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Osaka University
ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS IN WHICH SOCLE($G_\alpha$) IS NONSOLVABLE

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1. Introduction

Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ and $\alpha \in \Omega$. In [8], O'Nan has proved that $\text{socle}(G_\alpha) = A \times N$, where $A$ is an abelian group and $N$ is 1 or a nonabelian simple group. Here $\text{socle}(G_\alpha)$ is the product of all minimal normal subgroups of $G_\alpha$.

In the previous paper [4], we have studied doubly transitive permutation groups in which $N$ is isomorphic to $\text{PSL}(2,q)$, $\text{Sz}(q)$ or $\text{PSU}(3,q)$ with $q$ even. In this paper we shall prove the following:

Theorem. Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ with $|\Omega|$ even and let $\alpha \in \Omega$. If $G_\alpha$ has a normal simple subgroup $N^*$ isomorphic to $\text{PSL}(2,q)$, where $q$ is odd, then one of the following holds.

(i) $G_\Omega$ has a regular normal subgroup.
(ii) $G_\Omega \cong A_6$ or $S_6$, $N^* \cong \text{PSL}(2,5)$ and $|\Omega| = 6$.
(iii) $G_\Omega \cong M_{11}$, $N^* \cong \text{PSL}(2,11)$ and $|\Omega| = 12$.

In the case that $G_\alpha$ has a regular normal subgroup, by a result of Hering [3] we have $(|\Omega|, q) = (16, 9), (16, 5)$ or $(8, 7)$.

We introduce some notations:

$F(X)$: the set of fixed points of a nonempty subset $X$ of $G$
$X(\Delta)$: the global stabilizer of a subset $\Delta(\subseteq \Omega)$ in $X$
$X_\Delta$: the pointwise stabilizer of $\Delta$ in $X$
$X^\Delta$: the restriction of $X$ on $\Delta$
$m|n$: an integer $m$ divides an integer $n$
$X^H$: the set of $H$-conjugates of $X$
$|X|_p$: maximal power of $p$ dividing the order of $X$
$I(X)$: the set of involutions in $X$
$D_m$: dihedral group of order $m$

In this paper all sets and groups are finite.
2. Preliminaries

Lemma 2.1. Let \( G \) be a transitive permutation group on \( \Omega \), \( \alpha \in \Omega \) and \( N^\ast \) a normal subgroup of \( G_\alpha \) such that \( F(N^\ast) = \{ \alpha \} \). Let the subgroup \( X \leq N^\ast \) be conjugate in \( G_\alpha \) to every group \( Y \) which lies in \( N^\ast \) and which is conjugate to \( X \) in \( G \). Then \( N_\alpha(X) \) is transitive on \( \Delta = \{ \gamma \in \Omega \mid X \leq N^\ast \} \).

Proof. Let \( \beta \in \Delta \) and let \( g \in G \) such that \( \beta^g = \alpha \). Then, as \( X \leq N^\ast \), \( X^g \leq N^\ast \). By assumption, \( (X^g)^h = X \) for some \( h \in G_\alpha \). Hence \( gh \in N_\alpha(X) \) and \( \alpha^{(gh)^{-1}} = \alpha^{g^{-1}} = \beta \). Obviously \( N_\alpha(X) \) stabilizes \( \Delta \). Thus Lemma 2.1 holds.

Lemma 2.2. Let \( G \) be a doubly transitive permutation group on \( \Omega \) of even degree and \( N^\ast \) a nonabelian simple normal subgroup of \( G_\alpha \). If \( C_G(N^\ast) \neq 1 \), then \( N^\ast = N^\ast \cap N^\ast \) for \( \alpha \neq \beta \in \Omega \) and \( C_G(N^\ast) \) is semiregular on \( \Omega \).

Proof. See Lemma 2.1 of [4].

Lemma 2.3. Let \( G \) be a transitive permutation group on \( \Omega \), \( H \) a stabilizer of a point of \( \Omega \) and \( M \) a nonempty subset of \( G \). Then

\[
|F(M)| = |N_\alpha(M)| \times |M^G \cap H| / |H| .
\]

Here \( M^G \cap H = \{ g^{-1}Mg \mid g^{-1}Mg^H, g \in G \} \).

Proof. See Lemma 2.2 of [4].

Lemma 2.4. Let \( G \) be a doubly transitive permutation group on \( \Omega \) and \( N^\ast \) a normal subgroup of \( G_\alpha \) with \( \alpha \in \Omega \). Assume that a subgroup \( X \) of \( N^\ast \) satisfies \( X^G = X^N^\ast \). Then the following hold.

(i) \( |F(X) \cap N^\ast| = |F(X) \cap N^\ast| \) for \( \beta, \gamma \in \Omega - \{ \alpha \} \).

(ii) \( |F(X)| = 1 + |F(X) \cap N^\ast| \times r \), where \( r \) is the number of \( N^\ast \)-orbits on \( \Omega - \{ \alpha \} \).

Proof. Let \( \Gamma = \{ \Delta_1, \Delta_2, \ldots, \Delta_r \} \) be the set of \( N^\ast \)-orbits on \( \Omega - \{ \alpha \} \). Since \( G_\alpha \) is transitive on \( \Omega - \{ \alpha \} \) and \( G_\alpha \) is solvable, we have \( |\Delta_i| = |\Delta_j| \) for \( 1 \leq i, j \leq r \). By assumption, \( G_\alpha = N_\alpha(X)N^\ast \) and so \( N_\alpha(X) \) is transitive on \( \Gamma \). Hence for each \( i \) with \( 1 \leq i \leq r \) there exists \( g \in N_\alpha(X) \) such that \( (\Delta_i)^g = \Delta_i \). Therefore \( |F(X) \cap \Delta_i| = |F(X^g) \cap (\Delta_i)^g| = |F(X) \cap \Delta_i| \). Thus (i) holds and (ii) follows immediately from (i).

Lemma 2.5 (Huppert [5]). Let \( G \) be a doubly transitive permutation group on \( \Omega \). Suppose that \( \vartheta_2(G) \neq 1 \) and \( G_\alpha \) is solvable. Then for any involution \( z \) in \( G_\alpha \), \( |F(z)|^2 = |\Omega| \).

We list now some properties of \( PSL(2,q) \) with \( q \) odd which will be required
in the proof of our theorem.

Lemma 2.6 ([2], [6], [10]). Set \( N = PSL(2, q) \) and \( G = Aut(N) \), where \( q = p^r \) and \( p \) is an odd prime. Let \( z \) be an involution in \( N \). Then the following hold.

(i) \( |N| = (q-1)q(q+1)/2 \), \( I(N) = z^N \) and \( C_N(z) = D_{q-2} \), where \( q \equiv 0 \pmod{4} \).

(ii) If \( q \neq 3 \), \( N \) is a nonabelian simple group and a Sylow \( r \)-subgroup of \( N \) is cyclic when \( r \neq 2, \ p \).

(iii) If \( X \) and \( Y \) are cyclic groups of \( N \) and \( |X| = |Y| \neq 2, \ p \), then \( X \) is conjugate to \( Y \) in \( \langle X, Y \rangle \) and \( N_\langle X \rangle = D_{q-2} \).

(iv) If \( X \leq N \) and \( X = Z_2 \times Z_2 \), \( N_\langle X \rangle \) is isomorphic to \( A_4 \) or \( S_4 \).

(v) If \( |N| \geq 8 \), \( N \) has two conjugate classes of four-groups in \( N \).

(vi) There exist a field automorphism \( f \) of \( N \) of order \( n \) and a diagonal automorphism \( d \) of \( N \) of order \( 2 \) and if we identify \( N \) with its inner automorphism group, \( \langle f, d \rangle N = PGL(2, q) \), \( \langle f \rangle \langle d \rangle N = G \) and \( G[N] = Z_2 \times Z_n \).

(vii) \( C_N(d) = D_{q-2} \) and \( C_\langle d \rangle N(z) = D_{q-2} \).

(viii) Suppose \( n = mk \) for positive integers \( m, k \). Then \( C_N(f^m) = PGL(2, p^m) \) if \( k \) is odd and \( C_N(f^m) = PGL(2, p^n) \) if \( k \) is even.

(ix) Assume \( n \) is even and let \( u \) be a field automorphism of order \( 2 \). Then \( I(G) = I(N) \cup d^N \cup u^\langle d \rangle N \). If \( n \) is odd, \( I(G) = I(N) \cup d^N \).

Lemma 2.7. Let \( G, N, d \) and \( f \) be as defined in Lemma 2.6 and \( H \) an \( \langle f, d \rangle \)-invariant subgroup of \( N \) isomorphic to \( D_{q-2} \). Let \( W \) be a cyclic subgroup of \( \langle d \rangle N = PGL(2, q) \) and set \( Y = \langle d \rangle H \) of index \( 2 \) (cf. (vii) of Lemma 2.6). Then \( C_\langle d \rangle (Y) = W \cdot C_{\langle d \rangle}^\langle f \rangle (Y) \).

Proof. By (viii) of Lemma 2.6, we can take an involution \( t \) satisfying \( \langle d \rangle H = \langle f \rangle W \) and \([f, t] = 1\). Since \( N_\langle d \rangle Y = \langle f, d \rangle N_\langle Y \rangle = \langle f, d \rangle H \), \( C_\langle d \rangle Y = C_{\langle d \rangle}^\langle f \rangle \langle f \rangle \langle f \rangle^{-1} \).

Suppose \( \langle d \rangle H = \langle f \rangle W \) for some \( h \in \langle f \rangle \). Since \( t \) inverts \( Y \), \( h \) also inverts \( Y \) and so \( h^2 \) centralizes \( Y \). Hence some nontrivial 2-element \( g \in \langle h \rangle \) inverts \( Y \), so that \( C_\langle x \rangle (g) \) contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, \( G \Omega \) will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume \( G \Omega \) has no regular normal subgroup.
Notation. \( C^*=C_G(N^*) \), which is semi-regular on \( \Omega-\{\alpha\} \) by Lemma 2.2. Let \( r \) be the number of \( N^*\)-orbits on \( \Omega-\{\alpha\} \).

Since \( G_\beta \trianglerighteq N^* \), \( |\beta N^*| = |\beta N^*| \) for \( \beta, \gamma \in \Omega-\{\alpha\} \) and so \( |\Omega| = 1+r \times |\beta N^*| \).

Hence \( r \) is odd and \( N^*_\beta \) is a subgroup of \( N^* \) of odd index. Therefore \( N^*_\beta \) is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

**Lemma 2.8.** Let \( Z \) be a cyclic subgroup of \( N^*_\beta \) with \( |Z| = 1, p \). Then

(i) \( |Z|=2, \ |F(Z)|=1+(q-\epsilon)\left|I(N^*_\beta)\right|r/|N^*_\beta| \).

(ii) \( |Z|=2, \ |F(Z)|=1+|N^*_\beta(Z)|r/|N^*_\beta(Z)| \).

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

**Lemma 2.9.** Let \( N^*_\beta \triangleleft D_{q^s} \), and \( Z \) is a cyclic subgroup of \( N^*_\beta \) with \( |Z| = 1, p \) and \( N^*_G(Z)^{(Z)} \) is doubly transitive. Then \( C^*=1 \) and one of the following holds.

(i) \( N^*_G(Z)^{(Z)} \leq \text{AGL}(1, q^s) \) for some \( q^s \).

(ii) \( N^*_G(Z)^{(Z)} \leq \text{PSL}(2, p^1), r=1 \) and \( |F(Z)| = 1 = |N^*_\beta(Z): N^*_G(Z)| = p^1 \), where \( p^1 \) is a prime.

(iii) \( N^*_G(Z)^{(Z)} = R(3), \) the smallest Ree group, \( |F(Z)| = 28 \).

Proof. Set \( N^*_G(Z)=L \) and \( F(Z)=\Delta \). By Lemma 2.6(iii), \( L \cap N^*=D_{q^s} \) and \( L \cap N^* = \langle t \rangle Y \geq Y \geq Z \), where \( 0(t)=2, Y \simeq Z(q^{s+2})/2 \).

If \( L \cap N^*=1 \), then \( L \cap N^*=N^*_\beta \) because \( L \cap N^* \) is a maximal subgroup of \( N^* \). Since \( |L^*: N^*_\beta| \) is odd, \( L \cap N^*=N^*_\beta=D_{q^s} \), contrary to the assumption. Hence \( L \cap N^*=1 \) and as \( L_{a}$ G$L_{a} \cap N^* \) and \( L_{a} \geq Y, (L_{a})^* \) has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:

(a) \( L^* \) has a regular normal subgroup

(b) \( L^* \geq \text{PSL}(2, p_1), |L^*|=p_1+1 \), where \( p_1 \) is a prime

(c) \( L^* \geq \text{PSL}(3, p_1), p_1 \geq 3, |L^*|=(p_1)^3+1 \)

(d) \( L^* \geq R(3), |L^*|=28 \).

Suppose \( C^*=1 \). Then there exists a subgroup \( D \) of \( C^* \) of prime order such that \( (L_{a})^* \leq D \). Since \( [L_{a}, D] \leq D \cdot L_{a} \cap C^* = D(L_{a} \cap C^*) = D, D \) is a normal subgroup of \( L_{a} \). By (i) and (iii) of Lemma 2.6, \( G_{a}=L_{a} \cap N^* \) and so \( D \) is a normal subgroup of \( G_{a} \). By Theorem 3 of [1], \( G^a \) has a regular normal subgroup, contrary to the hypothesis. Thus \( C^*=1 \).

If (a) occurs, \( L^* \) is solvable because \( L_{a}/L \cap N^*=L_{a}N^*/N^* \leq \text{Out}(N^*) \) and \( L \cap N^*=D_{q^s} \). Hence by [5], (i) holds in this case.

If (b) occurs, we have \( Y^* \leq L_{a} \cap N^* \leq D_{q^s} \) and \( N^*=D_{q^s}, \) a contradiction. Hence \( 1=C_G(Z)^* \leq L^* \) and so \( C_G(Z)^* \geq \text{PSL}(2, p_1) \) and \( Y^* \leq Z_{p_1} \). Therefore \( |\Delta \cap \beta N^*| = p_1 \) and \( r=1 \) by Lemma 2.4 (ii). Since \( |\beta^*|=p_1 \), we have \( |\beta^{L_{a}} N^*|=p_1 \), so that \( L \cap N^*: L \cap N^*_\beta = p_1 \). Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of \( \text{PSL}(3, p_1) \),
a Sylow $p_1$-subgroup of $(L_0')$ is not cyclic, while $(L_0') \leq L \cap N^g = D_{q^2}$, a contradiction.

3. Case (I)

In this section we assume that $N_\beta^a \leq D_{q-r}$, where $\beta = \alpha, q = p^a$.

(3.1) (i) If $N_\beta^a = Z_2 \times Z_2$, $N_\alpha^a(N_\beta^a) = N_\beta^a$ and $|F(N_\beta^a)| = r + 1$.

(ii) If $N_\beta^a = Z_2 \times Z_2$, $N_\alpha^a(N_\beta^a) = A_4$ and $|F(N_\beta^a)| = 3r + 1$.

Proof. Put $X = N_\alpha^a(N_\beta^a)$. Let $S$ be a Sylow 2-subgroup of $N_\beta^a$ and $Y$ a cyclic subgroup of $N_\beta^a$ of index 2.

If $N_\beta^a = Z_2 \times Z_2$, then $|Y| > 2$ and so $Y$ is characteristic in $N_\beta^a$. Hence $X \leq N_\alpha^a(Y) = D_{q-r}$. From this $[N_\beta(X), S \cap Y] \leq S \cap Y$ and $0^2(N_\beta(S))$ stabilizes a normal series $S \geq S \cap Y \geq 1$, so that $0^2(N_\beta(S)) \leq C_\infty(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{\infty}(S) \leq S$ and hence $N_\beta(S) = S$. On the other hand by a Frattini argument, $X = N_\beta(S)N_\beta^a$ and so $X = N_\beta^a$.

By Lemma 2.6(ii), $(N_\beta^a)^{g^a} = (N_\beta^a)^{N_\beta^a}$ and so by Lemmas 2.3 and 2.4(ii), $|F(N_\beta^a)| = 1 + |F(N_\beta^a) \cap \beta^{N_\beta^a} \times r| = 1 + |N_\beta^a| \times r = r + 1$. Thus (i) holds.

If $N_\beta^a = Z_2 \times Z_2$, $N_\alpha^a(N_\beta^a) = A_4$ by Lemma 2.6(iv). Similarly as in the case $N_\beta^a = Z_2 \times Z_2$, we have $|F(N_\beta^a)| = 3r + 1$.

(3.2) $N_\beta^a/N_\alpha^a \cap N^g \leq Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^a = 1$. Suppose $C^a = 1$. Then $N_\beta^a/N_\alpha^a \cap N^g = N_\beta^a/N_\beta^a \leq \text{Out}(N^g) = Z_2 \times Z_2$ by Lemma 2.6(vii) and hence $(N_\beta^a)^{N^g} \leq N_\alpha^a \cap N^g$. Since $N_\beta^a$ is dihedral, $N_\beta^a(N_\beta^a)^{N^g} = Z_2 \times Z_2$, so that $N_\beta^a/N_\alpha^a \cap N^g \leq Z_2 \times Z_2$.

(3.3) Suppose $N_\beta^a = N_\alpha^a \cap N^g$ and let $U$ be a subgroup of $N_\alpha^a$ isomorphic to $Z_2 \times Z_2$. Then $|F(U)| = 3r + 1$ and $N_\alpha(U)^{F(U)}$ is doubly transitive.

Proof. Sex $X = N_\alpha(U), \Delta = F(N_\alpha^a)$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^g$-orbits on $\Omega - \{\alpha\}$. If $g^2N_\beta^a \leq G_{\alpha^a}$, then $g^{-1}N_\beta^a \leq N_\beta^a \cap N^g = N_\beta^a \cap N^g \leq N_\beta^a$, where $\gamma = \alpha^g$. By a Witt's theorem, $X^g$ is doubly transitive.

If $U$ is a Sylow 2-subgroup of $N_\alpha^a$, by a Witt's theorem, $N_\alpha(U)^{F(U)}$ is doubly transitive. Moreover $N_\alpha(U) = A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)| = 1 + |A_4| \times |N_\beta^a| \times r/|N^g| = 3r + 1$.

If $|N_\beta^a| > 4$, by Lemma 2.6(iv) and (v), $N_\alpha(U) = S_4$ and $N_\alpha^a$ has two conjugate classes of four-groups, say $\pi = \{K_1, K_2\}$. Set $X_{\pi} = M$. Then $M \geq N_\alpha^a$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_i = \phi$ for each $i$ and so $|F(U) \cap \Delta_i| = 3$ by Lemma 2.3. Hence $|F(U)| = 3r + 1$. Since $N_\alpha(U) = S_4$, we may assume $r > 1$. Hence by (3.1)(i) $|\Delta| = r + 1 \geq 4$, so that $M^a$ is doubly transitive. Since $M = N_\beta^a N_\alpha(U), M_\alpha^a(U)$ is also doubly transitive and so $N_\alpha(U)$ is transitive on $\Delta$—
\{a\}. As $|\Delta \cap \Delta_i|=1$, $\Delta \cap \Delta_i \subseteq F(U)$ and $N_{\Delta}(U)$ is transitive on $F(U) \cap \Delta_i$ for each $i$, $N_{\Delta}(U)^{F(U)}$ is doubly transitive.

(3.4) (i) $C^a=1$.

(ii) Let $U$ be a subgroup of $N^a$ isomorphic to $Z_2 \times Z_2$. If $N^a=N^a \cap N^b$, then $N_{\Delta}(U)^{F(U)}$ has a regular normal $2$-subgroup. In particular $|F(U)|=3r+1=2^b$ for positive integer $b$.

Proof. Since $N_{\Delta}(U)^{F(U)} \supseteq N^a(U)^{F(U)}=S_3$ or $Z_3$, by (3.3) and Theorem 3 of [1], $N_{\Delta}(U)^{F(U)}$ has a regular normal subgroup, $N_{\Delta}(U)^{F(U)}\supseteq PSU(3,3)$ or $N_{\Delta}(U)^{F(U)}=R(3)$.

Suppose $C^a \neq 1$. Let $D$ be a minimal characteristic subgroup of $C^a$. Clearly $G_{\Delta} \supseteq D$. If $N_{\Delta}(U)^{F(U)}=R(3)$, $D$ is cyclic. By Theorem 3 of [1], $C^a$ has a regular normal subgroup, contrary to the hypothesis. Hence $N_{\Delta}(U)^{F(U)}=R(3)$. Therefore $(N_{\Delta}(U)^{F(U)})'$ contains an element of order 9. Since $N_{\Delta}(U)/C^aN_{\Delta}(U)\cong N_{\Delta}(U)/C^aN_{\Delta}(U)$, by (vi) of Lemma 2.6 we have $(N_{\Delta}(U)^{F(U)})'=C^a \times N_{\Delta}(U)$. From this, $C^a$ contains an element of order 9 and so $C^a=Z_9$ or $M_3(3)$. In both cases, $C^a$ contains a characteristic subgroup of order 3. Since $G_{\Delta} \supseteq D$, by Theorem 3 of [1] $G^a$ has a regular normal subgroup, a contradiction. Thus $C^a=1$.

Let $R$ be a Sylow $3$-subgroup of $N_{\Delta}(U)$. Since $N_{\Delta}(U)/N^a(U)=N_{\Delta}(U)/N^a(U)\leq \Out(N^a)=Z_2 \times Z_3$, $R/R \cong N^a(U)$ is cyclic. Clearly $R \cap N^a(U)\cong Z_3$. Therefore $N_{\Delta}(U)^{F(U)}\supseteq PSU(3,3)$, $(3.4)$ holds.

Since $N^a_{\Delta}$ is dihedral, we set $N^a_{\Delta}=\langle \alpha \rangle W$ and $Y=W \cap N^a \cap N^b$, where $W$ is a cyclic subgroup of $N^a_{\Delta}$ of index 2 and $t$ is an involution in $N^a_{\Delta}$ which inverts $W$.

(3.5) (i) If $|Y| \geq 3$, $N_{\Delta}(Y)^{F(Y)}$ is doubly transitive.

(ii) If $|Y|<3$, $N^a_{\Delta}=Z_2 \times Z_2$ or $N^a_{\Delta}=D_6$ and $N^a \cap N^b \leq Z_2 \times Z_2$.

Proof. Suppose $|Y| \geq 3$. If $Y^g \leq G_{\Delta}$, $Y^g \leq N^a \cap G_{\Delta} \leq N^a$, where $\gamma=\alpha^g$. If $\gamma=\alpha$, obviously $Y^g \leq N^a$. If $\gamma \neq \alpha$, $N^a=\gamma$. Therefore, as $|Y| \geq 3$, $N^a$ has a unique cyclic subgroup of order $|Y|$. Hence $Y^g \leq N^a \cap N^a \leq N^a$, so that $Y^g \leq N^a$. Similarly $Y^g \leq N^b$. Thus $Y^g \leq N^a \cap N^b$ and so $Y^g=Y$. By a Witt’s theorem, $N_{\Delta}(Y)$ is doubly transitive on $F(Y)$.

Suppose $|Y|<3$. Since $|N^a \cap N^b|=2$, we have $N^a \cap N^b \leq Z_2 \times Z_2$. On the other hand, as $N^a$ is dihedral, $(N^a_{\Delta})'$ is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set $\Delta=F(N^a_{\Delta})$, $L=G(\Delta)$, $K=G_{\Delta}$ and suppose $N^a_{\Delta} \neq Z_2 \times Z_2$. Then $L^a \supseteq N^a_{\Delta}$, $(L_a')\leq N^a_{\Delta}$, $K'\leq N^a \cap N^b$ and $(L_a')'=Z_r$. If $r \neq 1$, $L^a$ is a doubly transitive Frobenius group of degree $r+1$.

Proof. By Corollary B1 of [7] and (i) of (3.1), $L^a$ is doubly transitive and
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Since \( N^* \cap L \geq N^* \cap K = N^*_b \), by (i) of (3.1), we have \( N^* \cap L = N^*_b \). Hence \( L_a \supseteq N^*_b \). By (i) of (3.4), \( L_a/N^*_b = L_a/N^* \leq \text{Out}(N^*) = Z \times Z \), and so \( (L_a)^a \supseteq Z \). If \( r \neq 1 \), then \( (L_a)^a \neq 1 \). On the other hand \( (L_a)^a = 1 \) as \( (L_a)^a \) is abelian. Hence \( L^a \) is a Frobenius group.

(3.7) Suppose \( |Y| \geq 3 \). Then there exists an involution \( z \) in \( N^*_b \cap Y \) such that \( Z(N^*_b) = \langle z \rangle \).

Proof. Suppose \( N^*_b \neq Z \). Since \( N^*_b \) is dihedral, we have \( \langle J(W) \rangle = Z(N^*_b) = Z_2 \) and \( N^*_b/N^* \gamma = Z_2 \). Let \( Z(N^*_b) = \langle z \rangle \) and suppose that \( z \) is not contained in \( Y \). By (3.2), \( (N^*_b)^\gamma \leq N^* \cap N^* = W = Y \) and so \( |(N^*_b)^\gamma| \) is odd. Hence \( |N^*_b| = 4 \) and \( q \equiv 3 \) or 5 (mod 8), so that \( n \) is odd. By (3.2) and (i) of (3.4), \( N^*_b/N^* \gamma \cap N^* = N^*_b/N^* \gamma = 1 \) or \( Z_2 \). If \( N^*_b = N^* \cap N^* \), then \( W = Y \) and so \( z \in Y \), contrary to the assumption. Therefore we have \( N^*_b \cap N^* = Z_2 \) and \( N^*_b = \langle z \rangle \times (N^* \cap N^*). \) Since \( n \) is odd and \( z \in N^*_b \cap N^*, \) by Lemma 2.6 (vi), (vii) and (ix), \( N^*_b \cap N^* = Z \). Hence \( N^* \cap N^* \neq Z_2 \), a contradiction.

(3.8) Suppose \( |Y| \geq 3 \). Then \( N^*_b = N^* \cap N^* \).

Proof. Suppose \( N^*_b \neq N^* \cap N^* \) and let \( \Delta, L, K \) be as defined in (3.6) and \( x \in L_a \) such that its order is odd and \( \langle x \rangle \) is transitive on \( \Delta - \{a\} \). As \( |Y| \geq 3 \), \( W \) is characteristic in \( N^* \) and hence by (3.6), \( x \) stabilizes a normal series \( L_a \supseteq N^*_b \supseteq W \supseteq (N^*_b)^\gamma \). By Theorem 5.3.2 of [2], \( [x, 0,(L_a/(N^*_b)^\gamma)] = 1 \). Since \( L_a/(N^*_b)^\gamma \) has a normal Sylow 2-subgroup and \( (N^*_b)^\gamma \leq K, \) we have \( [x, 0,(L_a/K)] = 1 \), so that \( x, N^*_b \leq K \leq N^* \cap N^* \) by (3.6). If \( r \neq 1 \), then \( \beta^r \neq \beta \) and \( \beta^r \in \Delta \), hence \( N^*_b = x^{-1}N^*_b = N^*_b \), where \( \gamma = \beta^r \). Since \( \gamma \in \Delta \) and \( \Delta = F(N_2), N^*_b \leq N^* \cap G \gamma = N^*_b \) and so \( N^*_b = N^* \). Similarly \( N^*_b = N^*_b \). Hence \( N^*_b = N^*_b \), which implies \( N^*_b \cap N^* \). By the doubly transitivity of \( G \), we have \( N^*_b = N^* \cap N^* \), contrary to the assumption. Therefore we obtain \( r = 1 \).

Let \( z \) be as defined in (3.7) and put \( k = (q-\varepsilon)/|N^*_b| \). By Lemma 2.8(i) we have \( |F(z)| = 1 + (q-\varepsilon)/|N^*_b| = (q-\varepsilon)/2 + k + 1 \). Similarly \( |F(Y)| = k + 1 \). As \( N^*_b \neq N^* \cap N^* \), there is an involution \( t \) in \( N^* \) which is not contained in \( N^* \). By Lemma 2.6 (i), \( t = z \gamma \) for some \( \gamma \in N^* \). Set \( \gamma = \beta^r \). Then \( \gamma \in F(z) \) and \( z \in N^* \). By Lemma 2.6 (vii), (viii) and (ix), \( C_{N^*}(z) = D_{q+1}, \) or \( PGL(2, \sqrt{q}) \). Assume \( C_{N^*}(z) = D_{q+1} \) and let \( R \) be a cyclic subgroup of \( C_{N^*}(z) \) of index 2. We note that \( R \) is semi-regular on \( \Omega - \{\alpha\} \). Set \( X = C_R(z) \). Since \( 2 \leq k + 1 \leq (q-\varepsilon)/2 \), we have \( (q-\varepsilon)/2 \leq k + 1 \) and so \( |\alpha^x| > k + 1 \). By (i) of (3.5) and (3.7), \( N^*_C(Y) = C_{N^*}(z) = X \) and \( \alpha^x \supseteq F(Y) \). It follows from Lemma 2.1 that \( \alpha^x = \{z \in N^* \} \gg \gamma \). Hence \( |F(z)| > |\alpha^x| > |F(Y)| + (q-\varepsilon)/2 = k + 1 + (q-\varepsilon)/2 + \varepsilon = |F(z)| + \varepsilon \). Therefore \( \varepsilon = 1 \) and \( \gamma^x = \{\gamma\} \), so that \( \gamma \in F(Y) \), a contradiction. Thus \( C_{N^*}(z) = PGL(2, \sqrt{q}), \) \( \varepsilon = 1, \) \( N^*_b \cap N^* = Z_2 \) and \( |\langle \alpha^x \cap G_a \rangle| = 2 \).
Set $\Delta_1 = \alpha^x$ and $\Delta_2 = F(z) - \Delta_1$. Let $\delta \in \Delta_2$ and $g$ an element of $G$ satisfying $\delta^g = z$. Then $x \in N_s^gN^N - N^N$ and so $x^\prime \in N_s^gN^N - N^N$, where $v = \alpha^x$. Since $\langle \delta^g \cap G \rangle : N^N = 2$ and $x \in G_\gamma - N^N$, it follows from Lemma 2.6 (ix) that $(\alpha^x)^h = z$ for some $h \in G_\gamma$. Hence $g \in X$ and $\delta^g = z$. Thus $\Delta_2 = \gamma^x$. Let $\delta \in \Delta_2$. Then $\delta \in \Delta_s$ and $\gamma \in G_s$ satisfying $\delta^g = \gamma^x$. Then $z \in N_s^g$ and $\gamma \in Z(N_s^g)$ by (3.7) and so $X \cap N_s^g = Z_3 \times Z_2$, which implies $|\delta^{c(x)}| = (q - 1)/4$. Hence $|(\Delta_1|, |\Delta_2)| = ((q - 1)/4 + k - 1)/2$. Let $P$ be a subgroup of $C_{N_s}(z)$ of order $\sqrt{q}$. Then $F(P) = \{z\}$ and $P$ is semi-regular on $\Omega - \{\gamma\}$. If $|\Delta_1| = (q - 1)/4$, then $\sqrt{q} | (q - 1)/4 - 1 = (q - 5)/4$ and $\sqrt{q} | (q - 1)/4 + k - 1$. From this, $q = 5^e, k = 3$, $|\Delta_1| = 10$ and $|\Delta_2| = 6$. Since $(C_\gamma(x))^g = (S_3, X^x \cong S_3) \times X$ and so $|X| \geq 3^k$. As $X$ acts on $\Delta_1$ and $|\Delta_1| \equiv 1 \mod{3}$, $|G_s| \geq |X| \geq 3^k$ and $\Delta_1 = \{x, \beta\}$. Hence $C_{N_s}(z)$ fixes $x$ and $\beta$, so that $PGL(2, 3) = C_{N_s}(\gamma) \leq N_s^\ast \cong D_9$, a contradiction.

(3.9) Suppose $|Y| \geq 3$. Then $r = 1$.

Proof. By (3.6), $r + 1 = 2^c$ for some integer $c \geq 0$. On the other hand $3r + 1 = 2^c$ by (3.8) and (ii) of (3.4). Hence $2r = 2(2^c - c) - 1$ and so $c = 1$ as $r$ is odd. Thus $r = 1$.

(3.10) Put $k = (q - \varepsilon)/|N_s^\ast|$. If $N_s^\ast = N^s \cap N^b$ and $r = 1$, then

$$q - \varepsilon + 2k + 2 | 2(2 + \varepsilon)(k + 1 - \varepsilon)(k + 1 - \varepsilon).$$

Proof. Set $S = \{[x, u] | x \in F(u), u \in z^g\}$, where $x$ is an involution in $N_s^g$. We now count the number of elements of $S$ in two ways. Since $N_s^\ast = N^s \cap N^b$, $F(z) = \{z \in \Omega | z \in N^N\}$ and hence $C_G(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S| = |\Omega| |z^g| = |z^g| |F(z)|$. Since $r = 1$, $|\Omega| = 1 + |N^s|: N_s^\ast = kg(q + \varepsilon) - 1/2 + 1$ and by Lemma 2.8 $|F(z)| = (q - \varepsilon)/2 + k + 1$. Since $G_s \geq |N^s|, z^g$ is contained in $N^s$ and so $|G_s|: C_G(z) = |N^s|: C_{N^s}(z) = q(q + \varepsilon)/2$. Hence $(q - \varepsilon)/2 + k + 1 | (kg(q + \varepsilon + 2)/2(q + \varepsilon)/2).$ On the other hand $|F(z)| = |C_G(z)|/2 | C_{G_s}(z)| \leq |G_s|/2 | C_{G_s}(z)| = |G_s|/2 | G_s| = |\Omega|/2$ because $|G_s|: C_{G_s}(z) = q(q + \varepsilon)/2 + 1 \mod{2}$. Hence $|q - \varepsilon + 2k + 2/2 | kq(q + \varepsilon + 2)/2$. Since $kq(q + \varepsilon + 2) = kg(k + 2)(\varepsilon - k - 2)$ $q - \varepsilon + 2k + 2 + 2((2 + \varepsilon)(k + 1 - \varepsilon)(k + 1 - \varepsilon)$ and $q + \varepsilon = (q + 2\varepsilon - 2k - 2)(q - \varepsilon + 2k + 2 + 2(2k + 2 - \varepsilon)(k + 1 - \varepsilon)$, we have (3.10).

(3.11) Suppose $|Y| \geq 3$. Then one of the following holds.

(i) $N_s^\ast = N^s \cap N^b \neq D_{q-\varepsilon}$.

(ii) $N_s^\ast = N_s^b \cap N^b \neq D_{q-\varepsilon}$ and $N_G(Y)^{F(Y)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(Y)} = R(3)$ or there exists a prime $p_1 \geq 5$ such that $C_G(Y)^{F(Y)} \geq PSL(2, p_1)$ and $V/Y \cong Z_{p_1}$, where $V = C_{N^s}(Y)$. By (i) of (3.1) and (3.9), $F(N_s^\ast) = \{x, \beta\}$. On the other hand, $(N_s^\ast)^{F(Y)} \cong N_s^\ast | Y \cong Z_2$. Hence $N_G(Y)^{F(Y)} \neq R(3)$ and $C_G(Y)^{F(Y)} \geq$
By (i) of (3.4) and Lemma 2.7, we have \( C_{G_a}(Y) = V \langle f_i \rangle \), where \( f_i \) is a field automorphism of \( N^a \). Let \( t \) be the order of \( f_i \), \( n = tm \) and let \( p^m \equiv \varepsilon_i \equiv \pm 1 \pmod{4} \). Clearly \( C_{G_a}(Y)^{\langle f_i \rangle} \geq V^{\langle f_i \rangle} \cong \mathbb{Z}_{n} \) and \( |C_{G_a}(Y)^{\langle f_i \rangle}| / t \), so that \( (p^m - 1)/2 | t \).

First we assume that \( t \) is even and set \( t = 2t_1 \). Then \( Y \leq C_{N^a}(f_i) = PGL(2, p^m) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \) is a cyclic subgroup of \( C_{N^a}(f_i) \) of order \( p^m - \varepsilon_1 \) and \( (p^m - 1)/2(p^m - \varepsilon_1) = p_1 \). Put \( s = \sum_{i=1}^{t_1} (p^m)^i \). Then \( (p^m + \varepsilon_1)s/2 = p_1 \), so that we have either (i) \( t_1 = 1 \) and \( p_1 = (p^m + \varepsilon_1)/2 \) or (ii) \( t_1 \geq 2 \), \( p^m = 3 \) and \( p_1 = s \). In the case (i), \( 2 \leq (p^m - 1)/2 = (p^m + \varepsilon_1 - 2)/4 \) if \( t_1 = 2 \). Hence \( (p_1, q) = (5, 3^3) \) or \((4, 11^2)\). Let \( z \) be as in (3.7). As mentioned in the proof of (3.10), \( |F(z)| = (q-1)/2 + k + 1 \), \( |\Omega| = kq(q+1)/2 + 1 \) and \( C_{G_a}(z) \) is transitive on \( F(z) \). If \( q = 3^3 \), then \( |F(z)| = 46 \) and \( |\Omega| = 2 \cdot 19 \cdot 23 \). Hence \( |C_{G_a}(z)| = |F(z)| / (C_{G_a}(z)N^a/N^a|N^a| = 46 \cdot 2^2 \cdot 80 = 5 \cdot 23 \) with \( 0 \leq i \leq 3 \).

Let \( P \) be a Sylow 23-subgroup of \( C_{G_a}(z) \) and \( Q \) a Sylow 5-subgroup of \( C_{G_a}(z) \). It follows from a Sylow's theorem that \( P \) is a normal subgroup of \( C_{G_a}(z) \) and so \( [P, Q] = 1 \). Therefore \( |F(Q)| \geq 23 \), contrary to \( 5 \nmid |N^a| \). If \( q = 11^2 \), then \( |F(z)| = 66 \) and \( |\Omega| = 2 \cdot 3 \cdot 6151 \). Let \( P \) be a Sylow 11-subgroup of \( C_{G_a}(z) \). Since \( 11 \nmid |\Omega| \), \( P \) is a subgroup of \( N^y \) for some \( y \in \Omega \) and \( F(P) = \{ y \} \). Hence \( y \in N^\varepsilon \), contrary to \( C_{N^a}(z) = D_{150} \). If \( t = 1 \), \( p_1 = 7 \) and \( q = 3^3 \), so that \( N^a = \mathbb{Z}_2 \times \mathbb{Z}_2 \), a contradiction.

Assume \( t \) is odd. Then \( Y \leq C_{N^a}(f_i) = PGL(2, p^m) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \cong \mathbb{Z}_{p^m - t_1} \) and \( (q-\varepsilon)/(p^m - \varepsilon_1) = p_1 \). Hence \( \sum_{i=1}^{t_1} (p^m)^i(\varepsilon_i)^{-1-i} = p_1 \) and \( (p^m - 1)/2 = (\sum_{i=1}^{t_1} (p^m)^i(\varepsilon_i)^{-1-i}) - 1)/2 \). In particular, \( 2t \geq (p^m)^{t-1} - (p^m)^{t-2} \geq 2(p^m)^{t-2} \geq 2(p^m)^{t-2} \). From this \( t = 3, m = 1, p_1 = 7 \) and \( q = 3^3 \), so that \( N^a = \mathbb{Z}_2 \times \mathbb{Z}_2 \), a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let \( G^a \) be a minimal counterexample to (3.12) and \( M \) a minimal normal subgroup of \( G \). By the hypothesis, \( G \) has no regular normal subgroup and hence \( M^a \neq 1 \). As \( M^a \) is a normal subgroup of \( G^a \), by (i) of (3.4), \( M^a \) contains \( N^a \). By (3.9), \( r = 1 \), hence \( M \) is doubly transitive on \( \Omega \). Therefore \( G = M \) and \( G \) is a nonabelian simple group.

Since \( N^a \cong D_{4-t}, k = 1 \) and so \( q - \varepsilon + 4 | 2((4 - \varepsilon)(2 - \varepsilon) + 1)(4 - \varepsilon)(2 - \varepsilon) \) by (3.10). Hence we have \( q = 7, 9, 11, 19, 27 \) or 43.

Let \( x \) be an element of \( N^a \). If \( |x| > 2 \), by Lemma 2.8, \( |F(x)| = 1 + |N^a| \times 1/|N^a| = 2 \) and if \( |x| = 2 \), similarly we have \( |F(x)| = (q - \varepsilon)/2 + 2 \). Assume \( q = 9 \) and let \( d \) be an involution in \( G^a - N^a \) such that \( \langle d \rangle N^a \) is isomorphic to \( PGL \).
(2, q). We may assume \( d \in G_{\alpha \beta} \). Since \( \langle d \rangle N^* \) is transitive on \( \Omega = \{ \alpha \} \), by Lemma 2.3 and 2.6 (vii), (ix), \( |F(d)| = 2(q-1)(q+1/2)/(q+1)+1=(q+1)/2 \), while \( |F(x)|=(q+1)/2+2 \) for \( x \in I(N^*) \). Hence \( d \) is an odd permutation, contrary to the simplicity of \( G \). Thus \( G_{\alpha \beta} = N^* \) if \( q \neq 9, 27 \) and \( |G_{\alpha \beta}/N^*| = 1, 3 \) if \( q = 27 \).

If \( g = 9 \), \( |\Omega| = 1 + 9 \cdot 10/2 = 27 \) and \( |G_{\alpha \beta}| = 2^7 |PSL(2, 9)| = 2^{2 + 3 + 5}.5 \) with \( 0 \leq a \leq 3 \). Let \( P \) be a Sylow 23-subgroup of \( G \). Since \( Aut(Z_{23}) \cong Z_2 \times Z_{11} \), \( \forall \alpha \in |N_{G}(P)| \), for otherwise \( P \) centralizes a nontrivial 3-element \( x \) and so \( P \cong F(x) \) because \( |F(x)| = 1 \), contrary to \( |F(P)| = 0 \). Similarly \( 5 \not\mid |N_{G}(P)| \).

Hence \( |P(z)| = 2^a \cdot 3^b \cdot 5 \) for some \( a \) with \( 0 \leq a \leq 6 \). By a Sylow’s theorem, \( 2^a \cdot 3^b \cdot 5 \equiv 1 \pmod{23} \), a contradiction.

If \( g = 27 \), \( |\Omega| = 1 + 27 \cdot 26/2 = 2^{11} \) and \( |G_{\alpha \beta}| = 2^5 |PSL(2, 27)| = 2^2 3^2 5^2 \) with \( 0 \leq a < 1 \). Let \( P \) be a Sylow 11-subgroup of \( G \). Since \( P \cong Z_{23} \) and \( Aut(Z_{23}) \cong Z_2 \times Z_{11} \), \( \forall \alpha \in N_{G}(P) \) by the similar argument as above. Hence \( |G : N_{G}(P)| = 2^a \cdot 3^b \cdot 7 \cdot 13 \) with \( 0 \leq a \leq 7 \) and \( 3 \leq b \leq 3 + i \). By a Sylow’s theorem, \( 2^2 \cdot 3^3 \cdot 7 \cdot 13 \equiv 2^a \cdot 3^b \cdot 5 \cdot 1 \equiv 1 \pmod{11} \). Hence \( a = 0, b = 4 \). Therefore \( N_{G}(P) \) contains a Sylow 2-subgroup \( S \) of \( G \). Let \( T \) be a Sylow 2-subgroup of \( N_{G}(P) \). Then \( T < S \). Then \( T \cong Z_{23} \) and \( |T| = 2^a \cdot 3^b \cdot 5 \) with \( 0 \leq a \leq 6 \). By a Sylow’s theorem, \( 2^a \cdot 3^b \cdot 5 \equiv 1 \pmod{23} \), a contradiction.

If \( g = 7, 11, 19 \) or \( 43 \), then \( G_{\alpha \beta} = N^* \) and \( q - 1 \). Set \( \Gamma = \{ (y, z) \mid y, z \in \Omega, y \neq z \} \). We consider the action of \( G \) on \( \Gamma \). Since \( G \) is doubly transitive, \( G \) is transitive and \( G : \Gamma = 1 \). Let \( 2 \) be an involution of \( Z(N^*) \). There exists an involution \( \tau \) such that \( \tau \cong \tau \). Since \( G_{\alpha \beta} = N^* \) and \( F(N^*) = \{ \alpha, \beta \} \), we have \( G_{\alpha \beta} = \langle \tau \rangle \cong \tau \). By Lemma 2.3, \( |\langle \tau \rangle N^* | = |C_G(\tau) \times | \langle \tau \rangle N^* \cap \alpha^c \cap |N^* | = |F(\tau) \times | \langle \tau \rangle N^* \cap \alpha^c \cap |N^* | = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1 \pmod{11} \). Hence \( a = 0, b = 4 \). Therefore \( N_{G}(P) \) contains a Sylow 2-subgroup \( S \) of \( G \). Let \( T \) be a Sylow 2-subgroup of \( N_{G}(P) \). Then \( T \cong Z_{23} \) and \( |T| = 2^a \cdot 3^b \cdot 5 \) with \( 0 \leq a \leq 6 \). By a Sylow’s theorem, \( 2^a \cdot 3^b \cdot 5 \equiv 1 \pmod{23} \), a contradiction.
5 \not| |N_c(Q)| and 11 \not| |N_c(Q)| by the similar argument as in the case \( q=9 \). Therefore \(|G: N_c(Q)| = 2^a \cdot 5 \cdot 11 \) for some \( a \) with \( 0 \leq a \leq 3 \). Hence \(|G: N_c(Q)| \equiv 1 \pmod{7}\), a contradiction. Thus \( \langle \rho \rangle \cong D_8 \).

Let \( U \) be a Sylow 2-subgroup of \( N^*_a \) and set \( L = N^*_a(U) \). It follows from (3.3) and Lemma 2.6 (iv) that \( L \cap N^*_a = A_4 \), \( L^F(U) = A_4 \) and \(|L|=2^3 \cdot 3\). Let \( T, \langle \sigma \rangle \) be Sylow 2- and 3-subgroup of \( L \), respectively. Obviously \( L \supset T \) and \( \langle \sigma \rangle \n cong A_4 \) and \(|L \cap \langle \sigma \rangle| = 2^4 \cdot 3 \).

Let \( \tau \) be a Sylow 3-subgroup of \( L \). Since \( \langle \tau \rangle \n cong A_4 \) and \(|L \cap \langle \tau \rangle| = 2^2 \cdot 3 \), \( \langle \sigma \rangle \cap \langle \tau \rangle \n cong A_4 \) and \(|L \cap \langle \sigma \rangle \langle \tau \rangle| = 2^3 \cdot 3 \).

Let \( P \) be a Sylow 61-subgroup of \( G \). Then \( P \cong Z_{61} \). As mentioned above, \( 5, 13 \n| C\langle P \rangle \) and so \(|G:C\langle P \rangle| = 2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv 0 \pmod{173} \), where \( 0 \leq a \leq 12 \). Hence \(|G:C\langle P \rangle| \equiv 1 \pmod{173} \), a contradiction.

If \( q=7^2 \), then \(|\Omega| = 2^4 \cdot 61 \) and \(|G_a| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 13 \) \((0 \leq i \leq 2)\). Let \( P \) be a Sylow 61-subgroup of \( G \). Then \( P \cong Z_{61} \). As mentioned above, \( 5, 13 \n| C\langle P \rangle \) and so \(|G:C\langle P \rangle| = 2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv 0 \pmod{173} \), where \( 0 \leq a \leq 10 \) and \( 0 \leq b, c \leq 1 \). But we can easily verify \(|G:C\langle P \rangle| \equiv 1 \pmod{61} \), contrary to a Sylow's theorem.

If \( q=7^2 \), then \(|\Omega| = 2^4 \cdot 919 \) and \(|G_a| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 7^2 \) \((0 \leq i \leq 2)\). Let \( P \) be a Sylow 919-subgroup of \( G \). By the similar argument as above, we obtain \( 5, 7 \n| N_c(P) \) and so \(|G:N_c(P)| = 2^a \cdot 3 \cdot 5 \cdot 7^2 \equiv 0 \pmod{919} \), where \( 0 \leq a \leq 8 \) and \( 0 \leq b \leq 1 \). Hence \(|G:N_c(P)| \equiv 1 \), a contradiction.

If \( q=7^2 \), then \(|\Omega| = 2^7 \cdot 173 \) and \(|G_a| = 2^{7+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \) \((0 \leq i \leq 2)\). Let \( P \) be a Sylow 173-subgroup of \( G \). Similarly we have \( 3, 5, 11, 61 \n| N_c(P) \) and so \(|G:N_c(P)| = 2^a \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv 0 \pmod{173} \), where \( 0 \leq a \leq 12 \). Hence \(|G:N_c(P)| \equiv 1 \), a contradiction.

If \( q=59 \), then \(|\Omega| = 2^6 \cdot 17 \cdot 151 \) and \(|G_a| = 2^{6+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59 \) \((0 \leq i \leq 1)\). Let \( P \) be a Sylow 17-subgroup of \( G \). By the similar argument as above, we obtain \( 3, 5, 29, 59 \n| N_c(P) \) and so \(|G:N_c(P)| = 2^a \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151^2 \equiv 0 \pmod{17} \), where \( 0 \leq a \leq 4 \) and \( 0 \leq b \leq 1 \). From this, we have a contradiction.

If \( q=71 \), then \(|\Omega| = 2^5 \cdot 233 \) and \(|G_a| = 2^{5+i} \cdot 3 \cdot 5 \cdot 7 \cdot 71 \) \((0 \leq i \leq 1)\). Let \( P \) be
a Sylow 233-subgroup of G. Since $3, 5, 7, 71 \equiv -3 \cdot 2^a \pmod{233}$, where $0 \leq a \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N^*_p \cong Z_2 \times Z_2$ or $N^*_p \cong D_8$ and $N^* \cap N^p \cong Z_2 \times Z_2$.

(3.14) The case that $N^*_p \cong Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N^*_p)$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N^*_p N^p)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^*|_2 = 4$, we have $q = p^s \equiv 3, 5, 7 \pmod{8}$ and so $n$ is odd. Hence $|G_{ab}|/|N^a|_2 \leq 2$ and $N^*_p / N^a \cap N^p \cong N^*_p N^p / N^p = 1$ or $Z_2$ by (3.2).

Suppose $N^*_p / N^a \cap N^p \cong Z_2$. Then $N^* G(N^*_p N^p)$ is a Sylow 2-subgroup of $G_{ab}$, hence $G_{ab}(N^*_p N^p)$ is doubly transitive by a Witt’s theorem. Since $N^*_p N^p \cong D_8$ and $|\Delta| = 2^a \times 3^b \times 5^c \times 7$, where $0 < a < 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N^*_p \cong Z_2 \times Z_2$ or $N^*_p \cong D_8$ and $N^* \cap N^p \cong Z_2 \times Z_2$.

Proof. Set $\Delta = F(N^*_p)$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N^*_p N^p)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^*|_2 = 4$, we have $q = p^s \equiv 3, 5, 7 \pmod{8}$ and so $n$ is odd. Hence $|G_{ab}|/|N^a|_2 \leq 2$ and $N^*_p / N^a \cap N^p \cong N^*_p N^p / N^p = 1$ or $Z_2$ by (3.2).

Suppose $N^*_p / N^a \cap N^p \cong Z_2$. Then $N^* G(N^*_p N^p)$ is a Sylow 2-subgroup of $G_{ab}$, hence $G_{ab}(N^*_p N^p)$ is doubly transitive by a Witt’s theorem. Since $N^*_p N^p \cong D_8$ and $|\Delta| = 2^a \times 3^b \times 5^c \times 7$, where $0 < a < 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N^*_p \cong Z_2 \times Z_2$ or $N^*_p \cong D_8$ and $N^* \cap N^p \cong Z_2 \times Z_2$.

(3.14) The case that $N^*_p \cong Z_2 \times Z_2$ does not occur.

Proof. Set $\Delta = F(N^*_p)$. Then $|\Delta| = 3r + 1$ and $\Delta = F(N^*_p N^p)$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N^*|_2 = 4$, we have $q = p^s \equiv 3, 5, 7 \pmod{8}$ and so $n$ is odd. Hence $|G_{ab}|/|N^a|_2 \leq 2$ and $N^*_p / N^a \cap N^p \cong N^*_p N^p / N^p = 1$ or $Z_2$ by (3.2).

Suppose $N^*_p / N^a \cap N^p \cong Z_2$. Then $N^* G(N^*_p N^p)$ is a Sylow 2-subgroup of $G_{ab}$, hence $G_{ab}(N^*_p N^p)$ is doubly transitive by a Witt’s theorem. Since $N^*_p N^p \cong D_8$ and $|\Delta| = 2^a \times 3^b \times 5^c \times 7$, where $0 < a < 9$. Similarly we get a contradiction.

Let $z$ be an involution in $N^*_p$ and $t \in \mathbb{Z}^p$ an involution such that $\alpha t = \beta$. Set $\Gamma = \{t \cdot \sigma \mid \sigma \in \Omega, \gamma \neq \delta\}$. We consider the action of the element $z$ on $\Gamma$. By the similar argument as in the proof of (3.12), $|F(z)|/|F(z)| = |C_{G_{ab}}|/|C_{G_{ab}}| \times |C_{G_{ab}}|$.

Hence $|G_{ab}|/|N^a|_2 \leq 2$ and $N^*_p / N^a \cap N^p \cong N^*_p N^p / N^p = 1$ or $Z_2$ by (3.2).

Suppose $N^*_p / N^a \cap N^p \cong Z_2$. Then $N^* G(N^*_p N^p)$ is a Sylow 2-subgroup of $G_{ab}$, hence $G_{ab}(N^*_p N^p)$ is doubly transitive by a Witt’s theorem. Since $N^*_p N^p \cong D_8$ and $|\Delta| = 2^a \times 3^b \times 5^c \times 7$, where $0 < a < 9$. Similarly we get a contradiction.

We argue that $r = 1$. Suppose false. Then $32s(3r - 4)(3r - 2) > 0$ and so $3r(q - \varepsilon) < 864r^2$. Therefore $288n + \varepsilon > q = p^s \geq 3^s$ and so $288n > 3^s$. Hence $(n, r, p) = (5, 3, 3, -1)$, $(3, 3, 3, -1)$ or $(3, 3, 5, 1)$, while none of these satisfy (3.14). Thus $r = 1$.
The case that $N^a_b\cong D_6$ and $N^a \cap N^β \leq Z_2 \times Z_2$ does not occur.

Proof. Let $Δ, L$ and $K$ be as defined in (3.6). By (3.6), there exists an element $x$ in $L_α$ such that its order is odd and $\langle x^α \rangle$ is regular on $Δ—\{a\}$. Since $(L_α)' \leq N^a_β$ by (3.6) and $N^a_β \cong D_6$, $x$ stabilizes a normal series $N^a_β \leq N^a_β$. Hence $x$ centralizes $N^a_βN^a_β$ by Theorem 5.3.2 of [2] and so $x^{-1}N^a_βx = N^a_β$. Put $γ = β^2$. If $r = 1$, then $β = γ$, so that $N^a_γ = N^a_β$. From this, $N^a_β = N^a_γ$. By the doubly transitivity of $G$, $N^a_β = N^a_γ$, hence $N^a_β = N^a_γ$, a contradiction. Therefore $r = 1$ and $Δ = \{α, β\}$.

Set $\langle x \rangle = Z(ΛΓZ)$, $Δ = α^{C_G(α)}$ and let $\{Δ_1, Δ_2, \ldots Δ_4\}$ be the set of $C_G(α)$-orbits on $F(z)$. Since $L \geq N^a \cap N^β$ and by (3.2), $N^a \cap N^β = 1$, $z$ is contained in $N^a \cap N^β$. Hence, by Lemma 2.1, $β \in Δ_1$ and $k$ is at least two. By Lemma 2.8, $|F(z)| = 1+(q-ε)/5|N^a_β| = 1+5(q-ε)/8$. Clearly $|C_{N^a_β}(z)|: N^a_β| = (q-ε)/8$ and so $|Δ_1| \geq 1+(q-ε)/8$. If $γ \in F(z) - Δ_1$, then $C_{N^a_β}(z) = Z_2 \times Z_2$, for otherwise $\langle x \rangle = Z(N^a_β) \subseteq N^a \cap N^γ$ and by Lemma 2.1 $γ \in Δ_1$, a contradiction. Hence one of the following holds.

(i) $k = 3$ and $|Δ_1| = 1+(q-ε)/8$, $|Δ_2| = |Δ_3| = (q-ε)/4$.
(ii) $k = 2$ and $|Δ_1| = 1+(q-ε)/8$, $|Δ_2| = (q-ε)/2$.
(iii) $k = 2$ and $|Δ_1| = 1+3(q-ε)/8$, $|Δ_2| = (q-ε)/4$.

Let $γ \in F(z) - Δ_1$. Then, $z \in G_γ - N^γ$ and so $C_{N^a_γ}(z) = D_q + r$ or $PGL(2, \sqrt{q})$ by Lemma 2.5 (vii), (viii), (ix). If $C_{N^a_γ}(z) = D_q + r$, then $(q+ε)/2 | |Δ_1|$ and so $q = 7$ and (ii) occurs. But $(q+ε)/2 = 3 | |Δ_2| - 1 - 1 = 1$, a contradiction. If $C_{N^a_γ}(z) = PGL(2, \sqrt{q})$, then (i) does not occur because $\sqrt{q} \not\equiv q - ε$. Hence $\sqrt{q} | |Δ_1|$ and $\sqrt{q} | |Δ_2| - 1$. From this, $q = 25$ and (iii) occurs. In this case, we have $|Δ_1| = 10$, so that an element of $C_{N^a_γ}(z)$ of order 3 is contained in $N^γ_δ$ for some $δ \in Δ_1$, contrary to $N^a_δ \leq N^a_β = D_6$.

4. Case (II)

In this section we assume that $N^a_β = PGL(2, p^m)$, where $n = 2mk$ and $k$ is odd. Since $n$ is even, $q = p^m \equiv 1 \mod 4$. We set $p^m \equiv ε \equiv \{±1\} \mod 4$. In section 7 we shall consider the case that $N^a_β = S_4$. Therefore we assume $(p, m) \neq (3, 1)$ in this section.

(4.1) The following hold.

(i) $N^a_β/N^a \cap N^β = \{1\} or Z_2$ and $N^a \cap N^β \geq (N^a_β)' \cong PSL(2, p^m)$.
(ii) If $(p, m) \neq (5, 1)$, there exists a cyclic subgroup $Y \leq (N^a_β)'$ such that $N_{N^a_β}(Y) = D_q - r$ and $N_{C_G(α)}(Y)^{F(γ)}$ is doubly transitive.

Proof. As $N^a_β \geq N^a \cap N^β$, either $N^a_β/N^a \cap N^β \leq Z_2$ or $N^a \cap N^β = 1$. If $N^a \cap N^β = 1$, by Lemma 2.2 and 2.6 (vi), $N^a_β = N^a_β/N^a \cap N^β = N^a_β/N^β = Z_2 \times Z_2$, a
Now we assume that \((p, m) \neq (3,1), (5,1)\) and let \(z\) be an involution in \((N^\alpha)^*\). Then \(C_{N^\alpha}(z) = D_{2(p^m-1)}\) by Lemma 2.6 (vii). Suppose \(C_{N^\alpha}(z)\) is not a 2-subgroup and put \(Y = 0(C_{N^\alpha}(z))\). Then, if \(C_{N^\alpha}(z)\) is not a 2-subgroup, we have \(Y^\gamma \leq N^\alpha \) and \(Y^\delta \leq N^\beta\), where \(\gamma = \alpha^\epsilon\) and \(\delta = \beta^\epsilon\). By Lemma 2.5, we have \(Y = Y^\alpha\) for some \(h \in N^\alpha \cap N^\beta\). Thus \(N_G(Y)^{F(Y)}\) is doubly transitive. Assume that \(C_{N^\alpha}(z)\) is a 2-subgroup and set \(X = C_{N^\alpha}^1(Y)\). We may assume that \(v \in (N^\alpha)^*\) and \(<u, v>\) is a Sylow 2-subgroup of \((N^\alpha)^*\). Since \(p^m \neq 3,5\), the order of \(v^2\) is at least four. On the other hand there is no element of order \(|u^2|\) in \(<u, v> = <u^2, v>\). Hence any element of order \(|u^2|\) which is contained in \(N^\alpha\) is necessarily an element of \(N^\alpha \cap N^\beta\). By the similar argument as above, \(N_G(Y)^{F(Y)}\) is doubly transitive.

\[\text{(4.2)}\] Let notations be as in (4.1). Suppose \((p, m) \neq (3,1), (5,1)\) and set \(\Delta = F(Y)\) and \(X = N_G(Y)\). Then \(|\Delta| = rs(p^m+\epsilon)/2 + 1\), where \(s = \sum_{i=0}^{k-1} p^{2mi}\), \(C_G(N^\alpha) = 1\) and one of the following holds.

(i) \(X^\Delta \leq \text{ATL}(1,2^c)\) for some integer \(c\).

(ii) \(X^\Delta \simeq \text{PSL}(2,p_1)\) or \(\text{PGL}(2,p_1)\), \(r = 1, k = 1\) and \(2p_1 = p^m + \epsilon\).

Proof. By Lemma 2.8 (ii), \(|\Delta| = 1 + |N^\alpha \cap X|/r|N^\alpha| < |N^\alpha|/r = 1 + (p^{2mk} - 1)/2(p^m - \epsilon) = rs(p^m + \epsilon)/2 + 1\). By (4.1) and Lemma 2.9, we have (i), (ii) or \(X^\Delta = R(3)\).

Assume that \(X^\Delta = R(3)\). Then \(rs(p^m + \epsilon)/2 + 1 = 28\), hence \(k = 1\) and \(r(p^m + \epsilon)/2 = 27\). Since \(r\) is odd and \(r|2m = n\), we have \(r = m = 1\) and \(q = 53\).

But a Sylow 3-subgroup of \(X_{a,\alpha}\) is cyclic because \(N^\alpha \cap X = D_{q^2}\) and \(X_{a,\alpha}/X \cap N^\alpha = X_{a,\alpha}N^\alpha/N^\alpha \leq Z_2 \times Z_2\), a contradiction. Thus (i) or (ii) holds.

\[\text{(4.3)}\] (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose \(X^\Delta \leq \text{ATL}(1,2^2)\) and put \(W = C_{N^\alpha}(Y)\). Then \(Y \leq W = Z_{p^m-2}\). Since \(C_{N^\alpha}(Y)\) is cyclic, \(W\) is a characteristic subgroup of \(C_{N^\alpha}(Y)\) and so \(W\) is a normal subgroup of \(X_{a,\alpha}\). Hence \(W \leq X^\Delta\) and \((X \cap N^\alpha)^{\alpha} = 1\) or \(Z_2\). By Lemmas 2.4 and 2.6, \(F(X \cap N^\alpha) = 1 + |X \cap N^\alpha|/|N^\alpha| = X \cap N^\alpha \times r|N^\alpha| = 1 + r\). Since \(1 + r < |\Delta|\), \((X \cap N^\alpha)^{\alpha} = Z_2\) and hence \((1 + r)^2 = rs(p^m + \epsilon)/2 + 1\) by Lemma 2.5. From this, \(r = s(p^m + \epsilon)/2 - 2|mk|\) and so \(p^m(k-1) + mk \leq 2\). Hence \(m = k = r = 1\) and \(q = 7^2\).

Let \(R\) be a Sylow 3-subgroup of \(N^\alpha\). Since \(N^\beta \simeq \text{PGL}(2,7)\), we have \(R = Z_3\). By Lemmas 2.4 and 2.6, \(|F(R)| = 1 + (7^2 - 1)|N^\alpha|/|N^\alpha| = 4\). Hence \(N_G(R)^{F(R)} = A_4\) or \(S_4\). But is a Sylow 3-subgroup of \(N_{Ga}(R)\) because \(N^\alpha = PSL(2,7)\), contrary to \(N_G(R)^{F(R)} = A_4\) or \(S_4\).

\[\text{(4.4)}\] (ii) of (4.2) does not occur.
Proof. Let notations be as in (4.2). Suppose $X^\alpha \geq PSL(2, p_1)$. By the similar argument as in (4.3), $C_{N^\beta}(Y) \leq X_\Delta$ and so $C_{N^\alpha}(Y) \geq Z_{p^1}$, and $N_{N^\alpha}(Y)^\beta = D_{2p^1}$. Hence $|X^\alpha| \geq \langle p_1 \rangle$. Since $X^\alpha \geq PSL(2, p_1)$, $p_1(p_1 - 1)/2 \mid |X^\alpha|$, hence $p_1 - 1 \parallel 8n$. As $k=1$ and $2p_1 = p^m + \epsilon$, we have $p^m + \epsilon - 2 = 32m$. From this, $(p, m, p_1) = (11, 1, 5)$, $(3, 2, 5)$ or $(3, 3, 13)$.

Let $R$ be a cyclic subgroup of $N^\alpha$ such that $R = \mathbb{Z}(p^m + \epsilon)/2$. By Lemma 2.6, $N_\alpha(R)^{F(R)}$ is doubly transitive and by Lemma 2.8 (ii), $|F(R)| = 1 + |N_\alpha(R)| = |N_{N^\beta}(R)| = 1 + (p^m - 1)/2(p^m + \epsilon) = (p^m - \epsilon)/2 + 1$.

If $(p, m, p_1) = (11, 1, 5)$, $|F(R)| = 7$ and so by [9], $|N_\alpha(R)^{F(R)}| = 42$ and $N_\alpha(R)^{F(R)} = Z_6$. Since $|N_\alpha(R) : N_{N^\beta}(R)| = 6$, $N_\alpha(R)^{F(R)} = N_\alpha(R)^{F(R)}$. Hence $N_{N^\alpha}(R)/(N_{N^\alpha}(R)) = Z_2 \times Z_2$, a contradiction.

If $(p, m, p_1) = (3, 2, 5)$, $|F(R)| = 5$ and so by [9], $|N_\alpha(R)^{F(R)}| = 20$ and $N_\alpha(R)^{F(R)} = Z_4$. Since $|N_{N^\alpha}(R) : N_{N^\beta}(R)| = 4$, $N_{N^\alpha}(R)^{F(R)} = Z_4$, contrary to $N_{N^\alpha}(R)^{F(R)} = Z_2 \times Z_2$. Hence $N_{N^\alpha}(R)/(N_{N^\alpha}(R)) = Z_2 \times Z_2$.

If $(p, m, p_1) = (3, 3, 13)$, $|F(R)| = 15$. By [9], $N_{N^\alpha}(R)^{F(R)}$ is not solvable, a contradiction.

(4.5) $p^m = 5$.

Proof. Assume that $p^m = 5$. Then $n = 2k$ with $k$ odd and $N^\alpha = PGL(2, 5) \simeq S_5$. First we argue that $N^\beta = N^\alpha \cap N^\beta$. Suppose false. Then $C_\alpha(N^\beta) = 1$ by Lemma 2.2, and $N^\alpha | N^\alpha \cap N^\beta = Z_2$ by (4.1). Since $N^\beta N^\beta | N^\beta \cap N^\beta = Z_2$ and the outer automorphism group of $S_5$ is trivial, we have $Z(N^\beta N^\beta) = Z_2$.

Let $w$ be the involution of $Z(N^\beta N^\beta)$ and let $w \in N^\beta - N^\alpha$. Since $C_\alpha(w) \geq N^\beta$, by Lemma 2.6 (viii) and (ix), $w$ acts on $N^\beta$ as a field automorphism of order 2 and $C_\alpha(w) \simeq PGL(2, 5^k)$. By Lemma 2.8 $|F(w)| = 1 + r(g - \epsilon) |I(N^\alpha)| = 1 + |N^\beta| = 1 + 5r(5^{2k} - 1)/24$. Let $P$ be a Sylow 5-subgroup of $C_\alpha(w)$. Then $|P| = 5^k$ and $|\gamma | = 5^{k-1}$ or $5^k$ for each $\gamma \in \Omega - \{\alpha\}$. Since $P$ acts on $F(w) - \{\alpha\}$, we have $5^{k-1} | 5r(5^{2k} - 1)/24$, so that $k = 1$ and $|F(w)| = 6 = r/k$. Hence $C_\alpha(w)^{F(w)} = S_5$ and so $C_\alpha(w)^{F(w)} = S_5$. But clearly $w \in N^\alpha \cap N^\beta$ by Lemma 2.1, a contradiction. Thus $N^\beta = N^\alpha \cap N^\beta$.

Let $V$ be a cyclic subgroup of $N^\beta$ of order 4. Since $N^\beta = N^\alpha \cap N^\beta = S_5$, $N_\alpha(V)^{F(V)}$ is doubly transitive and by Lemma 2.8, $|F(V)| = 1 + |N_{N^\alpha}(V)|/r|N_{N^\beta}(V)| = 1 + (5^{2k} - 1)r/8 = 3rs + 1$, where $s = \sum_{i=0}^{k-1} 25^i$. By Lemma 2.9, $C_\alpha(N^\alpha) = 1$ and (a) $N_\alpha(V)^{F(V)} = \Gamma L(1, 2^c)$ or (b) $N_\alpha(V)^{F(V)} = R(3)$.

Put $P = N_\alpha(V)$. Then $P = D_5$, $|F(P)| = 1 + |N_{N^\alpha}(P)| = |N^\alpha : N_{N^\beta}(P)| = r/|N^\beta| = r + 1$ and $P^F \simeq Z_2$. If (b) occurs, $k = 1$ and $r = 9$, hence $|F(P)| = 10$, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r+1)^2 = 3rs + 1$ and so $r = 3s - 2/k$. Hence $k = r = 1$ and $G_{\alpha}/N^\alpha \leq Z_2 \times Z_2$. Let $z$ be an involution in $N^\beta$. Then $|F(z)| = 1 + 24 \cdot 25/120 = 6$.
by Lemma 2.8 and \(|\Omega|=1+|N^a|\) as \(r=1\). By the similar argument as in the proof of (3.12), \(|F(z)|(\Omega|-1)/2+\Omega=|C_0(z)|\) and \(\langle t\rangle G_{ab}/\langle t\rangle G_{ab}\), where \(t\) is an involution such that \(\alpha^t=\beta\). Hence \(|z^\alpha\cap\langle t\rangle G_{ab}|=15|G_{ab}|/|C_0(z)|\). Set \(H=\langle t\rangle G_{ab}\) and let \(R\) be a Sylow 3-subgroup of \(N^a_\beta\).

By Lemma 2.8, \(|F(R)|=1+24\cdot120=3\). Set \(F(R)=\{\alpha, \beta, \gamma\}\). On the other hand, as \(N^a_\beta=S_5\) and \(\text{Out}(S_5)=1\), we have \(H=Z(H)\times N^a_\beta\) and \(|Z(H)|=2\), 4 or \(H=C_H(N^a_\beta)\times N^a_\beta\) and \(C_H(N^a_\beta)\sim D_6\). In the latter case \(G_{ab}=Z(G_{ab})\times N^a_\beta\) and \(Z(G_{ab})=Z_2\times Z_2\), contrary to Lemma 2.6 (ix). In the former case, we have \(|Z(H)|=2\).

For otherwise \(Z(H)<G\) and \(Z(H)\) is an involution such that \(a=\beta\). Hence \(|Z(G_{ab})|=15|G_{ab}|/|C_G(z)|=15\cdot120/24=75\), a contradiction.

### 5. Case (III)

In this section we assume that \(N^a_\beta=PSL(2,p^m)\), where \(n=mk\) and \(k\) is odd. Set \(p^m\equiv\varepsilon\pmod{4}\). Then \(q\equiv\varepsilon\pmod{4}\) as \(k\) is odd. In section 6 we shall consider the case that \(N^a_\beta=A_4\), so we assume \((p,m),(3,1)\) in this section.

From this \(N^a_\beta\) is a nonabelian simple group and so \(N^a_\beta=N^a\cap N^b\) or \(N^a\cap N^b=1\). If \(N^a\cap N^b=1\), then \(C_0(N^a)=1\) by Lemma 2.2 and \(N^a_\beta=N^a\cap N^b\). Hence \(|Z(H)|=2\).

Let \(z\) be an involution of \(N^a_\beta\). Suppose \(z^\varepsilon\in G_{ab}\) for some \(g\in G\) and set \(\gamma=\alpha^\varepsilon\). Then \(z^\varepsilon\in N^a_\beta\cap G_{ab}\leq N^a_\beta \cap N^b\leq N^a\cap N^b\) and so \(z^\varepsilon\in N^a_\beta\). Hence \(C_0(z)^{F(\varepsilon)}\) is doubly transitive and by Lemma 2.8 (i), \(|F(z)|=2\cdot|C_{N^b}(z)|=3\cdot1\) as \(p^m-\varepsilon\geq p^{2m}+1\).

In particular \(|F(z)|=3r+1\) as \((p^m-\varepsilon)/(p^m-\varepsilon)\geq p^{2m+1}+p^{m+1}+1>3\).

By Lemma 2.9, \(C_0(N^a)=1\) and one of the following holds.

(a) \(C_0(z)^{F(\varepsilon)}\leq\text{ATL}(1,2^r)\).

(b) \(C_0(z)^{F(\varepsilon)}\leq\text{PSL}(2,p^r)\) \((p^r\geq5)\), \(r=1\) and \(|N_{N^a}^a(z): C_0^a(z)|=p^l\).

(c) \(C_0(z)^{F(\varepsilon)}=R(3)\).

Let \(Y\) be a cyclic subgroup of \(C_{N^a_\beta}(z)\) of index 2. Since \(C_{G_{ab}}(z)\geq Y\), \(z\in Y\) and \(C_0(z)^{F(\varepsilon)}\) is doubly transitive, we have \(F(Y)=F(z)\). By the similar argument as in (3.1), \(N^a\cap N(C_{N^a_\beta}(z))=C_{N^a_\beta}(z)\) or \(N^a\cap N(C_{N^a_\beta}(z))=A_4\). Hence by Lemmas 2.3 and 2.4 \(|F(C_{N^a_\beta}(z))|=1+|C_{N^a_\beta}(z)|+|N^a_\beta|\) \(|C_{N^a_\beta}(z)|/|N^a_\beta|\) or \(1+\langle A_4\rangle\langle N^a_\beta\rangle\). Therefore \(|F(C_{N^a_\beta}(z))|=r+1\) or \(3r+1\). From this \(C_{N^a_\beta}(z)^{F(\varepsilon)}\geq Z_2\).

In the case (a), \((r+1)^2=1+(p^m-\varepsilon)r/(p^m-\varepsilon)\) by Lemma 2.5 and hence \(r=(p^m-\varepsilon)/(p^m-\varepsilon)-2\cdot mk\). Since \((p^m-\varepsilon)/(p^m-\varepsilon)\geq((p^m)^k+1)/(p^m+1)=\sum_{i=0}^{k-1}(-p^m)^i\) and \(k\geq3\), we have \(p^{m(k-1)}(p^{2m}+p^m+1)\leq mk\), hence \((p^m)^{k-3}/k(m/(p^{2m}+p^m+1))<1\).

Thus \(k=3\), \(m=1\) and \(p=3\), contrary to \(\varepsilon>3\).

In the case (b), \(r=1\), \(p_1=(p^m-\varepsilon)/(p^m-\varepsilon), p_1(p_1-1)/2\) and \(s=4mkp_1\), where \(s\) is the order of \(C_{G_{ab}}(z)^{F(\varepsilon)}\). Hence \(p_1-1=3\) or \(8mk\).

Since \(p_1-1=(p^m-\varepsilon)/(p^m-\varepsilon)-1\)
\( (p^n+1)/(p^n+1)-1 = \sum_{k=0}^{\infty} (-p^m)^k \geq p^{m(k-2)}(p^m-1) \), we have \( p^{m(k-2)}/2k \leq 4m|(p^n-1) \leq 1 \) because \( p^n \neq 3 \). Hence \( k=3 \) and \( p^n=5 \), so that \( p_1=1=30 \times 8mk=24 \), a contradiction.

In the case (c), \( r+1=4 \) and \( 1+(p^n-\varepsilon)r/(p^n-\varepsilon)=28 \) and so \( r=3 \) and \( (p^n-\varepsilon)/(p^n-\varepsilon)=9 \). Hence \( 9 \geq (p^m+1)/(p^m+1) \geq p^m-p^m+1 \), so that \( p^m=3 \), a contradiction.

6. Case (IV)

In this section we assume that \( N^*=A_4 \) and \( q=3,5 \) (mod 8). If \( N^* \cap N^{\beta}=1 \), by Lemma 2.2, \( C_G(N^*)=1 \) and so \( N^* \cap N^{\beta}=N^{\beta}N^*|N^*=Z_2 \times Z_2 \). Hence \( N^* \cap N^{\beta}=1 \) or \( Z_3 \), so that \( z^G \cap G_{ab}=z^{G} \cap N^{\beta}=zN^{\beta} \) for an involution \( z \in N^* \). Therefore \( C_G(z)^F(z) \) is doubly transitive. By Lemma 2.9, \( C_G(N^*)=1 \) and one of the following holds.

(a) \( C_G(z)^F(z) \leq A_7(1,2) \) for some integer \( c \geq 1 \).
(b) \( C_G(z)^F(z) \geq PSL(2, p_1) \left( p_1 \geq 5 \right) \), \( r=1 \) and \( |C_N(z) : C_{N^*}(z)| = p_1 \).
(c) \( C_G(z)^F(z) = R(3) \).

Let \( T \) be a Sylow 2-subgroup of \( N^* \). Then \( z \in T \) and by Lemmas 2.3 and 2.4, \( |F(T)| = 1+|N_{N^*}(T)|r/|N^*_b|=r+1 \). By Lemma 2.8 (i), \( |F(z)|=(q-\varepsilon)r/4+1 \). Hence \( T^F(z)=Z_2 \) if \( q=5 \). If \( q=5 \), as \( PSL(2,5) \neq PSL(2,4) \), (ii) of our theorem holds by [4]. Therefore we may assume \( q=5 \).

In the case (a), \( (r+1)^2=1+(q-\varepsilon)r/4 \) by Lemma 2.5. Hence \( r=(q-\varepsilon-8)/4 \) and \( r|n \), so that \( q=11 \) or 13 and \( r=1 \). Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \). Then \( R=Z_3 \) and \( R \leq N_{ab}^* \) because \( G_{ab}/N_{ab}^*=G_{ab}/N^* \cap N^{\beta}=1 \) or \( Z_2 \) and \( N_{ab}^*=A_4 \). By Lemma 2.8 (ii), \( |F(R)|=1+12/3=5 \) and \( N_G(R)^F(R) \) is doubly transitive. Since \( N_{G_{ab}}=D_{12} \) or \( D_{24} \) and \( |F(R)|=5 \), we have \( |N_G(R)|=5 \). Let \( S \) be a Sylow 3-subgroup of \( N_G(R) \). Then \( [S, R]=1 \) as \( N_G(R)/C_G(R) \leq Z_5 \). Since \( 5 \times |G_{ab}| \), \( |F(S)|=0 \) or 1. If \( |F(S)|=1, F(S) \leq F(R) \) and so \( 5 \times |F(R)|=1-4 \), a contradiction. Therefore \( S \) is semi-regular on \( \Omega \). But \( |\Omega|=1+|N^* : N^*_b|=56 \) or 92. This is a contradiction.

In the case (b), \( p_1(p_1-1)/2 \) or \( |2n(q-\varepsilon)/2|=4np_1 \), where \( s \) is the order of \( C_{G_{ab}}(z)^F(z) \). Hence \( p_1=18n \). Since \( p_1=(q-\varepsilon)/4 \), \( p^n-\varepsilon-4 \times 32n \) and so we have \( q=11, 13, 19, 27 \) or 37. If \( q=27 \), by Lemma 2.6, \( C_{G_{ab}}(z)=D_{12} \) or \( D_{24} \) and so \( C_{G_{ab}}(z)^F(z)=Z_2 \). Hence \( (p_1-1)/2=2 \). From this \( q=19 \). Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \). By the similar argument as in the case (a), \( N_G(R)^F(R) \) is doubly transitive and \( |F(R)|=1+18/3=7 \). Hence \( 7 \times |G| \). On the other hand \( |G|=|\Omega| \times |G_{ab}|=(1+|N^*: N^*_b|)|G_{ab}|=(1+18\times 19\times 20/2\times 12)\times 18\times 19 \times 20/2=24^2 \times 3^5 \times 5 \times 11 \times 13 \times 19 \) with \( 0 \leq t \leq 1 \), a contradiction. If \( q=27 \), then \( |C_G(z)|=|F(z)|_{2} \times |C_{G_{ab}}(z)|_{2}=8 \times |G_{ab}|_{2} \), while \( |\Omega|=1+|N^*: N^*_b|=1+26 \times 27 \times 28/2 \times 12=820=2^2 \times 5 \times 41 \) and so \( |G|_{2}=4|F(G)|_{2} \). Therefore \( |C_G(z)| \times |G| \), a contradiction.

In the case (c), \( r+1=4 \) and \( 1+(q-\varepsilon)r/4=28 \). Hence \( r=3 \) and \( q=37 \).
7. Case (V)

In this section we assume that \( N^*_\beta = S_4 \) and \( q \equiv 7, 9 \pmod{16} \). We note that \( 4 \nmid n \).

First we argue that \( N^*_\beta = N^* \cap N^\beta \). Suppose \( N^*_\beta \neq N^* \cap N^\beta \). Then \( C_\circ(N^*) = 1 \) by Lemma 2.2. Since \( N^*_\beta \cap N^\beta = N^*_\beta N^\beta / N^\beta \leq Z_2 \times Z_n \), we have \( N^* \cap N^\beta = A_4 \) and \( N^*_\beta \cap N^\beta = Z_2 \). Hence as \( \text{Out}(S_4) = 1 \), \( Z(N^*_\beta N^\beta) = Z_2 \). Set \( \langle t \rangle = Z(N^*_\beta N^\beta) \) and \( t \in I(N^*_\beta) - I(N^*) \). Since \( C_{N^*(t)} \geq N^*_\beta = S_4 \) and \( \langle t \rangle N^* = N^*_\beta N^* \), by Lemma 2.6, we have \( C_{N^*(t)} = \text{PGL}(2, \sqrt{q}) \) and \( |F(t)| = 1 + (q - \varepsilon)r/8 \) by Lemma 2.8.

Let \( P \) be a Sylow \( p \)-subgroup of \( C_{N^*(t)} \). Then \( |P| = \sqrt{q} \). If \( p \neq 3 \), \( P \) acts semi-regularly on \( F(t) - \{ \alpha \} \) and \( \sqrt{q} |3(q - \varepsilon)r/8 \). Therefore \( \sqrt{q} | r \) and so \( 5^s \leq n^2 \) as \( p \geq 5 \) and \( r \mid n \). But obviously \( 5^s > n^2 \) for any positive integer \( n \). This is a contradiction. If \( p = 3 \), \( |P| = \sqrt{q} \leq 3 \) or \( 3^s \). Hence \( \sqrt{q} /3 \leq 3(q - \varepsilon)r/8 \) and so \( q \mid 8r^2 \). In particular, \( 3^s = q \mid 8n^2 \). From this, \( n \leq 7 \). Since \( q = 3 \equiv 7 \) or 9 (mod 16), we have \( q = 3^2 \) or \( 3^s \). If \( q = 3^s \), \( |\Omega| = 1 + N^*: N^*_\beta | = 1 + 8 \cdot 9 \cdot 10/2 \cdot 24 = 16 \), a contradiction by [9]. If \( q = 3^s \), \( F(t) = 1 + 273r \) and \( |F(t) - \{ \alpha \} | \geq |C_{N^*(t)}| \geq |\text{PGL}(2, 3^s)|/8 = 2457 \) contrary to \( r \mid 3 \). Thus \( N^*_\beta = N^* \cap N^\beta \).

Let \( V \) be a cyclic subgroup of \( N^*_\beta \) of order 4 and let \( U \) be a Sylow 2-subgroup of \( N^*_\beta \) containing \( V \). Then \( U = N^*_\beta(V) \), \( |F(V)| = 1 + (q - \varepsilon)r/8 \) by Lemma 2.8 and \( |F(U)| = 1 + 8 \cdot 3r/24 = r + 1 \) by Lemmas 2.3 and 2.4. If \( q \neq 7, 9 \), then \( |F(U)| \leq |F(V)| \) and hence \( U(F(V)) \leq Z_2 \). Suppose \( q = 7 \) or 9. Then \( r = 1 \) as \( r \mid n \). Hence \( |\Omega| = 1 + |N^*; N^*_\beta| = 8 \) or 16. By [10], we have a contradiction. Therefore \( U(F(V)) \leq Z_2 \).

Suppose \( V^x \leq G_{ab} \) for some \( g \in G \) and set \( \gamma = \alpha^x \). Then \( V^x \leq g^{-1}N^a \cap G_{ab} \leq N^y \cap G_{ab} \leq N^* \cap N^\beta = N^*_\beta \). As \( N^*_\beta = S_4 \), \( V^x = V^h \) for some \( h \in N^*_\beta \). Hence \( C_\circ(V^x) \) is doubly transitive. By Lemma 2.9, \( C_\circ(N^*) = 1 \) and one of the following holds.

(a) \( N_\circ(V^x) \leq \text{AGL}(1, 2^7) \).

(b) \( N_\circ(V^x) \geq \text{PSL}(2, p_1), p_1 = (q - \varepsilon)/8 \geq 5 \).

(c) \( N_\circ(V^x) = R(3) \).

In the case (a), \( (r + 1)| = 1 + (q - \varepsilon)r/8 \) by Lemma 2.5 and \( r = (q - \varepsilon - 16)/8 \) and \( r \mid n \). From this \( q = 23 \) or 25 and \( r = 1 \). Since \( |\Omega| = 1 + |N^*; N^*_\beta| = 2 \cdot 127 \) or \( 2 \cdot 163 \), we have \( |G|_2 = 2 \cdot |G_{a1}| \) while \( |N_\circ(V)|_2 = |F(V)|_2 \cdot |G_{a1}(V)|_2 = 4 \cdot |G_{a1}|_2 \), contrary to \( |N_\circ(V)| \mid |G| \).

In the case (b), \( p_1(1 + 1)/2 \mid s \) and \( s |2n(q - \varepsilon)/4 = 4np_1 \), where \( s \) is the order of \( N_{G_{a1}}(V^x) \). Hence \( p_1 = 1/8n \). From this, \( p^s - \varepsilon - 8 \mid 64n \) and so \( q = 23, 41, 71 \) or 73. Since \( p_1 \) is a prime and \( p_1 = (q - \varepsilon)/8 \geq 5, q = 23, 71, 73 \). Therefore \( q = 41 \) and \( |\Omega| = 1 + |N^*; N^*_\beta| = 1 + 40 \cdot 41 \cdot 42 \cdot 2 \cdot 24 = 2^2 \cdot 359 \), so that \( |G|_2 = 4 \cdot |G_{a1}|_2 \).
Since $N^* = N^* \cap N^\beta$, $C_\alpha(z)^{F(z)}$ is transitive by Lemma 2.1. On the other hand $|F(z)| = 1 + 40 \cdot 9/24 = 16$ by Lemma 2.8 (i) and so $C_\alpha(z)|z = 16|C_\alpha(z)|z = 16|G_{\alpha}|$, contrary to $|C_\alpha(z)||G|$.

In the case (c), $r + 1 = 4$ and $1 + (q - \varepsilon)r/8 = 28$. Hence $r = 3$ and $q = 71$ or 73, contrary to $r | n$.

8. Case (VI)

In this section we assume that $N^* = A_5$ and $q \equiv 3, 5 \pmod{8}$. In particular, $n$ is odd. If $N^* \neq N^* \cap N^\beta$, then $N^* \cap N^\beta = 1$, $C_\alpha(N^*) = 1$ and so $N^* = N^* N^\beta/N^\beta < \text{Out}(N^\beta) \cong Z_2 \times Z_8$, a contradiction. Hence $N^* = N^* \cap N^\beta$. Let $z$ be an involution in $N^*$ and $T$ a Sylow 2-subgroup of $N^*$ containing $z$. Then, by Lemma 2.6 $|F(z)| = 1 + (q - \varepsilon)15r/60 = 1 + (q - \varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)| = 1 + 12 \cdot 5r/60 = 1 + r$. Since $N^* = N^* \cap N^\beta$, $z^G \cap G_\alpha = z^G \cap N^* = z^N^*$ and so $C_\alpha(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_\alpha(N^*) = 1$ and one of the following holds.

(a) $C_\alpha(z)^{F(z)} \leq A\Gamma L(1, 2^3)$.

(b) $C_\alpha(z)^{F(z)} \cong \text{PSL}(2, p_1)$, $p_1 = (q - \varepsilon)/4 \geq 5$.

(c) $C_\alpha(z)^{F(z)} = R(3)$.

In the case (a), by Lemma 2.5, $(q - \varepsilon)/4 = 1$ or $(r + 1)^2 / 4 = 1 + (q - \varepsilon)r/4$. Hence $q = 5$ or $r = (q - \varepsilon - 8)/4 | n$. If $q = 5$, then $N^* = N^*$, a contradiction. Therefore $p^* = \varepsilon - 8 | 4n$ and so $n = 1$ and $q = 11$ or 13. If $q = 13$, we have $5 | |G^*, |$, a contradiction. Hence $q = 11$ and $|\Omega| = 1 + |N^*| N^* | = 1 + 10 \cdot 11 \cdot 12/2 \cdot 60 = 12$. By [9], $C^\Omega \cong M_{11}$, $|\Omega| = 12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1 | (p_1 - 1)/2 | s$ and $s \geq 2n(q - \varepsilon)/2 = 4n p_1$, where $s$ is the order of $C_\alpha(z)^{F(z)}$. Hence $p_1 = 18n$ and so $p^* = \varepsilon - 4 | 32n$. From this $q = 19, 27$ or 37. Since $5 | |G^*, | \neq 27, 37$. Hence $q = 19$ and $|\Omega| = 1 + |N^*: N^* | = 1 + 18 \cdot 19 \cdot 20/2 \cdot 60 = 29$. Since $G_\alpha = \text{PSL}(2, 19)$ or $\text{PGL}(2, 19)$, $|G| = |\Omega| |G^*| = 2 \cdot 29 \cdot 2 \cdot 19 \cdot 20/2 = 2^{i+1} \cdot 3^{i} \cdot 5 \cdot 19 \cdot 2^j$ with $0 \leq i \leq 1$. Let $P$ be a Sylow 29-subgroup of $G$. Then $P$ is semi-regular on $\Omega$ and $3, 5, 19 \not{|} |G^*|$ because $N_\alpha(P)/C_\alpha(P) \cong Z_4 \times Z_7$. Hence $|G: N_\alpha(P)| = 2^i \cdot 3^j \cdot 5 \cdot 19$ with $0 \leq j \leq 4$, while $2^i \cdot 3^j \cdot 5 \cdot 19 \equiv 1 \pmod{29}$ for any $j$ with $0 \leq j \leq 4$, contrary to a Sylow's theorem.

If $C_\alpha(z)^{F(z)} = R(3)$, $r + 1 = 4$ and $1 + (q - \varepsilon)r/4 = 28$ and hence $r = 3$, $q = 37$, contrary to $r | n$.

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References

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