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Osaka University
ON SOME DOUBLY TRANSITIVE PERMUTATION
GROUPS IN WHICH SOCLE(Gα) IS NONSOLVABLE

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1. Introduction

Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ and $\alpha \in \Omega$. In [8], O'Nan has proved that $\text{socle}(G_\alpha) = A \times N$, where $A$ is an abelian group and $N$ is 1 or a nonabelian simple group. Here $\text{socle}(G_\alpha)$ is the product of all minimal normal subgroups of $G_\alpha$.

In the previous paper [4], we have studied doubly transitive permutation groups in which $N$ is isomorphic to $PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q$ even. In this paper we shall prove the following:

**Theorem.** Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ with $|\Omega|$ even and let $\alpha \in \Omega$. If $G_\alpha$ has a normal simple subgroup $N^*$ isomorphic to $PSL(2, q)$, where $q$ is odd, then one of the following holds.

(i) $G_\Omega$ has a regular normal subgroup.

(ii) $G_\alpha \simeq A_6$ or $S_6$, $N^* = PSL(2, 5)$ and $|\Omega| = 6$.

(iii) $G_\alpha \simeq M_{11}$, $N^* = PSL(2, 11)$ and $|\Omega| = 12$.

In the case that $G_\alpha$ has a regular normal subgroup, by a result of Hering [3] we have $(|\Omega|, q) = (16, 9)$, $(16, 5)$ or $(8, 7)$.

We introduce some notations:

- $F(X)$: the set of fixed points of a nonempty subset $X$ of $G$
- $X(\Delta)$: the global stabilizer of a subset $\Delta (\subseteq \Omega)$ in $X$
- $X_\Delta$: the pointwise stabilizer of $\Delta$ in $X$
- $X^\Delta$: the restriction of $X$ on $\Delta$
- $m|n$: an integer $m$ divides an integer $n$
- $X^H$: the set of $H$-conjugates of $X$
- $|X|_p$: maximal power of $p$ dividing the order of $X$
- $I(X)$: the set of involutions in $X$

In this paper all sets and groups are finite.
2. Preliminaries

**Lemma 2.1.** Let $G$ be a transitive permutation group on $\Omega$, $\alpha \in \Omega$ and $N^\alpha$ a normal subgroup of $G_\alpha$ such that $F(N^\alpha) = \{\alpha\}$. Let the subgroup $X \leq N^\alpha$ be conjugate in $G_\alpha$ to every group $Y$ which lies in $N^\alpha$ and which is conjugate to $X$ in $G$. Then $N_\alpha(X)$ is transitive on $\Delta = \{\gamma \in \Omega | X \leq N^\alpha\}$.

Proof. Let $\beta \in \Delta$ and let $g \in G$ such that $\beta^g = \alpha$. Then, as $X \leq N^\beta$, $X^\beta \leq N^\beta = N^\alpha$. By assumption, $(X^\beta)^h = X$ for some $h \in G_\alpha$. Hence $gh \in N_\alpha(X)$ and $\alpha^{(gh)^{-1}} = \alpha^{-1} = \beta$. Obviously $N_\alpha(X)$ stabilizes $\Delta$. Thus Lemma 2.1 holds.

**Lemma 2.2.** Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and $N^\alpha$ a nonabelian simple normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. If $C_G(N^\alpha) \neq 1$, then $N^\alpha = N^\alpha \cap N^\beta$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semiregular on $\Omega - \{\alpha\}$.

Proof. See Lemma 2.1 of [4].

**Lemma 2.3.** Let $G$ be a transitive permutation group on $\Omega$, $H$ a stabilizer of a point of $\Omega$ and $M$ a nonempty subset of $G$. Then

$$|F(M)| = |N_\alpha(M)| \times |M^G \cap H| / |H|.$$  

Here $M^G \cap H = \{g^{-1}Mg | g \in G \}$.

Proof. See Lemma 2.2 of [4].

**Lemma 2.4.** Let $G$ be a doubly transitive permutation group on $\Omega$ and $N^\alpha$ a normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. Assume that a subgroup $X$ of $N^\alpha$ satisfies $X^G = X^{N^\alpha}$. Then the following hold.

(i) $|F(X) \cap \beta^N^\alpha| = |F(X) \cap \gamma^N^\alpha|$ for $\beta, \gamma \in \Omega - \{\alpha\}$.

(ii) $|F(X)| = 1 + |F(X) \cap \beta^N^\alpha| \times r$, where $r$ is the number of $N^\alpha$-orbits on $\Omega - \{\alpha\}$.

Proof. Let $\Gamma = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\alpha$-orbits on $\Omega - \{\alpha\}$. Since $G_\alpha$ is transitive on $\Omega - \{\alpha\}$ and $G_\alpha \supseteq N^\alpha$, we have $|\Delta_i| = |\Delta_j|$ for $1 \leq i, j \leq r$. By assumption, $G_\alpha = N_{G_\alpha}(X)N^\alpha$ and so $N_{G_\alpha}(X)$ is transitive on $\Gamma$. Hence for each $i$ with $1 \leq i \leq r$, there exists $g \in N_{G_\alpha}(X)$ such that $\Delta_i^g = \Delta_i$. Therefore $|F(X) \cap \Delta_i| = |F(X^g) \cap (\Delta_i)^g| = |F(X) \cap \Delta_i|$. Thus (i) holds and (ii) follows immediately from (i).

**Lemma 2.5** (Huppert [5]). Let $G$ be a doubly transitive permutation group on $\Omega$. Suppose that $\vartheta_2(G) \neq 1$ and $G_\alpha$ is solvable. Then for any involution $x$ in $G_\alpha$, $|F(x)|^2 = |\Omega|$.

We list now some properties of $PSL(2,q)$ with $q$ odd which will be required
in the proof of our theorem.

**Lemma 2.6** ([2], [6], [10]). Set \( N = PSL(2, q) \) and \( G = Aut(N) \), where \( q = p^e \) and \( p \) is an odd prime. Let \( z \) be an involution in \( N \). Then the following hold.

(i) \( |N| = (q-1)q(q+1)/2, I(N) = \langle z \rangle \) and \( C_N(z) = D_{q-1} \), where \( q \equiv \epsilon \in \{ \pm 1 \} \) (mod 4).

(ii) If \( q \neq 3 \), \( N \) is a nonabelian simple group and a Sylow \( r \)-subgroup of \( N \) is cyclic when \( r \neq 2, p \).

(iii) If \( X \) and \( Y \) are cyclic groups of \( N \) and \( |X| = |Y| \neq 2, p \), then \( X \) is conjugate to \( Y \) in \( \langle X, Y \rangle \) and \( N_N(X) = D_{q-1} \).

(iv) If \( X \leq N \) and \( X = Z_2 \times Z_2 \) and \( N_N(X) \) is isomorphic to \( A_4 \) or \( S_4 \).

(v) If \( |N|_2 \geq 8 \), \( N \) has two conjugate classes of four-groups in \( N \).

(vi) There exist a field automorphism \( f \) of \( N \) of order \( n \) and a diagonal automorphism \( d \) of \( N \) of order 2 and if we identify \( N \) with its inner automorphism group, \( \langle d \rangle N = PGL(2, q), \langle f \rangle \langle d \rangle N = G \) and \( G|N = Z_2 \times Z_n \).

(vii) \( C_N(d) = D_{q-1} \) and \( C_{\langle d \rangle N}(z) = D_{q-1} \).

(viii) Suppose \( n = mk \) for positive integers \( m, k \). Then \( C_N(f^n) = PSL(2, p^m) \) if \( k \) is odd and \( C_N(f^n) = PGL(2, p^m) \) if \( k \) is even.

(ix) Assume \( n \) is even and let \( u \) be a field automorphism of order 2. Then \( I(G) = I(N) \cup d^N \cup u^d \). If \( n \) is odd, \( I(G) = I(N) \cup d^N \).

(x) If \( H \) is a subgroup of \( N \) of odd index, then one of the following holds:

1. \( H \) is a subgroup of \( C_N(z) \) of odd index for some involution \( z \in N \).
2. \( H = PGL(2, p^n) \), where \( n = 2mk \) and \( k \) is odd.
3. \( H = PSL(2, p^n) \), where \( n = mk \) and \( k \) is odd.
4. \( H = A_4 \) and \( q \equiv 3, 5 \) (mod 8).
5. \( H = S_4 \) and \( q \equiv 7, 9 \) (mod 16).
6. \( H = A_5, q \equiv 3, 5 \) (mod 8) and \( 5 | (q - 1)q/q+1 \).

**Lemma 2.7.** Let \( G, N, d \) and \( f \) be as defined in Lemma 2.6 and \( H \) an \( \langle f, d \rangle \)-invariant subgroup of \( N \) isomorphic to \( D_{q-1} \). Let \( W \) be a cyclic subgroup of \( \langle d \rangle H \) of index 2 (cf. (vii) of Lemma 2.6) and set \( Y = 0_d(W \cap H) \). Then \( C_G(Y) = W \cdot C_{\langle d \rangle}(Y) \).

Proof. By (viii) of Lemma 2.6, we can take an involution \( t \) satisfying \( \langle d \rangle H = \langle t \rangle W \) and \( [f, t] = 1 \). Since \( N_\langle Y \rangle = \langle f, d \rangle N_N(Y) = \langle f, d \rangle H \), \( C_G(Y) = C_{\langle f \rangle \langle d \rangle H}(Y) = W \cdot C_{\langle d \rangle \langle d \rangle}(Y) \). Suppose \( ht \in C(Y) \) for some \( h \in \langle f \rangle \). Since \( t \) inverts \( Y \), \( h \) also inverts \( Y \) and so \( h^2 \) centralizes \( Y \). Hence some nontrivial 2-element \( g \in \langle h \rangle \) inverts \( Y \), so that \( C_H(g) \) contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, \( G^\Omega \) will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume \( G^\Omega \) has no regular normal subgroup.
Notation. $C^a = C_G(N^a)$, which is semi-regular on $\Omega - \{\alpha\}$ by Lemma 2.2. Let $r$ be the number of $N^a$-orbits on $\Omega - \{\alpha\}$.

Since $G_\beta \supseteq N^a$, $|\beta^N^a| = |\gamma^N^a|$ for $\beta, \gamma \in \Omega - \{\alpha\}$ and so $|\Omega| = 1 + r \times |\beta^N^a|$. Hence $r$ is odd and $N_\beta^a$ is a subgroup of $N^a$ of odd index. Therefore $N_\beta^a$ is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

**Lemma 2.8.** Let $Z$ be a cyclic subgroup of $N^a_\beta$ with $|Z| \neq 1, p$. Then
(i) $\beta^Z \neq \gamma^Z$, $|\beta^Z| = 1 + (q - e)|I(N_\beta^a)|r/|N_\beta^a|$.
(ii) If $|Z| \neq 1$, $|F(Z)| = 1 + |N_{N^a}(Z)/r/N_{N^a}(Z)|$.

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

**Lemma 2.9.** If $N^a_\beta \neq D_{25}$ and $Z$ is a cyclic subgroup of $N^a_\beta$ with $|Z| = 1, p$ and $N_\beta(Z)^F(Z)$ is doubly transitive. Then $C^a = 1$ and one of the following holds.
(i) $N_\beta(Z)^F(Z) \leq AGL(1, q_1)$ for some $q_1$.
(ii) $C_\beta Z \geq PSL(2, p_1)$, $r = 1$ and $|F(Z)| = 1 + |N_{N^a}(Z)|/|N_{N^a}(Z)|$.
(iii) $N_\beta(Z)^F(Z) = R(3)$, the smallest Ree group, $|F(Z)| = 28$.

Proof. Set $N_\beta(Z) = L$ and $F(Z) = \Delta$. By Lemma 2.6 (iii), $L \cap N^a = D_{25}$ and $L \cap N^a = \langle t \rangle Y \geq Y \geq Z$, where $0(t) = 2$, $Y = Z(t=2)/2$.

If $(L \cap N^a)^a = 1$, then $L \cap N^a = L^a$ because $L \cap N^a$ is a maximal subgroup of $N^a$. Since $|N^a: N^a_\beta|$ is odd, $L \cap N^a = N^a_\beta = D_{25}$, contrary to the assumption. Hence $(L \cap N^a)^a = 1$ and as $L_a \supseteq L_a \cap N^a$ and $L_a \supseteq Y$, $(L_a)^a$ has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:
(a) $L^a$ has a regular normal subgroup
(b) $L^a \supseteq PSL(2, p_1)$, $|\Delta| = p_1 + 1$, where $p_1 (\geq 5)$ is a prime
(c) $L^a \supseteq PSL(3, p_1)$, $p_1 \geq 3$, $|\Delta| = (p_1^a)^3 + 1$
(d) $L^a = R(3)$, $|\Delta| = 28$.

Suppose $C^a = 1$. Then there exists a subgroup $D$ in $C^a$ of prime order such that $(L_a)^D \supseteq D^a$. Since $[L_a, D] \leq D \cdot L_a \cap C^a = D(L_a \cap C^a) = D$, $D$ is a normal subgroup of $L_a$. By (i) and (iii) of Lemma 2.6, $G_a = L_a \cdot N^a$ and so $D$ is a normal subgroup of $G_a$. By Theorem 3 of [1], $G^a$ has a regular normal subgroup, contrary to the hypothesis. Thus $C^a = 1$.

If (a) occurs, $L^a$ is solvable because $L_a/\mathfrak{L} N^a = L_a N^a/N^a \leq \text{Out}(N^a)$ and $L \cap N^a = D_{25}$. Hence by [5], (i) holds in this case.

If (b) occurs, we have $Y^a = 1$, for otherwise $(L \cap N^a)^a = 1$ and so $N^a_\beta = L \cap N^a = D_{25}$, a contradiction. Hence $1 = C_G(Z)^a \leq D_\alpha$ and so $C_G(Z)^a \supseteq PSL(2, p_1)$ and $Y^a = Z_{p_1}$. Therefore $|\Delta| = |\beta^N^a| = p_1$, $r = 1$ by Lemma 2.4 (ii). Since $|\beta^N^a| = p_1$, we have $|\beta^L \cap N^a^a| = p_1$, so that $L \cap N^a: L \cap N^a_\beta = p_1$. Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of $PSU(3, p_1)$,
a Sylow $p_1$-subgroup of $(L_\Delta)'$ is not cyclic, while $(L_\Delta)' \leq L \cap N^\alpha \leq D_\Delta$, a contradiction.

3. Case (I)

In this section we assume that $N_\beta^\alpha \leq D_{\Delta}^{r-1}$, where $\beta \neq \alpha$, $q = p^\alpha$.

(3.1) (i) If $N_\beta^\alpha \neq Z_2 \times Z_2$, $N_\beta^\alpha (N_\beta^\alpha) = N_\beta^\alpha$ and $|F(N_\beta^\alpha)| = r+1$.

(ii) If $N_\beta^\alpha = Z_2 \times Z_2$, $N_\beta^\alpha (N_\beta^\alpha) = A_4$ and $|F(N_\beta^\alpha)| = 3r+1$.

Proof. Put $X = N_\beta^\alpha (N_\beta^\alpha)$. Let $S$ be a Sylow 2-subgroup of $N_\beta^\alpha$ and $Y$ a cyclic subgroup of $N_\beta^\alpha$ of index 2.

If $N_\beta^\alpha = Z_2 \times Z_2$, then $|Y| > 2$ and so $Y$ is characteristic in $N_\beta^\alpha$. Hence $X \leq N_\beta^\alpha (Y) \cong D_{\Delta}^{r-1}$. From this $[N_X(S), S \cap Y] \leq S \cap Y$ and $0^\beta(N_X(S))$ stabilizes a normal series $S \supseteq S \cap Y \supseteq 1$, so that $0^\beta(N_X(S)) \leq C_{N^\alpha}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^\alpha}(S) \leq S$ and hence $N_X(S) = S$. On the other hand by a Frattini argument, $X = N_X(S)N_\beta^\alpha$ and so $X = N_\beta^\alpha$. By Lemma 2.6(i), $(N_\beta^\alpha)^{\gamma} = (N_\beta^\alpha)^{\gamma}$ and so by Lemmas 2.3 and 2.4(ii), $|F(N_\beta^\alpha)| = 1 + |F(N_\beta^\alpha) \cap \beta^\alpha| \times r = 1 + |N_\beta^\alpha| / |N_\beta^\alpha| = r+1$. Thus (i) holds.

If $N_\beta^\alpha = Z_2 \times Z_2$, $N_\beta^\alpha (N_\beta^\alpha) = A_4$ by Lemma 2.6(iv). Similarly as in the case $N_\beta^\alpha \neq Z_2 \times Z_2$, we have $|F(N_\beta^\alpha)| = 3r+1$.

(3.2) $N_\beta^\alpha / N_\beta \cap N_\beta^\alpha \leq Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^\alpha = 1$. Suppose $C^\alpha = 1$. Then $N_\beta^\alpha / N_\beta \cap N_\beta^\alpha \leq N_\beta^\alpha / N_\beta \leq \text{Out}(N^\alpha) \cong Z_2 \times Z_2$ by Lemma 2.6(vi) and hence $(N_\beta^\alpha)^{\gamma} = N_\beta^\alpha \cap N_\beta^\alpha$. Since $N_\beta^\alpha$ is dihedral, $(N_\beta^\alpha)^{\gamma} = Z_2 \times Z_2$, so that $N_\beta^\alpha / N_\beta \cap N_\beta^\alpha \leq Z_2 \times Z_2$.

(3.3) Suppose $N_\beta^\alpha = N^\alpha \cap N_\beta^\alpha$ and let $U$ be a subgroup of $N_\beta^\alpha$ isomorphic to $Z_2 \times Z_2$. Then $|F(U)| = 3r+1$ and $N_\beta(U)^{F(U)}$ is doubly transitive.

Proof. Sex $X = N_\beta(U)^{N_\beta}$, $\Delta = F(N_\beta)$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\alpha$-orbits on $\Omega - \{a\}$. If $g^{-1}N_\beta^\alpha g \leq G_{a_{\beta}}$, then $g^{-1}N_\beta^\alpha g \leq N_\Delta \cap N_\beta^\alpha = N_\alpha \cap N_\beta^\alpha \leq N_\beta^\alpha$, where $\gamma = \alpha^\beta$. By a Witt's theorem, $X^\Delta$ is doubly transitive.

If $U$ is a Sylow 2-subgroup of $N_\beta^\alpha$, by a Witt's theorem, $N_\beta(U)^{F(U)}$ is doubly transitive. Moreover $N_\beta(U) = A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)| = 1 + |A_4| \times |N_\beta^\alpha : N_\beta(U)| \times r / |N_\beta^\alpha| = 3r+1$.

If $|N_\beta^\alpha| \geq 4$, by Lemma 2.6(iv) and (v), $N_\beta(U) = S_4$ and $N_\beta^\alpha$ has two conjugate classes of four-groups, say $p = \{K_1, K_2\}$. Set $X_d = M$. Then $M \geq N_\beta^\alpha$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_1 = \phi$ for each $i$ and so $|F(U) \cap \Delta_1| = 3$ by Lemma 2.3. Hence $|F(U)| = 3r+1$. Since $N_\beta(U) = S_4$, we may assume $r > 1$. Hence by (3.1) (i) $|\Delta| = r+1 \geq 4$, so that $M^\Delta$ is doubly transitive. Since $M = N_\beta^\alpha N_\beta(U), N_\beta(U)^\Delta$ is also doubly transitive and so $N_\beta(U)$ is transitive on $\Delta$.
{\alpha}. As $|\Delta \cap \Delta_i| = 1$, $\Delta \cap \Delta_i \subseteq F(U)$ and $N_{N^*}(U)$ is transitive on $F(U) \cap \Delta_i$ for each $i$, $N_G(U)^F(U)$ is doubly transitive.

(3.4) (i) $C^*=1$.

(ii) Let $U$ be a subgroup of $N^*_G$ isomorphic to $Z_2 \times Z_2$. If $N^*_G = N^* \cap N^\beta$, then $N_G(U)^F(U)$ has a regular normal 2-subgroup. In particular $|F(U)| = 3r+1 = 2^b$ for positive integer $b$.

Proof. Since $N_{G_a}(U)^F(U) \supseteq N_{N^*}(U)^F(U) = S_3$ or $Z_3$, by (3.3) and Theorem 3 of [1], $N_G(U)^F(U)$ has a regular normal subgroup, $N_G(U)^F(U) \supseteq PSU(3,3)$ or $N_G(U)^F(U) = R(3)$.

Suppose $C^* \neq 1$. Let $D$ be a minimal characteristic subgroup of $C^*$. Clearly $G_{a\beta} \supset D$. If $N_G(U)^F(U) = R(3)$, $D$ is cyclic. By Theorem 3 of [1], $G^a$ has a regular normal subgroup, contrary to the hypothesis. Hence $N_G(U)^F(U) = R(3)$. Therefore $(N_{G_a}(U)^F(U))'$ contains an element of order 9. Since $N_{G_a}(U)^F(U) = N_{N^*}(U)$ is dihedral, by (vi) of Lemma 2.6 we have $(N_{G_a}(U))' \leq C^* \times N_{N^*}(U)$. From this, $C^*$ contains an element of order 9 and so $C^* = Z_9$ or $M_3(3)$. In both cases, $C^*$ contains a characteristic subgroup of order 3. Since $G_{a\beta} \supset D$, by Theorem 3 of [1] $G^a$ has a regular normal subgroup, a contradiction. Thus $C^* = 1$.

Let $R$ be a Sylow 3-subgroup of $N_{G_a}(U)$. Since $N_{G_a}(U)/N_{N^*}(U) = N_{G_a}(U)/N^* \subseteq \text{Out}(N^*) = Z_2 \times Z_9$, $R/R \cap N_{N^*}(U)$ is cyclic. Clearly $R \cap N_{N^*}(U) = Z_3$. Therefore $N_G(U)^F(U) \supseteq PSU(3,3)$, $R(3)$. Thus (3.4) holds.

Since $N^*_G$ is dihedral, we set $N^*_G = \langle \gamma \rangle W$ and $Y = W \cap N^* \cap N^\beta$, where $W$ is a cyclic subgroup of $N^*_G$ of index 2 and $t$ is an involution in $N^*_G$ which inverts $W$.

(3.5) (i) If $|Y| \geq 3$, $N_G(Y)^F(Y)$ is doubly transitive.

(ii) If $|Y| < 3$, $N^*_G = Z_2 \times Z_2$ or $N^*_G = D_4$ and $N^* \cap N^\beta \leq Z_2 \times Z_2$.

Proof. Suppose $|Y| \geq 3$. If $Y^\gamma = G_{a\beta}$, $Y^\gamma \leq N^* \cap G_{a\beta} \leq N^*_G$, where $\gamma = \alpha^\beta$. If $\gamma = \alpha$, obviously $Y^\gamma \leq N^*$. If $\gamma = \alpha$, $N^*_G = N^\beta$. Therefore, as $|Y| \geq 3$, $N^*_G$ has a unique cyclic subgroup of order $|Y|$. Hence $Y^\gamma \leq N^* \cap N^\gamma \leq N^*$, so that $Y^\gamma \leq N^*$. Similarly $Y^\gamma \leq N^\beta$. Thus $Y^\gamma \leq N^* \cap N^\beta$ and so $Y^\gamma = Y$. By a Witt's theorem, $N_G(Y)$ is doubly transitive on $F(Y)$.

Suppose $|Y| < 3$. Since $|N^* \cap N^\beta: Y| \leq 2$, we have $N^* \cap N^\beta \leq Z_2 \times Z_2$. On the other hand, as $N^* G$ is dihedral, $(N^*_G)'$ is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set $\Delta = F(N^*_G)$, $L = G(\Delta)$, $K = G_\Delta$ and suppose $N^*_G \not\subset Z_2 \times Z_2$. Then $L \supseteq N^*_G$, $(L_a)^* \leq N^*_G$, $K' \leq N^* \cap N^\beta$ and $(L_a)^* = Z_r$. If $r \neq 1$, $L^\Delta$ is a doubly transitive Frobenius group of degree $r+1$.

Proof. By Corollary B1 of [7] and (i) of (3.1), $L^\Delta$ is doubly transitive and
Since \( |\Delta| = r+1 \). Since \( N^a \cap L \supseteq N^a \cap K = N^a \beta \), by (i) of (3.1), we have \( N^a \cap L = N^a \beta \). Hence \( L_\alpha \supseteq N^a _\beta \). By (i) of (3.4), \( L_\alpha \cap N^a _\beta = L_\alpha N^a _\beta / N^a _\beta = \text{Out}(N^a _\beta) = Z_2 \times Z_2 \) and so \( (L_\alpha)^2 \leq N^a _\beta \) and \( (L_\alpha)^2 = Z_2 \). If \( r \neq 1 \), then \( (L_\alpha)^2 = 1 \). On the other hand \( (L_\alpha)^2 = 1 \) as \( (L_\alpha)^2 \) is abelian. Hence \( L^\alpha \) is a Frobenius group.

(3.7) Suppose \( |Y| \geq 3 \). Then there exists an involution \( z \) in \( N^a _\beta \cap Y \) such that \( Z(N^a _\beta) = \langle z \rangle \).

Proof. Suppose \( N^a _\beta \neq Z_2 \times Z_2 \), \( |N^a _\beta| \geq 2 \) and \( N^a _\beta \) is dihedral, we have \( \langle I(W) \rangle = Z(N^a _\beta) = Z_2 \) and \( N^a _\beta / (N^a _\beta)^2 = Z_2 \times Z_2 \). Let \( Z(N^a _\beta) = \langle z \rangle \) and suppose that \( z \) is not contained in \( Y \). By (3.2), \( (N^a _\beta)^2 \leq N^a \cap N^a \cap W = Y \) and so \( |(N^a _\beta)^2| \) is odd. Hence \( |N^a _\beta| = 4 \) and \( q \equiv p^2 = 3 \) or \( 5 \) (mod 8), so that \( n \) is odd. By (3.2) and (i) of (3.4), \( N^a _\beta / (N^a _\beta)^2 \cong N^a _\beta / N^a _\beta = 1 \) or \( Z_2 \). If \( N^a _\beta = N^a \cap N^a \), then \( W = Y \) and so \( z \in Y \), contrary to the assumption. Therefore we have \( N^a _\beta / N^a \cap N^a = Z_2 \) and \( N^a _\beta = \langle z \rangle \times (N^a \cap N^a) \). Since \( n \) is odd and \( z \subseteq N^a \cap N^a \), by Lemma 2.6 (vi), (vii) and (ix), \( N^a _\beta / N^a \cap N^a = D_q \) and \( C_{N^a}(z) = D_q \). But \( N^a \cap N^a \leq C_{N^a}(z) \) and besides it is isomorphic to a subgroup of \( D_{q-1} \). Hence \( N^a \cap N^a = Z_2 \) and \( N^a _\beta = Z_2 \times Z_2 \), a contradiction.

(3.8) Suppose \( |Y| \geq 3 \). Then \( N^a _\beta = N^a \cap N^a \).

Proof. Suppose \( N^a _\beta \neq N^a \cap N^a \) and let \( \Delta, L, K \) be as defined in (3.6) and \( x \in L_\alpha \) such that its order is odd and \( \langle x \rangle \) is transitive on \( \Delta - \{a\} \). As \( |Y| \geq 3 \), \( W \) is characteristic in \( N^a _\beta \) and hence by (3.6), \( x \) stabilizes a normal series \( L_\alpha \supseteq N^a _\beta \supseteq W \supseteq (N^a _\beta)^2 \). By Theorem 5.3.2 of [2], \( [x, 0_{(L_\alpha / (N^a _\beta)^2)}] = 1 \). Since \( L_\alpha / (N^a _\beta)^2 \) has a normal Sylow 2-subgroup and \( (N^a _\beta)^2 \leq K' \), we have \( [x, 0_{(L_\alpha / K')} = 1 \), so that \( [x, N^a _\beta] \leq K \leq N^a \cap N^a \) by (3.6). If \( r \neq 1 \), then \( \beta \in \Delta \) and \( \beta^\sigma \in \Delta \), hence \( N^a _\beta = x^{-1} N^a _\beta x = N^a _\beta \), where \( \gamma = \beta^\sigma \). Since \( \gamma \in \Delta \) and \( \Delta = F(N^a _\beta) \), \( N^a _\beta \leq N^a \cap G_Y \) and so \( N^a _\beta = N^a _\beta \). Similarly \( N^a _\beta = N^a _\beta \). Hence \( N^a _\beta = N^a _\beta \), which implies \( N^a _\beta = N^a \cap N^a \). By the doubly transitivity of \( G \), we have \( N^a _\beta = N^a \cap N^a \), contrary to the assumption. Therefore we obtain \( r = 1 \).

Let \( z \) be as defined in (3.7) and put \( k = (q-\varepsilon)|N^a _\beta| \). By Lemma 2.8(i) we have \( |F(z)| = 1 + (q-\varepsilon) |N^a _\beta| / 2 + 1 |N^a _\beta| = (q-\varepsilon) / 2 + k + 1 \). Similarly \( |F(Y)| = k + 1 \). As \( N^a _\beta \neq N^a \cap N^a \), there is an involution \( t \) in \( N^a _\beta \) which is not contained in \( N^a \). By Lemma 2.6 (i), \( t = z \) for some \( y \in N^a \). Set \( \gamma = \beta^\sigma \). Then \( \gamma \in F(z) \) and \( z \in N^a \). By Lemma 2.6 (vii), (viii) and (ix), \( C_{N^a}(z) = D_q^\oplus \), or \( PGL(2, \sqrt{q}) \). Assume \( C_{N^a}(z) = D_q^\oplus \) and let \( R \) be a cyclic subgroup of \( C_{N^a}(z) \) of index 2. We note that \( R \) is semi-regular on \( \Omega - \{\alpha\} \). Set \( X = C_\alpha(z) \). Since \( 2 \leq k + 1 \leq (q-\varepsilon) / |q-\varepsilon| + 1 \), we have \( (q+\varepsilon) / 2 \) \( k + 1 \) and so \( |\alpha X| = k + 1 \). By (i) of (3.5) and (3.7), \( N_\alpha(Y) \leq C_\alpha(z) = X \) and \( \alpha X \geq F(Y) \). It follows from Lemma 2.1 that \( \alpha X = \{z | z \in N^a \} \equiv \gamma \). Hence \( |F(z)| > |\alpha X| \geq |F(Y)| + (q+\varepsilon) / 2 = k + 1 + (q-\varepsilon) / 2 + \varepsilon = |F(z)| + \varepsilon \). Therefore \( \varepsilon = -1 \) and \( \gamma X = \{\gamma \} \), so that \( \gamma \in F(Y) \), a contradiction. Thus \( C_{N^a}(z) = PGL(2, \sqrt{q}) \), \( \varepsilon = 1 \), \( N^a _\beta / N^a \cap N^a = Z_2 \) and \( \langle \alpha^\varepsilon \cap G_\alpha \rangle : N^a \rangle = 2 \).
Set $\Delta_1 = \alpha^x$ and $\Delta_2 = F(z) - \Delta_1$. Let $\delta \in \Delta_2$ and $g$ an element of $G$ satisfying $\delta^g = \gamma$. Then $z \in N^*_g N^3 - N^4$ and so $z^g \in N^*_g N^3 - N^4$, where $v = \alpha^x$. Since $|\langle \delta^g \cap G^\gamma \rangle| = 2$ and $x \in G^\gamma - N^4$, it follows from Lemma 2.6 (ix) that $(z^g)^4 = z$ for some $h \in G^\gamma$. Hence $gh \in X$ and $\delta^gh = \gamma$. Thus $\Delta_2 = \gamma^x$. Let $\delta \in \Delta_2$. Then $z \in N^*_g$ and $z \in Z(N^*_g)$ by (3.7) and so $X \cap N^*_g = Z \times Z_2$, which implies $|\{\delta^x \mid x \in \Omega\}| = (q-1)/4$. Hence $|\Delta_1| = |\Delta_2| = ((q-1)/4 + k + 1, (q-1)/4)$ or $(k+1, (q-1)/2)$. Let $P$ be a subgroup of $C_N^\gamma(z)$ of order $\sqrt{q}$. Then $F(P) = \{\gamma\}$ and $P$ is semi-regular on $\Omega - \{\gamma\}$. If $|\Delta_2| = (q-1)/4$, then $\sqrt{q} \mid (q-1)/4 - 1 = (q-5)/4$ and $\sqrt{q} \mid (q-1)/4 + k + 1$. From this, $q = 5^e$, $k = 3$, $|\Delta_1| = 10$ and $|\Delta_2| = 6$. Since $(C_N^\gamma(z))^2 = S_3$, $X^2 \subset S_3$ and so $|X| \geq 3^2$. As $X$ acts on $\Delta_1$ and $|\Delta_1| = 11$ (mod 3), $G_{\Delta_1} \geq \{X, |X| \geq 3^2\}$. If $|\Delta_2| = (q-1)/2$, $\sqrt{q} \mid (q-1)/2 - 1 = (q-3)/2$. Since $q = 3^e$, $k = 1$, $N^*_g = D_2$ and $\Delta_1 = \{e, \beta\}$. Hence $C_N^\gamma(z)$ fixes $\alpha$ and $\beta$, so that $\gamma \in \Omega - \{\gamma\}$, which implies $|\Delta_2| = (q-1)/2$. Hence $2r = 2^e - c - 1$ and so $c = 1$ as $r$ is odd. Thus $r = 1$.

(3.9) Suppose $|Y| \geq 3$. Then $r = 1$.

Proof. By (3.6), $r + 1 = 2^e$ for some integer $c \geq 0$. On the other hand, $3r + 1 = 2^e$ by (3.8) and (ii) of (3.4). Hence $2r = 2^e - c - 1$ and so $c = 1$ as $r$ is odd. Thus $r = 1$.

(3.10) Put $k = (q-\epsilon)/|N^*_p|$. If $N^*_p = N^* \cap N^p$ and $r = 1$, then

$$q-\epsilon + 2k + 2 \mid (2k + 2) \mid G = kq(q+\epsilon)/2 + 1.$$

Proof. Set $S = \{\{\gamma, u \mid \gamma \in F(u), u \in z^o\}, \text{ where } z \text{ is an involution in } N^*_g. \text{ We now count the number of elements of } S \text{ in two ways. Since } N^*_p = N^* \cap N^p, \text{ and hence } C_G(z) \text{ is transitive on } F(z) \text{ by Lemma 2.1. Therefore } |S| = |\Omega| \mid z^o \mid = |z^o| \mid F(z)\mid. \text{ Since } r = 1, |\Omega| = 1 + |N^*: N^*_g| = kq(q+\epsilon)/2 + 1 \text{ and by Lemma 2.8 } |F(z)| = (q-\epsilon)/2 + k + 1$. Since $G_{z^o} \supseteq N^*$, $z^o \in N^*$ is contained in $N^*$ and so $|G_{z^o}| = C_G(z) = N^*: N^*_g| = q(q+\epsilon)/2$. Hence $q - \epsilon/2 + k + 1$. On the other hand, $|F(z)| = |C_G(z)| = G_{z^o}(z) = \Omega/2$ because $|G_{z^o}| = q(q+\epsilon)/2 = 1$ (mod 2). Hence $q - \epsilon + 2k + 2 \mid kq(q+\epsilon)/2$. Since $kq(q+\epsilon)/2 = (kq + 2k(\epsilon - k - 1)) (q - \epsilon + 2k + 2) + (2k + 2 - \epsilon) (k + 1 - \epsilon)$ and so $q(q+\epsilon) = (q+2\epsilon - 2k - 2) (q - \epsilon + 2k + 2) + 2(2k + 2 - \epsilon) (k + 1 - \epsilon)$, we have (3.10).

(3.11) Suppose $|Y| \geq 3$. Then one of the following holds.

(i) $N^*_p = N^* \cap N^p \neq D_2$, and $N_G(Y)^{F(Y)}$ has a regular normal subgroup.

(ii) $N^*_p = N^* \cap N^p \neq D_2$, and $N_G(Y)^{F(Y)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(Y)} = R(3)$ or there exists a prime $p_1 \geq 5$ such that $C_G(Y)^{F(Y)} \supseteq \text{PSL}(2, p_1)$ and $V/Y \cong Z_{p_1}$. Where $V = C_N(Y)$. By (i) of (3.1) and (3.9), $F(N^*_p) = \{\alpha, \beta\}$. On the other hand, $(N^*_p)^{F(Y)} \cong N^*_g/Y \cong Z_2$. Hence $N_G(Y)^{F(Y)} \neq R(3)$ and $C_G(Y)^{F(Y)} \neq$
By (i) of (3.4) and Lemma 2.7, we have \( C_{G_\alpha}(Y) = V\langle f_i \rangle \), where \( f_i \) is a field automorphism of \( N^\alpha \). Let \( t \) be the order of \( f_i \), \( n = tm \) and let \( p^n \equiv \varepsilon_i \pmod{4} \). Clearly \( C_{G_\alpha}(Y)^{(F)} \geq V^{(F)} \cong Z_{N_1} \) and \( |C_{G_\alpha}(Y)^{(F)}| \mid t \), so that \((p_1-1)/2 \mid t\).

First we assume that \( t \) is even and set \( t = 2t_1 \). Then \( Y \leq C_{N^\alpha}(f_i) = PGL(2, p^n) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \) is a cyclic subgroup of \( C_{N^\alpha}(f_i) \) of order \( p^n - \varepsilon_i \) and \( (p^n - 1)/2(p^n - \varepsilon_i) = p_1 \). Put \( s = \sum_{i=1}^{t_1} (p^n)^i \). Then \((p^n + \varepsilon_i)s/2 = p_1 \), so that we have either (i) \( t_1 = 1 \) and \( p_1 = (p^n + \varepsilon_i)/2 \) or (ii) \( t_1 \geq 2 \), \( p^n = 3 \) and \( p_1 = s \). In the case (i), \( 2 \leq (p_1 - 1)/2 = (p^n + \varepsilon_i - 2)/4 \mid 2t_1 = 2 \). Hence \((p_1, q) = (5, 3) \) or \((4, 11^2) \). Let \( z \) be as in (3.7). As mentioned in the proof of (3.10), \( |F(z)| = (q-1)/2+z+1, |\Omega| = kq(q+1)/2+1 \) and \( C_G(z) \) is transitive on \( F(z) \). If \( q = 3^4 \), then \(|F(z)| = 46 \) and \(|\Omega| = 2 \cdot 19 \cdot 23 \) and hence \( |C_G(z)| = |F(z)| = |C_G(z)N^\alpha N^\alpha| \geq |N^\alpha| = 46 \cdot 2^4 \cdot 80 = 2^5 \cdot 5 \cdot 23 \) with \( 0 \leq i \leq 3 \).

Let \( P \) be a Sylow 23-subgroup of \( C_G(z) \) and \( Q \) a Sylow 5-subgroup of \( C_G(z) \). It follows from a Sylow’s theorem that \( P \) is a normal subgroup of \( C_G(z) \) and so \( |PQ| = 1 \). Therefore \( |F(Q)| \geq 23 \), contrary to \( 5^\alpha \mid |N^\alpha| \). If \( q = 11^2 \), then \(|F(z)| = 66 \) and \(|\Omega| = 2 \cdot 3 \cdot 6151 \). Let \( P \) be a Sylow 11-subgroup of \( C_G(z) \). Since \( 11 \mid |\Omega| \), \( P \) is a subgroup of \( N_\gamma \) for some \( \gamma \in \Omega \) and \( F(P) = \{\gamma\} \). Hence \( \gamma \in F(z) \), so that \( z \in N_\gamma \), contrary to \( C_G(z) = D_{1520} \). In the case (ii), we have \((p_1 - 1)/2 = (\sum_{i=1}^{t_1} 9^i)/2 \mid t = 2t_1 \). From this, \( 9^{t_1-1} \leq 4t_1, \) hence \( t_1 = 1, \) a contradiction.

Assume \( t \) is odd. Then \( Y \leq C_{N^\alpha}(f_i) = PSL(2, p^n) \) by Lemma 2.6 (viii). As \(|V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \cong Z_{p^n - \varepsilon_i} \) and \((q-\varepsilon)(p^n - \varepsilon_i) = p_1 \). Hence \( \sum_{i=1}^{t_1} (p^n)^i (\varepsilon_i)^{t_1-1-i} = p_1 \) and \((p_1 - 1)/2 = (\sum_{i=1}^{t_1} (p^n)^i (\varepsilon_i)^{t_1-1-i})/2 \mid t \). In particular \( 2t \geq (p^n)^{t_1-1} - (p^n)^{t_2} \geq (p^n)^{t_2} \geq 2(p^n)^{t_2} \). From this \( t = 3, m = 1, p_1 = 7 \) and \( q = 3^5 \), so that \( N^\alpha = Z_2 \times Z_2 \), a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let \( G^\alpha \) be a minimal counterexample to (3.12) and \( M \) a minimal normal subgroup of \( G \). By the hypothesis, \( G \) has no regular normal subgroup and hence \( M^\alpha = \pm 1 \). As \( M_\alpha \) is a normal subgroup of \( G_\alpha \), by (i) of (3.4), \( M_\alpha \) contains \( N^\alpha \). By (3.9), \( r = 1 \), hence \( M \) is doubly transitive on \( \Omega \). Therefore \( G = M \) and \( G \) is a nonabelian simple group.

Since \( N_\gamma^\alpha = D_{1-t, k = 1} \) and so \( q - \varepsilon + 4 \mid 2((4 - \varepsilon)(2 - \varepsilon) + 1)(4 - \varepsilon)(2 - \varepsilon) \) by (3.10). Hence we have \( q = 7, 9, 11, 19, 27 \) or 43.

Let \( x \) be an element of \( N_\gamma^\alpha \). If \(|x| > 2 \), by Lemma 2.8, \(|F(x)| = 1/|N_\gamma^\alpha| \times 1/|N_\gamma^\alpha| = 2 \) and if \(|x| = 2 \), similarly we have \(|F(x)| = (q - \varepsilon)/2 \). Assume \( q \neq 9 \) and let \( d \) be an involution in \( G_\alpha - N^\alpha \) such that \( \langle d \rangle N^\alpha \) is isomorphic to \( PGL \).
(2, q). We may assume \( d \in G_{ab} \). Since \( \langle d \rangle N^a \) is transitive on \( \Omega - \{ \alpha \} \), by Lemmas 2.3 and 2.6 (vii), (ix), \( |F(d)| = 2(q - 1)(q + 1/2)/2(q + 1) + 1 = (q + 1)/2 \), while \( |F(\alpha)| = (q + 1)/2 + 2 \) for \( \alpha \in I(N^a) \). Hence \( d \) is an odd permutation, contrary to the simplicity of \( G \). Thus \( G_{ab} = N^a \) if \( q \neq 9, 27 \) and \( |G_{ab}/N^a| = 1, 3 \) if \( q = 27 \).

If \( q = 9 \) or \( q = 27 \) and \( |G_{ab}/N^a| = 1, 3 \) if \( q = 27 \).

If \( q = 9 \), \( \Omega = 1 + 9 \cdot 10/2 = 23 \cdot 11 \) and \( |G_{ab}| = 2^3 \cdot 3^2 \cdot 5 \) with \( 0 \leq i \leq 2 \). Let \( P \) be a Sylow 23-subgroup of \( G \). Since \( \text{Aut}(Z_{23}) \simeq Z_2 \times Z_{11} \), \( 3 \not| N_6(P) \), for otherwise \( P \) centralizes a nontrivial 3-element \( x \) and so \( F(P) \supseteq F(x) \) because \( |F(x)| = 1 \), contrary to \( |F(P)| = 0 \). Similarly \( 5 \not| N_6(P) \).

Hence \( |G : N_6(P)| = 2^3 \cdot 3^2 \cdot 5 \) for some \( a \) with \( 0 \leq a \leq 6 \). By a Sylow’s theorem, \( 2^3 \cdot 3^2 \cdot 5 \equiv -2^3 \equiv 1 \) (mod 23), a contradiction.

If \( q = 27 \), \( \Omega = 1 + 27 \cdot 26/2 = 2^3 \cdot 3^2 \cdot 7 \cdot 13 \) with \( 0 \leq i \leq 1 \). Let \( P \) be a Sylow 11-subgroup of \( G \). Since \( P \simeq Z_{11} \) and \( \text{Aut}(Z_{11}) \simeq Z_3 \times Z_5 \), 3 \not| N_6(P) \) by the similar argument as above. Hence \( |G : N_6(P)| = 2^3 \cdot 3^2 \cdot 7 \cdot 13 \) with \( 0 \leq a \leq 7 \). By a Sylow’s theorem, \( 2^3 \cdot 3^2 \cdot 7 \cdot 13 \equiv 2^3 \cdot 3^2 \cdot 7 \cdot 13 \equiv 1 \) (mod 11). Hence \( a = 0, b = 4 \). Therefore \( N_6(P) \) contains a Sylow 2-subgroup \( S \) of \( G \). Let \( T \) be a Sylow 2-subgroup of \( N_6(P) \) and \( g \) an element such that \( T^g \subseteq S \). Then \( T^g \cap C_6(P) \not= 1 \) as \( N_6(P)/C_6(P) \leq Z_2 \). Let \( u \) be an involution in \( T^g \cap C_6(P) \). Then \( |F(u)| = (27 + 1)/2 + 2 = 16 \), while \( 11 \not| |F(u)| \) because \( |P, u| = 1 \) and \( |F(P)| = 0 \), a contradiction.

If \( q = 7, 11, 19 \) or \( 43 \), then \( G_{ab} = N^a \) and \( \varepsilon = -1 \). Let \( \gamma \) be a centralizing of \( Z(N^a) \). Then \( \gamma \) is an involution of \( Z(N^a) \). There exists an involution \( t \) such that \( t \in z^o \) and \( \alpha = \beta \). Since \( G_{ab} = N^a \) and \( F(N^a) = \{ \alpha, \beta \} \) we have \( G_{ab} = \langle t \rangle N^a \).

By Lemma 2.3, \( \langle t \rangle N^a \) is isomorphic to \( Z_2 \times D_{12} \) or \( D_{25} \). Suppose \( \langle t \rangle N^a \simeq Z_2 \times D_{12} \). Then \( \langle t \rangle N^a \) contains fifteen involutions and so we can take \( u \in I(\langle t \rangle N^a) \) satisfying \( |F(u)| = 0 \) and \( \langle t \rangle N^a = \langle u \rangle N^a \). As \( |F(u)| = 0 \), \( |F(u^2)| = (q + 1)/2 = 28 \). By Lemma 2.3, \( 28 = |C_6(u)| \) and \( |\langle u \rangle N^a \cap \Omega| = 24 \) and hence \( |C_6(u)| = 2^3 \cdot 3 \) or \( 2^3 \cdot 3 \). Since \( \langle u \rangle N^a = N_6(R) \), we have \( |C_6(u)| = 2^3 \cdot 3 \) or \( 2^3 \cdot 3 \).

By a Sylow’s theorem, \( |C_6(u) : C_6(u) \cap N_6(R)| = 2^3 \cdot 3 \), so that \( |C_6(u)| = 2^3 \cdot 3 \).

Let \( Q \) be a Sylow 7-subgroup of \( C_6(u) \). Then \( |C_6(u) \cap N_6(Q)| = 2^3 \cdot 3 \) or \( 2^3 \cdot 3 \) by a Sylow’s theorem. Hence \( 2^3 \cdot 3 \) or \( 2^3 \cdot 3 \) by a Sylow’s theorem. Hence \( 2^3 \cdot 3 \) or \( 2^3 \cdot 3 \) by a Sylow’s theorem.
Some Doubly Transitive Permutation Groups

Let $U$ be a Sylow 2-subgroup of $N^a_β$ and set $L=N_G(U)$. It follows from (3.3) and Lemma 2.6 (iv) that $L \cap N^a_β=A_4$, $L^{S(U)}=A_4$ and $|L|=2^2 \cdot 3$. Let $T, <\tau>$ be Sylow 2- and 3-subgroup of $L$, respectively. Obviously $L\triangleright \Gamma$ and $C_τ(x)=1$.

On the other hand $T \triangleright L \cap <\tau>N^a_β=D_8$ and so $T'=Z_2 \times Z_2$ because $C_τ(x)=1$. By Theorem 5.4.5 of [2], $T$ is dihedral or semi-dihedral. Hence $N_G(T)/C_G(T)$ ($\leq \text{Aut}(T)$) is a 2-group, so that $C_τ(x)=T$ a contradiction.

(3.13) (ii) of (3.11) does not occur.

Proof. Let $G^a$ be a doubly transitive permutation group satisfying (ii) of (3.11). Let $x$ be an involution in $N^a_β$ with $x\in \Gamma$. Then $F(x^{F(Y)})=F(x)=F(x^Y)=\{\alpha, \beta\}$ by (i) of (3.1) and (3.9). Since $|F(Y)|=1+(q-6)/|N^a_β|=1+k\geq 4$, $x^{F(Y)}$ is an involution. By Lemma 2.5, $1+k=2^2$ and so $k=3$. By (3.11), $q-\varepsilon+8|2((8-\varepsilon)(4-\varepsilon)+3)+1-(8-\varepsilon)(4-\varepsilon)$. Hence $q+7|2^2 \cdot 3 \cdot 7$ if $\varepsilon=1$ and $q+9|2^4 \cdot 3^2 \cdot 5 \cdot 17$ if $\varepsilon=-1$. Since $k=3, q-\varepsilon-8, 3|q-\varepsilon+8$. From this $q+7|2^2 \cdot 7$ if $\varepsilon=1$ and $q+9|2^4 \cdot 5 \cdot 17$ if $\varepsilon=-1$. Therefore $q=5^2, 7^2, 11^2, 61, 59$ or 71.

Let $p_1$ be an odd prime such that $p_1| |\Omega|$ and $p_1 \not| \text{gcd} \{G_a\}$ and let $P$ be a Sylow $p_1$-subgroup of $G$. Clearly $P$ is semi-regular on $\Omega$ and so any element in $C_G(P)$ has at least $p_1$ fixed points. If $x$ is an element of $N^a_β$ and its order is at least three, $|F(x)|=|F(Y)|=4b^γ$ by Lemma 2.8. Since $|N^a_β|=|\Omega|=1+3(q+6)/2$.

If $q=5^2$, then $|\Omega|=2^4 \cdot 61$ and $|G_a|=2^4 \cdot 3 \cdot 5^2 \cdot 13$ $(0<\varepsilon<2)$. Let $P$ be a Sylow 61-subgroup of $G$. Then $P=Z_61$. As mentioned above, $5, 13 \not| \text{gcd} \{G_a\}$ and so $5^2, 13^2 \not| \text{gcd} \{N^a_β\}$. Hence $G:N_G(P)=2^2 \cdot 3^2 \cdot 5^2 \cdot 13$, where $0<\varepsilon<10$ and $0\leq b, c<1$. But we can easily verify $G:N_G(P)=1$ (mod 61), contrary to a Sylow's theorem.

If $q=7^2$, then $|\Omega|=2^8 \cdot 919$ and $|G_a|=2^{4+i} \cdot 3 \cdot 5^2 \cdot 7^2$ $(0<\varepsilon<2)$. Let $P$ be a Sylow 919-subgroup of $G$. By the similar argument as above, we obtain $5, 7 \not| \text{gcd} \{N^a_β\}$ and so $5, 7 \not| \text{gcd} \{G_a\}$. Hence $G:N_G(P)=2^2 \cdot 3 \cdot 5^2 \cdot 7^2 \equiv 2^2 \cdot 306$ or $-2^2$ (mod 919), where $0<\varepsilon<8$ and $0<\varepsilon<12$. Hence $G:N_G(P)=1$, a contradiction.

If $q=11^2$, then $|\Omega|=2^7 \cdot 173$ and $|G_a|=2^{4+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$ $(0<\varepsilon<2)$. Let $P$ be a Sylow 173-subgroup of $G$. Similarly we have $3, 5, 11 \not| \text{gcd} \{N^a_β\}$ and so $3, 5 \not| \text{gcd} \{G_a\}$. Hence $G:N_G(P)=2^2 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -5 \cdot 2^2$ (mod 173), where $0<\varepsilon<12$. Hence $G:N_G(P)=1$, a contradiction.

If $q=59$, then $|\Omega|=2^5 \cdot 171$ and $|G_a|=2^{4+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59$ $(0<\varepsilon<1)$. Let $P$ be a Sylow 173-subgroup of $G$. Similarly we have $3, 5, 29, 59 \not| \text{gcd} \{N^a_β\}$ and so $3, 5, 29, 59 \not| \text{gcd} \{G_a\}$. Hence $G:N_G(P)=2^2 \cdot 3 \cdot 5 \cdot 29 \cdot 59 \equiv 10 \cdot 2^2$ or $12 \cdot 2^2$ (mod 17), where $0<\varepsilon<4$ and $0<\varepsilon<1$. From this, we have a contradiction.

If $q=71$, then $|\Omega|=2^5 \cdot 233$ and $|G_a|=2^{4+i} \cdot 3 \cdot 5 \cdot 7 \cdot 71$ $(0<\varepsilon<1)$. Let $P$ be
a Sylow 233-subgroup of $G$. Since $3, 5, 7, 71 | N_G(P)$, $|G|: N_G(P)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71 = -3 \cdot 2^a$ (mod 233), where $0 \leq a \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N_\beta^o = Z_2 \times Z_2$ or $N_\beta^o = D_8$ and $N_\beta^o \cap N_\beta = Z_2 \times Z_2$.

(3.14) The case that $N_\beta^o = Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N_\beta^o)$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N_\beta^oN_\beta)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N_\beta^o| = 3^2 \cdot 5$ (mod 8) and so $n$ is odd. Hence $|G_\alpha|/N_\beta^o| = 2^n \cdot 3^2$ by (3.2).

Suppose $N_\beta^o \cap N_\beta = N_\beta^o \cap N_\beta = Z_2$. Then $N_\beta^oN_\beta$ is a Sylow 2-subgroup of $G_\alpha$, hence $N_\beta^o(N_\betaN_\betaN_\beta)^{1/2}$ is a doubly transitive by a Witt's theorem. Since $N_\beta^oN_\beta = D_8$ and $|\Delta|$ is even, $C_\alpha(N_\beta^oN_\beta)^{1/2}$ is also doubly transitive. Let $g$ be an element of $C_\alpha(N_\beta^oN_\beta)^{1/2}$ such that $g^p = g$ and $g^q = g$. Then $N_\beta = g^{-1}N_\beta^o g = N_\beta^o$ and hence $N_\beta^o = N_\beta \cap N_\beta$, a contradiction. Thus $N_\beta^o = N_\beta^o \cap N_\beta = Z_2 \times Z_2$.

Let $z$ be an involution in $N_\beta^o$ and $t \in \omega$ an involution such that $\alpha^t = \beta$. Set $\Gamma = \{\gamma, \delta\} | \gamma, \delta \in \Omega, \gamma \neq \delta\}$. We consider the action of the element $z$ on $\Gamma$. By the similar argument as in the proof of (3.12), $|F(z)| = |(F(z)| = 1 + 3q(q-e) + 3r(q-e)r/4$. Hence $1 + 3(q-e)r/4 | 8n(1 + q(q-e) + q+e/8)$. Put $n_r = r$. Then $3q - 3r + 4 | 4rs + q(q-e) + q+e/8 | 8n$. Clearly $|\Omega| = 1 + q(q-e) + q+e/8$ and by Lemma 2.8 (i), $|F(z)| = 1 + 3(q-e)r/4$. Therefore $m = \frac{8n(1 + q(q-e) + q+e/8)}{4}$. Hence $3q - 3e + 4 | 64(5+9e)n$ and $|F(z)| = 1 + 3(q-e) + q+e/8$. Since $q \geq 3e$ and so $288n > 3e$. Therefore $288n = q > p^o \geq 3^e$ and so $288n > 3e$. Hence $(n, r, p, e) = (5, 5, 3, -1), (3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (3.4). Thus $r = 1$.

We argue that $r = 1$. Suppose false. Then $32s(3e-r-4) (3e-r-2) > 0$ and so $3r(q-e) < 864 r^2$. Therefore $288n > 3e$. Hence $(n, p, e) = (5, 5, 3, -1), (3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (3.4). Thus $r = 1$.

Hence $3q - 3e + 4 | 64(5+9e)n$ and $|F(z)| = 1 + 3(q-e) + q+e/8$. Since $q \geq 3e$ and so $288n > 3e$. Therefore $288n = q > p^o \geq 3^e$ and so $288n > 3e$. Hence $(n, r, p, e) = (5, 5, 3, -1), (3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (3.4). Thus $r = 1$.

Set $m = |\omega| \cap \langle \omega \rangle G_\alpha^{1/2}$. As we mentioned above, $|G_\alpha^{1/2}| = \frac{|G(z)| |(F(z)| |(F(z)|}{1 + q(q-e) + q+e/8}$. Since $q \geq 3e$ and so $288n > 3e$. Therefore $m = (2q^2 + (2e + 9)e - 9e)/3q - 3e + 4$. It follows that $(q, m) = (19, 27/2), (37, 28), (83, 449/8)$ or $(149, 411/4)$. Since $m$ is an integer, we have $(q, m) = (37, 28)$. But $m \leq \langle \omega \rangle G_\alpha^{1/2} \leq 16$, a contradiction. Thus (3.14)
The case that \( N_8^* = D_8 \) and \( N^8 \cap N^8 \leq Z_2 \times Z_2 \) does not occur.

Proof. Let \( \Delta, L \) and \( K \) be as defined in (3.6). By (3.6), there is an element \( x \in L_\alpha \) such that its order is odd and \( \langle x^\alpha \rangle \) is regular on \( \Delta - \{ \alpha \} \). Since \( (L_\alpha)' \leq N^8_\beta \) by (3.6) and \( N^8 = D_8 \), \( x \) stabilizes a normal series \( N^8_\beta N^8_\alpha \geq N^8_\beta \geq 1 \). Hence \( x \) centralizes \( N^8_\beta N^8_\alpha \) by Theorem 5.3.2 of [2] and so \( x^{-1}N^8_\alpha = N^8_\beta \). Put \( \gamma = \beta^2 \). If \( r = 1 \), then \( \beta = \gamma \), so that \( N^8_\beta = N^8_\alpha \). From this, \( N^8_\beta = N^8_\alpha \). By the doubly transitivity of \( G \), \( N^8_\beta = N^8_\alpha \), hence \( N^8_\beta = N^8 \cap N^8 \), a contradiction. Therefore \( r = 1 \) and \( \Delta = \{ \alpha, \beta \} \).

Set \( \langle x \rangle = Z(N^8_\beta) \), \( \Delta_1 = \alpha C_{G}(x) \) and let \( \{ \Delta_1, \Delta_2 \cdots \Delta \} \) be the set of \( C_{G}(x) \)-orbits on \( F(z) \). Since \( L \geq N^8_\alpha \cap N^8 \) and by (3.2), \( N^8_\alpha \cap N^8 = 1 \), \( z \) is contained in \( N^8_\alpha \cap N^8 \). Hence, by Lemma 2.1, \( \beta \in \Delta_1 \) and \( k \) is at least two. By Lemma 2.8, \(| F(z) | = 1+(q-\varepsilon)/2 | N^8_\alpha | = 1+5(q-\varepsilon)/8 \). Clearly \( | C_{N^8_\alpha}(x) : N^8_\alpha | = (q-\varepsilon)/8 \) and so \( | \Delta_1 | \geq 1+(q-\varepsilon)/8 \). If \( \gamma \in F(z) - \Delta_1 \), then \( C_{N^8_\alpha}(x) \sim Z_2 \times Z_2 \), for otherwise \( \langle x \rangle = Z(N^8_\beta) \leq N^8_\alpha \cap N^8 \) and by Lemma 2.1 \( \gamma \in \Delta_1 \), a contradiction. Hence one of the following holds.

(i) \( k = 3 \) and \( | \Delta_1 | = 1+(q-\varepsilon)/8, | \Delta_2 | = | \Delta_3 | = (q-\varepsilon)/4 \).
(ii) \( k = 2 \) and \( | \Delta_1 | = 1+(q-\varepsilon)/8, | \Delta_2 | = (q-\varepsilon)/2 \).
(iii) \( k = 2 \) and \( | \Delta_1 | = 1+3(q-\varepsilon)/8, | \Delta_2 | = (q-\varepsilon)/4 \).

Let \( \gamma \in F(z) - \Delta_1 \). Then, \( z \in G_{N^8_\alpha} \cap N^8 \) and so \( C_{N^8_\alpha}(z) \sim D_8 \) or \( PGL(2, \sqrt{q}) \) by Lemma 2.6 (vi), (vii), (ix). If \( C_{N^8_\alpha}(z) \sim D_8 \), then \( (q+\varepsilon)/2 \leq | \Delta_1 | \) and so \( q = 7 \) and (iii) occurs. But \( (q+\varepsilon)/2 = 3 \mid | \Delta_2 | - 1 \mid 5 = q - 1 \), a contradiction. If \( C_{N^8_\alpha}(z) \sim PGL(2, \sqrt{q}) \), then (i) does not occur because \( \sqrt{q} \not| \Gamma - q - \varepsilon \). Hence \( \sqrt{q} \mid | \Delta_1 | \) and \( \sqrt{q} \mid | \Delta_2 | - 1 \). From this, \( q = 25 \) and (iii) occurs. In this case, we have \( | \Delta_1 | = 10 \), so that an element of \( C_{N^8}(x) \) of order 3 is contained in \( N^8_\delta \) for some \( \delta \in \Delta_1 \), contrary to \( N^8_\delta \sim N^8_\beta = D_8 \).

Case (II)

In this section we assume that \( N^8_\beta = PGL(2, p^m) \), where \( n = 2mk \) and \( k \) is odd. Since \( n \) is even, \( q = p^* \equiv 1 \mod 4 \). We set \( p^* \equiv \varepsilon \equiv \{ \pm 1 \} \mod 4 \). In section 7 we shall consider the case that \( N^8_\beta = S_4 \). Therefore we assume \( (p, m) \neq (3, 1) \) in this section.

(i) \( N^8_\beta/N^8_\alpha \cap N^8 \geq Z_2 \) or \( Z_2 \), and \( N^8 \cap N^8 \geq (N^8_\beta)' \sim PGL(2, p^m) \).

(ii) If \( (p, m) \neq (5, 1) \), there exists a cyclic subgroup \( Y \) of \( (N^8_\beta)' \) such that \( N_{N^8}(Y) \sim D_{4+\varepsilon} \) and \( N_{C_{G}(Y)}(Y)^{P(Y)} \) is doubly transitive.

Proof. As \( N^8_\beta \geq N^8_\alpha \cap N^8 \), either \( N^8_\beta/N^8_\alpha \cap N^8 \leq Z_2 \) or \( N^8 \cap N^8 = 1 \). If \( N^8 \cap N^8 = 1 \), by Lemma 2.2 and 2.6 (vi), \( N^8_\beta \sim N^8_\alpha \cap N^8 \sim N^8_\alpha N^8_\beta /N^8_\beta = Z_2 \times Z_2 \). a
contradiction. Therefore $N^e/\mathcal{S} \cap N^g = \{1\}$ or $N^e \cap N^g \geq (N^e)^* \sim PSL(2, p^m)$.

Now we assume that $(p, m) \neq (3, 1)$ and let $z$ be an involution in $(N^e)^*$. Then $C_{N^e}(z) = D_{2^m}$ by Lemma 2.6 (vii). Suppose $C_{N^e}(z)$ is not a 2-subgroup and put $Y = 0(C_{N^e}(z))$. Then, if $Y^z \leq G_{ab}$ for some $g \in G$, we have $Y^z \leq N^e$ and $Y^g \leq N^g$, where $g = \alpha^e$ and $h = \beta^g$. By (i) $Y^g \leq N^e \cap N^g$ and so $Y^z = Y^g$ for some $h \in N^e \cap N^g$. Thus $N_d(Y)^x$ is doubly transitive. Assume that $C_{N^e}(z)$ is a 2-subgroup and put $z = (u, v | u^2 = u^{-1}, v^2 = 1)$. We may assume that $v \in (N^e)^*$ and $\langle u, v \rangle$ is a Sylow 2-subgroup of $N^e$. Since $p^m = 3, 5$, the order of $u^2$ is at least four. On the other hand there is no element of order $|u^2|$ in $\langle u, v \rangle$. Hence any element of order $|u^2|$ which is contained in $N^e$ is necessarily an element of $N^e \cap N^g$. By the similar argument as above, $N_d(Y)^x$ is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose $(p, m) = (3, 1), (5, 1)$ and set $\Delta = F(Y)$ and $X = N_d(Y)$. Then $|\Delta| = rs(p^m + \epsilon)/2 + 1$, where $s = \sum \frac{k}{r} p^{2m}$, $C_g(N^e)$ = 1 and one of the following holds.

(i) $X^a \leq ALT(1, 2^c)$ for some integer $c$.

(ii) $X^a = PSL(2, p_1)$ or $PGL(2, p_1)$, $r = 1, k = 1$ and $2p_1 = p^m + \epsilon$.

Proof. By Lemma 2.8 (ii), $|\Delta| = 1 + |N^e \cap X| r/|N^e \cap X| = 1 + (p^{2m} - 1)$ $r/2(p^m - \epsilon) = rs(p^m + \epsilon)/2 + 1$. By (4.1) and Lemma 2.9, we have (i), (ii) or $X^a = R(3)$.

Assume that $X^a = R(3)$. Then $rs(p^m + \epsilon)/2 + 1 = 28$, hence $k = 1$ and $r(p^m + \epsilon)/2 = 7$. Since $r$ is odd and $r | 2m = n$, we have $r = m = 1$ and $q = 53^3$. But a Sylow 3-subgroup of $X^a$ is cyclic because $N^e \cap X = D_{2m}$ and $X^a \cap X \cap N^e = X^a N^e/N^e \leq Z_2 \times Z_2$, a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose $X^a \leq ALT(1, 2^c)$ and put $W = C_{N^g}(Y)$. Then $Y \leq W = Z_{2^m}$. Since $C_{N^e}(Y)$ is cyclic, $W$ is a characteristic subgroup of $C_{N^e}(Y)$ and so $W$ is a normal subgroup of $X^a$. Hence $W \leq X^a$ and $(X \cap N^e) = \{1\} \leq Z_2$. By Lemmas 2.4 and 2.6, $F(X \cap N^e) = 1 + |X \cap N^e| |N^e| = X \cap N^e |r/|N^e| = 1 + r$. Since $1 + r < |\Delta|$, $(X \cap N^e)^a = Z_2$ and hence $(1 + r) = rs(p^m + \epsilon)/2 + 1$ by Lemma 2.5. From this, $r = s(p^m + \epsilon)/2 - 2 |mk$ and so $p^{2m}(k-1) + mk \leq 2$. Hence $m = k = r = 1$ and $q = 7^2$.

Let $R$ be a Sylow 3-subgroup of $N^e$. Since $N^e \sim PGL(2, 7)$, we have $R = Z_3$. By Lemmas 2.4 and 2.6, $|F(R)| = 1 + (7^2 - 1) |N^e|$, $N^e(R) = |N^e| = 4$. Hence $N_d(R)^{F(R)} = A_4$ or $S_4$. But is a Sylow 3-subgroup of $N_{G_a}(R)$ because $N^e \sim PSL(2, 7)$, contrary to $N_{G_a}(R)^{F(R)} = A_3$ or $S_3$.

(4.4) (ii) of (4.2) does not occur.
Proof. Let notations be as in (4.2). Suppose $X^\alpha \trianglerighteq \text{PSL}(2, p_1)$. By the similar argument as in (4.3), $C_{N_\beta}(Y) \leq X_\Delta$ and so $C_{N_\beta}(Y)^\Delta = Z_{p_1}$, and $N_{N_\beta}(Y)^\Delta = D_{2p_1}$. Hence $|(X)^\alpha| = |2p_1| = 2n$. Since $X^\alpha \trianglerighteq \text{PSL}(2, p_1)$, $p_1(p_1 - 1)/2 | |(X)^\alpha|$, hence $p_1 - 1 | 8n$. As $k = 1$ and $2p_1 = p_\alpha + \epsilon$, we have $p_\alpha + \epsilon - 2 | 32m$. From this, $(p, m, p_1) = (11, 1, 5), (3, 2, 5)$ or $(3, 3, 13)$.

Let $R$ be a cyclic subgroup of $N_\beta^\alpha$ such that $R \simeq Z_{(p_\alpha + \epsilon)/2}$. By Lemma 2.6, $N_\alpha(R)_{F(R)}$ is doubly transitive and by Lemma 2.8 (ii), $|F(R)| = 1 + |N_\alpha(R)|/|N_\alpha \beta(R)| = 1 + (p_\alpha^\beta - 1)/2(p_\alpha^\beta + \epsilon) = (p_\alpha^\beta - \epsilon)/2 + 1$.

If $(p, m, p_1) = (11, 1, 5), |F(R)| = 7$ and so by [9] $|N_\alpha(R)_{F(R)}| = 12$ and $N_\alpha(P)_{F(R)} = Z_6$. Since $|N_\alpha = (R): N_\alpha(R)^\Delta| = 6$, $N_\alpha = (R)_{F(R)} = N_\alpha = (R)_{F(R)}$. Hence $N_\alpha = (R)/(K(N_\alpha(R))) = Z_2 \times Z_2$, a contradiction.

If $(p, m, p_1) = (3, 2, 5), |F(R)| = 5$ and so by [9], $|N_\alpha(R)_{F(R)}| = 20$ and $N_\alpha = (R)_{F(R)} = Z_4$. Since $|N_\alpha = (R): N_\alpha = (R)^\Delta| = 4$, $N_\alpha = (R)^\Delta = Z_4$, contrary to $N_\alpha = (R)/(N_\alpha = (R))_{F(R)} = Z_2 \times Z_2$.

If $(p, m, p_1) = (3, 3, 13), |F(R)| = 15$. By [9], $N_\alpha = (R)_{F(R)}$ is not solvable, a contradiction.

(4.5) $p_\alpha^\beta = 5$.

Proof. Assume that $p_\alpha^\beta = 5$. Then $n = 2k$ with $k$ odd and $N_\beta^\alpha = \text{PGL}(2, 5) \simeq S_5$. First we argue that $N_\beta^\alpha = N_\alpha \cap N_\beta$. Suppose false. Then $C_\alpha(N_\alpha^\alpha) = 1$ by Lemma 2.2, and $N_\alpha^\alpha/N_\alpha \cap N_\beta = Z_2$ by (4.1). Since $N_\beta^\alpha/N_\beta^\alpha = N_\beta^\alpha/N_\alpha \cap N_\beta = Z_2$ and the outer automorphism group of $S_5$ is trivial, we have $Z(N_\alpha^\alpha N_\beta^\alpha) = Z_2$.

Let $w_\alpha$ be the involution of $Z(N_\alpha^\alpha N_\beta^\alpha)$ and let $w_\alpha \in I(N_\alpha^\alpha) - I(N_\alpha^\beta)$. Since $C_\alpha(w_\alpha) \geq N_\alpha^\beta$, by Lemma 2.6 (viii) and (ix), $w_\alpha$ acts on $N_\alpha^\beta$ as a field automorphism of order 2 and $C_\alpha = (w_\alpha) \simeq \text{PGL}(2, 5^k)$. By Lemma 2.8 $|F(w_\alpha)| = 1 + (q - \epsilon)/|I(N_\alpha^\beta)|$ and $|N_\alpha^\beta| = 1 + (5^{2k} - 1)/24$. Let $P$ be a Sylow $5$-subgroup of $C_\alpha(w_\alpha)$. Then $|P| = 5^k$ and $|\gamma^\alpha^\beta| = 5^{k-1}$ or $5^k$ for each $\gamma \in \Omega - \{\alpha\}$. Since $P$ acts on $F(w_\alpha) - \{\alpha\}$, we have $5^{k-1}5^{(2k - 1)/5}/24$, so that $k = 1$ and $|F(w_\alpha)| = 6 = r/k$. Hence $C_\alpha(w_\alpha)_{F(w_\alpha)} = Z_5$ and so $C_\alpha(w_\alpha)_{F(w_\alpha)} = Z_6$. But clearly $w_\alpha \in N_\alpha^\alpha \cap N_\beta$ by Lemma 2.1, a contradiction. Thus $N_\beta^\alpha = N_\alpha \cap N_\beta$.

Let $V$ be a cyclic subgroup of $N_\beta^\alpha$ of order 4. Since $N_\beta^\alpha = N_\alpha \cap N_\beta = S_5$, $N_\alpha(V)_{F(V)}$ is doubly transitive and by Lemma 2.8, $|F(V)| = 1 + |N_\alpha(V)|/|N_\alpha \beta(V)| = 1 + (5^{2k} - 1)/8 = 3(5^{2k - 1}) + 1$, where $s = \sum_{i=0}^{k-1}25^i$. By Lemma 2.9, $C_\alpha(N_\alpha^\alpha) = 1$ and (a) $N_\alpha(V)_{F(V)} \leq A_5 \leq L(1, 2^e)$ or (b) $N_\alpha = (V)_{F(V)} = R(3)$.

Put $P = N_\alpha = (V)$. Then $P \trianglelefteq D_5$, $|F(P)| = 1 + |N_\alpha = (P)|/|N_\alpha^\beta = (P)|$ and $N_\alpha^\beta = (P)| = r + 1$ and $P_{F(V)} = Z_2$. If (b) occurs, $k = 1$ and $r = 9$, hence $|F(P)| = 10$, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r + 1)^2 = 35 + 1$ and so $r = 3(5^{2k - 1}) + 1$. Hence $k = r = 1$ and $G_\alpha/N_\beta^\alpha = Z_2 \times Z_2$. Let $z$ be an involution in $N_\beta^\alpha$. Then $|F(z)| = 1 + 25(24 - 2)/120 = 6$.
by Lemma 2.8 and \(|\Omega|=1+|N^\alpha|N^\alpha|=66\) as \(r=1\). By the similar argument as in the proof of (3.12), \(|F(z)|(|F(z)|-1)/2+(|\Omega|-|F(z)|)/2=|C_G(z)||z^\beta\cap \langle t \rangle G_{\alpha \beta}\rangle /|\langle t \rangle G_{\alpha \beta}\rangle|\), where \(t\) is an involution such that \(\alpha^t=\beta\). Hence \(|z^\beta\cap \langle t \rangle G_{\alpha \beta}\rangle|=15|G_{\alpha \beta}|/|C_G(z)|\). Set \(H=\langle t \rangle G_{\alpha \beta}\rangle\) and let \(R\) be a Sylow \(3\)-subgroup of \(N^\alpha\).

By Lemma 2.8, \(|F(R)|=1+24\cdot120=3\). Set \(F(R)=\{\alpha, \beta, \gamma\}\). On the other hand, as \(N^\alpha=S_5\) and \(\text{Out}(S_5)=1\), we have \(H=Z(H)\times N^\alpha\) and \(|Z(H)|=2, 4\) or \(H=C_{N^\alpha}(N^\alpha)\times N^\alpha\) and \(C_{N^\alpha}(N^\alpha)\times Z_2\times Z_2\), contrary to Lemma 2.6 (ix). In the former case, we have \(|Z(H)|=2\).

For otherwise \(Z(H)<G^\alpha\) and \(Z(H)\Gamma G\Phi\Phi\) and so letting \(u^Z(H)\), we have \(|F(u)|=31|F(\kappa)|=5\), a contradiction. Therefore \(Z(H)\Gamma Z\) and so \(|G\Gamma|<25+25=50\), while \(|G\Gamma|=15|G\beta|/15-120/24-75\), a contradiction.

5. Case (III)

In this section we assume that \(N^\alpha=PSL(2,p^n)\), where \(n=mk\) and \(k\) is odd. Set \(p^n\equiv \in \{\pm 1\} \pmod{4}\). Then \(q\equiv \in \{\pm 1\} \pmod{4}\) as \(k\) is odd. In section 6 we shall consider the case that \(N^\alpha=A_4\), so we assume \((p,\tauw)\neq(3,1)\) in this section.

From this \(N^\alpha\) is a nonabelian simple group and so \(N^\alpha=N^\alpha\cap N^\beta\) or \(N^\alpha\cap N^\beta=1\).

If \(N^\alpha\cap N^\beta=1\), then \(C_G(N^\alpha)=1\) by Lemma 2.2 and \(N^\alpha=N^\beta\cap N^\alpha\cap N^\alpha\cap N^\beta\cap N^\beta\approx Z_2\times Z_2\), a contradiction. Hence \(N^\alpha=N^\alpha\cap N^\beta\).

Let \(z\) be an involution of \(N^\alpha\). Suppose \(z^\alpha\in G_{\alpha \beta}\) for some \(g\in G\) and set \(\gamma=\alpha^z\). Then \(\alpha^z\in N^\alpha\cap G_{\alpha \beta}\approx N^\alpha\cap N^\beta\approx N^\alpha\cap N^\beta\) and so \(\alpha^z\in N^\beta\\). Hence \(C_G(z)^{F(\alpha)}\) is doubly transitive and by Lemma 2.8 (i), \(|F(z)|=(q-\varepsilon)r/(p^m-\varepsilon)+1\).

In particular \(|F(z)|>3r+1\) as \((p^m-\varepsilon)/(p^m-\varepsilon)\geq p^{2m+\varepsilon}p^m+1>3\).

By Lemma 2.9, \(C_G(N^\alpha)^{F(\alpha)}=1\) and one of the following holds.

(a) \(C_G(z)^{F(\alpha)}=\text{ATL}(1,2^r)\).
(b) \(C_G(z)^{F(\alpha)}\geq PSL(2,p_1)\) \((p_1\geq 5)\), \(r=1\) and \(|C_{N^\alpha}(z): C_{N^\alpha}(z)|=p_1\).
(c) \(C_G(z)^{F(\alpha)}=R(3)\).

Let \(Y\) be a cyclic subgroup of \(C_{N^\alpha}(z)\approx D_{p^m-\varepsilon}\) of index 2. Since \(C_{G_\alpha}(z)\geq Y\), \(z\in Y\) and \(C_G(z)^{F(\alpha)}\) is doubly transitive, we have \(F(Y)=F(z)\). By the similar argument as in (3.1), \(N^\alpha\cap N(C_{N^\alpha}(z))=C_{N^\alpha}(z)\) or \(N^\alpha\cap N(C_{N^\alpha}(z))=A_4\). Hence by Lemmas 2.3 and 2.4 \(|F(C_{N^\alpha}(z))|=1+(C_{N^\alpha}(z))/|N^\alpha|: C_{N^\alpha}(z)/|R|/|N^\alpha|\) or \(1+|A_4|/|N^\alpha|: C_{N^\alpha}(z)/|R|/|N^\alpha|\).

Therefore \(|F(C_{N^\alpha}(z))|=r+1\) or \(3r+1\). From this \(C_{N^\alpha}(z)^{F(\alpha)}\approx Z_2\).

In the case (a), \((r+1)^2=1+(p^m-\varepsilon)r/(p^m-\varepsilon)\) by Lemma 2.5 and hence \(r=(p^m-\varepsilon)/(p^m-\varepsilon)-2/\text{mk}\). Since \((p^m-\varepsilon)/(p^m-\varepsilon)\geq((p^m)^k+1)/(p^m+1)=\sum_{i=0}^{k-1}(-p^m)^i\) and \(k\geq 3\), we have \(p^{(m-1)}(p^{m+1})\leq \text{mk}\), hence \(((p^m)^{k-3}/k)/(p^{m-1}+p^m+1)<1\).

Thus \(k=3, m=1\) and \(p=3\), contrary to \((p,\tauw)=(3,1)\).

In the case (b), \(r=1, p_1=(p^m-\varepsilon)/(p^m-\varepsilon), p_1(p_1-1)/2\) and \(s|4mpk_1\), where \(s\) is the order of \(C_{G_\alpha}(z)^{F(\alpha)}\). Hence \(p_1-1=8s\).

Since \(p_1-1=(p^m-\varepsilon)/(p^m-\varepsilon)-1\)
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\((p^n+1)/(p^n+1)-1 = \sum_{k=0}^{n}(-p^k) \geq p^{m(k-2)}(p^m-1),\) we have \(p^{m(k-2)}/2k \leq 4m!(p^m-1) \leq 1\) because \(p^m \neq 3.\) Hence \(k=3\) and \(p^m = 5,\) so that \(p_1-1 = 30/8mk = 24,\) a contradiction.

In the case \((c), r+1 = 4 \text{ and } 1+(p^n-\varepsilon)r/(p^n-\varepsilon) = 28 \text{ and so } r = 3 \text{ and } (p^n-\varepsilon)/(p^n-\varepsilon) = 9.\) Hence \(9 \geq (p^m+1)/(p^m+1) \geq p^m-p^m+1,\) so that \(p^m = 3,\) a contradiction.

6. Case (IV)

In this section we assume that \(N^* = A_4\) and \(q = 3,5 \text{ (mod 8). If } N^* \cap N^\beta = 1,\) by Lemma 2.2, \(C_G(N^*) = 1\) and so \(N^\alpha N^\beta \neq N^\beta N^\beta \leq Z_2 \times Z_2.\) Hence \(N^\alpha N^\beta \cap N^\beta = 1 \text{ or } Z_3,\) so that \(z^G \cap G_{ab} = z^G \cap N^\alpha = Z^G \text{ for an involution } z \in N^G.\) Therefore \(C_G(z)^{F(x)}\) is doubly transitive. By Lemma 2.9, \(C_G(N^*) = 1\) and one of the following holds.

(a) \(C_G(z)^{F(x)} \leq \text{ATL}(1,2)\) for some integer \(c \geq 1.\)

(b) \(C_G(z)^{F(x)} \geq \text{PSL}(2, p_1) (p_1 \geq 5), r = 1 \text{ and } |C_N\alpha(z) : C_{N^\beta}(z)| = p_1.\)

(c) \(C_G(z)^{F(x)} = R(3).\)

Let \(T\) be a Sylow 2-subgroup of \(N^G.\) Then \(z \in T\) and by Lemmas 2.3 and 2.4, \(|F(T)| = 1+|N^\alpha(T)|/|N^\beta| = r+1.\) By Lemma 2.8 (i), \(|F(z)| = (q-\varepsilon)r/4+1.\) Hence \(T^{F(x)} \cong Z_2\) if \(q = 5.\) If \(q = 5,\) as \(PSL(2,5) = PSL(2,4),\) (ii) of our theorem holds by [4]. Therefore we may assume \(q = 5.\)

In the case (a), \((r+1)^2 = 1+(q-\varepsilon)r/4\) by Lemma 2.5. Hence \(r = (q-\varepsilon-8)/4\) and \(r = 1.\) Let \(R\) be a Sylow 3-subgroup of \(G_{ab}.\) Then \(R \cong Z_3 \text{ and } R \leq N^G = G_{ab}/N^G \neq G_{ab}N^G/N^G = 1 \text{ or } Z_2 \text{ and } N^G = A_4.\)

By Lemma 2.8 (ii), \(|F(R)| = 1+12/3 = 5\) and \(N_G(R)^{F(x)}\) is doubly transitive. Since \(N_G(R) \cong D_{12} \text{ or } D_{24}\) and \(|F(R)| = 5,\) we have \(N_G(R) = 5.\) Let \(S\) be a Sylow 5-subgroup of \(N_G(R).\) Then \([S, S]|R = 1 \text{ as } N_G(R)/C_G(R) \leq Z_2.\) Since \(5 \not| |G_{ab}|, |F(S)| = 0\) or 1. If \(|F(S)| = 1, F(S) \leq F(R)\) and so \(5 \not| |F(R)| = 1-4,\) a contradiction. Therefore \(S\) is semi-regular on \(\Omega.\) But \(|\Omega| = 1+|N^\alpha : N^\beta| = 56 \text{ or } 92.\) This is a contradiction.

In the case (b), \((p_1-1)/2 | s\) and \(s | 2n(q-\varepsilon)/2 = 4np_1,\) where \(s\) is the order of \(C_{G_{ab}}(z)^{F(x)}\). Hence \(p_1-1 = 8n.\) Since \(p_1 = (q-\varepsilon)/4, p^s = -\varepsilon-4 | 32n\) and so we have \(q = 11,13,19,27 \text{ or } 37.\) If \(q = 27,\) by Lemma 2.6, \(C_{G_{ab}}(z) = D_{24} \text{ or } D_{27} \text{ and so } C_{G_{ab}}(z)^{F(x)} \cong Z_2.\) Hence \((p_1-1)/2 = 2.\) From this \(q = 19.\) Let \(R\) be a Sylow 3-subgroup of \(G_{ab}.\) By the simmilar argument as in the case (a), \(N_G(R)^{F(x)}\) is doubly transitive and \(|F(R)| = 1+18/3 = 7.\) Hence \(7 \not| |G|.\) On the other hand \(|G| = |\Omega| |G_{ab}| = (1+|N^\alpha : N^\beta|) |G_{ab}| = (1+18 \cdot 19 \cdot 20/2 \cdot 12) \cdot 2^4 \cdot 18 \cdot 19 \cdot 20/2 = 2^3 \cdot 3^3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \text{ with } 0 \leq i \leq 1,\) a contradiction. If \(q = 27,\) then \(|C_G(z)| = |F(z)| \times |C_{G_{ab}}(z)| = 8 \times |G_{ab}|, while \(|\Omega| = 1+|N^\alpha : N^\beta| = 1+26 \cdot 27 \cdot 28 \cdot 2 \cdot 12 = 820 = 2^2 \cdot 5 \cdot 41\) and so \(|G_{12} = 4 | G_{12}|.\) Therefore \(|C_G(z)| |G|, a contradiction.

In the case (c), \(r+1 = 4 \text{ and } 1+(q-\varepsilon)r/4 = 28.\) Hence \(r = 3 \text{ and } q = 37,\)
contrary to \( r \mid n \).

7. Case (V)

In this section we assume that \( N_\beta^*=S_4 \) and \( q=7,9 \mod 16 \). We note that \( 4 \nmid n \).

First we argue that \( N_\beta^*=N^* \cap N_\beta^\prime \). Suppose \( N_\beta^*=N^* \cap N_\beta^\prime \). Then \( C_\infty(N_\beta^*)=1 \) by Lemma 2.2. Since \( N_\beta^*/N^* \cap N_\beta^\prime = N_\beta^*/N^* \leq Z_2 \times Z_2 \), we have \( N^* \cap N_\beta^\prime = A_4 \) and \( N_\beta^*/N^* \cap N_\beta^\prime = Z_2 \), so that \( N_\beta^*/N^* \cap N_\beta^\prime = N_\beta^*/N^* \cap N_\beta^\prime = Z_2 \). Hence as Out\( (S_4)=1 \), \( Z(N_\beta^*/N_\beta^\prime)\). Set \( \langle t \rangle = Z(N_\beta^*/N_\beta^\prime) \) and let \( t \in I(N_\beta^*) \). Since \( C_{N^*}(t) \geq N_\beta^*/N_\beta^\prime \), by Lemma 2.6, we have \( C_\infty(t) = PGL(2, \sqrt{q}) \) and \( |F(t)| = 1 + 3(q-\varepsilon)r/8 \) by Lemma 2.8.

Let \( P \) be a Sylow \( p \)-subgroup of \( C_\infty(\tau) \). Then \( |P| = \sqrt{q} \). If \( p=3 \), \( P \) acts semi-regularly on \( F(t) \cdot \{1\} \) and so \( \sqrt{q} \mid |P| \). But obviously \( 5^2 > n^2 \) for any positive integer \( n \). This is a contradiction. If \( p=3 \), \( |P| = \sqrt{q} \mid 3(q-\varepsilon)r/8 \). Therefore \( \sqrt{q} \mid r \) and so \( 5^2 \leq n^2 \) as \( p=5 \) and \( r \mid n \). But obviously \( 5^2 > n^2 \) for any positive integer \( n \).

Let \( V \) be a cyclic subgroup of \( N_\beta^* \) of order 4 and let \( U \) be a Sylow 2-subgroup of \( N_\beta^* \) containing \( V \). Then \( U=N_\beta^*(V) \), \( |F(V)| = 1 + (q-\varepsilon)r/8 \) by Lemma 2.8 and \( |F(U)| = 1 + 3(q-\varepsilon)r/8 \) by Lemmas 2.3 and 2.4. If \( q=7,9 \mod 16 \), we have \( q=3^2 \text{ or } 3^3 \). If \( q=3^2 \), \( |\Omega| = 1 + N^*: N_\beta^* = 1 + 8 \cdot 9 \cdot 10/2 \cdot 24 = 16 \), a contradiction by [9]. If \( q=3^3 \), \( |F(t)| = 1 + 273r \) and \( |F(t)| = 3^3(r-\varepsilon)/8 \). From this, \( n \leq 7 \). Since \( q=3^2 \equiv 7 \text{ or } 9 \mod 16 \), we have \(|\Omega| = 1 + N^*: N_\beta^* = 1 + 8 \cdot 9 \cdot 10/2 \cdot 24 = 16 \), a contradiction by [9] if \( q=3^3 \). If \( q=7 \), \( |\Omega| = 1 + |\Omega| \cdot |\Omega| = 8 \text{ or } 16 \). By [10], we have a contradiction. Therefore \( U=Z_2 \).

Suppose \( V^e \leq G_\alpha \) for some \( g \in G \) and set \( \gamma = \alpha^g \). Then \( \gamma = g^{-1}N^* \gamma \cap G_\alpha \leq N^* / (G_\alpha \cap N_\beta^*) \leq N^* / N_\beta^* = N_\beta^* \gamma \). As \( N_\beta^* = S_4 \), \( V^e = V^h \) for some \( h \in N_\beta^* \). Hence \( C_\infty(V^e) \) is doubly transitive. By Lemma 2.9, \( C_\infty(N^*) = 1 \) and one of the following holds.

(a) \( N_\infty(V^e) \leq A_4 L(1, 2) \).
(b) \( N_\infty(V^e) \geq PSL(2, p_1) \), \( p_1 = (q-\varepsilon)/8 \geq 5 \).
(c) \( N_\infty(V^e) = R(3) \).

In the case (a), \( (r+1)/2 = 1 + (q-\varepsilon)r/8 \) by Lemma 2.5 and so \( r = (q-\varepsilon-16)/8 \) and \( r \mid n \). From this \( q = 23 \) or 25 and \( r = 1 \). Since \( |\Omega| = 1 + |N^*: N_\beta^*| = 2 \cdot 127 \) or \( 2 \cdot 163 \), we have \( |G| = 2 \cdot |G_{\alpha}(V)| \mid 2 \) while \( |N_\infty(V)| = |F(V)| |2| \mid |G_{\alpha}(V)| = 4 \cdot |G_{\alpha}(2)| \), contrary to \( |G_{\alpha}(V)| \mid |G| \).

In the case (b), \( p_1(1) = 2s \) and \( s \mid 2n(q-\varepsilon)/4 = 4np_1 \), where \( s \) is the order of \( N_{\infty}(V^e) \). Hence \( p_1 = 1/8n \). From this, \( p^s = q-\varepsilon \geq 64n \) and so \( q = 23, 41, 71 \) or 73. Since \( p_1 \) is a prime and \( p_1 = (q-\varepsilon)/8 \geq 5 \), \( q = 23, 71, 73 \). Therefore \( q = 41 \) and \( |\Omega| = 1 + |N^*: N_\beta^*| = 1 + 40 \cdot 41 \cdot 42 \cdot 2 \cdot 24 = 2^2 \cdot 359 \).
Since $N^\alpha_\beta = N^\alpha \cap N^\beta$, $C_\alpha(z)^{F(\alpha)}$ is transitive by Lemma 2.1. On the other hand $|F(z)| = 1 + 40 \cdot 9 / 24 = 16$ by Lemma 2.8 (i) and so $|C_\alpha(z)|_2 = 16 |C_\alpha(z)|_2 = 16 |G|$, contrary to $|C_\alpha(z)|_2 = |G|$

In the case (c), $r+1=4$ and $1+(q-\varepsilon)r/8=28$. Hence $r=3$ and $q=71$ or 73, contrary to $r | n$.

8. Case (VI)

In this section we assume that $N^\alpha_\beta = A_5$ and $q \equiv 3, 5 \pmod{8}$. In particular, $n$ is odd. If $N^\alpha_\beta = N^\alpha \cap N^\beta$, then $N^\alpha \cap N^\beta = 1$, so $N^\alpha_\beta = N^\alpha \cap N^\beta \leq \text{Out}(N^\beta)/Z_2 \times Z_n$, a contradiction. Hence $N^\alpha_\beta = N^\alpha \cap N^\beta$. Let $z$ be an involution in $N^\alpha_\beta$ and let $T$ be a Sylow 2-subgroup of $N^\alpha_\beta$ containing $z$. Then, by Lemma 2.8 $|F(z)| = 1 + (q-\varepsilon)15r/60 = 1 + (q-\varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|C_\alpha(x)|_2 = 1 + 12 \cdot 5r/60 = 1 + r$. Since $N^\alpha_\beta = N^\alpha \cap N^\beta$, $z^G \cap G_\alpha = z^G \cap N^\alpha_\beta = z^G$ and $C_\alpha(z)^{F(\alpha)}$ is doubly transitive. By Lemma 2.9, $C_\alpha(N^\alpha_\beta) = 1$ and one of the following holds.

(a) $C_\alpha(z)^{F(\alpha)} \leq AGL(1, 2^4)$.

(b) $C_\alpha(z)^{F(\alpha)} \geq PSL(2, p_1)$, $p_1 = (q-\varepsilon)/4 \geq 5$.

(c) $C_\alpha(z)^{F(\alpha)} = R(3)$.

In the case (a), by Lemma 2.5, $(q-\varepsilon)/4 = 1$ or $(r+1)/2 = 1 + (q-\varepsilon)r/4$. Hence $q = 5$ or $r = (q-\varepsilon-8)4n$ and so $n = 1$ or $q = 11$ or 13. If $q = 13$, we have $5 \nmid |G_a|$, a contradiction. Hence $q = 11$ and $|\Omega| = 1 + |N^\alpha_\beta| = 1 + 10 \cdot 11 \cdot 12 / 2 \cdot 60 = 12$. By [9], $G^\alpha = M_{11}$, $|\Omega| = 12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1 = p_1 - 1 / 2 | s$ and $s | 2n(q-\varepsilon)/2 = 4np_1$, where $s$ is the order of $C_\alpha(a)^{F(\alpha)}$. Hence $p_1 = 8$ and so $p^\alpha = \alpha - 4 \cdot 32n$. From this $q = 19, 27$ or 37. Since $5 \nmid |G_a|$, $q = 27, 37$. Hence $q = 19$ and $|\Omega| = 1 + |N^\alpha_\beta| = 1 + 18 \cdot 19 \cdot 20 / 2 \cdot 60 = 2.29$. Since $G_a = PSL(2, 19)$ or $PGL(2, 19)$, $|G| = |\Omega| |G_a| = 2.29 \cdot 2^2 \cdot 18 \cdot 19 \cdot 20 / 2 = 2^2 i. 3^2 \cdot 5 \cdot 19 \cdot 29$ with $0 \leq i \leq 1$. Let $P$ be a Sylow 29-subgroup of $G$. Then $P$ is semi-regular on $\Omega$ and 3, 5, 19 $\not\equiv |N^\alpha(P)|$ because $N^\alpha(P)/C_\alpha(P) \leq Z_4 \times Z_7$. Hence $|G| = N^\alpha(P) = 2^i \cdot 3^2 \cdot 5 \cdot 19$ with $0 \leq i \leq 4$, while $2^i \cdot 3^2 \cdot 5 \cdot 19 \equiv 1 \pmod{29}$ for any $i$ with $0 \leq i \leq 4$, contrary to a Sylow's theorem.

If $C_\alpha(z)^{F(\alpha)} = R(3)$, $r+1=4$ and $1+(q-\varepsilon)r/4=28$ and hence $r=3$, $q=37$, contrary to $r | n$.

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References


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