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ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS IN WHICH SOCLE(G_α) IS NONSOLVABLE

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1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and $\alpha\in\Omega$. In [8], O'Nan has proved that socle $(G_\alpha)=A\times N$, where A is an abelian group and N is 1 or a nonabelian simple group. Here socle (G_α) is the product of all minimal normal subgroups of G_α.

In the previous paper [4], we have studied doubly transitive permutation groups in which N is isomorphic to $PSL(2,q)$, $Sz(q)$ or $PSU(3,q)$ with q even. In this paper we shall prove the following:

Theorem. *Let G be a doubly transitive permutation group on a finite set Ω with $|\Omega|$ even and let $\alpha\in\Omega$. If G_α has a normal simple subgroup N^α isomorphic to $PSL(2,q)$, where q is odd, then one of the following holds.*

(i) *G^Ω has a regular normal subgroup.*

(ii) *$G^\Omega\simeq A_6$ or S_6, $N^\alpha\simeq PSL(2,5)$ and $|\Omega|=6$.*

(iii) *$G^\Omega\simeq M_{11}$, $N^\alpha\simeq PSL(2,11)$ and $|\Omega|=12$.*

In the case that G^Ω has a regular normal subgroup, by a result of Hering [3] we have $(|\Omega|,q)=(16,9)$, $(16,5)$ or $(8,7)$.

We introduce some notations:

$F(X)$: the set of fixed points of a nonempty subset X of G

$X(\Delta)$: the global stabilizer of a subset $\Delta(\subseteq\Omega)$ in X

X_Δ : the pointwise stabilizer of Δ in X

X^Δ : the restriction of X on Δ

$m|n$: an integer m divides an integer n

X^H : the set of H-conjugates of X

$|X|_p$: maximal power of p dividing the order of X

$I(X)$: the set of involutions in X

D_m : dihedral group of order m

In this paper all sets and groups are finite.

2. Preliminairies

Lemma 2.1. *Let G be a transitive permutation group on Ω, $\alpha\in\Omega$ and N^{α} a normal subgroup of G_{α} such that $F(N^{\alpha})=\{\alpha\}$. Let the subgroup $X\leq N^{\alpha}$ be conjugate in G_{α} to every group Y which lies in N^{α} and which is conjugate to X in G. Then $N_G(X)$ is transitive on $\Delta=\{\gamma\in\Omega\,|\,X\leq N^{\gamma}\}$.*

Proof. Let $\beta\in\Delta$ and let $g\in G$ such that $\beta^{g}=\alpha$. Then, as $X\leq N^{\beta}$, $X^{g}\leq N^{\beta^{g}}=N^{\alpha}$. By assumption, $(X^{g})^{h}=X$ for some $h\in G_{\alpha}$. Hence $gh\in N_G(X)$ and $\alpha^{(gh)^{-1}}=\alpha^{g^{-1}}=\beta$. Obviously $N_G(X)$ stabilizes Δ. Thus Lemma 2.1 holds.

Lemma 2.2. *Let G be a doubly transitive permutation group on Ω of even degree and N^{α} a nonabelian simple normal subgroup of G_{α} with $\alpha\in\Omega$. If $C_G(N^{\alpha})\neq 1$, then $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$ for $\alpha\neq\beta\in\Omega$ and $C_G(N^{\alpha})$ is semiregular on $\Omega-\{\alpha\}$.*

Proof. See Lemma 2.1 of [4].

Lemma 2.3. *Let G be a transitive permutation group on Ω, H a stabilizer of a point of Ω and M a nonempty subset of G. Then*

$$|F(M)| = |N_G(M)|\times|M^{G}\cap H|/|H|\,.$$

Here $M^{G}\cap H=\{g^{-1}Mg\,|\,g^{-1}Mg\subseteq H,\ g\in G\}$.

Proof. See Lemma 2.2 of [4].

Lemma 2.4. *Let G be a doubly transitive permutation group on Ω and N^{α} a normal subgroup of G_{α} with $\alpha\in\Omega$. Assume that a subgroup X of N^{α} satisfies $X^{G_{\alpha}}=X^{N^{\alpha}}$. Then the following hold.*

(i) $|F(X)\cap\beta^{N^{\alpha}}|=|F(X)\cap\gamma^{N^{\alpha}}|$ *for $\beta,\gamma\in\Omega-\{\alpha\}$.*

(ii) $|F(X)|=1+|F(X)\cap\beta^{N}|\times r$, *where r is the number of N^{α}-orbits on $\Omega-\{\alpha\}$.*

Proof. Let $\Gamma=\{\Delta_1,\Delta_2,\cdots,\Delta_r\}$ be the set of N^{α}-obrits on $\Omega-\{\alpha\}$. Since G_{α} is transitive on $\Omega-\{\alpha\}$ and $G_{\alpha}\trianglerighteq N^{\alpha}$, we have $|\Delta_i|=|\Delta_j|$ for $1\leq i,j\leq r$. By assumption, $G_{\alpha}=N_{G_{\alpha}}(X)N^{\alpha}$ and so $N_{G_{\alpha}}(X)$ is transitive on Γ. Hence for each i with $1\leq i\leq r$ there exists $g\in N_{G_{\alpha}}(X)$ such that $(\Delta_1)^{g}=\Delta_i$. Therefore $|F(X)\cap\Delta_i|=|F(X^{g})\cap(\Delta_1)^{g}|=|F(X)\cap\Delta_1|$. Thus (i) holds and (ii) follows immediately from (i)

Lemma 2.5 (Huppert [5]). *Let G be a doubly transitive permutation group on Ω. Suppose that $0_2(G)\neq 1$ and G_{α} is solvable. Then for any involution z in G_{α}, $|F(z)|^{2}=|\Omega|$.*

We list now some properties of $PSL(2,q)$ with q odd which will be required in the proof of our theorem.

Lemma 2.6 ([2], [6], [10]). *Set $N=PSL(2,q)$ and $G=Aut(N)$, where $q=p^n$ and p is an odd prime. Let z be an involution in N. Then the following hold.*

(i) $|N|=(q-1)q(q+1)/2$, $I(N)=z^N$ *and* $C_N(z)\simeq D_{q-\varepsilon}$, *where* $q\equiv\varepsilon\in\{\pm 1\}$ (*mod* 4).

(ii) *If* $q\neq 3$, *N is a nonabelian simple group and a Sylow r-subgroup of N is cyclic when* $r\neq 2, p$.

(iii) *If X and Y are cyclic groups of N and* $|X|=|Y|\neq 2, p$, *then X is conjugate to Y in* $\langle X, Y\rangle$ *and* $N_N(X)\simeq D_{q\pm\varepsilon}$.

(iv) *If* $X\leq N$ *and* $X\simeq Z_2\times Z_2$, $N_N(X)$ *is isomorphic to* A_4 *or* S_4.

(v) *If* $|N|_2\geq 8$, *N has two conjugate classes of four-groups in N.*

(vi) *There exist a field automorphism f of N of order n and a diagonal automorphism d of N of order 2 and if we identify N with its inner automorphism group,* $\langle d\rangle N\simeq PGL(2,q)$, $\langle f\rangle\langle d\rangle N=G$ *and* $G/N\simeq Z_2\times Z_n$.

(vii) $C_N(d)\simeq D_{q+\varepsilon}$ *and* $C_{\langle d\rangle N}(z)\simeq D_{2(q-\varepsilon)}$.

(viii) *Suppose $n=mk$ for positive integers m, k. Then* $C_N(f^m)\simeq PSL(2,p^m)$ *if k is odd and* $C_N(f^m)\simeq PGL(2,p^m)$ *if k is even.*

(ix) *Assume n is even and let u be a field automorphism of order 2. Then* $I(G)=I(N)\cup d^N\cup u^{\langle d\rangle N}$. *If n is odd,* $I(G)=I(N)\cup d^N$.

(x) *If H is a subgroup of N of odd index, then one of the following holds:*

(1) *H is a subgroup of $C_N(z)$ of odd index for some involution* $z\in N$.

(2) $H\simeq PGL(2,p^m)$, *where* $n=2mk$ *and k is odd.*

(3) $H\simeq PSL(2,p^m)$, *where* $n=mk$ *and k is odd.*

(4) $H\simeq A_4$ *and* $q\equiv 3, 5$ (*mod* 8).

(5) $H\simeq S_4$ *and* $q\equiv 7, 9$ (*mod* 16).

(6) $H\simeq A_5$, $q\equiv 3, 5$ (*mod* 8) *and* $5\,|\,(q-1)q/q+1)$.

Lemma 2.7. *Let G, N, d and f be as defined in Lemma* 2.6 *and H an* $\langle f,d\rangle$*-invariant subgroup of N isomorphic to* $D_{q-\varepsilon}$. *Let W be a cyclic subgroup of* $\langle d\rangle H$ *of index* 2 (*cf.* (vii) *of Lemma* 2.6) *and set* $Y=0_2(W\cap H)$. *Then* $C_G(Y)=W\cdot C_{\langle f\rangle}(Y)$.

Proof. By (viii) of Lemma 2.6, we can take an involution t satisfying $\langle d\rangle H=\langle t\rangle W$ and $[f,t]=1$. Since $N_G(Y)=\langle f, d\rangle N_N(Y)=\langle f, d\rangle H$, $C_G(Y)=C_{\langle f\rangle\langle d\rangle H}(Y)=W\cdot C_{\langle f\rangle\times\langle t\rangle}(Y)$. Suppose $ht\in C(Y)$ for some $h\in\langle f\rangle$. Since t inverts Y, h also inverts Y and so h^2 centralizes Y. Hence some nontrivial 2-element $g\in\langle h\rangle$ inverts Y, so that $C_H(g)$ contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, G^Ω will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume G^Ω has no regular normal subgroup.

Notation. $C^{\alpha}=C_G(N^{\alpha})$, which is semi-regular on $\Omega-\{\alpha\}$ by Lemma 2.2. Let r be the number of N^{α}-orbits on $\Omega-\{\alpha\}$.

Since $G_{\alpha}\trianglerighteq N^{\alpha}$, $|\beta^{N^{\alpha}}|=|\gamma^{N^{\alpha}}|$ for $\beta,\gamma\in\Omega-\{\alpha\}$ and so $|\Omega|=1+r\times|\beta^{N^{\alpha}}|$. Hence r is odd and N^{α}_{β} is a subgroup of N^{α} of odd index. Therefore N^{α}_{β} is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

Lemma 2.8. *Let Z be a cyclic subgroup of N^{α}_{β} with $|Z|\neq 1,p$. Then*
(i) *If $|Z|=2$, $|F(Z)|=1+(q-\varepsilon)|I(N^{\alpha}_{\beta})|r/|N^{\alpha}_{\beta}|$.*
(ii) *If $|Z|\neq 2$, $|F(Z)|=1+|N_{N^{\alpha}}(Z)|r/|N_{N^{\alpha}_{\beta}}(Z)|$.*

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

Lemma 2.9. *If $N^{\alpha}_{\beta}\not\simeq D_{q-\varepsilon}$ and Z is a cyclic subgroup of N^{α}_{β} with $|Z|\neq 1,p$ and $N_G(Z)^{F(Z)}$ is doubly transitive. Then $C^{\alpha}=1$ and one of the following holds.*
(i) *$N_G(Z)^{F(Z)}\leq A\Gamma L(1,q_1)$ for some q_1.*
(ii) *$C_G(Z)^{F(Z)}\trianglerighteq PSL(2,p_1)$, $r=1$ and $|F(Z)|-1=|N_{N^{\alpha}}(Z): N_{N^{\alpha}_{\beta}}(Z)|=p_1$, where p_1 (≥ 5) is a prime.*
(iii) *$N_G(Z)^{F(Z)}=R(3)$, the smallest Ree group, $|F(Z)|=28$.*

Proof. Set $N_G(Z)=L$ and $F(Z)=\Delta$. By Lemma 2.6(iii), $L\cap N^{\alpha}\simeq D_{q\pm\varepsilon}$ and $L\cap N^{\alpha}=\langle t\rangle Y\trianglerighteq Y\geq Z$, where $0(t)=2$, $Y\simeq Z_{(q\pm\varepsilon)/2}$.

If $(L\cap N^{\alpha})^{\Delta}=1$, then $L\cap N^{\alpha}=N^{\alpha}_{\beta}$ because $L\cap N^{\alpha}$ is a maximal subgroup of N^{α}. Since $|N^{\alpha}: N^{\alpha}_{\beta}|$ is odd, $L\cap N^{\alpha}=N^{\alpha}_{\beta}\simeq D_{q-\varepsilon}$, contrary to the assumption. Hence $(L\cap N^{\alpha})^{\Delta}\neq 1$ and as $L_{\alpha}\trianglerighteq L_{\alpha}\cap N^{\alpha}$ and $L_{\alpha}\trianglerighteq Y$, $(L_{\alpha})^{\Delta}$ has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:

(a) L^{Δ} has a regular normal subgroup
(b) $L^{\Delta}\trianglerighteq PSL(2,p_1)$, $|\Delta|=p_1+1$, where $p_1(\geq 5)$ is a prime
(c) $L^{\Delta}\trianglerighteq PSL(3,p_1)$, $p_1\geq 3$, $|\Delta|=(p_1)^3+1$
(d) $L^{\Delta}=R(3)$, $|\Delta|=28$.

Suppose $C^{\alpha}\neq 1$. Then there exists a subgroup D of C^{α} of prime order such that $(L_{\alpha})^{\Delta}\trianglerighteq D^{\Delta}$. Since $[L_{\alpha}, D]\leq D\cdot L_{\Delta}\cap C^{\alpha}=D(L_{\Delta}\cap C^{\alpha})=D$, D is a normal subgroup of L_{α}. By (i) and (iii) of Lemma 2.6, $G_{\alpha}=L_{\alpha}\cdot N^{\alpha}$ and so D is a normal subgroup of G_{α}. By Theorem 3 of [1], G^{Ω} has a regular normal subgroup, contrary to the hypothesis. Thus $C^{\alpha}=1$.

If (a) occurs, L^{Δ} is solvable because $L_{\alpha}/L\cap N^{\alpha}\simeq L_{\alpha}N^{\alpha}/N^{\alpha}\leq \mathrm{Out}(N^{\alpha})$ and $L\cap N^{\alpha}\simeq D_{q\pm\varepsilon}$. Hence by [5], (i) holds in this case.

If (b) occurs, we have $Y^{\Delta}\neq 1$, for otherwise $(L\cap N^{\alpha})^{\Delta}=1$ and so $N^{\alpha}_{\beta}=L\cap N^{\alpha}\simeq D_{q-\varepsilon}$, a contradiction. Hence $1\neq C_G(Z)^{\Delta}\trianglelefteq L^{\Delta}$ and so $C_G(Z)^{\Delta}\trianglerighteq PSL(2,p_1)$ and $Y^{\Delta}\simeq Z_{p_1}$. Therefore $|\Delta\cap\beta^{N^{\alpha}}|=p_1$ and $r=1$ by Lemma 2.4 (ii). Since $|\beta^{Y}|=p_1$, we have $|\beta^{L\cap N^{\alpha}}|=p_1$, so that $|L\cap N^{\alpha}: L\cap N^{\alpha}_{\beta}|=p_1$. Thus (ii) holds in this case.

The case (c) does not cocur, for otherwise, by the structure of $PSU(3,p_1)$, a Sylow p_1-subgroup of $(L_\alpha^\Delta)'$ is not cyclic, while $(L_\alpha)'\leq L\cap N^\alpha\simeq D_{q\pm\varepsilon}$, a contradiction.

3. Case (I)

In this section we assume that $N_\beta^\alpha\leq D_{q-\varepsilon}$, where $\beta\neq\alpha$, $q=p^n$.

(3.1) (i) *If* $N_\beta^\alpha\not\simeq Z_2\times Z_2$, $N_{N^\alpha}(N_\beta^\alpha)=N_\beta^\alpha$ *and* $|F(N_\beta^\alpha)|=r+1$.
(ii) *If* $N_\beta^\alpha\simeq Z_2\times Z_2$, $N_{N^\alpha}(N_\beta^\alpha)\simeq A_4$ *and* $|F(N_\beta^\alpha)|=3r+1$.

Proof. Put $X=N_{N^\alpha}(N_\beta^\alpha)$. Let S be a Sylow 2-subgroup of N_β^α and Y a cyclic subgroup of N_β^α of index 2.

If $N_\beta^\alpha\not\simeq Z_2\times Z_2$, then $|Y|>2$ and so Y is characteristic in N_β^α. Hence $X\leq N_{N^\alpha}(Y)\simeq D_{q-\varepsilon}$. From this $[N_X(S), S\cap Y]\leq S\cap Y$ and $0^2(N_X(S))$ stabilizes a normal series $S\trianglerighteq S\cap Y\trianglerighteq 1$, so that $0^2(N_X(S))\leq C_{N^\alpha}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^\alpha}(S)\leq S$ and hence $N_X(S)=S$. On the other hand by a Frattini argument, $X=N_X(S)N_\beta^\alpha$ and so $X=N_\beta^\alpha$. By Lemma 2.6(i), $(N_\beta^\alpha)^{G_\alpha}=(N_\beta^\alpha)^{N^\alpha}$ and so by Lemmas 2.3 and 2.4 (ii), $|F(N_\beta^\alpha)|=1+|F(N_\beta^\alpha)\cap\beta^{N^\alpha}|\times r=1+|N_\beta^\alpha|r/|N_\beta^\alpha|=r+1$. Thus (i) holds.

If $N_\beta^\alpha\simeq Z_2\times Z_2$, $N_{N^\alpha}(N_\beta^\alpha)\simeq A_4$ by Lemma 2.6 (iv). Similarly as in the case $N_\beta^\alpha\not\simeq Z_2\times Z_2$, we have $|F(N_\beta^\alpha)|=3r+1$.

(3.2) $N_\beta^\alpha/N^\alpha\cap N^\beta\leq Z_2\times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^\alpha=1$. Suppose $C^\alpha=1$. Then $N_\beta^\alpha/N^\alpha\cap N^\beta\simeq N_\beta^\alpha N^\beta/N^\beta\leq\mathrm{Out}(N^\alpha)\simeq Z_2\times Z_n$ by Lemma 2.6 (vi) and hence $(N_\beta^\alpha)'\leq N^\alpha\cap N^\beta$. Since N_β^α is dihedral, $N_\beta^\alpha/(N_\beta^\alpha)'\simeq Z_2\times Z_2$, so that $N_\beta^\alpha/N^\alpha\cap N^\beta\leq Z_2\times Z_2$.

(3.3) *Suppose* $N_\beta^\alpha=N^\alpha\cap N^\beta$ *and let* U *be a subgroup of* N_β^α *isomorphic to* $Z_2\times Z_2$. *Then* $|F(U)|=3r+1$ *and* $N_G(U)^{F(U)}$ *is doubly transitive.*

Proof. Sex $X=N_G(N_\beta^\alpha)$, $\Delta=F(N_\beta^\alpha)$ and let $\{\Delta_1, \Delta_2, \cdots, \Delta_r\}$ be the set of N^α-orbits on $\Omega-\{\alpha\}$. If $g^{-1}N_\beta^\alpha g\leq G_{\alpha\beta}$, then $g^{-1}N_\beta^\alpha g\leq N_\alpha^\gamma\cap N_\beta^\gamma=N_\gamma^\alpha\cap N_\gamma^\beta\leq N_\beta^\alpha$, where $\gamma=\alpha^g$. By a Witt's theorem, X^Δ is doubly transitive.

If U is a Sylow 2-subgroup of N_β^α, by a Witt's theorem, $N_G(U)^{F(U)}$ is doubly transitive. Moreover $N_{N^\alpha}(U)\simeq A_4$ and so by Lemmas 2.3 and 2.4 (ii), $|F(U)|=1+|A_4|\times|N_\beta^\alpha: N_{N_\beta^\alpha}(U)|\times r/|N_\beta^\alpha|=3r+1$.

If $|N_\beta^\alpha|_2>4$, by Lemma 2.6 (iv) and (v), $N_{N^\alpha}(U)\simeq S_4$ and N_β^α has two conjugate classes of four-groups, say $\pi=\{K_1, K_2\}$. Set $X_\pi=M$. Then $M\trianglerighteq N_\beta^\alpha$ and $X/M\leq Z_2$. Clearly $F(U)\cap\Delta_i\neq\phi$ for each i and so $|F(U)\cap\Delta_i|=3$ by Lemma 2.3. Hence $|F(U)|=3r+1$. Since $N_{N^\beta}(U)\simeq S_4$, we may assume $r>1$. Hence by (3.1) (i) $|\Delta|=r+1\geq 4$, so that M^Δ is doubly transitive. Since $M=N_\beta^\alpha N_M(U)$, $N_M(U)^\Delta$ is also doubly transitive and so $N_{M_\alpha}(U)$ is transitive on $\Delta-$

$\{\alpha\}$. As $|\Delta\cap\Delta_i|=1$, $\Delta\cap\Delta_i\subseteq F(U)$ and $N_{N^\alpha}(U)$ is transitive on $F(U)\cap\Delta_i$ for each i, $N_G(U)^{F(U)}$ is doubly transitive.

(3.4) (i) $C^\alpha=1$.

(ii) *Let U be a subgroup of N^α_β isomorphic to $Z_2\times Z_2$. If $N^\alpha_\beta=N^\alpha\cap N^\beta$, then $N_G(U)^{F(U)}$ has a regular normal 2-subgroup. In particular $|F(U)|=3r+1=2^b$ for positive integer b.*

Proof. Since $N_{G_\alpha}(U)^{F(U)}\trianglerighteq N_{N^\alpha}(U)^{F(U)}\simeq S_3$ or Z_3, by (3.3) and Theorem 3 of [1], $N_G(U)^{F(U)}$ has a regular normal subgroup, $N_G(U)^{F(U)}\trianglerighteq PSU(3,3)$ or $N_G(U)^{F(U)}=R(3)$.

Suppose $C^\alpha\neq 1$. Let D be a minimal characteristic subgroup of C^α. Clearly $G_\alpha\rhd D$. If $N_G(U)^{F(U)}\neq R(3)$, D is cyclic. By Theorem 3 of [1], G^Ω has a regular normal subgroup, contrary to the hypothesis. Hence $N_G(U)^{F(U)}=R(3)$. Therefore $(N_{G_\alpha}(U)^{F(U)})'$ contains an element of order 9. Since $N_{G_\alpha}(U)/C^\alpha N_{N^\alpha}(U)\simeq N_{G_\alpha}(U)C^\alpha N^\alpha/C^\alpha N^\alpha\leq \mathrm{Out}(N^\alpha)$, by (vi) of Lemma 2.6 we have $(N_{G_\alpha}(U))'\leq C^\alpha\times N_{N^\alpha}(U)$. From this, C^α contains an element of order 9 and so $C^\alpha\simeq Z_9$ or $M_3(3)$. In both cases, C^α contains a caracteristic subgroup of order 3. Since $G_\alpha\rhd D$, by Theorem 3 of [1] G^Ω has a regular normal subgroup, a contradiction. Thus $C^\alpha=1$.

Let R be a Sylow 3-subgroup of $N_{G_\alpha}(U)$. Since $N_{G_\alpha}(U)/N_{N^\alpha}(U)\simeq N_{G_\alpha}(U)N^\alpha/N^\alpha\leq\mathrm{Out}(N^\alpha)\simeq Z_2\times Z_n$, $R/R\cap N_{N^\alpha}(U)$ is cyclic. Clearly $R\cap N_{N^\alpha}(U)\simeq Z_3$. Therefore $N_G(U)^{F(U)}\not\trianglerighteq PSU(3,3)$, $R(3)$. Thus (3.4) holds.

Since N^α_β is dihedral, we set $N^\alpha_\beta=\langle t\rangle W$ and $Y=W\cap N^\alpha\cap N^\beta$, where W is a cyclic subgroup of N^α_β of index 2 and t is an involution in N^α_β which inverts W.

(3.5) (i) *If $|Y|\geq 3$, $N_G(Y)^{F(Y)}$ is doubly transitive.*

(ii) *If $|Y|<3$, $N^\alpha_\beta\simeq Z_2\times Z_2$ or $N^\alpha_\beta\simeq D_8$ and $N^\alpha\cap N^\beta\leq Z_2\times Z_2$.*

Proof. Suppose $|Y|\geq 3$. If $Y^g\leq G_{\alpha\beta}$, $Y^g\leq N^\gamma\cap G_{\alpha\beta}\leq N^\gamma_\alpha$, where $\gamma=\alpha^g$. If $\gamma=\alpha$, obviously $Y^g\leq N^\alpha$. If $\gamma\neq\alpha$, $N^\gamma_\alpha\simeq N^\alpha_\beta$. Therefore, as $|Y|\geq 3$, N^γ_α has a unique cyclic subgroup of order $|Y|$. Hence $Y^g\leq N^\gamma\cap N^\alpha\leq N^\alpha$, so that $Y^g\leq N^\alpha$. Similarly $Y^g\leq N^\beta$. Thus $Y^g\leq N^\alpha\cap N^\beta$ and so $Y^g=Y$. By a Witt's theorem, $N_G(Y)$ is doubly transitive on $F(Y)$.

Suppose $|Y|<3$. Since $|N^\alpha\cap N^\beta\colon Y|\leq 2$, we have $N^\alpha\cap N^\beta\leq Z_2\times Z_2$. On the other hand, as N^α_β is dihedral, $(N^\alpha_\beta)'$ is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) *Set $\Delta=F(N^\alpha_\beta)$, $L=G(\Delta)$, $K=G_\Delta$ and suppose $N^\alpha_\beta\not\simeq Z_2\times Z_2$. Then $L_\alpha\trianglerighteq N^\alpha_\beta$, $(L_\alpha)'\leq N^\alpha_\beta$, $K'\leq N^\alpha\cap N^\beta$ and $(L_\alpha)^\Delta\simeq Z_r$. If $r\neq 1$, L^Δ is a doubly transitive Frobenius group of degree $r+1$.*

Proof. By Corollary B1 of [7] and (i) of (3.1), L^Δ is doubly transitive and

$|\Delta|=r+1$. Since $N^{\alpha}\cap L\trianglerighteq N^{\alpha}\cap K=N^{\alpha}_{\beta}$, by (i) of (3.1), we have $N^{\alpha}\cap L=N^{\alpha}_{\beta}$. Hence $L_{\alpha}\trianglerighteq N^{\alpha}_{\beta}$. By (i) of (3.4), $L_{\alpha}/N^{\alpha}_{\beta}\simeq L_{\alpha}N^{\alpha}/N^{\alpha}\leq \mathrm{Out}(N^{\alpha})\simeq Z_2\times Z_n$ and so $(L_{\alpha})'\leq N^{\alpha}_{\beta}$ and $(L_{\alpha})^{\Delta}\simeq Z_r$. If $r\neq 1$, then $(L_{\alpha})^{\Delta}\neq 1$. On the other hand $(L_{\alpha\beta})^{\Delta}=1$ as $(L_{\alpha})^{\Delta}$ is abelian. Hence L^{Δ} is a Frobenius group.

(3.7) *Suppose* $|Y|\geq 3$. *Then there exists an involution* z *in* $N^{\alpha}_{\beta}\cap Y$ *such that* $Z(N^{\alpha}_{\beta})=\langle z\rangle$.

Proof. Since $N^{\alpha}_{\beta}\not\simeq Z_2\times Z_2$, $|N^{\alpha}_{\beta}|_2\geq 2^2$ and N^{α}_{β} is dihedral, we have $\langle I(W)\rangle =Z(N^{\alpha}_{\beta})\simeq Z_2$ and $N^{\alpha}_{\beta}/(N^{\alpha}_{\beta})'\simeq Z_2\times Z_2$. Let $Z(N^{\alpha}_{\beta})=\langle z\rangle$ and suppose that z is not contained in Y. By (3.2), $(N^{\alpha}_{\beta})'\leq N^{\alpha}\cap N^{\beta}\cap W=Y$ and so $|(N^{\alpha}_{\beta})'|$ is odd. Hence $|N^{\alpha}_{\beta}|_2=4$ and $q\equiv p^n=3$ or $5 \pmod 8$, so that n is odd. By (3.2) and (i) of (3.4), $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq N^{\alpha}_{\beta}N^{\beta}/N^{\beta}\simeq 1$ or Z_2. If $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$, then $W=Y$ and so $z\in Y$, contrary to the assumption. Therefore we have $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq Z_2$ and $N^{\alpha}_{\beta}=\langle z\rangle \times(N^{\alpha}\cap N^{\beta})$. Since n is odd and $z\in N^{\alpha}_{\beta}N^{\beta}-N^{\beta}$, by Lemma 2.6 (vi), (vii) and (ix), $N^{\alpha}_{\beta}N^{\beta}\simeq PGL(2,q)$ and $C_{N^{\beta}}(z)\simeq D_{q+\varepsilon}$. But $N^{\alpha}\cap N^{\beta}\leq C_{N^{\beta}}(z)$ and besides it is isomorphic to a subgroup of $D_{q-\varepsilon}$. Hence $N^{\alpha}\cap N^{\beta}\simeq Z_2$ and $N^{\alpha}_{\beta}\simeq Z_2\times Z_2$, a contradiction.

(3.8) *Suppose* $|Y|\geq 3$. *Then* $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$.

Proof. Suppose $N^{\alpha}_{\beta}\neq N^{\alpha}\cap N^{\beta}$ and let Δ, L, K be as defined in (3.6) and $x\in L_{\alpha}$ such that its order is odd and $\langle x\rangle$ is transitive on $\Delta-\{\alpha\}$. As $|Y| \geq 3$, W is characteristic in N^{α}_{β} and hence by (3.6), x stabilizes a normal series $L_{\alpha}\trianglerighteq N^{\alpha}_{\beta}\trianglerighteq W\trianglerighteq(N^{\alpha}_{\beta})'$. By Theorem 5.3.2 of [2], $[x, 0_2(L_{\alpha}/(N^{\alpha}_{\beta})')]=1$. Since $L_{\alpha}/(N^{\alpha}_{\beta})'$ has a normal Sylow 2-subgroup and $(N^{\alpha}_{\beta})'\leq K'$, we have $[x, 0_2(L_{\alpha}/K')] =1$, so that $[x, N^{\alpha}_{\alpha}]\leq K'\leq N^{\alpha}\cap N^{\beta}$ by (3.6). If $r\neq 1$, then $\beta^x\neq\beta$ and $\beta^x\in\Delta$, hence $N^{\beta}_{\alpha}=x^{-1}N^{\beta}_{\alpha}x=N^{\gamma}_{\alpha}$, where $\gamma=\beta^x$. Since $\gamma\in\Delta$ and $\Delta=F(N^{\beta}_{\alpha})$, $N^{\beta}_{\alpha}\leq N^{\beta}\cap G_{\gamma}=N^{\beta}_{\gamma}$ and so $N^{\beta}_{\alpha}=N^{\beta}_{\gamma}$. Similarly $N^{\gamma}_{\alpha}=N^{\gamma}_{\beta}$. Hence $N^{\beta}_{\gamma}=N^{\gamma}_{\beta}$, which implies $N^{\beta}_{\gamma}=N^{\beta}\cap N^{\gamma}$. By the doubly transitivity of G, we have $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$, contrary to the assumption. Therefore we obtain $r=1$.

Let z be as defined in (3.7) and put $k=(q-\varepsilon)/|N^{\alpha}_{\beta}|$. By Lemma 2.8(i) we have $|F(z)|=1+(q-\varepsilon)(|N^{\alpha}_{\beta}|/2+1)/|N^{\alpha}_{\beta}|=(q-\varepsilon)/2+k+1$. Similarly $|F(Y)| =k+1$. As $N^{\alpha}_{\beta}\neq N^{\alpha}\cap N^{\beta}$, there is an involution t in N^{α}_{β} which is not contained in N^{β}. By Lemma 2.6 (i), $t^y=z$ for some $y\in N^{\alpha}$. Set $\gamma=\beta^y$. Then $\gamma\in F(z)$ and $z\notin N^{\gamma}$. By Lemma 2.6 (vii), (viii) and (ix), $C_{N^{\gamma}}(z)\simeq D_{q+\varepsilon}$ or $PGL(2,\sqrt{q})$. Assume $C_{N^{\gamma}}(z)\simeq D_{q+\varepsilon}$ and let R be a cyclic subgroup of $C_{N^{\gamma}}(z)$ of index 2. We note that R is semi-regular on $\Omega-\{\alpha\}$. Set $X=C_G(z)$. Since $2\leq k+1\leq (q-\varepsilon)/|q-\varepsilon|_2+1$, we have $(q+\varepsilon)/2\not| k+1$ and so $|\alpha^X|>k+1$. By (i) of (3.5) and (3.7), $N_G(Y)\leq C_G(z)=X$ and $\alpha^X\supseteq F(Y)$. It follows from Lemma 2.1 that $\alpha^X=\{\mu\mid z\in N^{\mu}\}\not\ni\gamma$. Hence $|F(z)|>|\alpha^X|\geq|F(Y)|+(q+\varepsilon)/2=k+1+(q-\varepsilon)/2+\varepsilon= |F(z)|+\varepsilon$. Therefore $\varepsilon=-1$ and $\gamma^X=\{\gamma\}$, so that $\gamma\in F(Y)$, a contradiction. Thus $C_{N^{\gamma}}(z)\simeq PGL(2,\sqrt{q})$, $\varepsilon=1$, $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq Z_2$ and $|\langle z^G\cap G_{\alpha}\rangle: N^{\alpha}|=2$.

Set $\Delta_1=\alpha^X$ and $\Delta_2=F(z)-\Delta_1$. Let $\delta\in\Delta_2$ and g an element of G satisfying $\delta^g=\gamma$. Then $z\in N^\alpha_\delta N^\delta-N^\delta$ and so $z^g\in N^\nu_\gamma N^\gamma-N^\gamma$, where $\nu=\alpha^g$. Since $|\langle z^G\cap G_\gamma\rangle: N^\gamma|=2$ and $z\in G_\gamma-N^\gamma$, it follows from Lemma 2.6 (ix) that $(z^g)^h=z$ for some $h\in G_\gamma$. Hence $gh\in X$ and $\delta^{gh}=\gamma$. Thus $\Delta_2=\gamma^X$. Let $\delta\in\Delta_2$. Then $z\in N^\alpha_\delta$ and $z\notin Z(N^\alpha_\delta)$ by (3.7) and so $X\cap N^\alpha_\delta\simeq Z_2\times Z_2$, which implies $|\delta^{(C_{N^\alpha}(z))}|=(q-1)/4$. Hence $(|\Delta_1|, |\Delta_2|)=((q-1)/4+k+1, (q-1)/4)$ or $(k+1, (q-1)/2)$. Let P be a subgroup of $C_{N^\gamma}(z)$ of order $\sqrt{q}$. Then $F(P)=\{\gamma\}$ and P is semi-regular on $\Omega-\{\gamma\}$. If $|\Delta_2|=(q-1)/4$, then $\sqrt{q}\,|(q-1)/4-1=(q-5)/4$ and $\sqrt{q}\,|(q-1)/4+k+1$. From this, $q=5^2$, $k=3$, $|\Delta_1|=10$ and $|\Delta_2|=6$. Since $(C_{N^\gamma}(z))^{\Delta_2}\simeq S_5$, $X^{\Delta_2}\simeq S_6$ and so $|X|_3\geq 3^2$. As X acts on Δ_1 and $|\Delta_1|\equiv 1 \pmod 3$, $|G_\alpha|_3\geq |X_\alpha|_3\geq 3^3$, contrary to $N^\alpha\simeq PSL(2,25)$. If $|\Delta_2|=(q-1)/2, \sqrt{q}\,|(q-1)/2-1=(q-3)/2$, so $q=3^2$, $k=1$, $N^\alpha_\beta\simeq D_8$ and $\Delta_1=\{\alpha, \beta\}$. Hence $C_{N^\gamma}(z)$ fixes α and β, so that $PGL(2,3)\simeq C_{N^\gamma}(z)\leq N^\gamma_\alpha\simeq N^\alpha_\beta\simeq D_8$, a contradiction.

(3.9) *Suppose* $|Y|\geq 3$. *Then* $r=1$.

Proof. By (3.6), $r+1=2^c$ for some integer $c\geq 0$. On the other hand $3r+1=2^b$ by (3.8) and (ii) of (3.4). Hence $2r=2^c(2^{b-c}-1)$ and so $c=1$ as r is odd. Thus $r=1$.

(3.10) *Put* $k=(q-\varepsilon)/|N^\alpha_\beta|$. *If* $N^\alpha_\beta=N^\alpha\cap N^\beta$ *and* $r=1$, *then*

$$q-\varepsilon+2k+2\,|\,2((2k+2-\varepsilon)(k+1-\varepsilon)k+1)(2k+2-\varepsilon)(k+1-\varepsilon)\,.$$

Proof. Set $S=\{(\gamma, u)\,|\,\gamma\in F(u), u\in z^G\}$, where z is an involution in N^α_β. We now count the number of elements of S in two ways. Since $N^\alpha_\beta=N^\alpha\cap N^\beta$, $F(z)=\{\gamma\,|\,z\in N^\gamma\}$ and hence $C_G(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S|=|\Omega|\,|z^{G_\alpha}|=|z^G|\,|F(z)|$. Since $r=1$, $|\Omega|=1+|N^\alpha: N^\alpha_\beta|=kq(q+\varepsilon)/2+1$ and by Lemma 2.8 $|F(z)|=(q-\varepsilon)/2+k+1$. Since $G_\alpha\trianglerighteq N^\alpha$, z^{G_α} is contained in N^α and so $|G_\alpha: C_{G_\alpha}(z)|=|N^\alpha: C_{N^\alpha}(z)|=q(q+\varepsilon)/2$. Hence $(q-\varepsilon)/2+k+1\,|\,(kq(q+\varepsilon/2+1)q(q+\varepsilon)/2$. On the other hand $|F(z)|_2=|C_G(z)|_2/|C_{G_\alpha}(z)|_2\leq |G|_2/|C_{G_\alpha}(z)|_2=|G|_2/|G_\alpha|_2=|\Omega|_2$ because $|G_\alpha: C_{G_\alpha}(z)|=q(q+\varepsilon)/2\equiv 1 \pmod 2$. Hence $|q-\varepsilon+2k+2|_2\leq |kq(q+\varepsilon)+2|_2$. Since $kq(q+\varepsilon)+2=(kq+2k(\varepsilon-k-1))(q-\varepsilon+2k+2)+2((2k+2-\varepsilon)(k+1-\varepsilon)k+1)$ and $q(q+\varepsilon)=(q+2\varepsilon-2k-2)(q-\varepsilon+2k+2)+2(2k+2-\varepsilon)(k+1-\varepsilon)$, we have (3.10).

(3.11) *Suppose* $|Y|\geq 3$. *Then one of the following holds.*
(i) $N^\alpha_\beta=N^\alpha\cap N^\beta\simeq D_{q-\varepsilon}$.
(ii) $N^\alpha_\beta=N^\alpha\cap N^\beta\not\simeq D_{q-\varepsilon}$ and $N_G(Y)^{F(Y)}$ *has a regular normal subgroup.*

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(Y)}=R(3)$ or there exists a prime $p_1\geq 5$ such that $C_G(Y)^{F(Y)}\trianglerighteq PSL(2,p_1)$ and $V/Y\simeq Z_{p_1}$, where $V=C_{N^\alpha}(Y)$. By (i) of (3.1) and (3.9), $F(N^\alpha_\beta)=\{\alpha, \beta\}$. On the other hand, $(N^\alpha_\beta)^{F(Y)}\simeq N^\alpha_\beta/Y\simeq Z_2$. Hence $N_G(Y)^{F(Y)}\neq R(3)$ and $C_G(Y)^{F(Y)}\trianglerighteq$

$PSL(2, p_1)$.

By (i) of (3.4) and Lemma 2.7, we have $C_{G_\alpha}(Y)=V\langle f_1\rangle$, where f_1 is a field automorphism of N^α. Let t be the order of f_1, $n=tm$ and let $p^m\equiv\varepsilon_1\in\{\pm 1\}$ (mod 4). Clearly $C_{G_\alpha}(Y)^{F(Y)}\trianglerighteq V^{F(Y)}\simeq Z_{p_1}$ and $|C_{G_{\alpha\beta}}(Y)^{F(Y)}|\,|\,t$, so that $(p_1-1)/2\,|\,t$.

First we assume that t is even and set $t=2t_1$. Then $Y\leq C_{N^\alpha}(f_1)\simeq PGL(2, p^m)$ by Lemma 2.6 (viii). As $|V/Y|=p_1$ and p_1 is a prime, Y is a cyclic subgroup of $C_{N^\alpha}(f_1)$ of order $p^m-\varepsilon_1$ and $(p^n-1)/2(p^m-\varepsilon_1 1)=p_1$. Put $s=\sum_{i=0}^{t_1-1}(p^{2m})^i$. Then $(p^m+\varepsilon_1)s/2=p_1$, so that we have either (i) $t_1=1$ and $p_1=(p^m+\varepsilon_1)/2$ or (ii) $t_1\geq 2$, $p^m=3$ and $p_1=s$. In the case (i), $2\leq(p_1-1)/2=(p^m+\varepsilon_1-2)/4\,|\,2t_1=2$. Hence $(p_1, q)=(5, 3^4)$ or $(4, 11^2)$. Let z be as in (3.7). As mentioned in the proof of (3.10), $|F(z)|=(q-1)/2+k+1$, $|\Omega|=kq(q+1)/2+1$ and $C_G(z)$ is transitive on $F(z)$. If $q=3^4$, then $|F(z)|=46$ and $|\Omega|=2\cdot 19^2\cdot 23$. Hence $|C_G(z)|=|F(z)|\,|C_{G_\alpha}(z)|=|F(z)|\,|C_{G_\alpha}(z)N^\alpha/N^\alpha|\,|C_{N^\alpha}(z)|=46\cdot 2^i\cdot 80=2^{5+i}\cdot 5\cdot 23$ with $0\leq i\leq 3$. Let P be a Sylow 23-subgroup of $C_G(z)$ and Q a Sylow 5-subgroup of $C_G(z)$. It follows from a Sylow's theorem that P is a normal subgroup of $C_G(z)$ and so $[P, Q]=1$. Theorefore $|F(Q)|\geq 23$, contrary to $5\not|\,|N^\alpha_\beta|$. If $q=11^2$, then $|F(z)|=66$ and $|\Omega|=2\cdot 3\cdot 6151$. Let P be a Sylow 11-subgroup of $C_G(z)$. Since $11\not|\,|\Omega|$, P is a subgroup of N^γ for some $\gamma\in\Omega$ and $F(P)=\{\gamma\}$. Hence $\gamma\in F(z)$, so that $z\in N^\gamma$, contrary to $C_{N^\gamma}(z)\simeq D_{120}$. In the case (ii), we have $(p_1-1)/2=(\sum_{i=1}^{t_1-1}9^i)/2\,|\,t=2t_1$. From this, $9^{t_1-1}\leq 4t_1$, hence $t_1=1$, a contradiction.

Assume t is odd. Then $Y\leq C_{N^\alpha}(f_1)\simeq PSL(2, p^m)$ by Lemma 2.6 (viii). As $|V/Y|=p_1$ and p_1 is a prime, $Y\simeq Z_{(p^m-\varepsilon_1)/2}$ and $(q-\varepsilon)/(p^m-\varepsilon_1)=p_1$. Hence $\sum_{i=0}^{t-1}(p^m)^i(\varepsilon_1)^{t-1-i}=p_1$ and $(p_1-1)/2=((\sum_{i=1}^{t-1}(p^m)^i(\varepsilon_1)^{t-1-i})-1)/2\,|\,t$. In pariticular $2t\geq(p^m)^{t-1}-(p^m)^{t-2}=(p^m-1)\,(p^m)^{t-2}\geq 2(p^m)^{t-2}$. From this $t=3$, $m=1$, $p_1=7$ and $q=3^3$, so that $N^\alpha_\beta\simeq Z_2\times Z_2$, a contradiction.

(3.12) (i) *of* (3.11) *does not occur.*

Proof. Let G^Ω be a minimal counterexample to (3.12) and M a minimal normal subgroup of G. By the hypothesis, G has no regular normal subgroup and hence $M_\alpha\neq 1$. As M_α is a normal subgroup of G_α, by (i) of (3.4), M_α contains N^α. By (3.9), $r=1$, hence M is doubly transitive on Ω. Therefore $G=M$ and G is a nonabelian simple group.

Since $N^\alpha_\beta\simeq D_{q-\varepsilon}$, $k=1$ and so $q-\varepsilon+4\,|\,2((4-\varepsilon)\,(2-\varepsilon)+1)\,(4-\varepsilon)\,(2-\varepsilon)$ by (3.10). Hence we have $q=7$, 9, 11, 19, 27 or 43.

Let x by an element of N^α_β. If $|x|>2$, by Lemma 2.8, $|F(x)|=1+|N^\alpha_\beta|\times 1/|N^\alpha_\beta|=2$ and if $|x|=2$, similarly we have $|F(x)|=(q-\varepsilon)/2+2$. Assume $q\neq 9$ and let d be an involution in $G_\alpha-N^\alpha$ such that $\langle d\rangle\, N^\alpha$ is isomorphic to PGL

$(2,q)$. We may assume $d\in G_{\alpha\beta}$. Since $\langle d\rangle N^{\alpha}$ is transitive on $\Omega-\{\alpha\}$, by Lemmas 2.3 and 2.6 (vii), (ix), $|F(d)|=2(q-1)(q+1/2)/2(q+1)+1=(q+1)/2$, while $|F(x)|=(q+1)/2+2$ for $x\in I(N^{\alpha})$. Hence d is an odd permutation, contrary to the simplicity of G. Thus $G_{\alpha}=N^{\alpha}$ if $q\neq 9,27$ and $|G_{\alpha}/N^{\alpha}|=1$, 3 if $q=27$.

If $q=9$, $|\Omega|=1+|N^{\alpha}: N^{\alpha}_{\beta}|=1+9\cdot 10/2=2\cdot 23$ and $|G_{\alpha}|=2^{i}|PSL(2,9)|=2^{3+i}\cdot 3^2\cdot 5$ with $0\leq i\leq 2$. Let P be a Sylow 23-subgroup of G. Since $\mathrm{Aut}(Z_{23})\simeq Z_2\times Z_{11}$, $3\not| \, |N_G(P)|$, for otherwise P centralizes a nontrivial 3-element x and so $F(P)\supseteq F(x)$ because $|F(x)|=1$, contrary to $|F(P)|=0$. Similarly $5\not| \, |N_G(P)|$. Hence $|G: N_G(P)|=2^a\cdot 3^b\cdot 5$ for some a with $0\leq a\leq 6$. By a Sylow's theorem, $2^a\cdot 3^2\cdot 5\equiv -2^a\equiv 1 \pmod{23}$, a contradiction.

If $q=27$, $|\Omega|=1+27\cdot 26/2=2^5\cdot 11$ and $|G_{\alpha}|=2^2\cdot 3^{3+i}\cdot 7\cdot 13$ with $0\leq i\leq 1$. Let P a Sylow 11-subgroup of G. Since $P\simeq Z_{11}$ and $\mathrm{Aut}(Z_{11})\simeq Z_2\times Z_5$, 3^{1+i}, 7, $13\not| \, |N_G(P)|$ by the similar argument as above. Hence $|G: N_G(P)|=2^a\cdot 3^b\cdot 7\cdot 13$ with $0\leq a\leq 7$ and $3\leq b\leq 3+i$. By a Sylow's theorem, $2^a\cdot 3^b\cdot 7\cdot 13=2^a\cdot 3^{b-3}\cdot 3^3\cdot 7\cdot 13\equiv 2^a\cdot 3^{b-3}\cdot 4\equiv 1 \pmod{11}$. Hence $a=0$, $b=4$. Therefore $N_G(P)$ contains a Sylow 2-subgroup S of G. Let T be a Sylow 2-subgroup of N^{α}_{β} and g an element such that $T^g\leq S$. Then $T^g\cap C_G(P)\neq 1$ as $N_S(P)/C_S(P)\leq Z_2$. Let u be an involution in $T^g\cap C_G(P)$. Then $|F(u)|=(27+1)/2+2=16$, while $11\,|\,|F(u)|$ because $[P, u]=1$ and $|F(P)|=0$, a contradiction.

If $q=7$, 11, 19 or 43, then $G_{\alpha}=N^{\alpha}$ and $\varepsilon=-1$. Set $\Gamma=\{\{\gamma,\delta\}\,|\,\gamma,\delta\in\Omega, \gamma\neq\delta\}$. We consider the action of G on Γ. Since G^{Ω} is doubly transitive, G^{Γ} is transitive and $G_{\Gamma}=1$. Let z be an involution of $Z(N^{\alpha}_{\beta})$. There exists an involution t such that $t\in z^G$ and $\alpha^t=\beta$. Since $G_{\alpha\beta}=N^{\alpha}_{\beta}$ and $F(N^{\alpha}_{\beta})=\{\alpha,\beta\}$ we have $G_{\{\alpha,\beta\}}=\langle t\rangle N^{\alpha}_{\beta}$. By Lemma 2.3, $|F(z^{\Gamma})|=|C_G(z)|\times|\langle t\rangle N^{\alpha}_{\beta}\cap z^G|/2|N^{\alpha}_{\beta}|=|F(z)|\times|C_{G_{\alpha}}(z)|\times|\langle t\rangle N^{\alpha}_{\beta}\cap z^G|/2|N^{\alpha}_{\beta}|=|F(z)|\times|\langle t\rangle N^{\alpha}_{\beta}\cap z^G|/2$. As $|F(z^{\Gamma})|=|F(z)|(|F(z)|-1)/2+(|\Omega|-|F(z)|)/2$, $|\langle t\rangle N^{\alpha}_{\beta}\cap z^G|=|F(z)|+|\Omega|/|F(z)|-2$. In particular $|F(z)|\,|\,|\Omega|$. Since $|F(z)|=(q+1)/2+2=(q+5)/2$ and $|\Omega|=1+q(q-1)/2=(q^2-q+2)/2$, we have $q=11$ and $|\langle t\rangle N^{\alpha}_{\beta}\cap z^G|=13$. Moreover $|\Omega|=56$, $|G_{\alpha}|=|PSL(2,11)|=2^2\cdot 3\cdot 5\cdot 11$ and $|G|=2^5\cdot 3\cdot 5\cdot 7\cdot 11$.

We now argue that $\langle t\rangle N^{\alpha}_{\beta}\simeq D_{24}$. Let R be the Sylow 3-subgroup of N^{α}_{β}. If t centralizes R, R acts on $F(t)$ and so $F(R)\subseteq F(t)$ as $|F(t)|=8$ and $|F(R)|=2$. Hence $\alpha^t=\alpha$, contrary to the choice of t. Therefore t inverts R and $\langle t\rangle N^{\alpha}_{\beta}$ is isomorphic to $Z_2\times D_{12}$ or D_{24}. Suppose $\langle t\rangle N^{\alpha}_{\beta}\simeq Z_2\times D_{12}$. Then $\langle t\rangle N^{\alpha}_{\beta}$ contains fifteen involutions and so we can take $u\in I(\langle t\rangle N^{\alpha}_{\beta})$ satisfying $|F(u)|=0$ and $\langle t\rangle N^{\alpha}_{\beta}=\langle u\rangle\times N^{\alpha}_{\beta}$. As $|F(u)|=0$, $|F(u^{\Gamma})|=|\Omega|/2=28$. By Lemma 2.3, $28=|C_G(u)|\times|\langle u\rangle N^{\alpha}_{\beta}\cap u^G|/24$ and hence $|C_G(u)|=2^4\cdot 3\cdot 7$ or $2^5\cdot 3\cdot 7$. Since $\langle u\rangle N^{\alpha}_{\beta}=N_G(R)$, we have $|C_G(u): C_G(u)\cap N_G(R)|=2\cdot 7$ or $2^2\cdot 7$. By a Sylow's theorem, $|C_G(u): C_G(u)\cap N_G(R)|=2^2\cdot 7$, so that $|C_G(u)|=2^5\cdot 3\cdot 7$. Let Q be a Sylow 7-subgroup of $C_G(u)$. Then $|C_G(u)\cap N_G(Q)|=2^5\cdot 3\cdot 7$ or $2^2\cdot 3\cdot 7$ by a Sylow's theorem. Hence $2^2\cdot 3\cdot 7\,|\,|N_G(Q)|$. Since $\mathrm{Aut}(Z_7)\simeq Z_2\times Z_3$,

$5\not|\,|N_G(Q)|$ and $11\not|\,|N_G(Q)|$ by the similar argument as in the case $q=9$. Therefore $|G: N_G(Q)|=2^a\cdot 5\cdot 11$ for some a with $0\leq a\leq 3$. Hence $|G: N_G(Q)|\not\equiv 1 \pmod 7$, a contradiction. Thus $\langle t\rangle N_\beta^\alpha\simeq D_{24}$.

Let U be a Sylow 2-subgroup of N_β^α and set $L=N_G(U)$. It follows from (3.3) and Lemma 2.6 (iv) that $L\cap N^\alpha\simeq A_4$, $L^{F(U)}\simeq A_4$ and $|L|=2^4\cdot 3$. Let T, $\langle x\rangle$ be Sylow 2- and 3-subgroup of L, respectively. Obviously $L\trianglerighteq T$ and $C_T(x)=1$. On the other hand $T\trianglerighteq L\cap\langle t\rangle N_\beta^\alpha\simeq D_8$ and so $T'\simeq Z_2\times Z_2$ because $C_T(x)=1$. By Theorem 5.4.5 of [2], T is dihedral or semi-dihedral. Hence $N_G(T)/C_G(T)$ ($\leq \mathrm{Aut}(T)$) is a 2-group, so that $C_T(x)=T$, a contradiction.

(3.13) (ii) *of* (3.11) *does not occur.*

Proof. Let G^Ω be a doubly transitive permutation group satisfying (ii) of (3.11). Let x be an involution in N_β^α with $x\notin Y$. Then $F(x^{F(Y)})=F(\langle x\rangle Y)=F(N_\beta^\alpha)=\{\alpha, \beta\}$ by (i) of (3.1) and (3.9). Since $|F(Y)|=1+(q-\varepsilon)/|N_\beta^\alpha|=1+k\geq 4$, $x^{F(Y)}$ is an involution. By Lemma 2.5, $1+k=2^2$ and so $k=3$. By (3.11), $q-\varepsilon+8\,|\,2((8-\varepsilon)(4-\varepsilon)\times 3+1)(8-\varepsilon)(4-\varepsilon)$. Hence $q+7\,|\,2^7\cdot 3\cdot 7$ if $\varepsilon=1$ and $q+9\,|\,2^4\cdot 3^2\cdot 5\cdot 17$ if $\varepsilon=-1$. Since $k=3\,|\,q-\varepsilon$, $3\not|\,q-\varepsilon+8$. From this $q+7\,|\,2^7\cdot 7$ if $\varepsilon=1$ and $q+9\,|\,2^4\cdot 5\cdot 17$ if $\varepsilon=-1$. Therefore $q=5^2, 7^2, 11^2$, 59 or 71.

Let p_1 be an odd prime such that $p_1\,|\,|\Omega|$ and $p_1\not|\,|G_\alpha|$ and let P be a Sylow p_1-subgroup of G. Clearly P is semi-regular on Ω and so any element in $C_{G_\alpha}(P)$ has at least p_1 fixed points. If x is an element of N_β^α and its order is at least three, $|F(x)|=|F(Y)|=4$ by Lemma 2.8. Since $|N_\beta^\alpha|=(q-\varepsilon)/3$, we have $|\Omega|=1+|N^\alpha: N_\beta^\alpha|=1+3q(q+\varepsilon)/2$.

If $q=5^2$, then $|\Omega|=2^4\cdot 61$ and $|G_\alpha|=2^{4+i}\cdot 3\cdot 5^2\cdot 13$ $(0\leq i\leq 2)$. Let P be a Sylow 61-subgroup of G. Then $P\simeq Z_{61}$. As mentioned above, 5, $13\not|\,|C_G(P)|$ and so 5^2, $13\not|\,|N_G(P)|$. Hence $|G: N_G(P)|=2^a\cdot 3^b\cdot 5^{c+1}\cdot 13$, where $0\leq a\leq 10$ and $0\leq b, c\leq 1$. But we can easily verify $|G: N_G(P)|\not\equiv 1 \pmod{61}$, contrary to a Sylow's theorem.

If $q=7^2$, then $|\Omega|=2^2\cdot 919$ and $|G_\alpha|=2^{4+i}\cdot 3\cdot 5^2\cdot 7^2$ $(0\leq i\leq 2)$. Let P be a Sylow 919–subgroup of G. By the similar argument as above, we obtain 5, $7\not|\,|N_G(P)|$ and so $|G: N_G(P)|=2^a\cdot 3^b\cdot 5^2\cdot 7^2\equiv 2^a\cdot 306$ or $-2^a \pmod{919}$, where $0\leq a\leq 8$ and $0\leq b\leq 1$. Hence $|G: N_G(P)|\not\equiv 1$, a contradiction.

If $q=11^2$, then $|\Omega|=2^7\cdot 173$ $|G_\alpha|=2^{3+i}\cdot 3\cdot 5\cdot 11^2\cdot 61$ $(0\leq i\leq 2)$. Let P be a Sylow 173–subgroup of G. Similarly we have 3, 5, 11, $61\not|\,|N_G(P)|$ and so $|G: N_G(P)|=2^a\cdot 3\cdot 5\cdot 11^2\cdot 61\equiv -5\cdot 2^a \pmod{173}$, where $0\leq a\leq 12$. Hence $|G:N_G(P)|\not\equiv 1$, a contradiction.

If $q=59$, then $|\Omega|=2\cdot 17\cdot 151$ and $|G_\alpha|=2^{2+i}\cdot 3\cdot 5\cdot 29\cdot 59$ $(0\leq i\leq 1)$. Let P be a Sylow 17–subgroup of G. Similarly we have $3,5,29,59\not|\,|N_G(P)|$ and so $|G: N_G(P)|=2^a\cdot 3\cdot 5\cdot 29\cdot 59\cdot 151^b\equiv 10\cdot 2^a$ or $12\cdot 2^a \pmod{17}$, where $0\leq a\leq 4$ and $0\leq b\leq 1$. From this, we have a contradiction.

If $q=71$, then $|\Omega|=2^5\cdot 233$ and $|G_\alpha|=2^{3+i}\cdot 3^2\cdot 5\cdot 7\cdot 71$ $(0\leq i\leq 1)$. Let P be a Sylow 233–subgroup of G. Since $3,5,7,71\nmid |N_G(P)|$, $|G: N_G(P)|=2^a\cdot 3^2\cdot 5\cdot 7\cdot 71\equiv -3\cdot 2^a \pmod{233}$, where $0\leq a\leq 9$. Similarly we get a contradiction.

We now consider the case $|Y|<3$. By (ii) of (3.5), $N^{\alpha}_{\beta}\simeq Z_2\times Z_2$ or $N^{\alpha}_{\beta}\simeq D_8$ and $N^{\alpha}\cap N^{\beta}\leq Z_2\times Z_2$.

(3.14) *The case that $N^{\alpha}_{\beta}\simeq Z_2\times Z_2$ does not occur.*

Proof. Set $\Delta=F(N^{\alpha}_{\beta})$. Then $|\Delta|=3r+1$ and $\Delta=F(N^{\alpha}_{\beta}N^{\beta}_{\alpha})$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^{\alpha}|_2=4$, we have $q=p^n\equiv 3, 5 \pmod 8$ and so n is odd. Hence $|G_{\alpha}/N^{\alpha}|_2\leq 2$ and $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq N^{\alpha}_{\beta}N^{\beta}/N^{\beta}\simeq 1$ or Z_2 by (3.2). Suppose $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq Z_2$. Then $N^{\alpha}_{\beta}N^{\beta}_{\alpha}$ is a Sylow 2-subgroup of G_{α}, hence $N_G(N^{\alpha}_{\beta}N^{\beta}_{\alpha})^{\Delta}$ is doubly transitive by a Witt's theorem. Since $N^{\alpha}_{\beta}N^{\beta}_{\alpha}\simeq D_8$ and $|\Delta|$ is even, $C_G(N^{\alpha}_{\beta}N^{\beta}_{\alpha})^{\Delta}$ is also doubly transitive. Let g be an element of $C_G(N^{\alpha}_{\beta}N^{\beta}_{\alpha})$ such that $\alpha^g=\beta$ and $\beta^g=\alpha$. Then $N^{\alpha}_{\beta}=g^{-g}N^{\alpha}_{\beta}g=N^{\beta}_{\alpha}$ and hence $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$, a contradiction. Thus $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}\simeq Z_2\times Z_2$.

Let z be an involution in N^{α}_{β} and $t\in z^G$ an involution such that $\alpha^t=\beta$. Set $\Gamma=\{\{\gamma, \delta\}\mid \gamma, \delta\in\Omega, \gamma\neq\delta\}$. We consider the action of the element z on Γ. By the similar argument as in the proof of (3.12), $|F(z)|(|F(z)|-1)/2+(|\Omega|-|F(z)|)/2=|F(z^{\Gamma})|=|C_G(z)|\,|z^G\cap\langle t\rangle G_{\alpha\beta}|/|\langle t\rangle G_{\alpha\beta}|$. Since $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$, by Lemma 2.6 (i), $z^G\cap G_{\alpha}=z^{G_{\alpha}}$ and so $|C_G(z)|=|F(z)|\times|C_{G_{\alpha}}(z)|$. Hence $|G_{\alpha\beta}|(|F(z)|(|F(z)|-1)+|\Omega|-|F(z)|)=|F(z)|\,|C_{G_{\alpha}}(z)|\,|z^G\cap\langle t\rangle G_{\alpha\beta}|$, so that $|G_{\alpha\beta}|\,|\Omega|\equiv 0 \pmod{|F(z)|}$. Since $|G_{\alpha\beta}/N^{\alpha}_{\beta}|=|G_{\alpha\beta}N^{\alpha}/N^{\beta}|\,|2n$, we have $|G_{\alpha\beta}|\,|8n$. Clearly $|\Omega|=1+q(q-\varepsilon)(q+\varepsilon)r/8$ and by Lemma 2.8 (i), $|F(z)|=1+3(q-\varepsilon)r/4$. Hence $1+3(q-\varepsilon)r/4\,|\,8n(1+q(q-\varepsilon)(q+\varepsilon)r/8)$. Put $n=rs$. Then $3qr-3\varepsilon r+4\,|\,(4rs(8+q(q-\varepsilon)(q+\varepsilon)r)3^3r=864\, r^2s+4s(3pq)(3pq-3\varepsilon r)(3qr+3\varepsilon r)$. Hence $3qr-3\varepsilon r+4\,|\,864r^2s+4s(3\varepsilon r-4)(3\varepsilon r-4-3\varepsilon r)(3\varepsilon r-4+3\varepsilon r)=8634r^2s-32s(3\varepsilon r-4)(3\varepsilon r-2)$. $(*)$

We argue that $r=1$. Suppose false. Then $32s(3\varepsilon r-4)(3\varepsilon r-2)>0$ and so $3r(q-\varepsilon)<864r^2s$. Therefore $288n+\varepsilon>q=p^n\geq 3^n$ and so $288n>3^n$. Hence $(n, r, p, \varepsilon)=(5, 5, 3, -1)$, $(3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy $(*)$. Thus $r=1$.

Hence $3q-3\varepsilon+4\,|\,64(5+9\varepsilon)n$ and $|F(z)|=1+3(q-\varepsilon)/4$, $|\Omega|=1+q(q-\varepsilon)(q+\varepsilon)/8$. If $\varepsilon=-1$, then $3\cdot 3^n<3q+7\,|\,256n$. Hence $n=1$ or $(n,p)=(5,3), (3,3)$. Since $3\cdot 3^5+7\nmid 256\cdot 5$ and $3\cdot 3^3+7\,|\,256\cdot 3$, $n=1$ and $3q+7\,|\,256$. From this, $q=19$ or 83. If $\varepsilon=1$, then $3\cdot 5^n<3q+1\,|\,896n$ and so $n=1$ or $(n,p)=(3,5)$. Since $3\cdot 5^3+1\nmid 896\cdot 3$, we have $n=1$ and $3q+1\,|\,896$. From this, $q=5$, 37 or 149. As $PSL(2,5)\simeq PSL(2,4)$, $q\neq 5$ by [4]. Thus $q=19$, 37, 83 or 149.

Set $m=|z^G\cap\langle t\rangle G_{\alpha\beta}|$. As we mentioned above, $|G_{\alpha\beta}|(|G(z)|(|F(z)|-1)+|\Omega|-|F(z)|)=|F(z)|\,|C_{G_{\alpha}}(z)|m$. Since $|G_{\alpha}/N^{\alpha}|=1$ or 2, $|C_{G_{\alpha}}(z)|/|G_{\alpha\beta}|=(q-\varepsilon)/4$. Therefore $m=(2q^2+(2\varepsilon+9)q-9\varepsilon)/(3q-3\varepsilon+4)$. It follows that $(q,m)=(19, 27/2)$, $(37, 28)$, $(83, 449/8)$ or $(149, 411/4)$. Since m is an integer, we have $(q,m)=(37, 28)$. But $m\leq|\langle t\rangle G_{\alpha\beta}|\leq 16$, a contradiction. Thus (3.14) holds.

(3.15) *The case that* $N^{\alpha}_{\beta}\simeq D_8$ *and* $N^{\alpha}\cap N^{\beta}\leq Z_2\times Z_2$ *does not occur.*

Proof. Let Δ, L and K be as defined in (3.6). By (3.6), there exists an element x in L_{α} such that its order is odd and $\langle x^{\Delta}\rangle$ is regular on $\Delta-\{\alpha\}$. Since $(L_{\alpha})'\leq N^{\alpha}_{\beta}$ by (3.6) and $N^{\alpha}_{\beta}\simeq D_8$, x stabilizes a normal series $N^{\beta}_{\alpha}N^{\alpha}_{\beta}\trianglerighteq N^{\alpha}_{\beta}\trianglerighteq 1$. Hence x centralizes $N^{\beta}_{\alpha}N^{\alpha}_{\beta}$ by Theorem 5.3.2 of [2] and so $x^{-1}N^{\beta}_{\alpha}x=N^{\beta}_{\alpha}$. Put $\gamma=\beta^{y}$. If $r\neq 1$, then $\beta\neq\gamma$, so that $N^{\gamma}_{\alpha}=N^{\beta}_{\alpha}$. From this, $N^{\gamma}_{\beta}=N^{\beta}_{\gamma}$. By the doubly transitivity of G, $N^{\alpha}_{\beta}=N^{\beta}_{\alpha}$, hence $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$, a contradiction. Therefore $r=1$ and $\Delta=\{\alpha,\beta\}$.

Set $\langle z\rangle=Z(N^{\alpha}_{\beta})$, $\Delta_1=\alpha^{C_G(z)}$ and let $\{\Delta_1,\Delta_2\cdots\Delta_k\}$ be the set of $C_G(z)$-orbits on $F(z)$. Since $L\trianglerighteq N^{\alpha}\cap N^{\beta}$ and by (3.2), $N^{\alpha}\cap N^{\beta}\neq 1$, z is contained in $N^{\alpha}\cap N^{\beta}$. Hence, by Lemma 2.1, $\beta\in\Delta_1$ and k is at least two. By Lemma 2.8, $|F(z)|=1+(q-\varepsilon)5/|N^{\alpha}_{\beta}|=1+5(q-\varepsilon)/8$. Clearly $|C_{N^{\alpha}}(z)\colon N^{\alpha}_{\beta}|=(q-\varepsilon)/8$ and so $|\Delta_1|\geq 1+(q-\varepsilon)/8$. If $\gamma\in F(z)-\Delta_1$, then $C_{N^{\alpha}_{\gamma}}(z)\simeq Z_2\times Z_2$, for otherwise $\langle z\rangle=Z(N^{\alpha}_{\gamma})\leq N^{\alpha}\cap N^{\gamma}$ and by Lemma 2.1 $\gamma\in\Delta_1$, a contradiction. Hence one of the following holds.

(i) $k=3$ and $|\Delta_1|=1+(q-\varepsilon)/8$, $|\Delta_2|=|\Delta_3|=(q-\varepsilon)/4$.

(ii) $k=2$ and $|\Delta_1|=1+(q-\varepsilon)/8$, $|\Delta_2|=(q-\varepsilon)/2$.

(iii) $k=2$ and $|\Delta_1|=1+3(q-\varepsilon)/8$, $|\Delta_2|=(q-\varepsilon)/4$.

Let $\gamma\in F(z)-\Delta_1$. Then, $z\in G_{\gamma}-N^{\gamma}$ and so $C_{N^{\gamma}}(z)\simeq D_{q+\varepsilon}$ or $PGL(2,\sqrt{q})$ by Lemma 2.6 (vii), (viii), (ix). If $C_{N^{\gamma}}(z)\simeq D_{q+\varepsilon}$, then $(q+\varepsilon)/2\,|\,|\Delta_1|$ and so $q=7$ and (iii) occurs. But $(q+\varepsilon)/2=3\,|\,|\Delta_2|-1-1=1$, a contradiction. If $C_{N^{\gamma}}(z)\simeq PGL(2,\sqrt{q})$, then (i) does not occur because $\sqrt{q}\nmid q-\varepsilon$. Hence $\sqrt{q}\,|\,|\Delta_1|$ and $\sqrt{q}\,|\,|\Delta_2|-1$. From this, $q=25$ and (iii) occurs. In this case, we have $|\Delta_1|=10$, so that an element of $C_{N^{\gamma}}(z)$ of order 3 is contained in N^{γ}_{δ} for some $\delta\in\Delta_1$, contrary to $N^{\gamma}_{\delta}\simeq N^{\alpha}_{\beta}\simeq D_8$.

4. Case (II)

In this section we assume that $N^{\alpha}_{\beta}\simeq PGL(2,p^{m})$, where $n=2mk$ and k is odd. Since n is even, $q=p^{n}\equiv 1 \pmod 4$. We set $p^{m}\equiv\varepsilon\in\{\pm 1\} \pmod 4$. In section 7 we shall consider the case that $N^{\alpha}_{\beta}\simeq S_4$. Therefore we assume $(p,m)\neq(3,1)$ in this section.

(4.1) *The following hold.*

(i) $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq 1$ *or* Z_2 *and* $N^{\alpha}\cap N^{\beta}\geq(N^{\alpha}_{\beta})'\simeq PSL(2,p^{m})$.

(ii) *If* $(p,m)\neq(5,1)$, *there exists a cyclic subgroup* Y *of* $(N^{\alpha}_{\beta})'$ *such that* $N_{N^{\alpha}}(Y)\simeq D_{q-\varepsilon}$ *and* $N_G(Y)^{F(Y)}$ *is doubly transitive.*

Proof. As $N^{\alpha}_{\beta}\trianglerighteq N^{\alpha}\cap N^{\beta}$, either $N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\leq Z_2$ or $N^{\alpha}\cap N^{\beta}=1$. If $N^{\alpha}\cap N^{\beta}=1$, by Lemma 2.2 and 2.6 (vi), $N^{\alpha}_{\beta}\simeq N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}\simeq N^{\alpha}_{\beta}N^{\beta}/N^{\beta}\simeq Z_2\times Z_n$, a contradiction. Therefore $N_\beta^\alpha/N^\alpha\cap N^\beta\simeq 1$ or Z_2 and $N^\alpha\cap N^\beta\geq(N_\beta^\alpha)'\simeq PSL(2,p^m)$.

Now we assume that $(p,m)\neq(5,1)$ and let z be an involution in $(N_\beta^\alpha)'$. Then $C_{N_\beta^\alpha}(z)\simeq D_{2(p^m-\varepsilon)}$ by Lemma 2.6 (vii). Suppose $C_{N_\beta^\alpha}(z)$ is not a 2-subgroup and put $Y=0(C_{N_\beta^\alpha}(z))$. Then, if $Y^g\leq G_{\alpha\beta}$ for some $g\in G$, we have $Y^g\leq N_\alpha^\gamma$ and $Y^g\leq N_\beta^\delta$, where $\gamma=\alpha^g$ and $\delta=\beta^g$. By (i) $Y^g\leq N^\alpha\cap N^\beta$ and so $Y^g=Y^h$ for some $h\in N^\alpha\cap N^\beta$. Thus $N_G(Y)^{F(Y)}$ is doubly transitive. Assume that $C_{N_\beta^\alpha}(z)$ is a 2-subgroup and set $C_{N_\beta^\alpha}(z)=\langle u, v\,|\,u^v=u^{-1}, v^2=1\rangle$. We may assume that $v\in(N_\beta^\alpha)'$ and $\langle u^2, v\rangle$ is a Sylow 2-subgroup of $(N_\beta^\alpha)'$. Since $p^m\neq 3,5$, the order of u^2 is at least four. On the other hand there is no element of order $|u^2|$ in $\langle u, v\rangle-\langle u^2, v\rangle$. Hence any element of order $|u^2|$ which is contained in N_β^α is necessarily an element of $N^\alpha\cap N^\beta$. By the similar argument as above, $N_G(Y)^{F(Y)}$ is doubly transitive.

(4.2) *Let notations be as in* (4.1). *Suppose* $(p,m)\neq(3,1)$, $(5,1)$ *and set* $\Delta=F(Y)$ *and* $X=N_G(Y)$. *Then* $|\Delta|=rs(p^m+\varepsilon)/2+1$, *where* $s=\sum_{i=0}^{k-1}p^{2mi}$, $C_G(N^\alpha)=1$ *and one of the following holds.*

(i) $X^\Delta\leq A\Gamma L(1,2^c)$ *for some integer c.*

(ii) $X^\Delta\simeq PSL(2,p_1)$ *or* $PGL(2,p_1)$, $r=1$, $k=1$ *and* $2p_1=p^m+\varepsilon$.

Proof. By Lemma 2.8 (ii), $|\Delta|=1+|N^\alpha\cap X|r/|N_\beta^\alpha\cap X|=1+(p^{2mk}-1)r/2(p^m-\varepsilon)=rs(p^m+\varepsilon)/2+1$. By (4.1) and Lemma 2.9, we have (i), (ii) or $X^\Delta=R(3)$.

Assume that $X^\Delta=R(3)$. Then $rs(p^m+\varepsilon)/2+1=28$, hence $k=1$ and $r(p^m+\varepsilon)/2=27$. Since r is odd and $r\,|\,2m=n$, we have $r=m=1$ and $q=53^2$. But a Sylow 3-subgroup of X_α is cyclic becuase $N^\alpha\cap X\simeq D_{q-\varepsilon}$ and $X_\alpha/X\cap N^\alpha\simeq X_\alpha N^\alpha/N^\alpha\leq Z_2\times Z_2$, a contradiction. Thus (i) or (ii) holds.

(4.3) (i) *of* (4.2) *does not occur.*

Proof. Let notations be as in (4.2). Suppose $X^\Delta\leq A\Gamma L(1,2^c)$ and put $W=C_{N_\beta^\alpha}(Y)$. Then $Y\leq W\simeq Z_{p^m-\varepsilon}$. Since $C_{N^\alpha}(Y)$ is cyclic, W is a characteristic subgroup of $C_{N^\alpha}(Y)$ and so W is a normal subgroup of X_α. Hence $W\leq X_\Delta$ and $(X\cap N_\beta^\alpha)^\Delta\simeq 1$ or Z_2. By Lemmas 2.4 and 2.6, $F(X\cap N_\beta^\alpha)=1+|X\cap N_\beta^\alpha|\,|N_\beta^\alpha\colon X\cap N_\beta^\alpha|\times r/|N_\beta^\alpha|=1+r$. Since $1+r<|\Delta|$, $(X\cap N_\beta^\alpha)^\Delta\simeq Z_2$ and hence $(1+r)^2=rs(p^m+\varepsilon)/2+1$ by Lemma 2.5. From this, $r=s(p^m+\varepsilon)/2-2\,|\,mk$ and so $p^{2m(k-1)}+mk\leq 2$. Hence $m=k=r=1$ and $q=7^2$.

Let R be a Sylow 3-subgroup of N_β^α. Since $N_\beta^\alpha\simeq PGL(2,7)$, we have $R\simeq Z_3$. By Lemmas 2.4 and 2.6, $|F(R)|=1+(7^2-1)|N_\beta^\alpha\colon N_{N_\beta^\alpha}(R)|/|N_\beta^\alpha|=4$. Hence $N_G(R)^{F(R)}\simeq A_4$ or S_4. But is a Sylow 3-subgroup of $N_{G_\alpha}(R)$ because $N^\alpha\simeq PSL(2,7^2)$, contrary to $N_{G_\alpha}(R)^{F(R)}\simeq A_3$ or S_3.

(4.4) (ii) *of* (4.2) *does not occur.*

Proof. Let notations be as in (4.2). Suppose $X^{\Delta} \trianglerighteq PSL(2,p_1)$. By the similar argument as in (4.3), $C_{N_{\beta}^{\alpha}}(Y) \leq X_{\Delta}$ and so $C_{N^{\alpha}}(Y)^{\Delta} \simeq Z_{p_1}$, and $N_{N^{\alpha}}(Y)^{\Delta} \simeq D_{2p_1}$. Hence $|(X_{\alpha})^{\Delta}| \mid 2p_1 \cdot 2n$. Since $X^{\Delta} \trianglerighteq PSL(2,p_1)$, $p_1(p_1-1)/2 \mid |(X_{\alpha})^{\Delta}|$, hence $p_1-1 \mid 8n$. As $k=1$ and $2p_1=p^m+\varepsilon$, we have $p^m+\varepsilon-2 \mid 32m$. From this, $(p,m,p_1)=(11,1,5)$, $(3,2,5)$ or $(3,3,13)$.

Let R be a cyclic subgroup of N_{β}^{α} such that $R \simeq Z_{(p^m+\varepsilon)/2}$. By Lemma 2.6, $N_G(R)^{F(R)}$ is doubly transitive and by Lemma 2.8 (ii), $|F(R)|=1+|N_{N^{\alpha}}(R)| / |N_{N_{\beta}^{\alpha}}(R)|=1+(p^{2m}-1)/2(p^m+\varepsilon)=(p^m-\varepsilon)/2+1$.

If $(p,m,p_1)=(11,1,5)$, $|F(R)|=7$ and so by [9] $|N_G(R)^{F(R)}|=42$ and $N_{G_{\alpha}}(R)^{F(R)} \simeq Z_6$. Since $|N_{N^{\alpha}}(R): N_{N_{\beta}^{\alpha}}(R)|=6$, $N_{N^{\alpha}}(R)^{F(R)}=N_{G_{\alpha}}(R)^{F(R)}$. Hence $N_{N^{\alpha}}(R)/K \simeq Z_6$, where $K=(N_{N^{\alpha}}(R))_{F(R)}$. But $N_{N^{\alpha}}(R)/(N_{N^{\alpha}}(R))' \simeq Z_2 \times Z_2$, a contradiction.

If $(p,m,p_1)=(3,2,5)$, $|F(R)|=5$ and so by [9], $|N_G(R)^{F(R)}|=20$ and $N_{G_{\alpha}}(R)^{F(R)} \simeq Z_4$. Since $|N_{N^{\alpha}}(R): N_{N_{\beta}^{\alpha}}(R)|=4$, $N_{N^{\alpha}}(R)^{\Delta} \simeq Z_4$, contrary to $N_{N^{\alpha}}(R) /(N_{N_{\alpha}}(R))' \simeq Z_2 \times Z_2$.

If $(p,m,p_1)=(3,3,13)$, $|F(R)|=15$. By [9], $N_{G_{\alpha}}(R)^{F(R)}$ is not solvable, a contradiction.

(4.5) $p^m \neq 5$.

Proof. Assume that $p^m=5$. Then $n=2k$ with k odd and $N_{\beta}^{\alpha} \simeq PGL(2,5) \simeq S_5$. First we argue that $N_{\beta}^{\alpha}=N^{\alpha} \cap N^{\beta}$. Suppose false. Then $C_G(N^{\alpha})=1$ by Lemma 2.2, and $N_{\beta}^{\alpha}/N^{\alpha} \cap N^{\beta} \simeq Z_2$ by (4.1). Since $N_{\alpha}^{\beta}N_{\beta}^{\alpha}/N_{\beta}^{\alpha} \simeq N_{\alpha}^{\beta}/N^{\alpha} \cap N^{\beta} \simeq Z_2$ and the outer automorphism group of S_5 is trivial, we have $Z(N_{\beta}^{\alpha}N_{\alpha}^{\beta}) \simeq Z_2$. Let w_1 be the involution of $Z(N_{\beta}^{\alpha}N_{\alpha}^{\beta})$ and let $w \in I(N_{\alpha}^{\beta})-I(N^{\alpha})$. Since $C_{N^{\alpha}}(w_1) \geq N_{\beta}^{\alpha}$, by Lemma 2.6 (viii) and (ix), w acts on N^{α} as a field automorphism of order 2 and $C_{N^{\alpha}}(w) \simeq PGL(2,5^k)$. By Lemma 2.8 $|F(w)|=1+r(q-\varepsilon)|I(N_{\beta}^{\alpha})| / |N_{\beta}^{\alpha}|=1+5r(5^{2k}-1)/24$. Let P be a Sylow 5-subgroup of $C_{N^{\alpha}}(w)$. Then $|P|=5^k$ and $|\gamma^P|=5^{k-1}$ or 5^k for each $\gamma \in \Omega-\{\alpha\}$. Since P acts on $F(w)-\{\alpha\}$, we have $5^{k-1} \mid 5r(5^{2k}-1)/24$, so that $k=1$ and $|F(w)|=6$ as $r \mid k$. Hence $C_{N^{\alpha}}(w)^{F(w)} \simeq S_5$ and so $C_G(w)^{F(w)} \simeq S_6$. But clearly $w \in N^{\alpha} \cap N^{\beta}$ by Lemma 2.1, a contradiction. Thus $N_{\beta}^{\alpha}=N^{\alpha} \cap N^{\beta}$.

Let V be a cyclic subgroup of N_{β}^{α} of order 4. Since $N_{\beta}^{\alpha}=N^{\beta} \cap N^{\beta} \simeq S_5$, $N_G(V)^{F(V)}$ is doubly transitive and by Lemma 2.8, $|F(V)|=1+|N_{N^{\alpha}}(V)|r/ |N_{N_{\beta}^{\alpha}}(V)|=1+(5^{2k}-1)r/8=3rs+1$, where $s=\sum_{i=0}^{k-1}25^i$. By Lemma 2.9, $C_G(N^{\alpha})=1$ and (a) $N_G(V)^{F(V)} \leq A\Gamma L(1,2^c)$ or (b) $N_G(V)^{F(V)}=R(3)$.

Put $P=N_{N_{\beta}^{\alpha}}(V)$. Then $P \simeq D_8$, $|F(P)|=1+|N_{N^{\alpha}}(P)|\,|N_{\beta}^{\alpha}: N_{N_{\beta}^{\alpha}}(P)|r/|N_{\beta}^{\alpha}| =r+1$ and $P^{F(V)} \simeq Z_2$. If (b) occurs, $k=1$ and $r=9$, hence $|F(P)|=10$, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r+1)^2=3rs+1$ and so $r=3s-2 \mid k$. Hence $k=r=1$ and $G_{\alpha}/N^{\alpha} \leq Z_2 \times Z_2$. Let z be an involution in N_{β}^{α}. Then $|F(z)|=1+24 \cdot 25/120=6$

by Lemma 2.8 and $|\Omega|=1+|N^\alpha: N^\alpha_\beta|=66$ as $r=1$. By the similar argument as in the proof of (3.12), $|F(z)|(|F(z)|-1)/2+(|\Omega|-|F(z)|)/2=|C_G(z)||z^G\cap \langle t\rangle G_{\alpha\beta}|/|\langle t\rangle G_{\alpha\beta}|$, where t is an involution such that $\alpha^t=\beta$. Hence $|z^G\cap\langle t\rangle G_{\alpha\beta}|=15|G_{\alpha\beta}|/|C_{G_\alpha}(z)|$. Set $H=\langle t\rangle G_{\alpha\beta}$ and let R be a Sylow 3-subgroup of N^α_β. By Lemma 2.8, $|F(R)|=1+24\cdot 10/120=3$. Set $F(R)=\{\alpha,\beta,\gamma\}$. On the other hand, as $N^\alpha_\beta\simeq S_5$ and $\mathrm{Out}(S_5)=1$, we have $H=Z(H)\times N^\alpha_\beta$ and $|Z(H)|=2, 4$ or $H=C_H(N^\alpha_\beta)\times N^\alpha_\beta$ and $C_H(N^\alpha_\beta)\simeq D_8$. In the latter case $G_{\alpha\beta}=Z(G_{\alpha\beta})\times N^\alpha_\beta$ and $Z(G_{\alpha\beta})\simeq Z_2\times Z_2$, contrary to Lemma 2.6 (ix). In the former case, we have $|Z(H)|=2$. For otherwise $Z(H)\le G_\gamma$ and $Z(H)\cap z^G\neq\phi$ and so letting $u\in Z(H)\cap z^G$, we have $|R|=3\,|\,|F(u)|-1=5$, a contradiction. Therefore $Z(H)\simeq Z_2$ and so $|z^G\cap H|\le 25+25=50$, while $|z^G\cap H|=15|G_{\alpha\beta}|/|C_{G_\alpha}(z)|=15\cdot 120/24=75$, a contradiction.

5. Case (III)

In this section we assume that $N^\alpha_\beta\simeq PSL(2,p^m)$, where $n=mk$ and k is odd. Set $p^m\equiv\varepsilon\in\{\pm 1\}$ (mod 4). Then $q\equiv\varepsilon$ (mod 4) as k is odd. In section 6 we shall consider the case that $N^\alpha_\beta\simeq A_4$, so we assume $(p,m)\neq(3,1)$ in this section. From this N^α_β is a nonabelian simple group and so $N^\alpha_\beta=N^\alpha\cap N^\beta$ or $N^\alpha\cap N^\beta=1$. If $N^\alpha\cap N^\beta=1$, then $C_G(N^\alpha)=1$ by Lemma 2.2 and $N^\alpha_\beta\simeq N^\alpha_\beta/N^\alpha\cap N^\beta\simeq N^\alpha_\beta N^\beta/N^\beta\simeq Z_2\times Z_n$, a contradiction. Hence $N^\alpha_\beta=N^\alpha\cap N^\beta$.

Let z be an involution of N^α_β. Suppose $z^g\in G_{\alpha\beta}$ for some $g\in G$ and set $\gamma=\alpha^g$, $\delta=\beta^g$. Then $z^g\in N^\gamma_\delta\cap G_{\alpha\beta}\le N^\gamma_\alpha\cap N^\delta_\beta\le N^\alpha\cap N^\beta$ and so $z^g\in z^{N^\alpha_\beta}$. Hence $C_G(z)^{F(z)}$ is doubly transitive and by Lemma 2.8 (i), $|F(z)|=(q-\varepsilon)r/(p^m-\varepsilon)+1$. In particular $|F(z)|>3r+1$ as $(p^n-\varepsilon)/(p^m-\varepsilon)\ge p^{2m}+\varepsilon p^m+1>3$.

By Lemma 2.9, $C_G(N^\alpha)=1$ and one of the following holds.

(a) $C_G(z)^{F(z)}\le A\Gamma L(1,2^c)$.

(b) $C_G(z)^{F(z)}\trianglerighteq PSL(2,p_1)$ $(p_1\ge 5)$, $r=1$ and $|C_{N^\alpha}(z): C_{N^\alpha_\beta}(z)|=p_1$.

(c) $C_G(z)^{F(z)}=R(3)$.

Let Y be a cyclic subgroup of $C_{N^\alpha_\beta}(z)\simeq D_{p^m-\varepsilon}$ of index 2. Since $C_{G_\alpha}(z)\trianglerighteq Y$, $z\in Y$ and $C_G(z)^{F(z)}$ is doubly transitive, we have $F(Y)=F(z)$. By the similar argument as in (3.1), $N^\alpha\cap N(C_{N^\alpha_\beta}(z))=C_{N^\alpha_\beta}(z)$ or $N^\alpha\cap N(C_{N^\alpha_\beta}(z))\simeq A_4$. Hence by Lemmas 2.3 and 2.4 $|F(C_{N^\alpha_\beta}(z))|=1+|C_{N^\alpha_\beta}(z)|\,|N^\alpha_\beta: C_{N^\alpha_\beta}(z)|r/|N^\alpha_\beta|$ or $1+|A_4|\,|N^\alpha_\beta: C_{N^\alpha_\beta}(z)|r/|N^\alpha_\beta|$. Therefore $|F(C_{N^\alpha_\beta}(z))|=r+1$ or $3r+1$. From this $C_{N^\alpha_\beta}(z)^{F(z)}\simeq Z_2$.

In the case (a), $(r+1)^2=1+(p^n-\varepsilon)r/(p^m-\varepsilon)$ by Lemma 2.5 and hence $r=(p^n-\varepsilon)/(p^m-\varepsilon)-2\,|\,mk$. Since $(p^n-\varepsilon)/(p^m-\varepsilon)\ge((p^m)^k+1)/(p^m+1)=\sum_{i=0}^{k-1}(-p^m)^i$ and $k\ge 3$, we have $p^{m(k-1)}(p^{2m}-p^m+1)\le mk$, hence $((p^m)^{k-3}/k)(m/(p^{2m}-p^m+1))<1$. Thus $k=3$, $m=1$ and $p=3$, cotrary to $(p,m)\neq(3,1)$.

In the case (b), $r=1$, $p_1=(p^n-\varepsilon)/(p^m-\varepsilon)$, $p_1(p_1-1)/2\,|\,s$ and $s\,|\,4mkp_1$, where s is the order of $C_{G_\alpha}(z)^{F(z)}$. Hence $p_1-1\,|\,8mk$. Since $p_1-1=(p^n-\varepsilon)/(p^m-\varepsilon)-1$

$\geq(p^n+1)/(p^m+1)-1=\sum_{i=0}^{k-1}(-p^m)^i\geq p^{m(k-2)}(p^m-1)$, we have $p^{m(k-2)}/2k\leq 4m/(p^m-1)\leq 1$ because $p^m\neq 3$. Hence $k=3$ and $p^m=5$, so that $p_1-1=30\nmid 8mk=24$, a contradiction.

In the case (c), $r+1=4$ and $1+(p^n-\varepsilon)r/(p^m-\varepsilon)=28$ and so $r=3$ and $(p^n-\varepsilon)/(p^m-\varepsilon)=9$. Hence $9\geq(p^{mk}+1)/(p^m+1)\geq p^{2m}-p^m+1$, so that $p^m=3$, a contradiction.

6. Case (IV)

In this section we assume that $N_\beta^\alpha\simeq A_4$ and $q=3,5 \pmod 8$. If $N^\alpha\cap N^\beta=1$, by Lemma 2.2, $C_G(N^\alpha)=1$ and so $N_\beta^\alpha/N^\alpha\cap N^\beta\simeq N_\beta^\alpha N^\beta/N^\beta\leq Z_2\times Z_n$. Hence $N_\beta^\alpha/N^\alpha\cap N^\beta\simeq 1$ or Z_3, so that $z^G\cap G_{\alpha\beta}=z^G\cap N_\beta^\alpha=z^{N_\beta^\alpha}$ for an involution $z\in N_\beta^\alpha$. Therefore $C_G(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha)=1$ and one of the following holds.

(a) $C_G(z)^{F(z)}\leq A\Gamma L(1,2^c)$ for some interger $c\geq 1$.

(b) $C_G(z)^{F(z)}\trianglerighteq PSL(2,p_1)$ ($p_1\geq 5$), $r=1$ and $|C_{N^\alpha}(z): C_{N_\beta^\alpha}(z)|=p_1$.

(c) $C_G(z)^{F(z)}=R(3)$.

Let T be a Sylow 2-subgroup of N_β^α. Then $z\in T$ and by Lemmas 2.3 and 2.4, $|F(T)|=1+|N_{N^\alpha}(T)|r/|N_\beta^\alpha|=r+1$. By Lemma 2.8 (i), $|F(z)|=(q-\varepsilon)r/4+1$. Hence $T^{F(z)}\simeq Z_2$ if $q\neq 5$. If $q=5$, as $PSL(2,5)\simeq PSL(2,4)$, (ii) of our theorem holds by [4]. Therefore we may assume $q\neq 5$.

In the case (a), $(r+1)^2=1+(q-\varepsilon)r/4$ by Lemma 2.5. Hence $r=(q-\varepsilon-8)/4$ and $r|n$, so that $q=11$ or 13 and $r=1$. Let R be a Sylow 3-subgroup of $G_{\alpha\beta}$. Then $R\simeq Z_3$ and $R\leq N_\beta^\alpha$ because $G_{\alpha\beta}/N_\beta^\alpha\simeq G_{\alpha\beta}N^\alpha/N^\alpha\simeq 1$ or Z_2 and $N_\beta^\alpha\simeq A_4$. By Lemma 2.8 (ii), $|F(R)|=1+12/3=5$ and $N_G(R)^{F(R)}$ is doubly transitive. Since $N_{G_\alpha}(R)\simeq D_{12}$ or D_{24} and $|F(R)|=5$, we have $|N_G(R)|_5=5$. Let S be a Sylow 5-subgroup of $N_G(R)$. Then $[S,R]=1$ as $N_G(R)/C_G(R)\leq Z_2$. Since $5\nmid|G_{\alpha\beta}|$, $|F(S)|=0$ or 1. If $|F(S)|=1$, $F(S)\subseteq F(R)$ and so $5\,|\,|F(R)|-1=4$, a contradiction. Therefore S is semi-regular on Ω. But $|\Omega|=1+|N^\alpha: N_\beta^\alpha|=56$ or 92. This is a contradiction.

In the case (b), $p_1(p_1-1)/2\,|\,s$ and $s\,|\,2n(q-\varepsilon)/2=4np_1$, where s is the order of $C_{G_\alpha}(z)^{F(z)}$. Hence $p_1-1\,|\,8n$. Since $p_1=(q-\varepsilon)/4$, $p^n-\varepsilon-4\,|\,32n$ and so we have $q=11,13,19,27$ or 37. If $q\neq 27$, by Lemma 2.6, $C_{G_\alpha}(z)\simeq D_{q-\varepsilon}$ or $D_{2(q-\varepsilon)}$ and so $C_{G_{\alpha\beta}}(z)^{F(z)}\simeq Z_2$. Hence $(p_1-1)/2=2$. From this $q=19$. Let R be a Sylow 3-subgroup of $G_{\alpha\beta}$. By the simmilar argument as in the case (a), $N_G(R)^{F(R)}$ is doubly transitive and $|F(R)|=1+18/3=7$. Hence $7\,|\,|G|$. On the other hand $|G|=|\Omega|\,|G_\alpha|=(1+|N^\alpha: N_\beta^\alpha|)|G_\alpha|=(1+18\cdot 19\cdot 20/2\cdot 12)\cdot 2^i\cdot 18\cdot 19\cdot 20/2=2^{3+i}\cdot 3^2\cdot 5\cdot 11\cdot 13\cdot 19$ with $0\leq i\leq 1$, a contradiction. If $q=27$, then $|C_G(z)|_2=|F(z)|_2\times|C_{G_\alpha}(z)|_2=8\times|G_\alpha|_2$, while $|\Omega|=1+|N^\alpha: N_\beta^\alpha|=1+26\cdot 27\cdot 28/2\cdot 12=820=2^2\cdot 5\cdot 41$ and so $|G|_2=4|G_\alpha|_2$. Therefore $|C_G(z)|\nmid|G|$, a contradiction.

In the case (c), $r+1=4$ and $1+(q-\varepsilon)r/4=28$. Hence $r=3$ and $q=37$, contrary to $r|n$.

7. Case (V)

In this section we assume that $N_\beta^\alpha \simeq S_4$ and $q\equiv 7,9 \pmod{16}$. We note that $4\not| n$.

First we argue that $N_\beta^\alpha = N^\alpha \cap N^\beta$. Suppose $N_\beta^\alpha \neq N^\alpha \cap N^\beta$. Then $C_G(N^\alpha) = 1$ by Lemma 2.2. Since $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta/N^\alpha \leq Z_2 \times Z_n$, we have $N^\alpha \cap N^\beta \simeq A_4$ and $N_\beta^\alpha/N^\alpha \cap N^\beta \simeq Z_2$, so that $N_\alpha^\beta N_\beta^\alpha/N_\beta^\alpha \simeq N_\alpha^\beta/N^\alpha \cap N^\beta \simeq Z_2$. Hence as $\mathrm{Out}(S_4)=1$, $Z(N_\beta^\alpha N_\alpha^\beta) \simeq Z_2$. Set $\langle t_1\rangle = Z(N_\beta^\alpha N_\alpha^\beta)$ and let $t \in I(N_\alpha^\beta) - I(N^\alpha)$. Since $C_{N^\alpha}(t_1) \geq N_\beta^\alpha \simeq S_4$ and $\langle t\rangle N^\alpha = N_\alpha^\beta N^\alpha$, by Lemma 2.6, we have $C_{N^\alpha}(t) \simeq PGL(2, \sqrt{q})$ and $|F(t)| = 1 + 3(q-\varepsilon)r/8$ by Lemma 2.8.

Let P be a Sylow p-subgroup of $C_{N^\alpha}(t)$. Then $|P| = \sqrt{q}$. If $p \neq 3$, P acts semi-regularly on $F(t) - \{\alpha\}$ and so $\sqrt{q}\,|\,3(q-\varepsilon)r/8$. Therefore $\sqrt{q}\,|\,r$ and so $5^n \leq n^2$ as $p \geq 5$ and $r|n$. But obviously $5^n > n^2$ for any positive integer n. This is a contradiction. If $p=3$, $|P: P_\gamma| = \sqrt{q}/3$ or $\sqrt{q}$ for each $\gamma \in \Omega - \{\alpha\}$. Hence $\sqrt{q}/3\,|\,3(q-\varepsilon)r/8$ and so $q\,|\,81r^2$. In particular, $3^n = q\,|\,81n^2$. From this, $n \leq 7$. Since $q = 3^n \equiv 7$ or $9 \pmod{16}$, we have $q = 3^2$ or 3^6. If $q=3^2$, $|\Omega| = 1 + |N^\alpha: N_\beta^\alpha| = 1 + 8\cdot 9\cdot 10/2\cdot 24 = 16$, a contradiction by [9] If $q = 3^6$, $|F(t)| = 1 + 273r$ and $|F(t) - \{\alpha\}| \geq |C_{N^\alpha}(t): C_{N_\beta^\alpha}(t)| \geq |PGL(2,3^3)|/8 = 2457$ contrary to $r|3$. Thus $N_\beta^\alpha = N^\alpha \cap N^\beta$.

Let V be a cyclic subgroup of N_β^α of order 4 and let U be a Sylow 2- subgroup of N_β^α containing V. Then $U = N_{N_\beta^\alpha}(V)$, $|F(V)| = 1 + (q-\varepsilon)r/8$ by Lemma 2.8 and $|F(U)| = 1 + 8\cdot 3r/24 = r+1$ by Lemmas 2.3 and 2.4. If $q \neq 7, 9$, then $|F(U)| < |F(V)|$ and hence $U^{F(V)} \simeq Z_2$. Suppose $q=7$ or 9. Then $r=1$ as $r|n$. Hence $|\Omega| = 1 + |N^\alpha: N_\beta^\alpha| = 8$ or 16. By [10], we have a contradiction. Therefore $U^{F(V)} \simeq Z_2$.

Suppose $V^g \leq G_{\alpha\beta}$ for some $g \in G$ and set $\gamma = \alpha^g$. Then $V^g \leq g^{-1}N^\alpha g \cap G_{\alpha\beta} \leq N^\gamma \cap G_{\alpha\beta} \leq N_\alpha^\gamma \cap N_\beta^\gamma \leq N^\alpha \cap N^\beta = N_\beta^\alpha$. As $N_\beta^\alpha \simeq S_4$, $V^g = V^h$ for some $h \in N_\beta^\alpha$. Hence $N_G(V)^{F(V)}$ is doubly transitive. By Lemma 2.9. $C_G(N^\alpha)=1$ and one of the following holds.

(a) $N_G(V)^{F(V)} \leq A\Gamma L(1, 2^c)$.

(b) $N_G(V)^{F(V)} \trianglerighteq PSL(2, p_1)$, $p_1 = (q-\varepsilon)/8 \geq 5$.

(c) $N_G(V)^{F(V)} = R(3)$.

In the case (a), $(r+1)^2 = 1 + (q-\varepsilon)r/8$ by Lemma 2.5 and so $r = (q-\varepsilon-16)/8$ and $r|n$. From this $q=23$ or 25 and $r=1$. Since $|\Omega| = 1 + |N^\alpha: N_\beta^\alpha| = 2\cdot 127$ or $2\cdot 163$, we have $|G|_2 = 2|G_\alpha|_2$ while $|N_G(V)|_2 = |F(V)|_2 |N_{G_\alpha}(V)|_2 = 4|G_\alpha|_2$, contrary to $|N_G(V)|\,|\,|G|$.

In the case (b), $p_1(p_1-1)/2\,|\,s$ and $s\,|\,2n(q-\varepsilon)/4 = 4np_1$, where s is the order of $N_{G_\alpha}(V)^{F(V)}$. Hence $p_1 - 1\,|\,8n$. From this, $p^n - \varepsilon - 8\,|\,64n$ and so $q = 23$, 41, 71 or 73. Since p_1 is a prime and $p_1 = (q-\varepsilon)/8 \geq 5$, $q \neq 23, 71, 73$. Therefore $q=41$ and $|\Omega| = 1 + |N^\alpha: N_\beta^\alpha| = 1 + 40\cdot 41\cdot 42/2\cdot 24 = 2^2\cdot 359$, so that $|G|_2 = 4|G_\alpha|_2$.

Since $N_\beta^\alpha=N^\alpha\cap N^\beta$, $C_G(z)^{F(z)}$ is transitive by Lemma 2.1. On the other hand $|F(z)|=1+40\cdot 9/24=16$ by Lemma 2.8 (i) and so $|C_G(z)|_2=16|C_{G_\alpha}(z)|_2=16|G_\alpha|_2$, contrary to $|C_G(z)|\,|\,|G|$.

In the case (c), $r+1=4$ and $1+(q-\varepsilon)r/8=28$. Hence $r=3$ and $q=71$ or 73, contrary to $r|n$.

8. Case (VI)

In this section we assume that $N_\beta^\alpha\simeq A_5$ and $q\equiv 3,5 \pmod 8$. In particular, n is odd. If $N_\beta^\alpha\neq N^\alpha\cap N^\beta$, then $N^\alpha\cap N^\beta=1$, $C_G(N^\alpha)=1$ and so $N_\beta^\alpha\simeq N_\beta^\alpha N^\beta/N^\beta \leq \mathrm{Out}(N^\beta)\simeq Z_2\times Z_n$, a contradiction. Hence $N_\beta^\alpha=N^\alpha\cap N^\beta$. Let z be an involution in N_β^α and T a Sylow 2-subgroup of N_β^α contraining z. Then, by Lemma 2.8 $|F(z)|=1+(q-\varepsilon)\ 15r/60=1+(q-\varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)|=1+12\cdot 5r/60=1+r$. Since $N_\beta^\alpha=N^\alpha\cap N^\beta$, $z^G\cap G_{\alpha\beta}=z^G\cap N_\beta^\alpha=z^{N_\beta^\alpha}$ and so $C_G(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_G(N^\alpha)=1$ and one of the following holds.

(a) $C_G(z)^{F(z)}\leq A\Gamma L(1,2^c)$.

(b) $C_G(z)^{F(z)}\trianglerighteq PSL(2,p_1)$, $p_1=(q-\varepsilon)/4\geq 5$.

(c) $C_G(z)^{F(z)}=R(3)$.

In the case (a), by Lemma 2.5, $(q-\varepsilon)/4=1$ or $(r+1)^2/=1+(q-\varepsilon)r/4$. Hence $q=5$ or $r=(q-\varepsilon-8)/4|n$. If $q=5$, then $N_\beta^\alpha=N^\alpha$, a contradiction. Therefore $p^n-\varepsilon-8|4n$ and so $n=1$ and $q=11$ or 13. If $q=13$, we have $5\nmid|G_\alpha|$, a contradiction. Hence $q=11$ and $|\Omega|=1+|N^\alpha: N_\beta^\alpha|=1+10\cdot 11\cdot 12/2\cdot 60=12$. By [9], $G^\Omega\simeq M_{11}$, $|\Omega|=12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1(p_1-1)/2|s$ and $s|2n(q-\varepsilon)/2=4np_1$, where s is the order of $C_{G_\alpha}(z)^{F(z)}$. Hence $p_1-1|8n$ and so $p^n-\varepsilon-4|32n$. From this $q=19$, 27 or 37. Since $5||G_\alpha|$, $q\neq 27$, 37. Hence $q=19$ and $|\Omega|=1+|N^\alpha: N_\beta^\alpha|=1+18\cdot 19\cdot 20/2\cdot 60=2\cdot 29$. Since $G_\alpha\simeq PSL(2,19)$ or $PGL(2,19)$, $|G|=|\Omega|\,|G_\alpha|=2\cdot 29\cdot 2^i\cdot 18\cdot 19\cdot 20/2=2^{3+i}\cdot 3^2\cdot 5\cdot 19\cdot 29$ with $0\leq i\leq 1$. Let P be a Sylow 29-subgroup of G. Then P is semi-regular on Ω and 3, 5, 19 $\nmid|N_G(P)|$ because $N_G(P)/C_G(P)\leq Z_4\times Z_7$. Hence $|G: N_G(P)|=2^j\cdot 3^2\cdot 5\cdot 19$ with $0\leq j\leq 4$, while $2^j\cdot 3^2\cdot 5\cdot 19\not\equiv 1 \pmod{29}$ for any j with $0\leq j\leq 4$, contrary to a Sylow's theorem.

If $C_G(z)^{F(z)}=R(3)$, $r+1=4$ and $1+(q-\varepsilon)r/4=28$ and hence $r=3$, $q=37$, contrary to $r|n$.

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