{a} = N^{a}\) and \(e = -1\). Let \(\Gamma = \{\gamma, \delta \mid |\gamma, \delta \in \Omega, \gamma \neq \delta|\) be the action of \(G\) on \(\Gamma\). Since \(G_{a}\) is doubly transitive, \(G_{a}\) is transitive and \(G_{a} = 1\). Let \(z\) be an involution of \(Z(N_{a})\). There exists an involution \(t\) such that \(t \in \pi^{a}\) and \(\alpha = \beta\). Since \(G_{ab} = N_{a}^{a}\) and \(F(N_{a})^{a} = \{\alpha, \beta\}\) we have \(G_{(a,b)} = \langle \alpha \rangle N_{a}^{a}\). By Lemma 2.3, \(|F(z)| = |C_{G}(z)\times |(t)\cdot N_{a}^{a}| = |F(z)| \times |C_{G}(z)| \times |(t)\cdot N_{a}^{a}| = |F(z)| \times |(t)\cdot N_{a}^{a}| /|Z_{2}| = |F(z)| \times |(t)\cdot N_{a}^{a}| /|Z_{2}|. As \(|F(x')| = |F(z)| \times |F(z)| - 1)/2 + (|\Omega| - |F(z)|) /|\Omega| + |\Omega| /|F(z)| - 2. In particular \(|F(z)| \subseteq |\Omega|\). Since \(|F(z)| = (q + 1)/2 = 2(q + 5)/2 + |\Omega| = 1 + q(q - 1)/2 = (q^{2} - q + 2)/2\), we have \(q = 11\) and \(|t)\cdot N_{a}^{a}| \subseteq |\Omega| = 13\). Moreover \(|\Omega| = 56, |G_{a}| = |PSL(2,11)| = 2 \cdot 3 \cdot 5 \cdot 11\) and \(|G_{a}| = 2 \cdot 3 \cdot 5 \cdot 7\).

We now argue that \(\langle t \rangle N_{a}^{a} \cong D_{24}\). Let \(R\) be the Sylow 3-subgroup of \(N_{a}^{a}\). If \(t\) centralizes \(R, R\) acts on \(F(t)\) and so \(F(R) \subseteq F(t)\) as \(|F(t)| = 8\) and \(|F(R)| = 2\). Hence \(\alpha = \gamma, \delta\), contrary to the choice of \(t\). Therefore \(t\) inverts \(R\) and \(\langle t \rangle N_{a}^{a}\) is isomorphic to \(Z_{2} \times D_{12}\) or \(D_{24}\). Suppose \(\langle t \rangle N_{a}^{a} \cong Z_{2} \times D_{12}\). Then \(\langle t \rangle N_{a}^{a}\) contains fifteen involutions and so we can take \(u \in I(\langle t \rangle N_{a}^{a})\) satisfying \(|F(u)| = 1\) and \(\langle t \rangle N_{a}^{a} = \langle u \rangle \times N_{a}^{a}\). As \(|F(u)| = 0, |F(u')| = |\Omega| /|\Omega| = 28\). By Lemma 2.3, \(28 = |C_{G}(u)| \times |\langle u \rangle \times N_{a}^{a}| /|\Omega| /|\Omega| = 28\). Hence \(|C_{G}(u)| = 2 \cdot 3 \cdot 7\) or \(2 \cdot 3 \cdot 7\). Since \(\langle u \rangle \cdot N_{a}^{a} = C_{G}(R)\), we have \(|C_{G}(u)| = C_{G}(u) \cdot N_{G}(R)| = 2 \cdot 7\) or \(2 \cdot 7\). By a Sylow’s theorem, \(|C_{G}(u)| = C_{G}(u) \cdot N_{G}(R)| = 2 \cdot 7\), so that \(|C_{G}(u)| = 2 \cdot 3 \cdot 7\). Let \(Q\) be a Sylow 7-subgroup of \(C_{G}(u)\). Then \(|C_{G}(u) \times N_{G}(Q)| = 2 \cdot 3 \cdot 7\) or \(2 \cdot 3 \cdot 7\) by a Sylow’s theorem. Hence \(2 \cdot 3 \cdot 7 \mid |N_{G}(Q)|\). Since \(\text{Aut}(Z_{2}) \cong Z_{2} \times Z_{3},\)
by the similar argument as in the case \( q=9 \). Therefore \( |G: N_c(Q)| = 2^a \cdot 5 \cdot 11 \) for some \( a \) with \( 0 \leq a \leq 3 \). Hence \( |G: N_c(Q)| \equiv 1 \pmod{7} \), a contradiction. Thus \( \langle \iota \rangle N_c^{*} \cong D_{24} \).

Let \( U \) be a Sylow 2-subgroup of \( N_c^{*} \) and set \( L = N_c(U) \). It follows from (3.3) and Lemma 2.6 (iv) that \( L \cap N_c^{*} = A_4 \), \( L^{\iota} = A_4 \) and \( |L| = 2^a \cdot 3 \). Let \( T, \langle \iota \rangle \) be Sylow 2- and 3-subgroup of \( L \), respectively. Obviously \( L \supseteq \Gamma \) and \( C_T(x) = 1 \).

On the other hand \( T, \langle \iota \rangle \), \( T \) is dihedral or semi-dihedral. Hence \( N_c(T)/C_T(T) \) is a 2-group, so that \( C_T(x) = \iota \), a contradiction.

(3.13) (ii) of (3.11) does not occur.

Proof. Let \( G_c^{*} \) be a doubly transitive permutation group satisfying (ii) of (3.11). Let \( x \) be an involution in \( N_c^{*} \) with \( x \in \Gamma \). Then \( F(x^F) = F(x) = \iota \) by (i) of (3.1) and (3.9). Since \( |F(Y)| = 1 + (q-\epsilon)/3 \), \( |N_c^{*}| = 1 + k \cdot 4, x^F \) is an involution. By Lemma 2.5, \( 1 + k = 2^a \) and so \( k = 3 \). By (3.11), \( q-\epsilon = 2((8-\epsilon)(4-\epsilon)+3) \). Hence \( q+7 \leq 2^a \cdot 3 \cdot 7 \) if \( \epsilon = 1 \) and \( q+9 \leq 2^a \cdot 3 \cdot 5 \cdot 17 \) if \( \epsilon = -1 \). From this \( q+7 \leq 2^a \cdot 7 \) if \( \epsilon = 1 \) and \( q+9 \leq 2^a \cdot 5 \cdot 17 \) if \( \epsilon = -1 \). Therefore \( q = 5^2, 7^2, 11^2, 59 \) or 71.

Let \( p_1 \) be an odd prime such that \( p_1 \mid |\Omega| \) and \( p_1 \mid |G_a| \) and let \( P \) be a Sylow \( p_1 \)-subgroup of \( G \). Clearly \( P \) is semi-regular on \( \Omega \) and so any element in \( C_G(P) \) has at least \( p_1 \) fixed points. If \( x \) is an element of \( N_c^{*} \) and its order is at least three, \( |p(x)| = 4 \) by Lemma 2.8. Since \( |\Omega| = 1 + 3(q+1)/2 \), we have \( |\Omega| = 1 + 3q(q+1)/2 \).

If \( q = 5^2 \), then \( |\Omega| = 2^a \cdot 61 \) and \( |G_a| = 2^{a+1} \cdot 3 \cdot 5 \cdot 13 \) (0 \( \leq a \leq 2 \)). Let \( P \) be a Sylow 61-subgroup of \( G \). Then \( P \supseteq Z_{61} \). As mentioned above, \( 5, 13 \not\mid |C_G(P)| \) and so \( 5^2, 13 \not\mid |G_a| \). Hence \( |G: N_c(P)| = 2^{a+1} \cdot 5 \cdot 13 \), where \( 0 \leq a \leq 10 \) and \( 0 \leq b, c \leq 1 \). But we can easily verify \( |G: N_c(P)| \equiv 1 \pmod{61} \), contrary to a Sylow's theorem.

If \( q = 7^2 \), then \( |\Omega| = 2^a \cdot 919 \) and \( |G_a| = 2^{a+1} \cdot 3 \cdot 5 \cdot 7 \cdot 27 \) (0 \( \leq a \leq 2 \)). Let \( P \) be a Sylow 919-subgroup of \( G \). By the similar argument as above, we obtain 5, \( 7 \not\mid |N_c(P)| \) and so \( |G: N_c(P)| = 2^{a+1} \cdot 3 \cdot 5 \cdot 7 \cdot 27 \equiv 2^{a+1} \cdot 7 \cdot 3 \cdot 5 \cdot 27 \equiv 2^{a+1} \cdot 3 \cdot 5 \) if \( 2^{a+1} \cdot 3 \cdot 5 \) or \(-2^{a} \pmod{919} \), where \( 0 \leq a \leq 8 \) and \( 0 \leq b, c \leq 1 \). Hence \( |G: N_c(P)| \equiv 1 \pmod{919} \), a contradiction.

If \( q = 11^2 \), then \( |\Omega| = 2^a \cdot 17 \cdot 13 \) and \( |G_a| = 2^{a+1} \cdot 3 \cdot 5 \cdot 13 \cdot 11^2 \cdot 61 \) (0 \( \leq a \leq 2 \)). Let \( P \) be a Sylow 173-subgroup of \( G \). Similarly we have 3, 5, 11, 61 \( \not\mid |N_c(P)| \) and so \( |G: N_c(P)| = 2^{a+1} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -2^{a} \pmod{173} \), where \( 0 \leq a \leq 12 \). Hence \( |G: N_c(P)| \equiv 1 \pmod{173} \), a contradiction.

If \( q = 59 \), then \( |\Omega| = 2^a \cdot 17 \cdot 151 \) and \( |G_a| = 2^{a+1} \cdot 3 \cdot 5 \cdot 29 \cdot 59 \) (0 \( \leq a \leq 1 \)). Let \( P \) be a Sylow 17-subgroup of \( G \). Similarly we have 3, 5, 29, 59 \( \not\mid |N_c(P)| \) and so \( |G: N_c(P)| = 2^{a+1} \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151^2 \equiv 10 \cdot 2^a \) or \( 12 \cdot 2^a \pmod{17} \), where \( 0 \leq a \leq 4 \) and \( 0 \leq b, c \leq 1 \). From this, we have a contradiction.

If \( q = 71 \), then \( |\Omega| = 2^{a+1} \cdot 3 \cdot 7 \cdot 233 \) and \( |G_a| = 2^{a+1} \cdot 3 \cdot 5 \cdot 7 \cdot 233 \) (0 \( \leq a \leq 1 \)). Let \( P \) be
a Sylow 233-subgroup of $G$. Since $3, 5, 7, 71 | N(G)$, $|G: N(G)| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 71^e$ (mod 233), where $0 \leq a, b, c, d, e \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N^{a} \approx Z \times Z$ or $N^{a} \approx D_8$ and $N^{a} \cap N^{b} \leq Z \times Z$.

(3.14) The case that $N^{a} \approx Z \times Z$ does not occur.

Proof. Set $\Delta = F(N^{b})$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N^{a}N^{b})$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^{a}| \geq n$, we have $q = p^{n} \equiv 3, 5 \pmod{8}$ and so $n$ is odd. Hence $|G_{a}N^{b}| \leq 2$ and $N^{a} \approx N^{b}$ or $N^{a} \approx N^{b}$. Then $N^{a}N^{b}$ is a Sylow 2-subgroup of $G_{a}$, hence $N_{c}(N^{a}N^{b})$ is doubly transitive by a Witt's theorem. Since $N_{c}(N^{a}N^{b}) = D_8$ and $|\Delta|$ is even, $C_{G}(N^{a}N^{b})$ is also doubly transitive. Let $g$ be an element of $C_{G}(N^{a}N^{b})$ such that $c' = b$ and $b' = a$. Then $N^{a} = g^{-1}N^{b}g = N^{a}$ and hence $N^{a} = N^{b}$, a contradiction. Thus $N^{a} = N^{b} \cap N^{b} = Z \times Z$.

Let $z$ be an involution in $N^{a}$ and $t \in z^{a}$ an involution such that $\alpha = b$. Set $\Gamma = \{\gamma, \delta \} | \gamma, \delta \in \Omega, \gamma \neq \delta\}$. We consider the action of the element $z$ on $\Gamma$. By the similar argument as in the proof of (3.12), $|F(z)| = (|F(z)| - 1)/2 + (|\Omega| - |F(z)|)/2 = |G(z)| = |G_{a}(z)| = z^{a} \times \langle t \rangle |G_{a}| = |\langle t \rangle |G_{a}|$. Since $N^{a} = N^{b} \cap N^{b}$, by Lemma 2.6 (i), $z^{a} \in G_{a} = z^{a}$ and so $|C_{G}(z)| = |F(z)| \times |C_{G}(z)|$. Hence $|G_{a}| = (|F(z)| - 1)/|\Omega| - |F(z)| = |F(z)| = |C_{G}(z)| = z^{a} \times \langle t \rangle |G_{a}|$, so that $|G_{a}| = |\Omega| \equiv 0 \pmod{|F(z)|}$. Since $G_{a}/N^{b} = G_{a}/N^{b}$, we have $|G_{a}| = 8n$. Clearly $|\Omega| = 1 + q(q - e) + (q + e)r/8$ and by Lemma 2.8 (i), $|F(z)| = 1 + 3(q - e)r/4$. Hence $1 + 3(q - e)r/4 | 8n(1 + q(q - e) + (q + e)r)/8$. Let $n = rs$. Then $3qr - 3er + 4 | (4rs + q(q - e) + (q + e)r)3^{r} = 864r^{3} + 4s(3pq) + 3(q - e)r(3(r - 3er) + 364r^{3} + 4s(3er - 3e + 3er) + 364r^{3} - 32s(3er - 3e - 3er - 2)$. (*) We argue that $r = 1$. Suppose false. Then $32s(3r - 4) + 3s(r - 4) > 0$ and so $3(q - e) < 864r^{3}$. Therefore $288m + e > q = p^{n} > 3^{n}$ and so $288n > 3^{n}$. Hence $(n, r, p, e) = (5, 5, 3, -1), (3, 3, -1)$, while none of these satisfy the inequality. Thus $r = 1$.

Hence $3q - 3e + 4 | (64 + 96)e - n$ and $|F(z)| = 1 + 3(q - e)r/4, |\Omega| = 1 + q(q - e) + (q + e)r/8$. If $s = -1$, then $3q - 3e < 3q + 7/256n$. Hence $s = 1$ or $(n, p) = (5, 3), (3, 3)$. Since $3 + 3^{3} + 7/256 - 3 + 3^{3} + 7/256 - 3, n = 1$ and $3q + 7/256$. From this, $q = 19$ or 83. If $s = 1$, then $3 + 3^{3} + 3q + 1/896n$ and so $n = 1$ or $(n, p) = (5, 3).$ Since $3 + 3^{3} + 1/896 - 3, n = 1$ and $3q + 1/896$. From this, $q = 19, 37, 149$. As $PSL(2, 5) = PSL(2, 4), q = 5$ by [4]. Thus $q = 19, 37, 83$ or 149.

Set $m = |z^{a} \cap \langle t \rangle |G_{a}|$. As we mentioned above, $|G_{a}| = (|G(z)| = |F(z)|)/4, |\Omega| = 1 + q(q - e) + (q + e)r/8$. Therefore $m = (2q^{2} + (2e + 9)q - 9e)/(3q - 3e + 4)$. It follows that $(q, m) = (19, 27, 2), (37, 28), (83, 449/8)$ or $(149, 411/4)$. Since $m$ is an integer, we have $(q, m) = (37, 28)$. But $m \leq |\langle t \rangle |G_{a}| = 1, 16$, a contradiction. Thus (3.14)
(3.15) The case that $N_n^g = D_6$ and $N^g \cap N^g \leq Z_3 \times Z_2$ does not occur.

Proof. Let $\Delta$, $L$ and $K$ be as defined in (3.6). By (3.6), there exists an element $x$ in $L$ such that its order is odd and $\langle x \rangle^L$ is regular on $\Delta$. Since $(L \Delta)^L \leq N_n^g$ by (3.6) and $N_n^g = D_6$, $x$ stabilizes a normal series $N_n^g N^g \geq N_n^g \geq 1$. Hence $x$ centralizes $N_n^g N^g$ by Theorem 5.3.2 of [2] and so $x^{-1}N_n^g x = N_n^g$. Put $\gamma = \beta$. If $r = 1$, then $\beta = \gamma$, so that $N_n^g = N_n^g$. From this, $N_n^g = N_n^g$. By the doubly transitivity of $G$, $N_n^g = N_n^g$, hence $N_n^g = N_n^g \cap N^g$, a contradiction. Therefore $r = 1$ and $\Delta = \{\alpha, \beta\}$.

Set $\langle x \rangle = Z(N_n^g)$, $\Delta_1 = \alpha^{g(t)}$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_3\}$ be the set of $C_G(x)$-orbits on $F(z)$. Since $L \geq N_n^g \cap N^g$ and by (3.2), $N_n^g \cap N^g \neq 1$, $z$ is contained in $N_n^g \cap N^g$. Hence, by Lemma 2.1, $\beta \in \Delta_1$ and $k$ is at least two. By Lemma 2.8, $|F(z)| = 1 + (q - \varepsilon)5/|N_n^g| = 1 + (q - \varepsilon)/8$. Clearly $|C_{N_n^g}(z)| N_n^g = (q - \varepsilon)/8$ and so $|\Delta_1| \geq 1 + (q - \varepsilon)/8$. If $\gamma \in F(z) - \Delta_1$, then $C_N(z) = Z(z) \times Z_2$, for otherwise $\langle x \rangle = Z(N_n^g) \leq N_n^g \cap N^g$ and by Lemma 2.1 $\gamma \in \Delta_1$, a contradiction. Hence one of the following holds.

(i) $k = 3$ and $|\Delta_1| = 1 + (q - \varepsilon)/8$, $\Delta_2$ and $|\Delta_3| = (q - \varepsilon)/4$.
(ii) $k = 2$ and $|\Delta_1| = 1 + (q - \varepsilon)/8$, $\Delta_2$ and $|\Delta_3| = (q - \varepsilon)/2$.
(iii) $k = 2$ and $|\Delta_1| = 1 + 3(q - \varepsilon)/8$, $\Delta_2$ and $|\Delta_3| = (q - \varepsilon)/4$.

Let $\gamma \in F(z) - \Delta_1$. Then, $z \in G_{\gamma} - N^g$ and so $C_N(z) = D_{n+\varepsilon}$ or $PGL(2, \sqrt{q})$ by Lemma 2.6 (vii), (viii), (ix). If $C_N(z) = D_{n+\varepsilon}$, then $(q + \varepsilon)/2 | | \Delta_1 |$ and so $q = 7$ and (ii) occurs. But $(q + \varepsilon)/2 = 3 | | \Delta_2 | - 1 - 1 = 1$, a contradiction. If $C_N(z) = PGL(2, \sqrt{q})$, then (i) does not occur because $\sqrt{q} \wedge q - \varepsilon$. Hence $\sqrt{q} | | \Delta_1 |$ and $\sqrt{q} | | \Delta_2 | - 1$. From this, $q = 25$ and (iii) occurs. In this case, we have $|\Delta_1| = 10$, so that an element of $C_N(z)$ of order 3 is contained in $N_n^g$ for some $\delta \in \Delta_1$, contrary to $N_n^g = N_n^g = D_6$.

4. Case (II)

In this section we assume that $N_n^g = PGL(2, p^m)$, where $n = 2mk$ and $k$ is odd. Since $n$ is even, $q = p^m \equiv 1 \pmod{4}$. We set $p^m \equiv \varepsilon \equiv \{ \pm 1 \} \pmod{4}$. In section 7 we shall consider the case that $N_n^g = S_6$. Therefore we assume $(p, m) = (3, 1)$ in this section.

(4.1) The following hold.

(i) $N_n^g \cap N^g \leq Z_2$ and $N_n^g \cap N^g \geq (N_n^g)' = PSL(2, p^m)$.
(ii) If $(p, m) = (5, 1)$, there exists a cyclic subgroup $Y$ of $(N_n^g)'$ such that $N_{N^g}(Y) = D_{n+\varepsilon}$ and $N_G(Y)^{F(\gamma)}$ is doubly transitive.

Proof. As $N_n^g \geq N_n^g \cap N^g$, either $N_n^g \cap N^g \leq Z_2$ or $N_n^g \cap N^g = 1$. If $N_n^g \cap N^g = 1$, by Lemma 2.2 and 2.6 (vi), $N_n^g = N_n^g \cap N^g = (N_n^g)^g = Z_2 \times Z_2$, a
contradiction. Therefore $N_a^*/N^a \cap N^a \simeq 1$ or $Z_2$ and $N^a \cap N^b \simeq (N_b^*)' \simeq PSL(2, p^m)$.

Now we assume that $(p, m) \neq (3,1), (5,1)$ and let $z$ be an involution in $(N_b^*)'$. Then $C_{N_b^*}(z) = D_{2(p^m-1)}$ by Lemma 2.6 (vii). Suppose $C_{N^a}(z)$ is not a 2-subgroup and put $Z = 0(C_{N^a}(z))$. Then, if $Z \leq N^a \cap N^b$, where $\gamma = \alpha^x$ and $\delta = \beta^y$. By (i) $Y^x \leq N^a \cap N^b$ and so $Y^x = Y^a$ for some $h \in N^a \cap N^b$. Thus $N_\alpha(Y)^{F(Y)}$ is doubly transitive. Assume that $C_{N^a}(z)$ is a 2-subgroup and set $C_{N^a}(z) = \langle u, v \mid u^2 = u^{-1}, v^2 = 1 \rangle$. We may assume that $v \in (N_a^*)'$ and $\langle u, v \rangle$ is a Sylow 2-subgroup of $(N_b^*)'$. Since $p^m \neq 3, 5$, the order of $u^2$ is at least four. On the other hand, there is no element of order $|u^2|$ in $\langle u, v \rangle - \langle u^2, v \rangle$. Hence any element of order $|u^2|$ which is contained in $N^a \cap N^b$ is necessarily an element of $N^a \cap N^b$. By the similar argument as above, $N_\alpha(Y)^{F(Y)}$ is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose $(p, m) \neq (3,1), (5,1)$ and set

\[ \Delta = F(Y) \text{ and } X = N_\alpha(Y). \]

Then $|\Delta| = rs(p^m + \varepsilon)/2 + 1$, where $s = \sum_{i=0}^{k-1} 2p^{2mi}$, $C_\alpha(N^a) = 1$ and one of the following holds.

(i) $X^\Delta \leq \text{ATL}(1, 2^c)$ for some integer $c$.

(ii) $X^\Delta = \text{PSL}(2, p_1)$ or $\text{PGL}(2, p_1)$, $r = 1, k = 1$ and $2p_1 = p^m + \varepsilon$.

Proof. By Lemma 2.8 (ii), $|\Delta| = 1 + |N^a \cap X| |r| |N_b^* \cap X| = 1 + (p^{2mk} - 1)/r(2p^m - \varepsilon) = rs(p^m + \varepsilon)/2 + 1$. By (4.1) and Lemma 2.9, we have (i), (ii) or $X^\Delta = R(3)$.

Assume that $X^\Delta = R(3)$. Then $rs(p^m + \varepsilon)/2 + 1 = 28$, hence $k = 1$ and $r(2p^m - \varepsilon)/2 = 27$. Since $r$ is odd and $r | 2m = n$, we have $r = m = 1$ and $q = 3^2$. But a Sylow 3-subgroup of $X_a^a$ is cyclic because $N^a \cap X \simeq D_{27}$, and $X_a^a / X \cap N^a = X_a^a N^a / N^a \leq Z_2 \times Z_2$, a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose $X^\Delta \leq \text{ATL}(1, 2^c)$ and put $W = C_{N_b^*}(Y)$. Then $Y \leq W \simeq Z_{p^m-1}$. Since $C_{N^a}(Y)$ is cyclic, $W$ is a characteristic subgroup of $C_{N^a}(Y)$ and so $W$ is a normal subgroup of $X_a$. Hence $W \leq X_\Delta$ and $(X \cap N_b^*) = 1$ or $Z_2$. By Lemmas 2.4 and 2.6, $F(X \cap N_b^*) = 1 + |X \cap N_b^*| |N_b^*: X \cap N_b^*| \times r |N_b^*| = 1 + r$. Since $1 + r < |\Delta|$, $(X \cap N_b^*) = Z_2$ and hence $(1 + r)^2 = rs(p^m + \varepsilon)/2 + 1$ by Lemma 2.5. From this, $r = s(p^m + \varepsilon)/2 - 2 |mk$ and so $p^{2m}(k-1) + mk \leq 2$. Hence $m = k = r = 1$ and $q = 7^2$.

Let $R$ be a Sylow 3-subgroup of $N_b^*$. Since $N_b^* = \text{PGL}(2, 7)$, we have $R = Z_3$. By Lemmas 2.4 and 2.6, $|F(R)| = 1 + (7^2 - 1) |N_b^*| / |N_b^*| = 4$. Hence $N_\alpha(R)^{F(R)} = A_4$ or $S_4$. But is a Sylow 3-subgroup of $N_{Ga}(R)$ because $N^a = \text{PSL}(2, 7)$, contrary to $N_{Ga}(R)^{F(R)} = A_4$ or $S_4$.

(4.4) (ii) of (4.2) does not occur.
Proof. Let notations be as in (4.2). Suppose \( X^\Delta \supseteq PSL(2, p_1) \). By the similar argument as in (4.3), \( C_{N_0}(Y) \leq X_\Delta \) and so \( C_{N_0}(Y)_\beta = D_{2p_1} \). Hence \( |(X)_\Delta| = |2p_1 - 2n| \). Since \( X^\Delta \supseteq PSL(2, p_1) \), \( p_1(p_1 - 1)|2|(X}_\Delta| \), hence \( p_1 = 1 | 8n \). As \( k = 1 \) and \( 2p_1 = p^m + \varepsilon \), we have \( p^m + \varepsilon - 2 | 32m \). From this, \( (p, m, p_1) = (11, 1, 5), (3, 2, 5) \) or \( (3, 3, 13) \).

Let \( R \) be a cyclic subgroup of \( N_0^* \) such that \( R = Z(p^m + \varepsilon) \). By Lemma 2.6, \( N_0(R) F(R) \) is doubly transitive and by Lemma 2.8 (ii), \( |F(R)| = 1 + |N_0(R)| \)

\[ |N_0(R)| = 1 + (p^m - 1)/(p^m + \varepsilon) = (p^m - \varepsilon)/2 + 1. \]

If \( (p, m, p_1) = (11, 1, 5), |F(R)| = 7 \) and so by [9], \( |N_0(R)| = 42 \) and \( N_0(R) F(R) = Z_6 \). Since \( |N_0(R)| = 6 \), \( N_0(R) F(R) = N_0(R) F(R) \). Hence \( N_0(R) / K = Z_6 \), where \( K = (N_0(R)) \). But \( N_0(R) / (N_0(R)) = Z_2 \times Z_2 \), a contradiction.

If \( (p, m, p_1) = (3, 2, 5), |F(R)| = 5 \) and so by [9], \( |N_0(R)| = 20 \) and \( N_0(R) F(R) = Z_4 \). Since \( |N_0(R)| = 4 \), \( N_0(R) = Z_4 \), contrary to \( N_0(R) \).

If \( (p, m, p_1) = (3, 3, 13), |F(R)| = 15 \). By [9], \( N_0(R) \) is not solvable, a contradiction.

\( (4.5) \ p^m = 5. \)

Proof. Assume that \( p^m = 5 \). Then \( n = 2k \) with \( k \) odd and \( N_0^* = PGL(2, 5) \). First we argue that \( N_0^* = N^\alpha \cap N^\beta \). Suppose false. Then \( C_0(N^\alpha) = 1 \) by Lemma 2.2, and \( N^\alpha / N_0 \cap N_0 \) by (4.1). Since \( N_0^* N_0 \) is an outer automorphism group of \( S_5 \), \( N_0^* / N_0 \) is trivial, we have \( Z(N_0^* N_0^*) = Z_2 \).

Let \( w \) be the involution of \( Z(N_0^* N_0^*) \) and let \( w \in I(N_0^* - I(N^\alpha) \). Since \( C_0(N^\alpha) \geq N_0^*, \) by Lemma 2.6 (viii) and (ix), \( w \) acts on \( N^\alpha \) as a field automorphism of order 2 and \( C_0(w) = PGL(2, 5) \). By Lemma 2.8 \( |F(w)| = 1 + r(q - \varepsilon) / I(N^\alpha)| / |N_0^*| = 1 + 5r(5^2 - 1)/24. \) Let \( P \) be a Sylow 5-subgroup of \( C_0(w) \). Then \( |P| = 5^k \) and \( |\gamma| = 5^{k - 1} \) for each \( \gamma \in \Omega - \{w\} \). Since \( P \) acts on \( F(w) \), we have \( 5^{k - 1} |(5^2 - 1)/24 \), so that \( k = 1 \) and \( |F(w)| = 6 = r / k \). Hence \( C_0(w) F(w) = Z_6 \) and so \( C_0(w) F(w) = Z_6 \). But clearly \( w \in N_0^* \cap N_0^* \) by Lemma 2.1, a contradiction. Thus \( N_0^* = N^\alpha \cap N^\beta \).

Let \( V \) be a cyclic subgroup of \( N_0^* \) of order 4. Since \( N_0^* = N^\alpha \cap N_0 \), \( N_0(V) \) is doubly transitive and by Lemma 2.8, \( |F(V)| = 1 + |N_0(V)| / |N_0(V)| = 1 + (5^2 - 1)r/8 = 3rs + 1, \) where \( s = \sum_{i=0}^{k-1} 25^i \). By Lemma 2.9, \( C_0(N^\alpha) = 1 \) and (a) \( N_0(V) F(V) \leq ATL(1, 2) \) or (b) \( N_0(V) F(V) = R(3) \).

Put \( P = N_0(V) \). Then \( P = D_5, |F(P)| = 1 + |N_0(P)| / |N_0^*(P)| = r + 1 \) and \( P F(V) = Z_2 \). If (b) occurs, \( k = 1 \) and \( r = 9 \), hence \( |F(P)| = 10 \), a contradiction. Therefore (a) holds.

By Lemma 2.5, \( (r+1)^2 = 3rs + 1 \) and so \( r = 3s - 2 / k \). Hence \( k = r = 1 \) and \( G_0 N_0 \leq Z_2 \times Z_2 \). Let \( z \) be an involution in \( N_0^* \). Then \( |F(z)| = 1 + 24 \cdot 25/120 = 6 \)
by Lemma 2.8 and $|\Omega|=1+|N^*: N^*| = 66$ as $r=1$. By the similar argument as in the proof of (3.12), \[
|F(z)|/(|F(z)| - 1)/2 + (|\Omega| - |F(z)|)/2 = |C_G(z)| = |z^\alpha \cap \langle t \rangle G_{ab^\alpha} | / |\langle t \rangle G_{ab^\alpha} |
\]
where $t$ is an involution such that $\alpha = \beta$. Hence $|z^\alpha \cap \langle t \rangle G_{ab^\alpha} | = 15 |G_{ab^\alpha} | / |C_G(z)|$. Set $H = \langle t \rangle G_{ab^\alpha}$ and let $R$ be a Sylow 3-subgroup of $N^*_{ab}$. By Lemma 2.8, $|F(R)| = 1 + 24 \cdot 120 = 3$. Set $F(R) = \{\alpha, \beta, \gamma\}$. On the other hand, as $N^*_{ab} = S_5$ and $\text{Out}(S_5) = 1$, we have $H = Z(H) \times N^*_{ab}$ and $|Z(H)| = 2, 4$ or $H = C_{H}(N^*_{ab}) \times N^*_{ab}$ and $Z(G_{ab^\alpha}) = Z_3 \times Z_2$, contrary to Lemma 2.6 (ix). In the former case, we have $|Z(H)| = 2$. For otherwise $Z(H) \leq G_2$ and $Z(H) \cap z^\alpha = \phi$ and so letting $u \in Z(H) \cap z^\alpha$, we have $|R| = 3 \cdot |F(u)| - 1 = 5$, a contradiction. Therefore $Z(H) = Z_2$ and so $|z^\alpha \cap H| = 25 + 25 = 50$, while $|z^\alpha \cap H| = 15 |G_{ab^\alpha} | / |C_G(z)| = 15 \cdot 120/24 = 75$, a contradiction.

5. Case (III)

In this section we assume that $N^*_{ab} = PSL(2, p^n)$, where $n = mk$ and $k$ is odd. Set $p = \{+1\} \pmod{4}$. Then $q = \{+1\} \pmod{4}$ as $k$ is odd. In section 6 we shall consider the case that $N^*_{ab} = A_4$, so we assume $(p, q) = (3, 1)$ in this section.

From this $N^*_{ab}$ is a nonabelian simple group and so $N^*_{ab} = N^* \cap N^*$ or $N^* \cap N^* = 1$. If $N^* \cap N^* = 1$, then $C_G(N^*) = 1$ by Lemma 2.2 and $N^* = N^* \cap N^* \cap N^* = N^* \cap N^* / N^* \cap N^* \cap N^*$ is the order of $G_{ab^\alpha}$. Hence $N^*_{ab} = N^* \cap N^*$.

Let $x$ be an involution of $N^*_{ab}$. Suppose $z^\alpha \in G_{ab^\alpha}$ for some $g \in G$ and set $\gamma = \alpha^x$, $\delta = \beta^x$. Then $z^\alpha \in N^* \cap G_{ab^\alpha} \leq N^* \cap N^* \leq N^* \cap N^*$ and so $z^\alpha \in z^N_{ab^\alpha}$. Hence $C_G(z)^{F(x)}$ is doubly transitive and by Lemma 2.8 (i), $|F(z)| = (q-\delta)/(|p^n - \delta|) + 1$.

In particular $|F(z)| > 3 + 1$ as $(p^n - \delta)/(|p^n - \delta|) > p^{2m} + \delta p^{m+1} + 1 > 3$.

By Lemma 2.9, $C_G(N^*) = 1$ and one of the following holds.

(a) $C_G(z)^{F(x)} = \text{ATL}(1, 2^2)$.
(b) $C_G(z)^{F(x)} \geq PSL(2, p_1)$ ($p_1 \geq 5$), $r = 1$ and $|C_{N^*}(z) : C_{N^*}(z)| = p_1$.
(c) $C_G(z)^{F(x)} = R(3)$.

Let $Y$ be a cyclic subgroup of $C_{N^*_{ab}}(z) = D_{p^m-2}$ of index 2. Since $C_{G_{ab}^\alpha}(z) \geq Y$, $z \in Y$ and $C_G(z)^{F(x)}$ is doubly transitive, we have $F(Y) = F(z)$. By the similar argument as in (3.1), $N^* \cap N(C_{N^*_{ab}}(z)) = N_{ab^\alpha}^* (z)$ or $N^* \cap N(C_{N^*_{ab}}(z)) = A_4$. Hence by Lemmas 2.3 and 2.4 $|F(C_{N^*_{ab}}(z))| = 1 + |C_{N^*_{ab}}(z)| / |N^*_{ab}| : |C_{N^*_{ab}}(z)| / |N^*_{ab}|$ or $1 + |A_4| / |N^*_{ab}| : C_{N^*_{ab}}(z) / |N^*_{ab}|$ or $1 + |A_4| / |N^*_{ab}| : C_{N^*_{ab}}(z) / |N^*_{ab}|$. Therefore $|F(C_{N^*_{ab}}(z))| = r + 1$ or $3r + 1$. From this $C_{N^*_{ab}}(z)^{F(x)} \geq Z_2$.

In the case (a), $(r + 1)^2 = 1 + (p^n - \delta)/(|p^n - \delta|) - 2$ by Lemma 2.5 and hence $r = (p^n - \delta)/(|p^n - \delta|) - 2$ if $m$. Since $(p^n - \delta)/(|p^n - \delta|) \geq ((p^n)^{k-1} + 1)/(|p^n| + 1) = \lambda_{p-1}^{k-1}$ if $k \geq 3$, we have $p^{m(k-1)} - p^{m+1} \leq m$, hence $(p^{m(k-3)} / k)(m/(p^{m+1} + 1)) < 1$. Thus $k = 3$, $m = 1$ and $p = 3$, contrary to $(p, m) \neq (3, 1)$.

In the case (b), $r = 1$, $p_1 = (p^n - \delta)/(|p^n - \delta|)$, $p(p_{1} - 1)/s$ and $s | 4mpk_1$, where $s$ is the order of $C_{G_{ab}^\alpha}(z)^{F(x)}$. Hence $p_{1} - 1 = (p^n - \delta)/(|p^n - \delta|) - 1$
\[
\geq \frac{(p^n+1)/(p^n+1) - 1}{\sum_{i=0}^{k} (-1)^i} \geq p^{m(k-2)}(p^m - 1), \text{ we have } p^{m(k-2)}/2k \leq 4m/(p^m - 1) \leq 1 \text{ because } p^m \neq 3. \text{ Hence } k = 3 \text{ and } p^m = 5, \text{ so that } p_1 - 1 = 30 \sqrt[8]{8mk} = 24, \text{ a contradiction.}
\]

In the case (c), \( r + 1 = 4 \) and \( 1 + (p^m - \varepsilon)r/(p^m - \varepsilon) = 28 \) and so \( r = 3 \) and \( (p^m - \varepsilon)/(p^m - \varepsilon) = 9 \). Hence \( 9 \geq (p^m + 1)/(p^m + 1) \geq p^m - p^m + 1, \text{ so that } p^m = 3, \text{ a contradiction.}

6. Case (IV)

In this section we assume that \( N_\alpha = A_4 \) and \( q = 3, 5 \) (mod 8). If \( N_\alpha \cap N_\beta = 1, \) by Lemma 2.2, \( C_\alpha(N_\alpha) = 1 \) and so \( N_\alpha/N_\alpha \cap N_\beta = N_\beta/N_\beta \leq Z_2 \times Z_2. \) Hence \( N_\alpha/N_\alpha \cap N_\beta = 1 \) or \( Z_3, \) so that \( z^6 \cap G_{ab} = z^6 \cap N_\alpha = z^6 \) for an involution \( z \in N_\alpha. \) Therefore \( C_\alpha(z)^{F(z)} \) is doubly transitive. By Lemma 2.9, \( C_\alpha(N_\alpha) = 1 \) and one of the following holds.

(a) \( C_\alpha(z)^{F(z)} \leq A_4(1,3') \) for some integer \( c \geq 1. \)

(b) \( C_\alpha(z)^{F(z)} \geq PSL(2, p_1) (p_1 \geq 5), r = 1 \) and \( |C_\alpha(z)|: C_\alpha(z)| = p_1. \)

(c) \( C_\alpha(z)^{F(z)} = R(3). \)

Let \( T = \text{a Sylow 2-subgroup of } N_\alpha. \) Then \( z \in T \) and by Lemmas 2.3 and 2.4, \( |F(T)| = 1 + |N_\alpha(T)| \cdot r = 1 + |N_\alpha(z)| \cdot r = 1 + 1. \) By Lemma 2.8 (i), \( |F(z)| = (q - \varepsilon)r/4 + 1. \) Hence \( T^{F(z)} = Z_2 \) if \( q = 5. \) If \( q = 5, \) as \( PSL(2, 5) = PSL(2, 4), \) (ii) of our theorem holds by [4]. Therefore we may assume \( q = 5. \)

In the case (a), \( (r + 1)^2 = 1 + (q - \varepsilon)r/4 \) by Lemma 2.5. Hence \( r = (q - \varepsilon - 8)/4 \) and \( r = 1. \) Let \( R = \text{a Sylow 3-subgroup of } G_{ab}. \) Then \( R \leq N_\alpha \) because \( G_{ab}/N_\alpha \cong G_{ab}/N_\alpha = 1 \) or \( Z_2 \) and \( N_\alpha = A_4. \) By Lemma 2.8 (ii), \( |F(R)| = 1 + 12/3 = 5 \) and \( N_\alpha(R)^{F(R)} \) is doubly transitive. Since \( N_\alpha(R) = D_{12} \) or \( D_{24} \) and \( |F(R)| = 5, \) we have \( |N_\alpha(R)| = 5. \) Let \( S = \text{a Sylow 5-subgroup of } N_\alpha(R). \) Then \( [S, R] = 1 \) as \( N_\alpha(R)/C_\alpha(R) \leq Z_2. \) Since \( 5 \nmid |G_{ab}|, \) \( |F(S)| = 0 \) or 1. If \( |F(S)| = 1, \) \( F(S) \subseteq F(R) \) and so \( 5 \mid |F(R)| - 1 = 4, \) a contradiction. Therefore \( S \) is semi-regular on \( \Omega. \) But \( |\Omega| = 1 + |N_\alpha: N_\alpha| = 56 \) or 92. This is a contradiction.

In the case (b), \( p_1(p_1 - 1)/2 \) is \( s \) and \( s|2n(q - \varepsilon)/2 = 4np_1, \) where \( s \) is the order of \( C_{ga}(z)^{F(z)}. \) Hence \( p_1 = 1, 8n. \) Since \( p_1 = (q - \varepsilon)/4, p^s - \varepsilon - 4 \mid 32 \text{ and so we have } q = 11, 13, 19, 27 \) or 37. If \( q = 27, \) by Lemma 2.6, \( C_{ga}(z) = D_{24} \) or \( D_{24} = z \) and so \( C_{ga}(z)^{F(z)} = Z_2. \) Hence \( (p_1 - 1)/2 = 2. \) From this \( q = 19. \) Let \( R = \text{a Sylow 3-subgroup of } G_{ab}. \) By the similar argument as in the case (a), \( N_\alpha(R)^{F(R)} \) is doubly transitive and \( |F(R)| = 1 + 18/3 = 7. \) Hence \( |G| \mid |\Omega| \mid G_{\alpha} = (1 + |N_\alpha: N_\alpha|) \mid G_{\alpha} = (1 + 18 \cdot 19/2 \cdot 12) \cdot 2^4 \cdot 18 \cdot 19 \cdot 20/2 = 3^3 \cdot 5^3 \cdot 11 \cdot 13 \cdot 19 \text{ with } 0 \leq i \leq 1, \) a contradiction. If \( q = 27, \) then \( C_{ga}(z) = |F(z)| \cdot |G_{ga}(z)| = 8 \cdot |G_{ga}(z)|, \) while \( |\Omega| = 1 + |N_\alpha: N_\alpha| = 1 + 26 \cdot 27 \cdot 28/2 \cdot 12 = 820 = 2^3 \cdot 5 \cdot 41 \) and so \( |G| = 4 |G_{ga}(z)|. \) Therefore \( C_{ga}(z) \mid |G|, \) a contradiction.

In the case (c), \( r + 1 = 4 \) and \( 1 + (q - \varepsilon)r/4 = 28. \) Hence \( r = 3 \) and \( q = 37, \)
contrary to \( r \mid n \).

7. Case (V)

In this section we assume that \( N^*_\beta=S_4 \) and \( q \equiv 7,9 \pmod{16} \). We note that \( 4 \not\mid n \).

First we argue that \( N^*_\beta=N^* \cap N^\beta \). Suppose \( N^*_\beta \neq N^* \cap N^\beta \). Then \( C_G(N^*)=1 \) by Lemma 2.2. Since \( N^*_\beta/N^* \cap N^\beta \cong N^*_\beta/N^\beta \leq Z_2 \times Z_2 \), we have \( N^* \cap N^\beta = A_4 \) and \( N^*_\beta/N^* \cap N^\beta = Z_2 \), so that \( N^*_\beta/N^*_\beta \cong N^*_\beta/N^* \cap N^\beta \cong Z_2 \). Hence as \( \text{Out}(S_4)=1 \), \( Z(N^*_\beta N^*_\beta) \cong Z_2 \). Set \( \langle t_i \rangle = Z(N^*_\beta N^*_\beta) \) and let \( t_i \in I(N^*_\beta) - I(N^*) \). Since \( C_{N^*(t)} \cong N^*_\beta \) and \( t \in I(N^*_\beta) - I(N^*) \), by Lemma 2.6, we have \( C_{N^*(t)}=PGL(2, \sqrt{q}) \)

and \( |F(t)| = 1+3(q-6) \dfrac{r}{8} \) by Lemma 2.8.

Let \( P \) be a Sylow \( p \)-subgroup of \( C_{N^*(t)} \). Then \( |P| = \sqrt{q} \) and so \( \sqrt{q} \mid r \) and so \( 5^a \leq n^2 \) as \( p \geq 5 \) and \( r \mid n \). But obviously \( 5^a > n^2 \) for any positive integer \( n \). This is a contradiction. If \( p = 3 \), \( |P: P_t| = \sqrt{q}/3 \) or \( \sqrt{q}/3 \) for each \( \gamma \in \Omega - \{ \alpha \} \). Hence \( \sqrt{q}/3 = 3(q-6) \dfrac{r}{8} \) and so \( q \mid 81r^2 \). In particular, \( 3^a = q \mid 81n^2 \). From this, \( n \leq 7 \). Since \( q = 3^a \equiv 7 \) or \( 9 \pmod{16} \), we have \( q = 3^a \) or \( 3^b \) or \( 3^c \) or \( 3^d \). If \( q = 3^a \), \( |\Omega| = 1+|N^*: N^*_\beta|=1+8\cdot 9\cdot 10/2 \cdot 24 = 16 \), a contradiction by [9]. If \( q = 3^b \), \( |F(t)| = 1+273r \) and \( |F(t) - \{ \alpha \}| \geq |C_{N^*(t)}| \cong |PGL(2, 3^c)|/8 = 2457 \) contrary to \( r \mid 3 \). Thus \( N^*_\beta = N^* \cap N^\beta \).

Let \( V \) be a cyclic subgroup of \( N^*_\beta \) of order 4 and let \( U \) be a Sylow 2-subgroup of \( N^*_\beta \) containing \( V \). Then \( U = N_{G_{ab}}(V) \), \( |F(V)| = 1+(q-6) \dfrac{r}{8} \) by Lemma 2.8 and \( |F(U)| = 1+8 \cdot 3r/24 = r+1 \) by Lemmas 2.3 and 2.4. If \( q \equiv 7,9 \pmod{16} \), then \( |F(U)| < |F(V)| \) and hence \( U \not\leq Z_2 \). Suppose \( q = 7 \) or \( 9 \). Then \( r = 1 \) as \( r \mid n \). Hence \( |\Omega| = 1+|N^*: N^*_\beta| = 8 \) or \( 16 \). By [10], we have a contradiction. Therefore \( U \not\leq Z_2 \).

Suppose \( V^g \leq G_{ab} \) for some \( g \in G \) and set \( \gamma = \gamma^g \). Then \( V^g \leq G_{v^g} \). Since \( G_{v^g} \leq N^* \cap G_{ab} \leq N^* \cap N^\beta \leq N^* \cap N^\beta = N^*_\beta \). As \( N^*_\beta = S_4 \), \( V^g = V^h \) for some \( h \in N^*_\beta \). Hence \( C_G(V^g) \) is doubly transitive. By Lemma 2.9, \( C_G(N^*) = 1 \) and one of the following holds.

(a) \( N_G(V^g) \leq AGL(1, 2) \).

(b) \( N_G(V^g) \simeq PSL(2, p) \), \( p_1=(q-6)/8 \geq 5 \).

(c) \( N_G(V^g) = R(3) \).

In the case (a), \( (r+1) = 1+(q-6) \dfrac{r}{8} \) by Lemma 2.5 and so \( r = (q-6-16)/8 \) and \( r \mid n \). From this \( q = 23 \) or \( 25 \) and \( r = 1 \). Since \( |\Omega| = 1+|N^*: N^*_\beta| = 1+2 \cdot 127 = 2 \cdot 163 \), we have \( |G| = 2 \cdot |G_{a_2}| = 1 \cdot |N_G(V)| = |F(V)| = 1 \cdot |C_{G_{a_2}}(V)| = 3 \cdot |G_{a_2}| = 4 \cdot |G_{a_2}| \), contrary to \( |C_G(V)| = 1 \).

In the case (b), \( p_1 = 1/8n \). From this, \( p_1 = 1/8 \cdot 64 \cdot 4 \cdot 4 = 4n p_1 \), where \( s \) is the order of \( N_{G_{a_2}}(V)^g \). Hence \( p_1 = 1/8n \). From this, \( r = (q-6)/8 \) or \( q = 23, 41, 71, 73 \) or \( 83 \). Since \( p_1 \) is a prime and \( p_1 = (q-6)/8 \), \( q = 23, 41, 71, 73 \). Therefore \( q = 41 \) and \( r = 60 \cdot 41 \cdot 42/24 = 2^2 \cdot 359 \), so that \( |G| = 4 \cdot |G_{a_2}| \).
Since $N^*_a = N^a \cap N^b$, $C_G(z)^F(z)$ is transitive by Lemma 2.1. On the other hand $|F(z)| = 1 + 40 - 9/24 = 16$ by Lemma 2.8 (i) and so $|C_G(z)| = 16|C_G(z)|_2$ is transitive by Lemma 2.9, contrary to $|C_G(z)|_2 = |G|$. In the case (c), $r + 1 = 4$ and $1 + (q - \epsilon)r/8 = 28$. Hence $r = 3$ and $q = 71$ or 73, contrary to $r | n$.

8. Case (VI)

In this section we assume that $N^*_b = A_5$ and $q \equiv 3, 5 \pmod{8}$. In particular, $n$ is odd. If $N^*_a \neq N^a \cap N^b$, then $N^a \cap N^b = 1$, $C_G(N^a) = 1$ and so $N^*_b = N^a N^b / N^b \leq \text{Out}(N^b) \cong Z_2 \times Z_n$, a contradiction. Hence $N^*_b = N^a \cap N^b$. Let $z$ be an involution in $N^*_b$ and $T$ a Sylow 2-subgroup of $N^*_b$ containing $z$. Then, by Lemma 2.8, $|F(z)| = 1 + (q - \epsilon)15 = 1 + (q - \epsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)| = 1 + 12r/60 = 1 + r$. Since $N^*_a = N^a \cap N^b$, $z^G \cap G_{ab} = z^G \cap N^b = z^N$ and so $C_G(z)^F(z)$ is doubly transitive. By Lemma 2.9, $C_G(N^a) = 1$ and one of the following holds.

(a) $C_G(z)^F(z) = \text{Alt}(1, 2^a)$.
(b) $C_G(z)^F(z) \supseteq \text{PSL}(2, p_1)$, $p_1 = (q - \epsilon)/4 \geq 5$.
(c) $C_G(z)^F(z) = R(3)$.

In the case (a), by Lemma 2.5, $(q - \epsilon)/4 = 1$ or $(r + 1)/2 = 1 + (q - \epsilon)r/4$. Hence $q = 5$ or $r = (q - \epsilon - 8)/4 | n$. If $q = 5$, then $N^*_a = N^a$, a contradiction. Therefore $p^a - \epsilon - 8 \not| 4n$ and so $n = 1$ and $q = 11$ or 13. If $q = 13$, we have $5 < |G_a|$, a contradiction. Hence $q = 11$ and $|\Omega| = 1 + |N^a| = 1 + 10 \cdot 11 \cdot 12 / 2 \cdot 60 = 12$. By [9], $G^a = N_{11}$, $|\Omega| = 12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1(p_1 - 1)/2 \leq s$ and $2n(q - \epsilon)/2 = 4mp_1$, where $s$ is the order of $C_G(z)^F(z)$. Hence $p_1 - 1 | 8n$ and so $p^a = q - 4 + 32n$. From this $q = 19, 27$ or 37. Since $5 | |G_a|$, $q = 27, 37$. Hence $q = 19$ and $|\Omega| = 1 + |N^a| = 1 + 18 \cdot 19 \cdot 20 / 2 \cdot 60 = 2.29$. Since $G_a \cong \text{PSL}(2, 19)$ or $\text{PGL}(2, 19)$, $|G_a| = |\Omega| = |G_a| = 2 \cdot 2^1 \cdot 18 \cdot 19 \cdot 20 / 2 = 2^{i+1} \cdot 3.5.19.29$ with $0 \leq i \leq 1$. Let $P$ be a Sylow 29-subgroup of $G$. Then $P$ is semi-regular on $\Omega$ and 3, 5, 19 $\not| \langle N_G(P) \rangle$ because $N_G(P) / C_G(P) \leq Z_s \times Z_t$. Hence $|G : N_G(P)| = 2^i \cdot 3.5.19$ with $0 \leq j \leq 4$, while $2^i \cdot 3^2 . 5 . 19 \not| 29$ (mod 29) for any $j$ with $0 \leq j \leq 4$, contrary to a Sylow’s theorem.

If $C_G(z)^F(z) = R(3)$, $r + 1 = 4$ and $1 + (q - \epsilon)r/4 = 28$ and hence $r = 3$, $q = 37$, contrary to $r | n$.

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References

[3] C. Hering: *Transitive linear groups and linear groups which contain irreducible


