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ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS IN WHICH SOCLE(G_α) IS NONSOLVABLE

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1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and α∈Ω. In [8], O'Nan has proved that socle(G_α)=A×N, where A is an abelian group and N is 1 or a nonabelian simple group. Here socle (G_α) is the product of all minimal normal subgroups of G_α.

In the previous paper [4], we have studied doubly transitive permutation groups in which N is isomorphic to PSL(2,q), Sz(q) or PSU(3,q) with q even. In this paper we shall prove the following:

**Theorem.** Let G be a doubly transitive permutation group on a finite set Ω with |Ω| even and let α∈Ω. If G_α has a normal simple subgroup N* isomorphic to PSL(2,q), where q is odd, then one of the following holds.

(i) G_Ω has a regular normal subgroup.
(ii) G_Ω=A_6 or S_6, N*=PSL(2,5) and |Ω|=6.
(iii) G_Ω=M_{11}, N*=PSL(2,11) and |Ω|=12.

In the case that G_α has a regular normal subgroup, by a result of Hering [3] we have (|Ω|, q)=(16,9), (16,5) or (8,7).

We introduce some notations:

F(X): the set of fixed points of a nonempty subset X of G
X(Δ): the global stabilizer of a subset Δ(⊆Ω) in X
X_Δ : the pointwise stabilizer of Δ in X
X^Δ : the restriction of X on Δ
m|n : an integer m divides an integer n
X^H : the set of H-conjugates of X
|X|_p : maximal power of p dividing the order of X
I(X) : the set of involutions in X
D_m : dihedral group of order m

In this paper all sets and groups are finite.
2. Preliminaries

Lemma 2.1. Let $G$ be a transitive permutation group on $\Omega$, $\alpha \in \Omega$ and $N^*$ a normal subgroup of $G_\alpha$ such that $F(N^*)=\{\alpha\}$. Let the subgroup $X \leq N^*$ be conjugate in $G_\alpha$ to every group $Y$ which lies in $N^*$ and which is conjugate to $X$ in $G$. Then $N_\alpha(X)$ is transitive on $\Delta=\{\gamma \in \Omega \mid X \leq N^*\}$.

Proof. Let $\beta \in \Delta$ and let $g \in G$ such that $\beta^g=\alpha$. Then, as $X \leq N^*$, $X^g \leq N^g=N^*$. By assumption, $(X^g)^h=X$ for some $h \in G_\alpha$. Hence $gh \in N_\alpha(X)$ and $\alpha^{(gh)^{-1}}=\alpha^{-1}=\beta$. Obviously $N_\alpha(X)$ stabilizes $\Delta$. Thus Lemma 2.1 holds.

Lemma 2.2. Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and $N^*$ a nonabelian simple normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. If $C_G(N^*) \neq 1$, then $N^* \cap N^\beta$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^*)$ is semiregular on $\Omega-\{\alpha\}$.

Proof. See Lemma 2.1 of [4].

Lemma 2.3. Let $G$ be a transitive permutation group on $\Omega$, $H$ a stabilizer of a point of $\Omega$ and $M$ a nonempty subset of $G$. Then

$$|F(M)| = |N_\alpha(M)| \times |M^G \cap H|/|H|.$$ 

Here $M^G \cap H=\{g^{-1}Mg \mid g^{-1}Mg \subseteq H, g \in G\}$.

Proof. See Lemma 2.2 of [4].

Lemma 2.4. Let $G$ be a doubly transitive permutation group on $\Omega$ and $N^*$ a normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. Assume that a subgroup $X$ of $N^*$ satisfies $X^G=X^{N^*}$. Then the following hold.

(i) $|F(X) \cap N^\gamma|=|F(X) \cap \gamma^\gamma|$ for $\beta, \gamma \in \Omega-\{\alpha\}$.

(ii) $|F(X)|=1+\sum_{\beta} |F(X) \cap \beta^N|$ for $\beta \in \Omega-\{\alpha\}$.

Proof. Let $\Gamma=\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\alpha$-orbits on $\Omega-\{\alpha\}$. Since $G_\alpha$ is transitive on $\Omega-\{\alpha\}$ and $G_\alpha \supseteq N^*$, we have $|\Delta_i|=|\Delta_j|$ for $1 \leq i, j \leq r$. By assumption, $G_\alpha=N_{G_\alpha}(X)N^*$ and so $N_{G_\alpha}(X)$ is transitive on $\Gamma$. Hence for each $i$ with $1 \leq i \leq r$ there exists $g \in N_{G_\alpha}(X)$ such that $(\Delta_i)^g=\Delta_i$. Therefore $|F(X) \cap \Delta_i|=\sum_{\beta} |F(X^\beta) \cap (\Delta_i)^\beta|=|F(X) \cap \Delta_i|$. Thus (i) holds and (ii) follows immediately from (i)

Lemma 2.5 (Huppert [5]). Let $G$ be a doubly transitive permutation group on $\Omega$. Suppose that $\theta_2(G) \neq 1$ and $G_\alpha$ is solvable. Then for any involution $z$ in $G_\alpha$, $|F(z)|^2=|\Omega|$.

We list now some properties of $PSL(2,q)$ with $q$ odd which will be required
Lemma 2.6 ([2], [6], [10]). Set \( N = PSL(2, q) \) and \( G = Aut(N) \), where \( q = p^n \) and \( p \) is an odd prime. Let \( z \) be an involution in \( N \). Then the following hold.

(i) \(|N| = (q-1)q(q+1)/2\), \( I(N) = \{ z^N \} \) and \( C_N(z) = D_{q-1} \), where \( q \equiv 1 \in \{ \pm 1 \} \) (mod 4).

(ii) If \( q \equiv 3 \), \( N \) is a nonabelian simple group and a Sylow \( r \)-subgroup of \( N \) is cyclic when \( r \neq 1, p \).

(iii) If \( X \) and \( Y \) are cyclic groups of \( N \) and \( |X| = |Y| = 2, p \), then \( X \) is conjugate to \( Y \) in \( \langle X, Y \rangle \) and \( N_X(X) = D_{q-1} \).

(iv) If \( X \leq N \) and \( X = Z_2 \times Z_2 \), \( N_X(X) \) is isomorphic to \( A_4 \) or \( S_4 \).

(v) If \( |N| \geq 8 \), \( N \) has two conjugate classes of four-groups in \( N \).

(vi) There exist a field automorphism \( f \) of \( N \) of order \( n \) and a diagonal automorphism \( d \) of \( N \) of order 2 and if we identify \( N \) with its inner automorphism group, \( \langle d \rangle N = PGL(2, q) \), \( \langle f \rangle \langle d \rangle N = G \) and \( G|N = Z_2 \times Z_2 \).

(vii) \( C_N(d) = D_{q+1} \) and \( C_{\langle d \rangle N}(z) = D_{q-1} \).

(viii) Suppose \( n = mk \) for positive integers \( m, k \). Then \( C_N(f^m) = PSL(2, p^m) \) if \( k \) is odd and \( C_N(f^m) = PGL(2, p^m) \) if \( k \) is even.

(ix) Assume \( n \) is even and let \( u \) be a field automorphism of order 2. Then \( I(G) = I(N) \cup d^N \). If \( n \) is odd, \( I(G) = I(N) \cup d^N \).

(x) If \( H \) is a subgroup of \( N \) of odd index, then one of the following holds:

1. \( H \) is a subgroup of \( C_N(z) \) of odd index for some involution \( z \in N \).
2. \( H = PGL(2, p^m) \), where \( n = 2mk \) and \( k \) is odd.
3. \( H = PSL(2, p^m) \), where \( n = mk \) and \( k \) is odd.
4. \( H = A_4 \) and \( q \equiv 3, 5 \) (mod 8).
5. \( H = S_4 \) and \( q \equiv 7, 9 \) (mod 16).
6. \( H = A_5 \), \( q \equiv 3, 5 \) (mod 8) and \( 5 \mid (q-1)q(q+1) \).

Lemma 2.7. Let \( G, N, d \) and \( f \) be as defined in Lemma 2.6 and \( H \) an \( \langle f, d \rangle \)-invariant subgroup of \( N \) isomorphic to \( D_{q+1} \). Let \( W \) be a cyclic subgroup of \( \langle d \rangle H \) of index 2 (cf. (vii) of Lemma 2.6) and set \( Y = 0_d(W \cap H) \). Then \( C_G(Y) = W \cdot C_{\langle f \rangle}(Y) \).

Proof. By (viii) of Lemma 2.6, we can take an involution \( t \) satisfying \( \langle d \rangle H = \langle t \rangle W \) and \( [f, t] = 1 \). Since \( N_0(Y) = \langle f, d \rangle N_X(Y) = \langle f, d \rangle H \), \( C_G(Y) = C_{\langle f \rangle \langle d \rangle H}(Y) = W \cdot C_{\langle f \rangle}(Y) \). Suppose \( ht \in C(Y) \) for some \( h \in \langle f \rangle \). Since \( t \) inverts \( Y \), \( h \) also inverts \( Y \) and so \( h^2 \) centralizes \( Y \). Hence some nontrivial 2-element \( g \in \langle h \rangle \) inverts \( Y \), so that \( C_H(g) \) contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, \( G^0 \) will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume \( G^0 \) has no regular normal subgroup.
Notation. \( C^a = C_G(N^a) \), which is semi-regular on \( \Omega - \{\alpha\} \) by Lemma 2.2. Let \( r \) be the number of \( N^a \)-orbits on \( \Omega - \{\alpha\} \).

Since \( G_\alpha \supseteq N^a \), \( |\beta^{N^a}| = |\gamma^{N^a}| \) for \( \beta, \gamma \in \Omega - \{\alpha\} \) and so \( |\Omega| = 1 + r \times |\beta^{N^a}| \). Hence \( r \) is odd and \( N^a \) is a subgroup of \( N^\alpha \) of odd index. Therefore \( N^a \) is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

**Lemma 2.8.** Let \( Z \) be a cyclic subgroup of \( N^a \) with \( |Z| = \pm 1 \), \( p \). Then

(i) \( |Z| = 2 \), \( |F(Z)| = 1 + (q - e) |I(N^a)|/|N^a| \).

(ii) \( |Z| = 2 \), \( |F(Z)| = 1 + |N_{N^a}(Z)|/|N_{N^a}(Z)| \).

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

**Lemma 2.9.** If \( N^a \neq D_q \) and \( Z \) is a cyclic subgroup of \( N^a \) with \( |Z| = \pm 1 \), \( p \) and \( N_G(Z) \) is doubly transitive. Then \( C^a = 1 \) and one of the following holds.

(i) \( N_G(Z) \leq GL(1, q) \) for some \( q_i \).

(ii) \( C_G(Z) \geq PSL(2, p_i), r = 1 \) and \( |F(Z)| = 1 + |N_{N^a}(Z)|/|N_{N^a}(Z)| \).

(iii) \( N_G(Z) = R(3) \), the smallest Ree group, \( |F(Z)| = 28 \).

Proof. Set \( N_G(Z) = L \) and \( F(Z) = \Delta \). By Lemma 2.6(iii), \( L \cap N^a = D_q \) and \( L \cap N^a = \langle t \rangle Y \geq Y \geq Z \), where \( 0(t) = 2 \), \( Y \simeq Z_{(q^2)}/2 \).

If \( (L \cap N^a)^a = 1 \), then \( L \cap N^a = N^a \) because \( L \cap N^a \) is a maximal subgroup of \( N^a \). Since \( \beta \leq N^a \), \( N^a \) is odd, \( L \cap N^a = N^a = D_q \), contrary to the assumption. Hence \( (L \cap N^a)^a = 1 \) and as \( L_{a} \geq L_{a} \cap N^a \) and \( L_{a} \cap Y \) (\( L_{a} \)) has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:

(a) \( L^a \) has a regular normal subgroup

(b) \( L^a \geq PSL(2, p_i), |\Delta| = p_i + 1 \), where \( p_i (\geq 5) \) is a prime

(c) \( L^a \geq PSL(3, p_i), p_i \geq 3, |\Delta| = (p_i)^3 + 1 \)

(d) \( L^a = R(3), |\Delta| = 28 \).

Suppose \( C^a = 1 \). Then there exists a subgroup \( D \) of \( C^a \) of prime order such that \( (L_a)^D = D^a \). Since \( [L_a, D] \subseteq D \cdot L_a \cap C^a = D(L_a \cap C^a) = D \), \( D \) is a normal subgroup of \( L_a \). By (i) and (iii) of Lemma 2.6, \( G_a = L_{a} \cap N^a \) and so \( D \) is a normal subgroup of \( G_a \). By Theorem 3 of [1], \( G^a \) has a regular normal subgroup, contrary to the hypothesis. Thus \( C^a = 1 \).

If (a) occurs, \( L^a \) is solvable because \( L_a/L \cap N^a = L_{a}N^a/N^a \leq \text{Out}(N^a) \) and \( L \cap N^a = D_q \). Hence by [5], (i) holds in this case.

If (b) occurs, we have \( Y^a = 1 \), for otherwise \( (L \cap N^a)^a = 1 \) and so \( N^a = L \cap N^a = D_q \), a contradiction. Hence \( 1 \neq C_G(Z)^a \leq L^a \) and so \( C_G(Z)^a \geq PSL(2, p_i) \) and \( Y^a = Z_{p_i} \). Therefore \( |\Delta| = |\beta^{N^a}| = p_i \) and \( r = 1 \) by Lemma 2.4 (ii). Since \( |\beta^{N^a}| = p_i \), we have \( |\beta^{L \cap N^a}| = p_i \), so that \( |L \cap N^a: L \cap N^a| = p_i \). Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of \( PSU(3, p_i) \),
a Sylow $p_1$-subgroup of $(L_a)'$ is not cyclic, while $(L_a)' \leq L \cap N_a = D_{q^2}$, a contradiction.

3. Case (I)

In this section we assume that $N_a^a \leq D_{q-r}$, where $\beta \neq \alpha$, $q = p^a$.

(3.1) (i) If $N_a^a \neq Z_2 \times Z_2$, $N \eta(N_a^a) = N_a^a$ and $|F(N_a^a)| = r + 1$.

(ii) If $N_a^a = Z_2 \times Z_2$, $N \eta(N_a^a) = A_4$ and $|F(N_a^a)| = 3r + 1$.

Proof. Put $X = N \eta(N_a^a)$. Let $S$ be a Sylow 2-subgroup of $N_a^a$ and $Y$ a cyclic subgroup of $N_a^a$ of index 2. If $N_a^a \neq Z_2 \times Z_2$, then $|Y| > 2$ and so $Y$ is characteristic in $N_a^a$. Hence $X \leq N \eta(Y) = D_{q-r}$. From this $[N \eta(S), S \cap Y] \leq S \cap Y$ and $0^\beta(N \eta(S))$ stabilizes a normal series $S \trianglerighteq S \cap Y \trianglerighteq 1$, so that $0^\beta(N \eta(S)) \leq C_{N \eta}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N \eta}(S) \leq S$ and hence $N \eta(S) = S$. On the other hand by a Frattini argument, $X = N \eta(S)N_a^a$ and so $X = N_a^a$. By Lemma 2.6(ii), $(N_a^a)^{N_a} = (N_a^a)^{N_a}$ and so by Lemmas 2.3 and 2.4(ii), $|F(N_a^a)| = 1 + |F(N_a^a) \cap \beta N_a^a| \times r = 1 + |N_a^a|/r = r + 1$. Thus (i) holds.

If $N_a^a = Z_2 \times Z_2$, $N \eta(N_a^a) = A_4$ by Lemma 2.6(iv). Similarly as in the case $N_a^a \neq Z_2 \times Z_2$, we have $|F(N_a^a)| = 3r + 1$.

(3.2) $N_a^a \cap N_a^a \leq Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^a = 1$. Suppose $C^a = 1$. Then $N_a^a \cap N_a^a = N_a^a/N_a^a \leq \text{Out}(N_a^a) = Z_2 \times Z_2$ by Lemma 2.6(vi) and hence $(N_a^a)^{N_a^a} \leq N_a^a \cap N_a^a$. Since $N_a^a$ is dihedral, $N_a^a(N_a^a)^{N_a^a} = Z_2 \times Z_2$, so that $N_a^a \cap N_a^a \leq Z_2 \times Z_2$.

(3.3) Suppose $N_a^a = N_a^a \cap N_a^a$ and let $U$ be a subgroup of $N_a^a$ isomorphic to $Z_2 \times Z_2$. Then $|F(U)| = 3r + 1$ and $N \xi(U)^{F(U)}$ is doubly transitive.

Proof. Sex $X = N \xi(N_a^a)$, $F = F(N_a^a)$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N_a^a$-orbits on $\Omega - \{\alpha\}$. If $g^{-1}N_a^a g \leq G_{a^b}$, then $g^{-1}N_a^a g \leq N_a^a \cap N_a^a = N_a^a \cap N_a^a \leq N_a^a$, where $\gamma = \alpha^\xi$. By a Witt's theorem, $X^\xi$ is doubly transitive.

If $U$ is a Sylow 2-subgroup of $N_a^a$, by a Witt's theorem, $N \xi(U)^{F(U)}$ is doubly transitive. Moreover $N \eta(U) = A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)| = 1 + |A_4| \times |N_a^a| = N \eta(U)| \times r = 3r + 1$.

If $|N_a^a| > 4$, by Lemma 2.6(iv) and (v), $N \eta(U) = S_4$ and $N_a^a$ has two conjugate classes of four-groups, say $\pi = \{K_1, K_2\}$. Set $X_\pi = M$. Then $M \geq N_a^a$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_i \neq \emptyset$ for each $i$ and so $|F(U) \cap \Delta_i| = 3$ by Lemma 2.3. Hence $|F(U)| = 3r + 1$. Since $N \eta(U) = S_4$, we may assume $r > 1$. Hence by (3.1) (i) $|\Delta| = r + 1 \geq 4$, so that $M^a$ is doubly transitive. Since $M = N_a^a N_a^a(U)$, $N_a^a(U)^{N_a^a}$ is also doubly transitive and so $N_a^a(U)$ is transitive on $\Delta$—
\[ \{ \alpha \}. \text{ As } |\Delta \cap \Delta_i| = 1, \Delta \cap \Delta_i \subseteq F(U) \text{ and } N_{N^\alpha}(U) \text{ is transitive on } F(U) \cap \Delta_i \text{ for each } i, N_G(U)^F(U) \text{ is doubly transitive}. \]

(3.4) (i) \( C^\alpha = 1 \).

(ii) Let \( U \) be a subgroup of \( N^\alpha \) isomorphic to \( Z_2 \times Z_2 \). If \( N^\alpha = N^\alpha \cap N^\beta \), then \( N_G(U)^F(U) \) has a regular normal 2-subgroup. In particular \( |F(U)| = 3r + 1 = 2^r \) for positive integer \( b \).

Proof. Since \( N_G(U)^F(U) \) is doubly transitive, by (3.3) and Theorem 3 of [1], \( N_G(U)^F(U) \) has a regular normal subgroup, \( N_G(U)^F(U) \cap PSU(3,3) \) or \( N_G(U)^F(U) = R(3) \).

Suppose \( C^\alpha \neq 1 \). Let \( D \) be a minimal characteristic subgroup of \( C^\alpha \). Clearly \( G^\alpha \supset D \). If \( N_G(U)^F(U) + R(3), D \) is cyclic. By Theorem 3 of [1], \( C^\alpha \) has a regular normal subgroup, contrary to the hypothesis. Hence \( N_G(U)^F(U) = R(3) \). Therefore \((N_G(U)^F(U))^\prime\) contains an element of order 9. Since \( N_G(U)^F(U) \cap C^\alpha N^\alpha(U) \approx N_G(U)^F(U) \cap Z_2 \times Z_2 \), we have \((N_G(U)^F(U))^\prime \leq C^\alpha \times N^\alpha(U) \). From this, \( C^\alpha \) contains an element of order 9 and so \( C^\alpha = Z_9 \) or \( M_3(3) \). In both cases, \( C^\alpha \) contains a characteristic subgroup of order 3. Since \( G^\alpha \supset D \), by Theorem 3 of [1] \( C^\alpha \) has a regular normal subgroup, a contradiction. Thus \( C^\alpha = 1 \).

Let \( R \) be a Sylow 3-subgroup of \( N_G(U)^F(U) \). Since \( N_G(U)^F(U) / N^\alpha(U) \approx N_G(U)^F(U) \cap Z_2 \times Z_2 \), \( R \) is cyclic. Clearly \( R \subseteq N^\alpha(U) \). Therefore \( N_G(U)^F(U) \) is doubly transitive on \( F(Y) \).

(3.5) (i) If \( |Y| > 3 \), \( N_G(Y)^F(Y) \) is doubly transitive.

(ii) If \( |Y| < 3 \), \( N^\alpha_B = Z_2 \times Z_2 \) or \( N^\beta_B = D_4 \) and \( N^\alpha \cap N^\beta \leq Z_2 \times Z_2 \).

Proof. Suppose \( |Y| > 3 \). If \( Y^\varepsilon \leq G^\alpha, Y^\varepsilon \leq N^\gamma \cap G^\alpha, \varepsilon \geq N^\gamma \), where \( \gamma = \alpha^\varepsilon \). If \( \gamma = \alpha \), obviously \( Y^\varepsilon \leq N^\gamma \). If \( \gamma + \alpha \), \( N^\gamma \), \( N^\gamma \) has a unique cyclic subgroup of order \( |Y| \). Hence \( Y^\varepsilon \leq N^\gamma \) and \( N^\gamma \leq N^\alpha \). Similarly \( Y^\varepsilon \leq N^\beta \). Thus \( Y^\varepsilon \leq N^\alpha \cap N^\beta \) and so \( Y^\varepsilon = Y \). By a Witt’s theorem, \( N_G(Y) \) is doubly transitive on \( F(Y) \).

Suppose \( |Y| < 3 \). Since \( N^\alpha \cap N^\beta : Y \leq 2 \), we have \( N^\alpha \cap N^\beta \leq Z_2 \times Z_2 \). On the other hand, as \( N^\alpha \) is dihedral, \((N^\alpha)^\prime \) is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set \( \Delta = F(N^\alpha_B), L = G(\Delta), K = G_{\Delta} \) and suppose \( N^\alpha_B \neq Z_2 \times Z_2 \). Then \( L^\alpha \supseteq N^\alpha_B, (L^\alpha)^\prime \leq N^\beta, K^\prime \leq N^\alpha \cap N^\beta \) and \((L^\alpha)^\alpha \leq Z_r \). If \( r + 1, L^\alpha \) is a doubly transitive Frobenius group of degree \( r + 1 \).

Proof. By Corollary B1 of [7] and (i) of (3.1), \( L^\alpha \) is doubly transitive and
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Since \( |\Delta| = r + 1 \). Since \( N^a \cap L \geq N^a \cap K = N^a_\beta \), by (i) of (3.1), we have \( N^a \cap L = N^a_\beta \). Hence \( L_a \geq N^a_\beta \). By (i) of (3.4), \( L_a/N^a_\beta \approx L_a/N^a/\text{Out}(N^a) = Z_2 \times Z_2 \) and so \( (L_a)^r \leq N^a_\beta \) and \( (L_a)^r = Z_r \). If \( r \neq 1 \), then \( (L_a)^r \neq 1 \). On the other hand \( (L_a)^r = 1 \) as \( (L_a)^r \) is abelian. Hence \( L^a \) is a Frobenius group.

(3.7) Suppose \( |Y| \geq 3 \). Then there exists an involution \( z \) in \( N^a_\beta \cap Y \) such that \( Z(N^a_\beta) = \langle z \rangle \).

Proof. Suppose \( N^a_\beta \neq Z_2 \times Z_2 \), \( |N^a_\beta| \geq 2 \) and \( N^a \) is dihedral, we have \( \langle I(W) \rangle = Z(N^a_\beta) = Z_2 \) and \( N^a_\beta/(N^a_\beta)^r = Z_2 \times Z_2 \). Let \( Z(N^a_\beta) = \langle z \rangle \) and suppose that \( z \) is not contained in \( Y \). By (3.2), \( (N^a_\beta)^r \leq N^a \cap N^a_\beta \cap W = Y \) and so \( (N^a_\beta)^r \) is odd. Hence \( |N^a_\beta|_2 = 4 \) and \( q \equiv p^2 \equiv 3 \) or \( 5 \) (mod 8), so that \( n \) is odd. By (3.2) and (i) of (3.4), \( N^a_\beta/N^a \cap N^a = N^a/N^a_\beta = 1 \) or \( Z_2 \). If \( N^a_\beta = N^a \cap N^a_\beta \), then \( W = Y \) and so \( x \in Y \), contrary to the assumption. Therefore we have \( N^a_\beta \cap N^a \cap N^a_\beta = Z_2 \) and \( N^a_\beta = \langle z \rangle \times (N^a \cap N^a_\beta) \). Since \( n \) is odd and \( z \in N^a \cap N^a_\beta \), by Lemma 2.6 (vi), (vii) and (ix), \( N^a_\beta/N^a \cap N^a_\beta = \langle z \rangle \) and besides it is isomorphic to a subgroup of \( D_{q - 1} \). Hence \( N^a_\beta \cap N^a \cap N^a_\beta \), \( \text{C}(\gamma) \) and \( \langle z \rangle \), a contradiction.

(3.8) Suppose \( |Y| \geq 3 \). Then \( N^a_\beta = N^a \cap N^a_\beta \).

Proof. Suppose \( N^a_\beta \neq N^a_\beta \cap N^a \) and let \( \Delta, L, K \) be as defined in (3.6) and \( x \in L_a \) such that its order is odd and \( \langle x \rangle \) is transitive on \( \Delta = \{a\} \). As \( |Y| \geq 3 \), \( W \) is characteristic in \( N^a \) and hence by (3.6), \( x \) stabilizes a normal series \( L_a/N^a_\beta \geq W \geq (N^a_\beta)^r \). By Theorem 5.3.2 of [2], \([x, 0 \langle L_a/\langle N^a_\beta \rangle \rangle] = 1\). Since \( L_a/(N^a_\beta)^r \) has a normal Sylow 2-subgroup and \( (N^a_\beta)^r \leq K \), we have \( [x, 0 \langle L_a/K \rangle] = 1 \), so that \( [x, N^a_\beta] \leq K \leq N^a \cap N^a_\beta \) by (3.6). If \( r \neq 1 \), then \( \beta^s \neq \beta \) and \( \beta^s \in \Delta \), hence \( N^a_\beta = x^{-1} N^a x = N^a_\beta \), where \( \gamma = \beta^s \). Since \( \gamma \in \Delta \) and \( \Delta = F(N^a_\beta) \), \( N^a_\beta \leq N^a_\beta \cap G = N^a_\beta \) and so \( N^a_\beta = N^a_\beta \). Similarly \( N^a_\beta = N^a_\beta \). Hence \( N^a_\beta = N^a_\beta \), which implies \( N^a_\beta = N^a_\beta \cap N^a \). By the doubly transitivity of \( G \), we have \( N^a_\beta = N^a \cap N^a_\beta \), contrary to the assumption. Therefore we obtain \( r = 1 \).

Let \( z \) be as defined in (3.7) and put \( k = (q - \varepsilon)/|N^a_\beta| \). By Lemma 2.8(i) we have \( |F(z)| = 1 + (q - \varepsilon) (|N^a_\beta|/2 + 1)/|N^a_\beta| = (q - \varepsilon)/2 + k + 1 \). Similarly \( |F(Y)| = k + 1 \). As \( N^a_\beta \neq N^a \cap N^a_\beta \), there is an involution \( t \) in \( N^a_\beta \) which is not contained in \( N^a \). By Lemma 2.6 (i), \( t^r = z \) for some \( y \in N^a \). Set \( \gamma = \beta^s \). Then \( \gamma \in F(z) \) and \( z \in N^a \). By Lemma 2.6 (vii), (viii) and (ix), \( C_N(z) \approx D_{q + 1} \) or \( PGL(2, \sqrt{q}) \). Assume \( C_N(z) \approx D_{q + 1} \) and let \( R \) be a cyclic subgroup of \( C_N(z) \) of index 2. We note that \( R \) is semi-regular on \( \Omega = \{\alpha\} \). Set \( X = C_\alpha(z) \). Since \( 2 \leq k + 1 \leq (q - \varepsilon)/|q - \varepsilon|/2 + k + 1 \), we have \( (q + \varepsilon)/2 = k + 1 \) and so \( |\alpha^x| > k + 1 \). By (i) of (3.5) and (3.7), \( N_\alpha(Y) \leq C_\alpha(z) = X \) and \( \alpha^x \cap F(Y) \). It follows from Lemma 2.1 that \( \alpha^x = \{z \in N^a_\beta \} \neq \gamma \). Hence \( |F(z)| > |\alpha^x| > |F(Y)| + (q + \varepsilon)/2 + k + 1 + (q - \varepsilon)/2 + \varepsilon = |F(z)| + \varepsilon \). Therefore \( \varepsilon = 1 \) and \( \gamma^x = \{y\} \), so that \( \gamma \in F(Y) \), a contradiction. Thus \( C_N(z) \approx PGL(2, \sqrt{q}) \), \( \varepsilon = 1 \), \( N^a_\beta/N^a \cap N^a = Z_2 \) and \( |\langle z \rangle \cap G_\alpha \rangle; N^a_\beta = 2 \).
Set $\Delta_1=\alpha^x$ and $\Delta_2=F(z)-\Delta_1$. Let $\delta \in \Delta_2$ and $g$ an element of $G$ satisfying $\delta^g=\gamma$. Then $z \in N^*_gN^3-N^3$ and so $z^g \in N^*_gN^g-N^g$, where $v=\alpha^x$. Since $|z^g \cap G_\gamma|\geq \frac{q}{4}$ and $z \in G_\gamma-N_\gamma$, it follows from Lemma 2.6 (ix) that $(z^g)^h=z$ for some $h \in G_\gamma$. Hence $gh \in X$ and $\delta^g=\gamma$. Thus $\Delta_2=\gamma^x$. Let $\delta \in \Delta_2$. Then $z \neq N^*_g$ and $z \in Z(N^*_g)$ by (3.7) and so $X \cap N^*_g=Z_2 \times Z_2$, which implies $|\delta^{(g \circ z^g(\delta))}|=(q-1)/4$. Hence $|\Delta_1|=|\Delta_2|=((q-1)/4+k+1, (q-1)/4)$ or $(k+1, (q-1)/2)$. Let $P$ be a subgroup of $C_N^g(z)$ of order $\sqrt{q}$. Then $F(P)=\{\gamma\}$ and $P$ is semi-regular on $\Omega$. If $|\Delta_2|=(q-1)/4$, then $\sqrt{q}|(q-1)/4-1=(q-5)/4$ and $\sqrt{q}|(q-1)/4+k+1$. From this, $q=5^k$, $k=3$, $|\Delta_1|=10$ and $|\Delta_2|=6$. Since $(C_N^g(z))^2=S_3$, $X^{2z}=S_3$ and so $|X| \geq 3^2$. As $X$ acts on $\Delta_1$ and $|\Delta_1|=1 (mod 3)$, $|G_\alpha| \leq |X^\alpha| \geq 3^2$, contrary to $N^*_g=PSL(2,25)$. If $|\Delta_2|=(q-1)/2$, then $\sqrt{q}|(q-1)/2-1=(q-3)/2$, so $q=3^t$, $k=1$, $N^*_g=D_8$ and $\Delta_1=\{\alpha, \beta\}$. Hence $C_N^g(z)$ fixes $\alpha$ and $\beta$, so that $PGL(2,3)=C_N^g(\alpha)$, which implies $|\Delta_2|=1/4$. Hence $(I \Delta_1 I, I \Delta_2 I)=(q-1)/4+1, (q-1)/2)$ or $(q-1/4, q+1)$.

(3.9) Suppose $|Y| \geq 3$. Then $r=1$.

Proof. By (3.6), $r+1=2^c$ for some integer $c \geq 0$. On the other hand $3r+1=2^k$ by (3.8) and (ii) of (3.4). Hence $2r=2^r(2^{c-r}-1)$ and so $c=1$ as $r$ is odd. Thus $r=1$.

(3.10) Put $k=(q-\delta)/|N^*_g|$. If $N^*_g=N^* \cap N^g$ and $r=1$, then

$$q-\delta+2k+2(2k+2-\delta)(k+1-\delta)k+1) (2k+2-\delta)(k+1-\delta).$$

Proof. Set $S=\{\gamma, u) \in F(u), u \in z^\circ\}$, where $z$ is an involution in $N^*_g$. We now count the number of elements of $S$ in two ways. Since $N^*_g=N^* \cap N^g$, $F(z)=\{\gamma \in F(u) \}$ and hence $C_N^g(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S|=|D_2||z^\circ|=|z^\circ||F(z)|$. Since $r=1$, $|\Omega|=1+|N^*_g|$, $N^*_g=|F(z)|=(q+\delta)/2+1$ and by Lemma 2.8 $|F(z)|=(q+\delta)/2+k+1$. Since $G_\alpha \geq N^*_g$, $z^\circ$ is contained in $N^*$ and so $|G_\alpha|=|N^*_g|=|C_N^g(z)=q(q+\delta)/2|$. Hence $(q-\delta)/2+k+1$ $(kq+q+\delta)/2_k(q+\delta)/2$. On the other hand $|F(z)|=|C_N^g(z)|=|C_N^g(z)| \leq |G_\alpha|/2$ and $|G_\alpha|/2$. Hence $|q-\delta+2k+2| \leq k(q+\delta)/2+1/2$. Since $k(q+\delta)/2=2k(q+\delta)/2$ and $q(q+\delta+2(2k+2-\delta)(k+1-\delta)k+1) (q-\delta+2k+2)+2(2k+2-\delta)(k+1-\delta)k+1)$ and $q(q+\delta+2e-2k-2)(q-\delta+2k+2)+2(2k+2-\delta)(k+1-\delta)$, we have (3.10).

(3.11) Suppose $|Y| \geq 3$. Then one of the following holds.

(i) $N^*_g=N^* \cap N^g=D_{q-1}$

(ii) $N^*_g=N^* \cap N^g \neq D_{q-1}$ and $N_\alpha(Y)^{F(\alpha)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_\alpha(Y)^{F(\alpha)}=R(3)$ or there exists a prime $p \geq 5$ such that $C_N^g(Y)^{F(\alpha)} \geq PSL(2,p)$ and $V/Y \cong Z_{p^2}$, where $V=C_N^g(Y)$. By (i) of (3.1) and (3.9), $F(N^*_g)=\{\alpha, \beta\}$. On the other hand, $(N^*_g)^{F(\alpha)}=N^*_g/Y \cong Z_2$. Hence $N_\alpha(Y)^{F(\alpha)} \neq R(3)$ and $C_N^g(Y)^{F(\alpha)} \geq$
PSL(2, \(p^m\)).

By (i) of (3.4) and Lemma 2.7, we have \(C_{G^\alpha}(Y) = V(f_1)\), where \(f_1\) is a field automorphism of \(N^\alpha\). Let \(t\) be the order of \(f_1\), \(n = tm\) and let \(p^m \equiv \epsilon_t \equiv \{\pm 1\} \pmod{4}\). Clearly \(C_{G^\alpha}(Y)^{F(Y)} \simeq V^{F(Y)} \simeq Z_{p^m}\). Hence \(|C_{G^\alpha}(Y)^{F(Y)}| \mid t\), so that \((p_1-1)/2 \mid t\).

First we assume that \(t\) is even and set \(t = 2t_1\). Then \(Y \leq C_{N^\alpha}(f_1) = PGL(2, p^m)\) by Lemma 2.6 (viii). As \(|V/Y| = p_1\) and \(p_1\) is a prime, \(Y\) is a cyclic subgroup of \(C_{N^\alpha}(f_1)\) of order \(p^m - \epsilon_t\) and \((p^m - 1)/2(p^m - \epsilon_t) = p_1\). Put \(s = \sum_{i=0}^{t_1}(p_i^m)^i\). Then \((p^m + \epsilon_t)s/2 = p_1\), so that we have either (i) \(t_1 = 1\) and \(p_1 = (p^m + \epsilon_t)/2\) or (ii) \(t_1 \geq 2\), \(p^m = 3\) and \(p_1 = s\). In the case (i), \(2 \leq (p_1 - 1)/2 = (p^m + \epsilon_t - 2)/4 \mid 2t_1 = 2\). Hence \((p_1, q) = (5, 3^2)\) or \((4, 11^2)\). Let \(z\) be as in (3.7). As mentioned in the proof of (3.10), \(|F(z)| = (q-1)/2 + k + 1, |\Omega| = kq(q+1)/2 + 1\) and \(C_{G}(z)\) is transitive on \(F(z)\). If \(q = 3^2\), then \(|F(z)| = 46\) and \(|\Omega| = 2 \cdot 19\cdot 23\). Hence \(|C_{G}(z)| = |F(z)|\) \(|C_{G^\alpha}(z) N^\alpha| N^\alpha| N^\alpha| = 46 \cdot 2 \cdot 80 = 2 \cdot 19 \cdot 23\) with \(0 \leq i \leq 3\). Let \(P\) be a Sylow 23-subgroup of \(C_{G}(z)\) and \(Q\) a Sylow 5-subgroup of \(C_{G}(z)\). Since \(11 \not| \Omega\), \(P\) is a subgroup of \(N^\gamma\) for some \(\gamma \in \Omega\) and \(F(P) = \{\gamma\}\). Hence \(\gamma \in F(z)\), so that \(z \in N^\gamma\), contrary to \(C_{N^\gamma}(z) = D_{120}\). In the case (ii), we have \((p_1 - 1)/2 = (\sum_{i=0}^{t_1-1}9^i)/2 \mid t_1 = 2t_1\). From this, \(9^i - 1 \leq 4t_1\), hence \(t_1 = 1\), a contradiction.

Assume \(t\) is odd. Then \(Y \leq C_{N^\alpha}(f_1) = PGL(2, p^m)\) by Lemma 2.6 (viii). As \(|V/Y| = p_1\) and \(p_1\) is a prime, \(Y \simeq Z_{(p^m - 1)/2}\) and \((q - \epsilon)/(p^m - \epsilon_t) = p_1\). Hence \(\sum_{i=0}^{t_1}(p_i^m)^i(-\epsilon_t)^{i-1} = p_1\) and \((p_1 - 1)/2 = (\sum_{i=1}^{t_1}(p_i^m)^i(-\epsilon_t)^{i-1}) - 1)/2 \mid t\). In particular \(2t \geq (p^m)^{t-1} - (p^m)^{-2}(p^m)^{t-2} \geq 2(p^m)^{t-2}\). From this \(t = 3, m = 1, p_1 = 7\) and \(q = 3^2\), so that \(N^\alpha \simeq Z_2 \times Z_2\), a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let \(G^\alpha\) be a minimal counterexample to (3.12) and \(M\) a minimal normal subgroup of \(G\). By the hypothesis, \(G\) has no regular normal subgroup and hence \(M_\alpha \not= 1\). As \(M_\alpha\) is a normal subgroup of \(G_\alpha\), by (i) of (3.4), \(M_\alpha\) contains \(N^\alpha\). By (3.9), \(r = 1\), hence \(M\) is doubly transitive on \(\Omega\). Therefore \(G = M\) and \(G\) is a nonabelian simple group.

Since \(N^\alpha_\alpha \simeq D_4, k = 1\) and so \(q - \epsilon + 4 \mid 2((4 - \epsilon)(2 - \epsilon) + 1)(4 - \epsilon)(2 - \epsilon)\) by (3.10). Hence we have \(q = 7, 9, 11, 19, 27\) or 43.

Let \(x\) be an element of \(N^\alpha\). If \(|x| > 2\), by Lemma 2.8, \(|F(x)| = 1 + |N^\alpha| \times 1/|N^\alpha| = 2\) and if \(|x| = 2\), similarly we have \(|F(x)| = (q - \epsilon)/2 + 2\). Assume \(q \not= 9\) and let \(d\) be an involution in \(G_\alpha - N^\alpha\) such that \(\langle d \rangle N^\alpha\) is isomorphic to \(PGL\).
(2, q). We may assume \( d \subseteq G_{\alpha^2} \). Since \( \langle d \rangle N^* \) is transitive on \( \Omega - \{ \alpha \} \), by Lemmas 2.3 and 2.6 (vii), (ix), \( |F(d)| = 2(q-1)(q+1/2)q \) while \( |F(\omega)| = (q+1)/2+2 \) for \( \omega \in I(N^*) \). Hence \( d \) is an odd permutation, contrary to the simplicity of \( G \). Thus \( G_{\alpha} = N^* \) if \( q \neq 9, 27 \) and \( |G_{\alpha}/N^*| = 1, 3 \) if \( q = 27 \).

If \( q = 9, |\Omega| = 1 + 9\cdot 10/2 = 23 \) and \( |G_{\alpha}| = 2^1 |PSL(2, 9)| = 2^4 \cdot 3 \cdot 5 \) with \( 0 \leq a \leq 2 \). Let \( P \) be a Sylow 3-subgroup of \( G \). Since Aut\( (Z_{23}) \cong Z_2 \times Z_{11}, \exists \lambda \mid N_{\alpha}(P) \rangle \), for otherwise \( P \) centralizes a nontrivial 3-element \( \omega \) and so \( F(P) \supseteq F(\omega) \) because \( |F(\omega)| = 1 \), contrary to \( |F(P)| = 0 \). Similarly \( \exists \lambda \mid N_{\alpha}(P) \rangle \).

Hence \( |G: N_{\alpha}(P)| = 2^7 \cdot 3 \cdot 5 \) for some \( a \) with \( 0 \leq a \leq 6 \). By a Sylow’s theorem, \( 2^7 \cdot 3 \cdot 5 = 2^2 \mod 33 \), a contradiction.

If \( q = 27, |\Omega| = 1 + 27 \cdot 10/2 = 25 \cdot 11 \) and \( |G_{\alpha}| = 2^{2^9} \cdot 3 \cdot 5 \) with \( 0 \leq a \leq 3 \). Let \( P \) be a Sylow 11-subgroup of \( G \). Since \( P \cong Z_7 \) and Aut\( (Z_{11}) \cong Z_2 \times Z_11 \), \( \exists \lambda \mid N_{\alpha}(P) \rangle \) by the similar argument as above. Hence \( |G: N_{\alpha}(P)| = 2^{2^9} \cdot 3 \cdot 5 \) with \( 0 \leq a \leq 3 \).

As \( |G: N_{\alpha}(P)| = 2^{2^9} \cdot 3 \cdot 5 \), we know that \( 2^{2^9} \cdot 3 \cdot 5 \) is divisible by 23. Hence \( a = 0, b = 4 \). Therefore \( N_{\alpha}(P) \) contains a Sylow 2-subgroup \( S \) of \( G \). Let \( T \) be a Sylow 2-subgroup of \( N_{\alpha}(P) \) and \( g \) an element such that \( T \subseteq S \). Then \( T \cap N_{\alpha}(P) \neq 1 \) as \( N_{\alpha}(P)/\langle \alpha \rangle = Z_2 \).

Let \( t^* \) be an involution in \( T \). Then \( |F(t^*)| = (q+1)/2+2 = 16 \), while \( 11 \mid |F(t^*)| \) because \( [\alpha, t] = 1 \) and \( |F(\alpha)| = 0 \), a contradiction.

If \( q = 7, 11, 19 \) or 43, then \( G_{\alpha} = N^* \) and \( \beta = -1 \). Set \( \Gamma = \{ \gamma, \delta \mid \gamma, \delta \in \Omega, \gamma \neq \delta \} \). We consider the action of \( G \) on \( \Gamma \). Since \( G^\alpha \) is doubly transitive, \( G^\Gamma \) is transitive and \( G^\Gamma = 1 \). Let \( \omega \) be an involution of \( Z(N_{\alpha}) \). There exists an involution \( \omega \) such that \( \omega^2 = \omega \) and \( \omega^2 \neq 1 \) as \( N_{\alpha}(P)/\omega = Z_2 \).

We now argue that \( \langle t \rangle N_{\alpha}^* = D_{24} \). Let \( R \) be the Sylow 3-subgroup of \( N_{\alpha}^* \). If \( t \) centralizes \( R \), then \( R \) acts on \( F(t) \) and so \( F(R) = F(t) \) as \( |F(t)| = 8 \) and \( |F(R)| = 2^2 - 1 \).

Hence \( \omega \neq \alpha \), contrary to the choice of \( t \). Therefore \( t \) inverts \( R \) and \( \langle t \rangle N_{\alpha}^* \) is isomorphic to \( Z_2 \times D_{24} \). Suppose \( \langle t \rangle N_{\alpha}^* \cong Z_2 \times D_{24} \). Then \( \langle t \rangle N_{\alpha}^* \) contains fifteen involutions and so we can take \( u \in I(\langle t \rangle N_{\alpha}^* \) satisfying \( |F(u)| = 0 \) and \( \langle t \rangle N_{\alpha}^* = \langle u \rangle \times N_{\alpha}^* \). As \( |F(u)| = 0, |F(u^2)| = |\Omega|/2 = 28 \). By Lemma 2.3, \( 28 = |C(u)| \times |\langle u \rangle N_{\alpha}^* \cap u^2 \rangle \) and hence \( |C(u)| = 2^2 \cdot 3 \cdot 7 \) or \( 2^3 \cdot 7 \). Since \( \langle u \rangle N_{\alpha}^* = N_{\alpha}(R) \), we have \( |C(u)| = 2^2 \cdot 3 \cdot 7 \) or \( 2^3 \cdot 7 \).

By a Sylow’s theorem, \( |C(u)| = 2^2 \cdot 3 \cdot 7 \) or \( 2^3 \cdot 7 \), so that \( |C(u)| = 2^2 \cdot 3 \cdot 7 \).

Let \( Q \) be a Sylow 7-subgroup of \( C(u) \). Then \( |C(u) \cap N_{\alpha}(Q)| = 2^2 \cdot 3 \cdot 7 \) or \( 2^3 \cdot 7 \) by a Sylow’s theorem. Hence \( 2^2 \cdot 3 \cdot 7 \mid |N_{\alpha}(Q)| \). Since Aut\( (Z_{23}) \cong Z_2 \times Z_{23} \),
Let \( U \) be a Sylow 2-subgroup of \( N^a \) and set \( L=N_\beta U \). It follows from (3.3) and Lemma 2.6 (iv) that \( L \cap N^a=A_4 \), \( L^F(U)=A_4 \) and \( |L|=2^4 \cdot 3 \). Let \( T, <x> \) be Sylow 2- and 3-subgroup of \( L \), respectively. Obviously \( L > T \) and \( C_T(x)=1 \). On the other hand \( T \cap C_T(N^a) \) is not contained in \( T \) and so \( T \neq Z_2 \times Z_2 \) because \( C_T(x)=1 \). By Theorem 5.4.5 of \([2]\), \( T \) is dihedral or semi-dihedral. Hence \( N_\beta G(T)/C_T(T) \) is a 2-group, so that \( C_T(x)=1 \), a contradiction.

(3.13) (ii) of (3.11) does not occur.

Proof. Let \( G^a \) be a doubly transitive permutation group satisfying (ii) of (3.11). Let \( x \) be an involution in \( N^a \) with \( x \in Y \). Then \( F(x^F(Y))=F(x^F(Y)) \) by (i) of (3.1) and (3.9). Since \( |F(Y)|=1+(q-6)|N^a|=1+k \geq 4 \), \( x^F(Y) \) is an involution. By Lemma 2.5, \( 1+k=2^2 \) and so \( k=3 \). By (3.11), \( q-\epsilon+8|2((8-\epsilon)(4-\epsilon)+3)+1| \). Hence \( q+7|2^2 \cdot 3 \cdot 7 \) if \( \epsilon=1 \) and \( q+9|2^4 \cdot 3^2 \cdot 5 \cdot 17 \) if \( \epsilon=-1 \). From this \( q+7|2^2 \cdot 7 \) if \( \epsilon=1 \) and \( q+9|2^2 \cdot 5 \cdot 17 \) if \( \epsilon=-1 \). Therefore \( q=5^2, 7^2, 11^2, 59 \) or 71.

Let \( p \) be an odd prime such that \( p \mid |\Omega| \) and \( p \mid |G^a| \) and let \( P \) be a Sylow \( p \)-subgroup of \( G \). Clearly \( P \) is semi-regular on \( \Omega \) and so any element in \( C_G(P) \) has at least \( p \gamma \) fixed points. If \( x \) is an element of \( N^a \) and its order is at least three, \( |F(x)|=|F(Y)|=4p \gamma \) by Lemma 2.8. Since \( \Omega=p \gamma \cdot |N^a|=1+3(q+\epsilon)/2 \).

If \( q=5^2 \), then \( |\Omega|=2^4 \cdot 61 \) and \( |G^a|=2^{4+i} \cdot 3 \cdot 5^2 \cdot 13 \). Let \( P \) be a Sylow 61-subgroup of \( G \). Then \( P \cong Z_{61} \). As mentioned above, 5, 13 \( \mid |C_G(P)| \) and so \( 5^2, 13 \mid |N^a| \). Hence \( |G: N^a(P)|=2^a \cdot 3^4 \cdot 5^3 \cdot 13 \), where \( 0 \leq a \leq 10 \) and \( 0 \leq b, c \leq 1 \). But we can easily verify \( |G: N^a(P)| \equiv 1 \) (mod 61), contrary to a Sylow's theorem.

If \( q=7^2 \), then \( |\Omega|=2^2 \cdot 919 \) and \( |G^a|=2^{4+i} \cdot 3 \cdot 5^2 \cdot 7^2 \). Let \( P \) be a Sylow 919-subgroup of \( G \). By the similar argument as above, we obtain 5, 7 \( \mid |N^a(P)| \) and so \( |G: N^a(P)|=2^a \cdot 3^4 \cdot 5^3 \cdot 7^2 \equiv 2^a \cdot 306 \) or \( -2^a \) (mod 919), where \( 0 \leq a \leq 8 \) and \( 0 \leq b \leq 1 \). Hence \( |G: N^a(P)| \equiv 1 \), a contradiction.

If \( q=11^2 \), then \( |\Omega|=2^7 \cdot 173 \) and \( |G^a|=2^{4+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \). Let \( P \) be a Sylow 173-subgroup of \( G \). Similarly we have 3, \( 5, 11, 61 \mid |N^a(P)| \) and so \( |G: N^a(P)|=2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -5 \cdot 2^a \) (mod 173), where \( 0 \leq a \leq 12 \). Hence \( |G: N^a(P)| \equiv 1 \), a contradiction.

If \( q=59 \), then \( |\Omega|=2^7 \cdot 17 \cdot 151 \) and \( |G^a|=2^{4+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59 \). Let \( P \) be a Sylow 17-group of \( G \). Similarly we have 3, \( 5, 29, 59 \mid |N^a(P)| \) and so \( |G: N^a(P)|=2^a \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151 \equiv 10 \cdot 2^a \) or \( 12 \cdot 2^a \) (mod 17), where \( 0 \leq a \leq 4 \) and \( 0 \leq b \leq 1 \). From this, we have a contradiction.

If \( q=71 \), then \( |\Omega|=2^5 \cdot 233 \) and \( |G^a|=2^{4+i} \cdot 3 \cdot 5 \cdot 7 \cdot 71 \). Let \( P \) be
a Sylow 233-subgroup of $G$. Since $3,5,7,71 \not| N_\alpha(P)$, $|G| N_\alpha(P)| = 2^a 3^b 5^c 7^d 71^e$ (mod 233), where $0 \leq a \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N_\beta^e = Z_2 \times Z_2$ or $N_\beta^e = D_8$ and $N_\beta^e \cap N_\beta \leq Z_2 \times Z_2$.

(3.14) The case that $N_\beta^e = Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N_\beta^e)$. Then $|\Delta| = 3r+1$ and $\Delta = F(N_\beta^e N_\beta^e)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N_\beta^e| = 4$, we have $q = p^e = 3, 5$ (mod 8) and so $n$ is odd. Hence $|G_\alpha|/|N_\beta^e| < 2$ and $N_\beta^e \cap N_\beta = N_\beta^e N_\beta^e \cap N_\beta = 1$ or $Z_2$ by (3.2). Suppose $N_\beta^e \cap N_\beta = Z_2$. Then $N_\beta^e N_\beta^e$ is a Sylow 2-subgroup of $G_\alpha$, hence $G_\alpha (N_\beta^e N_\beta^e)^\alpha$ is doubly transitive by a Witt's theorem. Since $N_\beta^e N_\beta^e D_8$ and $|\Delta|$ is even, $G_\alpha (N_\beta^e N_\beta^e)^\alpha$ is also doubly transitive. Let $g$ be an element of $G_\alpha (N_\beta^e N_\beta^e)^\alpha$ such that $\alpha^g = \beta$ and $\beta^g = \alpha$. Then $N_\beta^e = g^{-1} N_\beta^e g = N_\beta^e$ and hence $N_\beta^e = N_\beta^e \cap N_\beta^e$, a contradiction. Thus $N_\beta^e = N_\beta^e \cap N_\beta^e = Z_2 \times Z_2$.

Let $z$ be an involution in $N_\beta^e$ and $t \in z^G$ an involution such that $\alpha^t = \beta$. Set $\Gamma = \{\gamma, \delta\} | \gamma, \delta \in \Omega, \gamma \neq \delta\}$. We consider the action of the element $z$ on $\Gamma$.

By the similar argument as in the proof of (3.12), $|F(z)|((|F(z)| - 1)/2 + (|\Omega| - |F(z)|))/2 = |G_\delta| |z^G \cap \langle t \rangle G_\alpha^z|/|\langle t \rangle G_\alpha^z|$. Since $N_\beta^e = N_\beta^e \cap N_\beta^e$, by Lemma 2.6 (i), $z^G \cap G_\delta = z^{G_\delta}$ and so $|C_{G_\delta}(z)| = |F(z)| \times |C_{G_\alpha}(z)|$. Hence $|G_\alpha^z|((|F(z)| - 1)/2 + (|\Omega| - |F(z)|))/2 = |G_\delta| |C_{G_\delta}(z)|/|z^G \cap \langle t \rangle G_\alpha^z|$, so that $|G_\alpha^z|/|\Omega| \equiv 0$ (mod $|F(z)|$). Since $|G_\alpha^z|/|N_\beta^e| = |G_\alpha^z N_\beta^e|/|N_\beta^e| |2n|, we have $|G_\alpha^z|/|8n|$. Clearly $|\Omega| = 1 + q(q - e)(q + e)/8$ and by Lemma 2.8 (i), $|F(z)| = 1 + 3(q - e)r/4$. Hence $1 + 3(q - e)r/4 | 8n(1 + q(q - e)(q + e)r)/8$. Put $n = rs$. Then $3qr - 3er + 4 | 4rs(8 + q(q - e)(q + e)r)3r = 864 r^3 + 4s(3pq) (3pq - 3er) (3qr + 3er)$. Hence $3qr - 3er + 4 | 864 r^3 + 4s(3er - 4) (3er - 4 + 3er) (3er - 4 + 3er) = 8634r^3 - 32s(3er - 4) (3er - 2)$. (2)

We argue that $r = 1$. Suppose false. Then $32s(3er - 4) (3er - 2) > 0$ and so $3r(q - e) < 864 r^3 s$. Therefore $288n + e > q = p^e \geq 3^e$ and so $288n > 3^e$. Hence $(n, r, p, e) = (5, 5, 3, -1), (3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (2). Thus $r = 1$.

Hence $3q - 3e + 4 | 64(5 + 9e) n$ and $|F(z)| = 1 + 3(q - e)/4, |\Omega| = 1 + q(q - e)(q + e)/8$. If $e = 1$, then $3q^3 < 3q + 7 | 256n$. Hence $n = 1$ or $(n, p) = (5, 3), (3, 3)$. Since $3q^3 + 7 | 256 \cdot 5$ and $3q^3 + 7 | 256 \cdot 3, n = 1$ and $3q + 7 | 256$. From this, $q = 19$ or 83. If $e = 1$, then $3q^3 < 3q + 1 | 896 n$ and so $n = 1$ or $(n, p) = (3, 5)$. Since $3q^3 + 1 | 896 \cdot 3$, we have $n = 1$ and $3q + 1 | 896$. From this, $q = 5, 37$ or 149. As $PSL(2, 5) = PSL(2, 4), q \neq 5$ by [4]. Thus $q = 19, 37, 83$ or 149.

Set $m = |z^G \cap \langle t \rangle G_\alpha|$. As we mentioned above, $|G_\alpha^z| ((|G_\delta| |\langle t \rangle G_\alpha^z| - 1) + |\Omega| - |F(z)|)) = |F(z)| |z^G \cap \langle t \rangle G_\alpha^z| m$. Since $|G_\alpha|/|N_\beta^e| = 1$ or 2, $|C_{G_\alpha}(z)|/|G_\alpha^z| = (q - e)/4$. Therefore $m = (2q^2 + (2e + 9q - 9e)(q - 3e + 4))$. It follows that $(q, m) = (19, 27/2), (37, 28), (83, 449/8)$ or $(149, 411/4)$. Since $m$ is an integer, we have $(q, m) = (37, 28)$. But $m \leq (|G_\alpha^z| \leq 16$, a contradiction. Thus (3.14)
holds.

(3.15) The case that \( N_\alpha^a = D_6 \) and \( N^a \cap N^b \leq Z_2 \times Z_2 \) does not occur.

Proof. Let \( \Delta, L \) and \( K \) be as defined in (3.6). By (3.6), there exists an element \( x \) in \( L_\alpha \) such that its order is odd and \([x^a]_L \) is regular on \( \Delta - \{a\} \).

Since \( (L_\alpha)' \leq N_\alpha^a \) by (3.6) and \( N_\alpha^a = D_6 \), \( x \) stabilizes a normal series \( N_\alpha^a N_\beta^a N_\delta^a \geq N_\beta^a \geq 1 \). Hence \( x \) centralizes \( N_\alpha^a N_\beta^a \) by Theorem 5.3.2 of [2] and so \( x^{-1}N_\alpha^a x = N_\alpha^a \). Put \( \gamma = \beta^2 \). If \( r \neq 1 \), then \( \beta \neq \gamma \), so that \( N_\alpha^a = N_\alpha^a \). From this, \( N_\beta^a = N_\beta^a \). By the doubly transitivity of \( G \), \( N_\beta^a = N_\beta^a \), hence \( N_\beta^a = N^a \cap N^b \), a contradiction. Therefore \( r = 1 \) and \( \Delta = \{\alpha, \beta\} \).

Set \( <x> = Z(N_\alpha^a), \Delta_1 = \alpha \sigma(t) \) and let \( \Delta_1, \Delta_2, \ldots, \Delta_4 \) be the set of \( C_n(z) \)-orbits on \( F(z) \). Since \( L \geq N^a \cap N^b \) and by (3.2), \( N^a \cap N^b = 1 \), \( z \) is contained in \( N^a \cap N^b \). Hence, by Lemma 2.1, \( \beta \in \Delta_1 \) and \( k \) is at least two. By Lemma 2.8, \(|F(z)| = 1 + (q-\varepsilon)/8|N_\alpha^a| = 1 + 3(q-\varepsilon)/8 \). Clearly \( |C_\gamma(z)| = (q-\varepsilon)/8 \) and so \( \Delta_1 \geq 1 + (q-\varepsilon)/8 \). If \( \gamma \in F(z) - \Delta_1 \), then \( C_{N^a}(\gamma) \approx Z_2 \times Z_2 \), for otherwise \( <x> = Z(N_\alpha^a) \leq N^a \cap N^b \) and by Lemma 2.1 \( \gamma \in \Delta_1 \), a contradiction. Hence one of the following holds.

(i) \( k = 3 \) and \( |\Delta_1| = 1 + (q-\varepsilon)/8 \), \( |\Delta_2| = |\Delta_3| = (q-\varepsilon)/4 \).

(ii) \( k = 2 \) and \( |\Delta_1| = 1 + (q-\varepsilon)/8 \), \( |\Delta_2| = (q-\varepsilon)/2 \).

(iii) \( k = 2 \) and \( |\Delta_1| = 1 + 3(q-\varepsilon)/8 \), \( |\Delta_2| = (q-\varepsilon)/4 \).

Let \( \gamma \in F(z) - \Delta_1 \). Then, \( z \in G_{\gamma} - N^b \) and so \( C_{N^b}(\gamma) \approx D_{q+\varepsilon} \) or \( PGL(2, \sqrt{q}) \) by Lemma 2.5 (vii), (viii), (ix). If \( C_{N^b}(\gamma) \approx D_{q+\varepsilon} \), then \( (q+\varepsilon)/2 \) \( |\Delta_1| \) and so \( q = 7 \) and (iii) occurs. But \( (q+\varepsilon)/2 = 3 |\Delta_1| - 1 \) and \( \Delta_1 \approx 1 \), a contradiction. If \( C_{N^b}(\gamma) \approx PGL(2, \sqrt{q}) \), then (i) does not occur because \( \sqrt{q} \not\equiv q-\varepsilon \). Hence \( \sqrt{q} | |\Delta_1| \) and \( \sqrt{q} | |\Delta_2| - 1 \). From this, \( q = 25 \) and (iii) occurs. In this case, we have \( |\Delta_1| = 10 \), so that an element of \( C_{N^b}(\gamma) \) of order 3 is contained in \( N_\delta^a \) for some \( \delta \in \Delta_1 \), contrary to \( N_\delta^a = N_\beta^a \).

4. Case (II)

In this section we assume that \( N_\alpha^a = PGL(2, p^m) \), where \( n = 2mk \) and \( k \) is odd. Since \( n \) is even, \( q = p^m \equiv 1 \) (mod 4). We set \( p^m \equiv 1 \) (mod 4). In section 7 we shall consider the case that \( N_\alpha^a = S_4 \). Therefore we assume \( (p, m) \neq (3, 1) \) in this section.

(4.1) The following hold.

(i) \( N_\alpha^a/N^a \cap N^b \leq Z_2 \) or \( Z \) and \( N^a \cap N^b \geq (N_\alpha^a)' \approx PSL(2, p^m) \).

(ii) If \( (p, m) = (5, 1) \), there exists a cyclic subgroup \( Y \) of \( (N_\alpha^a)' \) such that \( N_{N^a}(Y) \approx D_{q+\varepsilon} \) and \( N_{C_\gamma}(Y)^{(p, m)} \) is doubly transitive.

Proof. As \( N_\alpha^a \\ N^a \cap N^b \), either \( N_\alpha^a/N^a \cap N^b \leq Z_2 \) or \( N^a \cap N^b = 1 \). If \( N^a \cap N^b = 1 \), by Lemma 2.2 and 2.6 (vi), \( N_\beta^a = N_\alpha^a/N^a \cap N^b = N_\alpha^a/N^b \times Z_2 \times Z_2 \), a
contradiction. Therefore \( N^a_\beta / N^a \cap N^\beta = 1 \) or \( N^a \cap N^\beta \leq (N^a_\beta) = P(2, p^m) \).

Now we assume that \((p, m) \neq (3, 1) \) and let \( z \) be an involution in \((N^a_\beta) \). Then \( C_{N^a_\beta}(z) \cong D_{2(\beta^m-1)} \) by Lemma 2.6 (vii). Suppose \( C_{N^a_\beta}(z) \) is not a 2-subgroup and put \( Y = 0(C_{N^a_\beta}(z)) \). Then, if \( Y^z \leq G_{\alpha} \) and \( Y^\delta = Y^\beta \) for some \( \delta \in N^\beta \), then \( Y^z \leq N^a \cap N^\beta \) and so \( Y^z = Y^\beta \) for some \( h \in N^a \cap N^\beta \). Thus \( N_G(Y)^F(Y) \) is doubly transitive. Assume that \( C_{N^a_\beta}(z) \) is a 2-subgroup and set \( C_{N^a_\beta}(z) = \langle u, v \rangle \) with \( u^2 = u^{-1}, v^2 = 1 \). We may assume that \( v \in (N^a_\beta)', \langle u, v \rangle \) is a Sylow 2-subgroup of \((N^a_\beta)'\). Since \( \beta \neq 3,5 \), the order of \( u^2 \) is at least four. On the other hand there is no element of order \( |u^2| \) in \( \langle u, v \rangle \). Hence any element of order \( |u^2| \) which is contained in \( N^a \cap N^\beta \) is necessarily an element of \( N^a \cap N^\beta \). By the similar argument as above, \( N_G(Y)^F(Y) \) is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose \((p, m) \neq (3, 1), (5, 1) \) and set 
\[ \Delta = F(Y) \text{ and } X = N_G(Y). \] Then \( |\Delta| = rs(p^m + \epsilon)/2 + 2 \), where \( s = \sum_{i=0}^{k-1} p^{2mi} \), \( C_G(N^a) = 1 \) and one of the following holds.

(i) \( X^a = \text{ATL}(1, 2^c) \) for some integer \( c \).
(ii) \( X^a = PSL(2, p_i) \) or \( PGL(2, p_i) \), \( r = 1 \) and \( 2p_i = p^m + \epsilon \).

Proof. By Lemma 2.8 (ii), \( |\Delta| = 1 + |N^a \cap X| r/|N^a_\beta \cap X| = 1 + (p^{2mk} - 1) r/2(p^m - \epsilon) = rs(p^m + \epsilon)/2 + 1 \). By (4.1) and Lemma 2.9, we have (i), (ii) or \( X^a = R(3) \).

Assume that \( X^a = R(3) \). Then \( rs(p^m + \epsilon)/2 + 1 = 28 \), hence \( k = 1 \) and \( r(p^m + \epsilon)/2 = 27 \). Since \( r \) is odd and \( r_2 = m_n \), we have \( r = m = 1 \) and \( q = 53^2 \). But a Sylow 3-subgroup of \( X^a \) is cyclic because \( N^a \cap X \cong D_{2^{k-1}} \) and \( X^a / X \cap N^a = X^a N^a / N^a \leq Z_2 \times Z_2 \), a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose \( X^a = \text{ATL}(1, 2^c) \) and put \( W = C_{N^a_\beta}(Y) \). Then \( Y \leq W = Z_{p^{m-1}} \). Since \( C_{N^a}(Y) \) is cyclic, \( W \) is a characteristic subgroup of \( C_{N^a}(Y) \) and so \( W \) is a normal subgroup of \( X^a \). Hence \( W \leq X^a \) and \( (X \cap N^a_\beta) = 1 \) or \( Z_2 \). By Lemmas 2.4 and 2.6, \( F(X \cap N^a_\beta) = 1 + |X \cap N^a_\beta| \) and \( X \cap N^a_\beta \times r / |N^a_\beta| = 1 + r \). Since \( 1 + r < |\Delta| \), \( (X \cap N^a_\beta) = Z_2 \) and hence \( r = s(p^m + \epsilon)/2 - 2 |mk \) and so \( p^m = (k-1) + mk \leq 2 \). Hence \( m = k = r = 1 \) and \( q = 7^2 \).

Let \( R \) be a Sylow 3-subgroup of \( N^a_\beta \). Since \( N^a_\beta = PGL(2, 7) \), we have \( R = Z_3 \). By Lemmas 2.4 and 2.6, \( |F(R)| = 1 + (7^2 - 1)|N^a_\beta| = N^a_\beta \cap |N^a_\beta| = 4 \). Hence \( N_G(R)^F(R) = A_4 \) or \( S_4 \). But is a Sylow 3-subgroup of \( N_G(R) \) because \( N^a = PSL(2, 7) \), contrary to \( N_G(R)^F(R) = A_4 \) or \( S_4 \).

(4.4) (ii) of (4.2) does not occur.
Proof. Let notations be as in (4.2). Suppose $X^\alpha \succeq PSL(2, p_1)$. By the similar argument as in (4.3), $C_{N^\alpha}(Y) \leq X_\Delta$ and so $C_{N^\alpha}(Y^\alpha) = Z_{p_1}$, and $N_{N^\alpha}(Y^\alpha) = D_{2p_1}$. Hence $|(X^\alpha)\Delta| = 2p_1 \cdot 2m$. Since $X^\alpha \succeq PSL(2, p_1)$, $p_1(\text{mod } 2)||(X^\alpha)\Delta|$, hence $p_1 - 1 \mid 8m$. As $k = 1$ and $2p_1 = p^\alpha + \varepsilon$, we have $p^\alpha + \varepsilon - 2 = 32m$. From this, $(p, m, p_1) = (11, 1, 5), (3, 2, 5)$ or $(3, 3, 13)$.

Let $R$ be a cyclic subgroup of $N_{\beta}^\alpha$ such that $R = Z(p^\alpha + \varepsilon)$. By Lemma 2.6, $N_\alpha(R)^{F(\alpha)}$ is doubly transitive and by Lemma 2.8 (ii), $|F(R)| = 1 + |N_\alpha(R)|$ and $N_{N_{\beta}^\alpha}(R) = Z_{2}\times Z_{2}$.

If $(p, m, p_1) = (11, 1, 5)$, $|F(R)| = 7$ and by [9], $N_\alpha(R)^{F(\alpha)} = 42$.

If $(p, m, p_1) = (3, 2, 5)$, $|F(R)| = 5$ and so by [9], $N_\alpha(R)^{F(\alpha)} = 20$.

If $(p, m, p_1) = (3, 3, 13)$, $|F(R)| = 15$. By [9], $N_\alpha(R)^{F(\alpha)}$ is not solvable, a contradiction.

(4.5) $p^\alpha = 5$.

Proof. Assume that $p^\alpha = 5$. Then $n = 2k$ with $k$ odd and $N_{\beta}^\alpha = PGL(2, 5) = S_5$. First we argue that $N_{\beta}^\alpha = N_{\beta} \cap N^\beta$. Suppose false. Then $C_\beta(N_{\beta}^\alpha) = 1$ by Lemma 2.2, and $N_{\beta}^\alpha / N_{\beta} \cap N^\beta = Z_2$ by (4.1). Since $N_{\beta}^\alpha / N_{\beta} = N_{\beta}^\alpha / N^\beta \cap N^\beta = Z_2$ and the outer automorphism group of $S_5$ is trivial, we have $Z(N_{\beta}^\alpha N_{\beta}) = Z_2$.

Let $\omega$ be the involution of $Z(N_{\beta}^\alpha N_{\beta})$ and let $w \in I(N_{\beta}^\alpha) - I(N_{\beta}^\alpha)$. Since $C_{N_{\beta}^\alpha}(w)$ is not solvable, (a) $N_{\beta}^\alpha = N_{\beta} \cap N^\beta$.

Let $V$ be a cyclic subgroup of $N_{\beta}^\alpha$ of order 4. Since $N_{\beta}^\alpha = N_{\beta} \cap N^\beta = S_5$, $N_\alpha(V)^{F(V)}$ is doubly transitive and by Lemma 2.8, $|F(V)| = 1 + |N_\alpha(V)| / |N_{N_{\beta}^\alpha}(V)| = 1 + (p^\alpha - 1)/2(p^\alpha + \varepsilon) = (p^\alpha - \varepsilon)/2 + 1$.

Put $P = N_{N_{\beta}^\alpha}(V)$. Then $N_{\beta}^\alpha = D_5$, $|F(P)| = 1 + |N_{\beta}(P)| / |N_{\beta}^\alpha: N_{\beta}(P)| = r + 1$ and $P^F \simeq Z_2$. If (b) occurs, $k = 1$ and $r = 9$, hence $|F(P)| = 10$, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r + 1)^2 = 3rs + 1$ and so $r = 3s - 2/k$. Hence $k = r = 1$ and $G_{\alpha} / N_{\beta} \leq Z_2 \times Z_2$. Let $z$ be an involution in $N_{\beta}^\alpha$. Then $|F(z)| = 1 + 24 \cdot 25/120 = 6$.
by Lemma 2.8 and \(|\Omega|=1+|N^\alpha: N^\alpha_\beta|=66\) as \(r=1\). By the similar argument as in the proof of (3.12), \(|F(z)|(\{F(z)|-1\})/2+(\{|\Omega|-|F(z)|\})/2=|C_G(z)| \times \alpha \in \langle t\rangle G_{ab}\times \langle t\rangle G_{ab}\rangle, \text{ where } t \text{ is an involution such that } \alpha \in \beta. \) Hence \(|z^\beta \in \langle t\rangle G_{ab}\rangle | =15|G_{ab}\rangle | /|C_G(z)|. \) Set \(H=\langle t\rangle G_{ab}\rangle \) and let \(R\) be a Sylow 3-subgroup of \(N^\alpha_\beta\). By Lemma 2.8, \(|F(R)|=1+24\cdot10/120=3\). Set \(F(R)=\{\alpha,\beta,\gamma\}. \) On the other hand, as \(N^\alpha_\beta=S_5\) and \(\text{Out}(S_5)=1\), we have \(H=Z(H) \times N^\alpha_\beta\) and \(|Z(H)|=2, 4 \) or \(8\). \(G_{ab}\rangle =Z(G_{ab}\rangle ) \times N^\alpha_\beta\) and \(\text{Out}(S_5)\) in the latter case \(G_{ab}\rangle =Z(G_{ab}\rangle ) \times N^\alpha_\beta\) and \(Z(G_{ab}\rangle )=Z_2 \times Z_2\), contrary to Lemma 2.6 (ix). In the former case, we have \(|Z(H)|=2\). For otherwise \(|Z(H)|<G\rangle \text{ and } |Z(H)|=1\) and so letting \(u|Z(H)\rangle \), we have \(|Z(H)|=31 |F(\kappa)| -1=5\), a contradiction. Therefore \(|Z(H)|=Z(H)\rangle Z\rangle 2\) and so \(|\ast\rangle G\rangle \ast \rangle \mid <25+25=50\), while \(|\ast\rangle H \mid =15 |G_{ab}\rangle | /|C_G(z)| =15 \cdot120/24=75\), a contradiction.

5. Case (III)

In this section we assume that \(N^\alpha_\beta=PSL(2, p_m)\), where \(n=mk\) and \(k\) is odd.

Set \(p^m=\varepsilon \in \{\pm 1\} \pmod{4}\). Then \(q=\varepsilon \pmod{4}\) as \(k\) is odd. In section 6 we shall consider the case that \(\Gamma^\alpha_\beta=4\), so we assume \((p,m)=3,1\) in this section. From this \(N^\alpha_\beta\) is a nonabelian simple group and so \(N^\alpha_\beta=\text{Aut}(N^\alpha_\beta)\) or \(\Gamma^\alpha_\beta=1\). If \(\Gamma^\alpha_\beta \in 1\), then \(C_G(N^\alpha)=1\) by Lemma 2.2 and \(N^\alpha_\beta=\text{Aut}(N^\alpha_\beta)\) or \(\Gamma^\alpha_\beta=1\). If \(\Gamma^\alpha_\beta \in 1\), then \(C_G(N^\alpha)=1\) by Lemma 2.2 and \(N^\alpha_\beta=\text{Aut}(N^\alpha_\beta)\) or \(\Gamma^\alpha_\beta=1\).

Let \(z\) be an involution of \(N^\alpha_\beta\). Suppose \(z^\in G_{ab}\rangle \) for some \(g \in G\) and set \(\gamma=\alpha^z, \delta=\beta^z\). Then \(z^\in N^\alpha_\beta\) \(G_{ab}\rangle \leq N^\alpha_\beta \leq N^\alpha_\beta \leq N^\alpha_\beta \leq N^\alpha_\beta \) and so \(z^\in z^\in N^\alpha_\beta\). Hence \(C_G(z)^{F(z)}\) is doubly transitive and by Lemma 2.8 (i), \(|F(z)|=(q-\varepsilon)r/(p^m-\varepsilon)+1\).

In particular \(|F(z)|>3r+1\) as \((p^m-\varepsilon)/(p^m-\varepsilon) \geq p^{p+1}+p^{p+1}+1>3\).

By Lemma 2.9, \(C_G(N^\alpha)=1\) and one of the following holds.

(a) \(C_G(z)^{F(z)} \leq \text{ATL}(1,2)\).

(b) \(C_G(z)^{F(z)} \geq \text{PSL}(2, p_1)\) \((p_1 \geq 5), r=1\) and \(|C_N(z)|: C_{N^\alpha_\beta}(z)|=p_1\).

(c) \(C_G(z)^{F(z)} = R(3)\).

Let \(Y\) be a cyclic subgroup of \(C_{N^\alpha_\beta}(z)=D_{p^m-1}\) of index 2. Since \(C_G(z) \geq Y, z \in Y \) and \(C_G(z)^{F(z)}\) is doubly transitive, we have \(F(Y)=F(z)\). By the similar argument as in (3.1), \(N^\alpha \cap N(C_{N^\alpha}(z))=C_{N^\alpha}(z)\) or \(N^\alpha \cap N(C_{N^\alpha}(z))=A_4\). Hence by Lemmas 2.3 and 2.4 \(|F(C_{N^\alpha}(z))|=1+|C_{N^\alpha}(z)| \mid |N^\alpha_\beta: C_{N^\alpha_\beta}(z)| \mid |N^\alpha_\beta| =1+|A_4| \mid |N^\alpha_\beta: C_{N^\alpha_\beta}(z)| \mid |N^\alpha_\beta|\). Therefore \(|F(C_{N^\alpha_\beta}(z))|=r+1 \text{ or } 3r+1\). From this \(C_{N^\alpha_\beta}(z)^{F(z)}=Z_2\).

In the case (a), \((r+1)^2=1+(p^m-\varepsilon)r/(p^m-\varepsilon)\) by Lemma 2.5 and hence \(r=(p^m-\varepsilon)/(p^m-\varepsilon)-2 \mid mk\). Since \((p^m-\varepsilon)/(p^m-\varepsilon) \geq ((p^m)^k+1)/((p^m)^k+1)=\sum_{m=k}^{k}(-p^m)^i\) and \(k \geq 3\), we have \(p^{m(k-1)}(p^{m-1}+mk) \leq mk\), hence \((p^m)^k/(k)(m/(p^{m-1}+mk))<1\). Thus \(k=3, m=1 \) and \(p=3\), contrary to \((p,m)=3,1\).

In the case (b), \(r=1, p_1=(p^m-\varepsilon)/(p^m-\varepsilon), p_1(p_1-1)/2 \) and \(s \mid 4mkp_1\), where \(s\) is the order of \(C_G(z)^{F(z)}\). Hence \(p_1-1 \mid 8mk\). Since \(p_1-1=(p^m-\varepsilon)/(p^m-\varepsilon)-1\)
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\[ (p^n+1)/(p^n+1) - 1 = \sum_{k=0}^{n-1} (-1)^k (p^{n-k}) \geq p^{e_1} (p^n-1), \]
where \( p \) is prime and \( n \geq e_1 = 1 \) because \( p^n \geq 3 \). Hence \( k = 3 \) and \( p^n = 5 \), so that \( p_1 - 1 = 30 \) and \( m = 24 \), a contradiction.

In the case (c), \( r + 1 = 4 \) and \( 1 + (p^n - \varepsilon) r / (p^n - \varepsilon) = 28 \) and so \( r = 3 \) and \( (p^n - \varepsilon) / (p^n - \varepsilon) = 9 \). Hence \( 9 \geq (p^n + 1) / (p^n + 1) \geq p^n - 1, p^n = 3 \), so that \( p^n = 3 \), a contradiction.

6. Case (IV)

In this section we assume that \( N^\circ = A_4 \) and \( q = 3, 5 \) (mod 8). If \( N^\circ \cap N^\circ = 1 \), by Lemma 2.2, \( C_G(N^\circ) = 1 \) and so \( N^\circ / N^\circ = N^\circ / N^\circ \leq Z_2 \times Z_3 \). Hence \( N^\circ / N^\circ \cap N^\circ = 1 \) or \( Z_3 \), so that \( z^g \cap G_{ab} = z^g \cap N^\circ = z^N_{ab} \) for an involution \( z \in N^\circ \).

Therefore \( C_G(z)^{G_{ab}} \) is doubly transitive. By Lemma 2.9, \( C_G(N^\circ) = 1 \) and one of the following holds.

(a) \( C_G(z)^{G_{ab}} \geq \text{PGL}(2, q) \) for some integer \( c \geq 1 \).
(b) \( C_G(z)^{G_{ab}} \geq \text{PSL}(2, q) \) for \( q = 5 \), \( r = 1 \) and \( |C_G(z)| = q_1 \).
(c) \( C_G(z)^{G_{ab}} \geq R(3) \).

Let \( T \) be a Sylow 2-subgroup of \( N^\circ \). Then \( z \in T \) and by Lemmas 2.3 and 2.4,
\[ |F(T)| = 1 + |N_G(T)| r = 1 + 1 + |N_G(T)| r = r + 1. \]
By Lemma 2.8 (i), \( |F(z)| = (q-\varepsilon) r / 4 + 1 \). Hence \( T^{G_{ab}} = Z_2 \) if \( q = 5 \).

If \( q = 5 \), as \( \text{PGL}(2, 5) = \text{PSL}(2, 4) \), (ii) of our theorem holds by [4]. Therefore we may assume \( q = 5 \).

In the case (a), \( (r + 1)^2 = 1 + (q - \varepsilon) r / 4 \) by Lemma 2.5. Hence \( r = (q - \varepsilon - 8) / 4 \) and \( r | n \), so that \( q = 11 \) or 13 and \( r = 1 \). Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \).

Then \( R \subseteq Z_3 \) and \( R \leq N^\circ_3 \) because \( G_{ab}/N^\circ_3 \cong G_{ab} N^\circ / N^\circ = 1 \) or \( Z_2 \) and \( N^\circ_3 = A_4 \).

By Lemma 2.8 (ii), \( |F(R)| = 1 + 12 / 3 = 5 \) and \( N_G(R)^{G_{ab}} \) is doubly transitive. Since \( N_{G_{ab}}(R) = D_{12} \) or \( D_{16} \) and \( |F(R)| = 5 \), we have \( |N_G(R)| = 5 \). Let \( S \) be a Sylow 5-subgroup of \( N_G(R) \).

Then \( S, R \) = 1 as \( N_G(R) / C_G(R) \subseteq Z_2 \). Since \( 5 \not| |G_{ab}| \), \( |F(S)| = 0 \) or 1. If \( |F(S)| = 1 \), \( F(S) \subseteq F(R) \) and so \( 5 \not| |F(R)| = 1 \), a contradiction. Therefore \( S \) is semi-regular on \( \Omega \). But \( |\Omega| = 1 + |N^\circ_3| = 56 \) or 92. This is a contradiction.

In the case (b), \( p_1 (p_1 - 1) / 2 \) and \( s | 2n(q - \varepsilon) / 2 = 4np_1 \), where \( s \) is the order of \( C_G(z)^{G_{ab}} \).

Hence \( p_1 - 1 | 8n \). Since \( p_1 = (q - \varepsilon) / 4 \), \( p_1 - \varepsilon - 4 \) | 32n and so we have \( q = 11, 13, 19, 27 \) or 37. If \( q = 27 \), by Lemma 2.6, \( C_{G_{ab}}(z) = D_{12} \) or \( D_{24} \) and so \( C_{G_{ab}}(z)^{G_{ab}} = Z_2 \). Hence \( (p_1 - 1) / 2 = 2 \). From this \( q = 19 \). Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \).

By the similar argument as in the case (a), \( N_G(R)^{G_{ab}} \) is doubly transitive and \( |F(R)| = 1 + 18 / 3 = 7 \). Hence \( 7 \not| |G| \). On the other hand \( |G| = |\Omega| \cdot |G_1| = (1 + |N^\circ_3| \cdot N^\circ_3) |G_1| = (1 + 18 \cdot 19 / 2) \cdot 2^3 \cdot 3^5 \cdot 11 \cdot 13 \cdot 19 \) with \( 0 \leq i \leq 1 \), a contradiction. If \( q = 27 \), then \( |C_G(z)| = |F(z)| \cdot |C_G(z)_1| = 8 \cdot |G_1| \), while \( |\Omega| = 1 + |N^\circ_3| \cdot N^\circ_3 = 1 + 26 \cdot 27 \cdot 28 / 2 = 820 = 2^3 \cdot 5 \cdot 41 \) and so \( |G| = 4 |G_1| \). Therefore \( |C_G(z)| \not| |G| \), a contradiction.

In the case (c), \( r = 1 + 4 \) and \( 1 + (q - \varepsilon) r / 4 = 28 \). Hence \( r = 3 \) and \( q = 37 \),
7. Case (V)

In this section we assume that \( N^*_{β} = S_4 \) and \( q \equiv 7, 9 \pmod{16} \). We note that \( 4 \not| n \).

First we argue that \( N^*_α = N^* \cap N^β \). Suppose \( N^*_α \leq N^* \cap N^β \). Then \( C_0(N^*_α) \leq 1 \) by Lemma 2.2. Since \( N^*_β/N^* \cap N^β \cong N^*_α/N^* \) by Lemma 2.1, we have \( N^*_α \cap N^β \cong A_4 \) and \( N^*_β/N^* \cap N^β \cong Z_2 \). Hence \( N^*_α \cap N^β \cong \{e\} \).

As \( Out(S_4) = 1 \), \( Z(N^*_βN^*_α) \cong Z_2 \). Set \( \langle t \rangle = Z(N^*_αN^*_β) \) and let \( t \in I(N^*_β) - I(N^*_α) \). Since \( C_{N^*_α}(t) \geq N^*_α = S_4 \) and \( \langle t \rangle N^* = N^*_βN^*_α \), by Lemma 2.6, we have \( C_{N^*_α}(t) = PGL(2, \sqrt{q}) \) and \( |F(t)| = 1 + (q-8)/2 \).

Let \( P \) be a Sylow \( p \)-subgroup of \( C_{N^*_α}(t) \). Then \( |P| = \sqrt{q} \). If \( p \neq 3, \) \( P \) acts semi-regularly on \( (t) \) and so \( \sqrt{q} = 1 + (q-8)/2 \) by Lemma 2.8 and \( |F(U)| = 1 + (q-8)/2 \) by Lemmas 2.3 and 2.4. If \( q \neq 7, 9 \), then \( |F(U)| < |F(V)| \) and hence \( U^{F(V)} \cong Z_2 \). Suppose \( q = 7 \) or \( 9 \). Then \( \sqrt{q} = 1 + 27\sqrt{r} \) and \( |F(t) - \langle t \rangle| \geq |C_{N^*_α}(t) : C_{N^*_β}(t)| \geq |PGL(2, 3)|/8 = 2457 \) contrary to \( r \mid 3 \).

Thus \( N^*_α = N^* \cap N^β \).

Let \( V \) be a cyclic subgroup of \( N^*_β \) of order 4 and let \( U \) be a Sylow 2-subgroup of \( N^*_α \) containing \( V \). Then \( U = N^*_β(V) \), \( |F(V)| = 1 + (q - 8)/2 \) by Lemma 2.8 and \( |F(U)| = 1 + 8\sqrt{r}/24 = r + 1 \) by Lemmas 2.3 and 2.4. If \( q \neq 7, 9 \), then \( |F(U)| < |F(V)| \) and hence \( U^{F(V)} \cong Z_2 \). Suppose \( q = 7 \) or \( 9 \). Then \( r = 1 \) as \( r \mid n \). Hence \( \sqrt{q} = 1 + |N^*: N^*| = 8 \) or \( 16 \). By [10], we have a contradiction. Therefore \( U^{F(V)} \cong Z_2 \).

Suppose \( V^g \leq G_{ab} \) for some \( g \in G \) and set \( γ = α^g \). Then \( V^g \leq g^{-1}N^αg \cap G_{ab} \leq N^γ \cap G_{ab} \leq N^* \cap N^β = N^*_β \). As \( N^*_β = S_4 \), \( V^g = V^h \) for some \( h \in N^*_β \). Hence \( N_0(V^h) \) is doubly transitive. By Lemma 2.9, \( C_0(N^*_α) = 1 \) and one of the following holds.

(a) \( N_0(V^h) \leq AGL(1, 2) \).
(b) \( N_0(V^h) \geq PSL(2, p_1), \) \( p_1 = (q - 8)/2 \geq 5 \).
(c) \( N_0(V^h) = R(3) \).

In the case (a), \( (r + 1)/2 \leq 1 + (q - 8)/2 \) by Lemma 2.5 and so \( r = (q - 8)/2 \) and \( r \mid n \). From this \( q = 23 \) or \( 25 \) and \( r = 1 \). Since \( |N^*: N^*| = 2 \cdot 127 \) or \( 2 \cdot 163 \), we have \( |G|_2 = 2 |G_α|_2 \) while \( |N_0(V)|_2 = |F(V)|_2 |G_α(V)|_2 = 4 |G_α|_2 \), contrary to \( |N_0(V)| \mid |G| \).

In the case (b), \( p_1(1/2) \leq 2 |2n(q - 8)/4 = 4n \), where \( s \) is the order of \( N_0(V^h) \). Hence \( p_1 = 1/8n \). From this, \( p_1(1/8) = 64n \) and so \( q = 23, 41, 71 \) or \( 73 \). Since \( p_1 \) is a prime and \( p_1 = (q - 8)/8 \geq 5, \) \( q = 23, 71 \) or \( 73 \). Therefore \( q = 23 \) and \( |N^*: N^*| = 1 + 40 \cdot 41 \cdot 42 \cdot 2.24 = 2^2 \cdot 359, \) so that \( |G|_2 = 4 |G_α|_2 \).
Since \( N^\beta_\ast = N^\ast \cap N^\beta \), \( C_\ast(z)^{F(z)} \) is transitive by Lemma 2.1. On the other hand \(|F(z)| = 1 + 40 - 9/24 = 16\) by Lemma 2.8 (i) and so \(|C_\ast(z)| = 16|C_\ast(\ast)| = 16|G_{a_l}|_2\), contrary to \(|C_\ast(\ast)| = |G|_2\).

In the case (c), \( r+1=4 \) and \( 1+(q-\epsilon)r/8=28 \). Hence \( r=3 \) and \( q=71 \) or 73, contrary to \( r | n \).

### 8. Case (VI)

In this section we assume that \( N^\beta_\ast = A_5 \) and \( q \equiv 3, 5 \mod 8 \). In particular, \( n \) is odd. If \( N^\beta_\ast \neq N^\ast \cap N^\beta \), then \( N^\ast \cap N^\beta = 1 \) and so \( N^\beta_\ast = N^\ast \cap N^\beta \leq \text{Out}(N^\beta) = Z_2 \times Z_2 \), a contradiction. Hence \( N^\beta_\ast = N^\ast \cap N^\beta \). Let \( z \) be an involution in \( N^\beta_\ast \) and \( T \) a Sylow 2-subgroup of \( N^\beta_\ast \) containing \( z \). Then, by Lemma 2.8 \(|F(z)| = 1+(q-\epsilon)15r/60 = 1+(q-\epsilon)r/4 \) and by Lemmas 2.3 and 2.4 \(|F(T)| = 1+12 \cdot 5r/60 = 1+r \). Since \( N^\beta_\ast = N^\ast \cap N^\beta \), \( z^G \cap N^\beta = z^G \cap N^\beta = z^N \) and so \( C_\ast(z)^{F(z)} \) is doubly transitive. By Lemma 2.9, \( C_\ast(N^\ast) = 1 \) and one of the following holds.

(a) \( C_\ast(z)^{F(z)} \leq AGL(1, 2^5) \).

(b) \( C_\ast(z)^{F(z)} \cong PSL(2, p_1), p_1 = (q-\epsilon)/4 \geq 5 \).

(c) \( C_\ast(z)^{F(z)} = R(3) \).

In the case (a), by Lemma 2.5, \( (q-\epsilon)/4 = 1 \) or \( (r+1)/4 = 1+(q-\epsilon)r/4 \). Hence \( q = 5 \) or \( r = (q-\epsilon-8)/4 | n \). If \( q = 5 \), then \( N^\beta_\ast = N^\ast \), a contradiction. Therefore \( p^\ast = \epsilon - 8/4n \) and so \( n = 1 \) and \( q = 11 \) or 13. If \( q = 13 \), we have \( 5 | |G_\ast| \), a contradiction. Hence \( q = 11 \) and \( |\Omega| = 1+|N^\ast|: N^\beta_\ast = 1+10 \cdot 11 \cdot 12/2 \cdot 60 = 12 \).

By [9], \( C^\ast \cong M_{11}, |\Omega| = 12 \) and so (iii) of our theorem holds.

In the case (b), we have \( p_1(p_1-1)/2 | s \) and \( s | 2n(q-\epsilon)/2 = 4np_1 \), where \( s \) is the order of \( C_\ast(z)^{F(z)} \). Hence \( p_1 = 18n \) and so \( p^\ast = \epsilon - 4/32n \). From this \( q = 19, 27 \) or 37. Since \( 5 | |G_\ast|, q = 27, 37 \). Hence \( q = 19 \) and \( |\Omega| = 1+|N^\ast|: N^\beta_\ast = 1+18 \cdot 19 \cdot 20/2 \cdot 60 = 2 \cdot 29 \). Since \( G_\ast \cong PSL(2, 19) \) or \( PGL(2, 19), |G| = |\Omega| \in |G_\ast| = 2 \cdot 19 \cdot 20/2 = 2/3 \cdot 5 \cdot 19 \cdot 29 \) with \( 0 \leq i \leq 1 \). Let \( P \) be a Sylow 29-subgroup of \( G \). Then \( P \) is semi-regular on \( \Omega \) and 3, 5, 19 \( / \in |N_\ast(P)| \) because \( N_\ast(P)/C_\ast(P) \leq Z_4 \times Z_4 \). Hence \( |G|: N_\ast(P) = 2^i \cdot 3 \cdot 5 \cdot 19 \) with \( 0 \leq j \leq 4 \), while \( 2^i \cdot 3 \cdot 5 \cdot 19 \equiv 1 \) (mod 29) for any \( j \) with \( 0 \leq j \leq 4 \), contrary to a Sylow's theorem.

If \( C_\ast(z)^{F(z)} = R(3), r+1=4 \) and \( 1+(q-\epsilon)r/4=28 \) and hence \( r=3, q=37 \), contrary to \( r | n \).

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References


[3] C. Hering: Transitive linear groups and linear groups which contain irreducible


