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Osaka University
ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS IN WHICH SOCLE(G_α) IS NONSOLVABLE

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1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and α ∈ Ω. In [8], O'Nan has proved that socle(G_α) = A × N, where A is an abelian group and N is 1 or a nonabelian simple group. Here socle(G_α) is the product of all minimal normal subgroups of G_α.

In the previous paper [4], we have studied doubly transitive permutation groups in which N is isomorphic to PSL(2,q), Sz(q) or PSU(3,q) with q even. In this paper we shall prove the following:

Theorem. Let G be a doubly transitive permutation group on a finite set Ω with |Ω| even and let α ∈ Ω. If G_α has a normal simple subgroup N* isomorphic to PSL(2,q), where q is odd, then one of the following holds.

(i) G_Ω has a regular normal subgroup.

(ii) G_Ω = A_6 or S_6, N* = PSL(2,5) and |Ω| = 6.

(iii) G_Ω = M_{11}, N* = PSL(2,11) and |Ω| = 12.

In the case that G_α has a regular normal subgroup, by a result of Hering [3] we have (|Ω|, q) = (16, 9), (16, 5) or (8, 7).

We introduce some notations:

F(X): the set of fixed points of a nonempty subset X of G
X(Δ): the global stabilizer of a subset Δ(⊆ Ω) in X
X_Δ: the pointwise stabilizer of Δ in X
X^\Delta: the restriction of X on Δ
m|n: an integer m divides an integer n
X^H: the set of H-conjugates of X
|X|_p: maximal power of p dividing the order of X
I(X): the set of involutions in X
D_m: dihedral group of order m

In this paper all sets and groups are finite.
2. Preliminaries

Lemma 2.1. Let \( G \) be a transitive permutation group on \( \Omega, \alpha \in \Omega \) and \( N^\alpha \) a normal subgroup of \( G^\alpha \) such that \( F(N^\alpha) = \{\alpha\} \). Let the subgroup \( X \leq N^\alpha \) be conjugate in \( G^\alpha \) to every group \( Y \) which lies in \( N^\alpha \) and which is conjugate to \( X \) in \( G \). Then \( N^\alpha(X) \) is transitive on \( \Delta = \{\gamma \in \Omega \mid X \leq N^\gamma\} \).

Proof. Let \( \beta \in \Delta \) and let \( g \in G \) such that \( \beta g = \alpha \). Then, as \( X \leq N^\beta \), \( X^g \leq N^{\beta g} = N^\alpha \). By assumption, \( (X^g)^h = X \) for some \( h \in G^\alpha \). Hence \( gh \in N^\alpha(X) \) and \( \alpha^{(gh)^{-1}} = \alpha^{-1} = \beta \). Obviously \( N^\alpha(X) \) stabilizes \( \Delta \). Thus Lemma 2.1 holds.

Lemma 2.2. Let \( G \) be a doubly transitive permutation group on \( \Omega \) of even degree and \( N^\star \) a nonabelian simple normal subgroup of \( G^\star \) with \( \alpha \in \Omega \). If \( C_G(N^\star) \neq 1 \), then \( N^\star \cap N^\beta \) for \( \alpha \neq \beta \in \Omega \) and \( C_G(N^\star) \) is semiregular on \( \Omega \) except \( \{\alpha\} \).

Proof. See Lemma 2.1 of [4].

Lemma 2.3. Let \( G \) be a transitive permutation group on \( \Omega \), \( H \) a stabilizer of a point of \( \Omega \) and \( M \) a nonempty subset of \( G \). Then
\[
|F(M)| = |N^\alpha(M)| \times |M^\alpha \cap H| \times |H|.
\]
Here \( M^\alpha \cap H = \{g^{-1}Mg, g^{-1}Mg \subseteq H, g \in G\} \).

Proof. See Lemma 2.2 of [4].

Lemma 2.4. Let \( G \) be a doubly transitive permutation group on \( \Omega \) and \( N^\alpha \) a normal subgroup of \( G^\alpha \) with \( \alpha \in \Omega \). Assume that a subgroup \( X \) of \( N^\alpha \) satisfies \( X^{G^\alpha} = X^{N^\alpha} \). Then the following hold.

(i) \(|F(X) \cap \gamma N^\alpha| = |F(X) \cap \gamma N^\alpha| \) for \( \beta, \gamma \in \Omega \) for \( \alpha \neq \beta \in \Omega \).

(ii) \(|F(X)| = 1 + |F(X) \cap \beta N^\alpha| \times r \), where \( r \) is the number of \( N^\alpha \)-orbits on \( \Omega \) for \( \{\alpha\} \).

Proof. Let \( \Gamma = \{\Delta_1, \Delta_2, \ldots, \Delta_r\} \) be the set of \( N^\alpha \)-orbits on \( \Omega \) for \( \{\alpha\} \). Since \( G^\alpha \) is transitive on \( \Omega \) for \( \{\alpha\} \) and \( G^\alpha \trianglelefteq N^\alpha \), we have \(|\Delta_i| = |\Delta_j| \) for \( 1 \leq i, j \leq r \). By assumption, \( G^\alpha = N^{G^\alpha}(X)N^\alpha \) and so \( N^{G^\alpha}(X) \) is transitive on \( \Gamma \). Hence for each \( i \) with \( 1 \leq i \leq r \) there exists \( g \in N^{G^\alpha}(X) \) such that \( (\Delta_i)^g = \Delta_j \). Therefore \(|F(X) \cap \Delta_i| = |F(X^\alpha) \cap (\Delta_i)^g| = |F(X) \cap \Delta_j| \). Thus (i) holds and (ii) follows immediately from (i).

Lemma 2.5 (Huppert [5]). Let \( G \) be a doubly transitive permutation group on \( \Omega \). Suppose that \( \theta(G) \neq 1 \) and \( G^\alpha \) is solvable. Then for any involution \( z \in G^\alpha \), \( |F(z)|^2 = |\Omega| \).

We list now some properties of \( PSL(2, q) \) with \( q \) odd which will be required
Lemma 2.6 ([2], [6], [10]). Set $N = \text{PSL}(2, q)$ and $G = \text{Aut}(N)$, where $q = p^*$ and $p$ is an odd prime. Let $z$ be an involution in $N$. Then the following hold.

(i) $|N| = (q - 1)q(q + 1)/2$, $I(N) = z^N$ and $C_N(z) = D_{q - 1}$, where $q \equiv \varepsilon \in \{ \pm 1 \} (\text{mod } 4)$.

(ii) If $q \pm 3$, $N$ is a nonabelian simple group and a Sylow $r$-subgroup of $N$ is cyclic when $r \neq 2, p$.

(iii) If $X$ and $Y$ are cyclic groups of $N$ and $|X| = |Y| = 2, p$, then $X$ is conjugate to $Y$ in $\langle X, Y \rangle$ and $N_N(X) = D_{q^2}$.

(iv) If $X \leq N$ and $X = Z_2 \times Z_2$, $N_N(X)$ is isomorphic to $A_4$ or $S_4$.

(v) If $|N|_2 = 8$, $N$ has two conjugate classes of four-groups in $N$.

(vi) There exist a field automorphism $f$ of $N$ of order $n$ and a diagonal automorphism $d$ of $N$ of order 2 and if we identify $N$ with its inner automorphism group, $\langle d \rangle N = \text{PGL}(2, q)$, $\langle f \rangle \langle d \rangle N = G$ and $G|N = Z_2 \times Z_2$.

(vii) $C_N(d) = D_{q^2}$ and $C_{dN}(z) = D_{q^2}$.

(viii) Suppose $n = mk$ for positive integers $m, k$. Then $C_N(f^m) = \text{PSL}(2, p^m)$ if $k$ is odd and $C_N(f^m) = \text{PGL}(2, p^m)$ if $k$ is even.

(ix) Assume $n$ is even and let $u$ be a field automorphism of order 2. Then $I(G) = I(N) \cup d^n \cup u \langle d \rangle N$. If $n$ is odd, $I(G) = I(N) \cup d^n$.

Lemma 2.7. Let $G, N, d$ and $f$ be as defined in Lemma 2.6 and $H$ an $\langle f, d \rangle$-invariant subgroup of $N$ isomorphic to $D_{q - 1}$. Let $W$ be a cyclic subgroup of $\langle d \rangle H$ of index 2 (cf. (vii) of Lemma 2.6) and set $Y = \langle f \rangle W \cap H$. Then $C_H(Y) = W \cdot C_{\langle f \rangle}(Y)$.

Proof. By (viii) of Lemma 2.6, we can take an involution $t$ satisfying $\langle d \rangle H = \langle t \rangle W$ and $[f, t] = 1$. Since $N_G(Y) = \langle f, d \rangle N_N(Y) = \langle f, d \rangle H$, $C_G(Y) = C_{\langle f, d \rangle H}(Y) = W \cdot C_{\langle f \rangle}(Y)$. Suppose $ht \in C(Y)$ for some $h \in \langle f \rangle$. Since $t$ inverts $Y$, $h$ also inverts $Y$ and so $h^2$ centralizes $Y$. Hence some nontrivial 2-element $g \in \langle h \rangle$ inverts $Y$, so that $C_H(g)$ contains no element of order 4, contrary to (viii) of Lemma 2.6.

Throughout the rest of the paper, $G^0$ will always denote a doubly transitive permutation group satisfying the hypothesis of our theorem and we assume $G^0$ has no regular normal subgroup.
Notation. \( C^*=C_G(N^*) \), which is semi-regular on \( \Omega-\{\alpha\} \) by Lemma 2.2. Let \( r \) be the number of \( N^* \)-orbits on \( \Omega-\{\alpha\} \).

Since \( G_\beta \geq N^* \), \( |\beta N^*| = |\gamma N^*| \) for \( \beta, \gamma \in \Omega-\{\alpha\} \) and so \( |\Omega|=1+r \times |\beta N^*| \). Hence \( r \) is odd and \( N^*_\beta \) is a subgroup of \( N^* \) of odd index. Therefore \( N^*_\beta \) is isomorphic to one of the groups listed in (x) of Lemma 2.6. Accordingly the proof of our theorem will be divided in six cases.

**Lemma 2.8.** Let \( Z \) be a cyclic subgroup of \( N^*_\beta \) with \( |Z| \neq 1, p \). Then

(i) If \( |Z|=2 \), \( |F(Z)|=1+(q-\varepsilon)|I(N^*_\beta)|/|N^*_\beta| \).

(ii) If \( |Z|=2 \), \( |F(Z)|=1+|N^*_\beta(Z)|/|N^*_\beta(Z)| \).

Proof. It follows from Lemma 2.3, 2.4 and 2.6 (i), (iii).

**Lemma 2.9.** If \( N^*_\beta \neq D_{q+1} \) and \( Z \) is a cyclic subgroup of \( N^*_\beta \) with \( |Z| \neq 1, p \) and \( N_G(Z)^{F(Z)} \) is doubly transitive. Then \( C^*=1 \) and one of the following holds.

(i) \( N_G(Z)^{F(Z)} \leq AGL(1,q_i) \) for some \( q_i \).

(ii) \( C_G(Z)^{F(Z)} \geq PSL(2,p_i), r=1 \) and \( |F(Z)|=1+|N^*_\beta(Z)|/|N^*_\beta(Z)| \).

(iii) \( N_G(Z)^{F(Z)}=R(3), \) the smallest Ree group, \( |F(Z)|=28 \).

Proof. Set \( N_G(Z)=L^* \) and \( F(Z)=\Delta \). By Lemma 2.6 (iii), \( L \cap N^*=D_{q+1} \) and \( L \cap N^*=\langle t \rangle Y \geq Y \geq Z \), where \( 0(t)=2, Y \simeq Z_{(q+1)/2} \).

If \( (L \cap N^*)^A = 1 \), then \( L \cap N^*=N^*_\beta \) because \( L \cap N^* \) is a maximal subgroup of \( N^* \). Since \( |N^*: N^*_\beta| \) is odd, \( L \cap N^*=N^*=D_{q+1} \), contrary to the assumption. Hence \( (L \cap N^*)^\Delta \neq 1 \) and as \( L_a \geq L_a \cap N^* \) and \( L_a \geq Y, (L_a)^A \) has a nontrivial cyclic normal subgroup. By Theorem 3 of [1], one of the following occurs:

(a) \( L^* \) has a regular normal subgroup

(b) \( L^* \geq PSL(2,p_i), |\Delta|=p_i+1 \), where \( p_i \geq 5 \) is a prime

(c) \( L^* \geq PSL(3,p_i), p_i \geq 3, |\Delta|=(p_i)^3+1 \)

(d) \( L^*=R(3), |\Delta|=28 \).

Suppose \( C^* \neq 1 \). Then there exists a subgroup \( D \) of \( C^* \) of prime order such that \( (L_a)^D \geq D^* \). Since \( [L_a, D] \leq D \cdot D_a \cap C^*=D(L_a \cap C^*)=D, D \) is a normal subgroup of \( L_a \). By (i) and (iii) of Lemma 2.6, \( G_a=L_a \cdot N^* \) and so \( D \) is a normal subgroup of \( G_a \). By Theorem 3 of [1], \( G^a \) has a regular normal subgroup, contrary to the hypothesis. Thus \( C^*=1 \).

If (a) occurs, \( L^* \) is solvable because \( L_a/L \cap N^*=L_a N^*/N^* \leq \text{Out}(N^*) \) and \( L \cap N^*=D_{q+1} \). Hence by [5], (i) holds in this case.

If (b) occurs, we have \( Y^A \neq 1 \), for otherwise \( (L \cap N^*)^A = 1 \) and \( N^*_\beta = L \cap N^*=D_{q+1} \), a contradiction. Hence \( 1 \neq C_G(Z)^A \leq L^* \) and so \( C_G(Z)^A \geq PSL(2,p_i) \) and \( Y^A \geq Z_{p_i} \). Therefore \( |\Delta \cap \beta N^*|=p_i \) and \( r=1 \) by Lemma 2.4 (ii). Since \( |\beta^A|=p_i \), we have \( |\beta^{L \cap N^*}|=p_i \), so that \( L \cap N^*: L \cap N^*_\beta=p_i \). Thus (ii) holds in this case.

The case (c) does not occur, for otherwise, by the structure of \( PSL(3,p_i) \),
a Sylow $p_1$-subgroup of $(L_a)'$ is not cyclic, while $(L_a)' \leq L \cap N^\ast=D_{2^e}$, a contradiction.

3. **Case (I)**

In this section we assume that $N^\ast_\beta \leq D_{2^e}$, where $\beta \neq \alpha$, $q=p^e$.

(3.1) (i) If $N^\ast_\beta \neq Z_2 \times Z_2$, $N_\beta N^\ast_\beta=N^\ast_\beta$ and $|F(N^\ast_\beta)|=r+1$.
(ii) If $N^\ast_\beta = Z_2 \times Z_2$, $N_\beta N^\ast_\beta = A_4$ and $|F(N^\ast_\beta)|=3r+1$.

Proof. Put $X=N_\beta N^\ast_\beta(N^\ast_\beta)$. Let $S$ be a Sylow 2-subgroup of $N^\ast_\beta$ and $Y$ a cyclic subgroup of $N^\ast_\beta$ of index 2.

If $N^\ast_\beta \neq Z_2 \times Z_2$, then $|Y|>2$ and so $Y$ is characteristic in $N^\ast_\beta$. Hence $X \leq N_\beta N^\ast_\beta(Y) \cong D_{2^e}$. From this [$N_\beta(S)$, $S \cap Y$] $\leq S \cap Y$ and $0^2(N_\beta(S))$ stabilizes a normal series $S \geq S \cap Y \geq 1$, so that $0^2(N_\beta(S)) \leq C_{N^\ast_\beta}(S)$ by Theorem 5.3.2 of [2]. By Lemma 2.6(i), $C_{N^\ast_\beta}(S) \leq S$ and hence $N_\beta(S)=S$. On the other hand by a Frattini argument, $X=N_\beta(S)N^\ast_\beta$ and so $X=N^\ast_\beta$. By Lemma 2.6(i), $(N^\ast_\beta)^\ast=(N^\ast_\beta)^\ast$ and so by Lemmas 2.3 and 2.4(ii), $|F(N^\ast_\beta)|=1+|F(N^\ast_\beta) \cap \beta^\ast| \times r=1+|N^\ast_\beta|/|N^\ast_\beta|=r+1$. Thus (i) holds.

If $N^\ast_\beta = Z_2 \times Z_2$, $N_\beta N^\ast_\beta = A_4$ by Lemma 2.6(iv). Similarly as in the case $N^\ast_\beta \neq Z_2 \times Z_2$, we have $|F(N^\ast_\beta)|=3r+1$.

(3.2) $N^\ast_\beta N^\ast_\beta \cap N^\ast = Z_2 \times Z_2$.

Proof. By Lemma 2.2, it suffices to consider the case $C^\ast=1$. Suppose $C^\ast=1$. Then $N^\ast_\beta/N^\ast \cap N^\ast = N^\ast_\beta N^\ast/N^\ast \leq \text{Out}(N^\ast)=Z_2 \times Z_2$ by Lemma 2.6(vi) and hence $(N^\ast_\beta)^\ast \leq N^\ast \cap N^\ast$. Since $N^\ast_\beta$ is dihedral, $N^\ast_\beta(N^\ast_\beta)^\ast = Z_2 \times Z_2$, so that $N^\ast_\beta/N^\ast \cap N^\ast \leq Z_2 \times Z_2$.

(3.3) Suppose $N^\ast_\beta = N_\beta \cap N^\ast$ and let $U$ be a subgroup of $N^\ast_\beta$ isomorphic to $Z_2 \times Z_2$. Then $|F(U)|=3r+1$ and $N_\gamma(U)^{F(U)}$ is doubly transitive.

Proof. Sex $X=N_\gamma(N^\ast_\beta)$, $\Delta=F(N^\ast_\beta)$ and let $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of $N^\ast$-orbits on $\Omega-\{\alpha\}$. If $g^{-1}N^\ast_\beta g \leq G_\gamma$, then $g^{-1}N^\ast_\beta g \leq N_\delta \cap N^\ast_\beta = N_\gamma \cap N^\ast_\beta \leq N_\beta$, where $\gamma=\alpha^\ast$. By a Witt's theorem, $X^\ast$ is doubly transitive.

If $U$ is a Sylow 2-subgroup of $N^\ast_\beta$, by a Witt's theorem, $N_\gamma(U)^{F(U)}$ is doubly transitive. Moreover $N_\beta N^\ast_\beta(U)=A_4$ and so by Lemmas 2.3 and 2.4(ii), $|F(U)|=1+|A_4| \times |N^\ast_\beta: N_\gamma(U)| \times r/|N^\ast_\beta|=3r+1$.

If $|N^\ast_\beta|\geq 4$, by Lemma 2.6(iv) and(v), $N_\beta N^\ast_\beta(U)=S_4$ and $N^\ast_\beta$ has two conjugate classes of four-groups, say $\pi=\{K_1, K_2\}$. Set $X^\ast=M$. Then $M \geq N^\ast_\beta$ and $X/M \leq Z_2$. Clearly $F(U) \cap \Delta_\gamma=\emptyset$ for each $i$ and so $|F(U) \cap \Delta_\gamma|=3$ by Lemma 2.3. Hence $|F(U)|=3r+1$. Since $N_\beta N^\ast_\beta(U)=S_4$, we may assume $r>1$. Hence by (3.1) (i) $|\Delta|=r+1 \geq 4$, so that $M^\ast$ is doubly transitive. Since $M=M^\ast N_\beta N^\ast_\beta(U)$, $N_\beta N^\ast_\beta(U)^{A_4}$ is also doubly transitive and so $N_\beta N^\ast_\beta(U)$ is transitive on $\Delta$—
\{a\}. As \(|\Delta \cap \Delta_i|=1, \Delta \cap \Delta_i \subseteq F(U)\) and \(N_{,\beta}(U)\) is transitive on \(F(U) \cap \Delta_i\) for each \(i\), \(N_G(U)^{F(U)}\) is doubly transitive.

(3.4) (i) \(C^\circ=1\).

(ii) Let \(U\) be a subgroup of \(N_{\alpha}\) isomorphic to \(Z_2 \times Z_2\). If \(N_{\alpha}^\circ=N^\circ \cap N_{\beta}\), then \(N_G(U)^{F(U)}\) has a regular normal 2-subgroup. In particular \(|F(U)|=3r+1=2^b\) for positive integer \(b\).

Proof. Since \(N_G(U)^{F(U)}\) is doubly transitive, by (3.3) and Theorem 3 of [1], \(N_G(U)^{F(U)}\) has a regular normal subgroup, \(N_G(U)^{F(U)} \supseteq \text{PSU}(3,3)\) or \(N_G(U)^{F(U)}=R(3)\).

Suppose \(C^\circ \neq 1\). Let \(D\) be a minimal characteristic subgroup of \(C^\circ\). Clearly \(G_{,\alpha}D\). If \(N_G(U)^{F(U)} \supseteq R(3)\), \(D\) is cyclic. By Theorem 3 of [1], \(C^\circ\) has a regular normal subgroup, contrary to the hypothesis. Hence \(N_G(U)^{F(U)} \supseteq \text{PSU}(3,3)\) or \(N_G(U)^{F(U)}=R(3)\). Thus (3.4) holds.

(3.5) (i) If \(|Y| \geq 3\), \(N_G(Y)^{F(Y)}\) is doubly transitive.

(ii) If \(|Y| < 3\), \(N_{\beta}^\circ=Z_2 \times Z_2\) or \(N_{\beta}^\circ=Z_2\) and \(N^\circ \cap N_{\beta}^\circ \leq Z_2 \times Z_2\).

Proof. Suppose \(|Y| \geq 3\). If \(Y^\varepsilon \leq G_{,\alpha}\), \(Y^\varepsilon \leq N^\circ \cap G_{,\alpha}\leq N_{\beta}^\circ\), where \(\gamma=\alpha^\varepsilon\). If \(\gamma=\alpha\), obviously \(Y^\varepsilon \leq N^\circ\). If \(\gamma \neq \alpha\), \(N_{\beta}^\circ=N^\circ\). Therefore, as \(|Y| \geq 3\), \(N_{\beta}^\circ\) has a unique cyclic subgroup of order \(|Y|\). Hence \(Y^\varepsilon \leq N^\circ \cap N_{\beta}^\circ \leq N^\circ\), so that \(Y^\varepsilon \leq N^\circ\). Similarly \(Y^\varepsilon \leq N_{\beta}^\circ\). Thus \(Y^\varepsilon \leq N^\circ \cap N_{\beta}^\circ\) and so \(Y^\varepsilon = Y\). By a Witt's theorem, \(N_G(Y)^{F(Y)}\) is doubly transitive on \(F(Y)\).

Suppose \(|Y| < 3\). Since \(|N^\circ \cap N_{\beta}^\circ; Y| \leq 2\), we have \(N^\circ \cap N_{\beta}^\circ \leq Z_2 \times Z_2\). On the other hand, as \(N_{\beta}^\circ\) is dihedral, \((N_{\beta}^\circ)'\) is cyclic. Hence (ii) follows immediately from (3.2).

(3.6) Set \(\Delta=F(N_{\beta}^\circ), L=G(\Delta), K=G_{,\Delta}\) and suppose \(N_{\beta}^\circ \neq Z_2 \times Z_2\). Then \(L_G \supseteq N_{\beta}^\circ\), \((L_{,\alpha})' \leq N_{\beta}^\circ, K' \leq N^\circ \cap N_{\beta}^\circ\) and \((L_{,\alpha})'=Z_{,r}\). If \(r \neq 1\), \(L_{,\alpha}\) is a doubly transitive Frobenius group of degree \(r+1\).

Proof. By Corollary B1 of [7] and (i) of (3.1), \(L_{,\alpha}\) is doubly transitive and
\(|\Delta| = r + 1\). Since \(N^a \cap L \geq N^a \cap K = N^a_s\), by (i) of (3.1), we have \(N^a \cap L = N^a_s\). Hence \(L_a \geq N^a_s\). By (i) of (3.4), \(L_a/N^a_s = L_a/N^a_s \cap \text{Out}(N^a) = Z_2 \times Z_2\) and so \((L_a)^2 \leq N^a_s\) and \((L_a)^2 = Z_2\). If \(r \neq 1\), then \((L_a)^2 = 1\). On the other hand \((L_a)^2 = 1\) as \((L_a)^2\) is abelian. Hence \(L^a\) is a Frobenius group.

(3.7) Suppose \(|Y| \geq 3\). Then there exists an involution \(z\) in \(N^a_{\beta} \cap Y\) such that \(Z(N^a_\beta) = \langle z \rangle\).

Proof. Suppose \(N^a_{\beta} \neq Z_2 \times Z_2, |N^a_{\beta}|_2 \geq 2\) and \(N^a_{\beta}\) is dihedral, we have \(\langle I(W) \rangle = Z(N^a_{\beta}) = Z_2\) and \(N^a_{\beta}/N^a_{\beta}' = Z_2 \times Z_2\). Let \(Z(N^a_{\beta}) = \langle z \rangle\) and suppose that \(z\) is not contained in \(Y\). By (3.2), \((N^a_{\beta})' \leq N^a \cap N^a \cap W = Y\) and so \(|N^a_{\beta}'|\) is odd. Hence \(|N^a_{\beta}|_2 = 4\) and \(q \equiv \beta^2 = 3\) or 5 (mod 8), so that \(n\) is odd. By (3.2) and (i) of (3.4), \(N^a_{\beta}/N^a \cap N^a = N^a_{\beta}/N^a_{\beta}', N^a = 1\). If \(N^a_{\beta} = N^a \cap N^a\), then \(W = Y\) and so \(z \in Y\), contrary to the assumption. Therefore we have \(N^a_{\beta} \cap N^a = Z_2\) and \(N^a_{\beta} = \langle z \rangle \times (N^a \cap N^a)\). Since \(n\) is odd and \(z \in N^a_{\beta} \cap N^a\), by Lemma 2.6 (vi), (vii) and (ix), \(N^a_{\beta} \cap N^a = N^a_{\beta} \cap N^a\), and \(C_N(z) = D_{q+4}\). But \(N^a \cap N^a \subseteq C_N(z)\) and besides it is isomorphic to a subgroup of \(D_{q+4}\). Hence \(N^a \cap N^a = Z_2\) and \(N^a_{\beta} = Z_2 \times Z_2\), a contradiction.

(3.8) Suppose \(|Y| \geq 3\). Then \(N^a_{\beta} = N^a \cap N^a\).

Proof. Suppose \(N^a_{\beta} \neq N^a \cap N^a\) and let \(\Delta, L, K\) be as defined in (3.6) and \(x \in L_a\) such that its order is odd and \(\langle x \rangle\) is transitive on \(\Delta - \{\alpha\}\). As \(|Y| \geq 3\), \(W\) is characteristic in \(N^a_{\beta}\) and hence by (3.6), \(x\) stabilizes a normal series \(L_a \supseteq N^a_{\beta} \supseteq W \supseteq (N^a_{\beta}')\). By Theorem 5.3.2 of [2], \([x, 0_L(L_a/(N^a_{\beta}'))] = 1\). Since \(L_a/(N^a_{\beta}')\) has a normal Sylow 2-subgroup and \((N^a_{\beta}') \leq K'\), we have \([x, 0_L(L_a/K')] = 1\). So \([x, N^a_{\beta}] = K \leq N^a \cap N^a\) by (3.6). If \(r \neq 1\), then \(\beta^2 = \beta\) and \(\beta^2 \in \Delta\), hence \(N^a_{\beta} = x^{-1}N^a_\beta x = N^a_\beta\), where \(\gamma = \beta^2\). Since \(\gamma \in \Delta\) and \(\Delta = F(N^a_{\beta}), N^a_{\beta} \leq N^a \cap N^a \subseteq G = N^a_{\beta}\). Similarly \(N^a_{\beta} = N^a_{\beta}\). Hence \(N^a_{\beta} = N^a_{\beta}\), which implies \(N^a_{\beta} = N^a \cap N^a\). By the doubly transitivity of \(G\), we have \(N^a_{\beta} = N^a \cap N^a\), contrary to the assumption. Therefore we obtain \(r = 1\).

Let \(z\) be as defined in (3.7) and put \(k = (q - \varepsilon)/|N^a_{\beta}|\). By Lemma 2.8(i) we have \(|F(z)| = 1 + (q - \varepsilon)(|N^a_{\beta}|/2 + 1)/|N^a_{\beta}| = (q - \varepsilon)/2 + k + 1\). Similarly \(|F(Y)| = k + 1\). As \(N^a_{\beta} \neq N^a \cap N^a\), there is an involution \(t\) in \(N^a_{\beta}\) which is not contained in \(N^a\). By Lemma 2.6 (i), \(t^z = z\) for some \(y \in N^a_{\beta}\). Set \(\gamma = \beta^2\). Then \(\gamma \in F(z)\) and \(z \in N^a_{\beta}\). By Lemma 2.6 (vii), (viii) and (ix), \(C_{N^a}(z) = D_{q+4}\) or \(PGL(2, \sqrt{q})\). Assume \(C_{N^a}(z) = D_{q+4}\) and let \(R\) be a cyclic subgroup of \(C_{N^a}(z)\) of index 2. We note that \(R\) is semi-regular on \(\Omega - \{\alpha\}\). Set \(X = C_{N^a}(z)\). Since \(2 \leq k + 1 \leq (q - \varepsilon)/|q - \varepsilon| + 1\), we have \((q + \varepsilon)/2 \geq k + 1\) and so \(|\alpha^z| > k + 1\). By (i) of (3.5) and (3.7), \(N_{\gamma}(Y) \subseteq C_{N^a}(X) = X\) and \(\alpha^X \subseteq F(Y)\). It follows from Lemma 2.1 that \(\alpha^z = \{x \in N^a_{\beta} \mid x \in N^a_{\beta}\} \neq \gamma\). Hence \(|F(z)| > |\alpha^x| > |F(Y)| + (q + \varepsilon)2 = k + 1 + (q - \varepsilon)2 + \varepsilon = |F(z)| + \varepsilon\). Therefore \(\varepsilon = 1\) and \(\gamma^x = \{\gamma\}\), so that \(\gamma \in F(Y)\), a contradiction. Thus \(C_{N^a}(z) \subseteq PGL(2, \sqrt{q}), \varepsilon = 1, N^a_{\beta}/N^a \cap N^a = Z_2\) and \(|\langle \alpha^x \cap G_{\tilde{\alpha}} \rangle| = N^a_{\beta}| = 2\).
Set $\Delta_1=\alpha^x$ and $\Delta_2=F(z)-\Delta_1$. Let $\delta\in\Delta_2$ and $g$ an element of $G$ satisfying $\delta^g=\gamma$. Then $x\in N_5^\alpha N^3-N^4$ and so $x^g\in N_5^\alpha N^3-N^4$, where $v=\alpha^x$. Since $|\langle x^g \cap G_\gamma \rangle| = |N^3| = 2$ and $x\in G_\gamma-N^4$, it follows from Lemma 2.6 (ix) that $(x^g)^h=x$ for some $h\in G_\gamma$. Hence $gh\in X$ and $\delta^gh=\gamma$. Thus $\Delta_2=\gamma^x$. Let $\delta\in\Delta_2$. Then $z=\gamma^x$ and so $z\in N^4$ by (3.7) and so $X\cap N^4=Z_2\times Z_2$, which implies $|\delta^G(z)|=(q-1)/4$. Hence $(|\Delta_1|, |\Delta_2|)=((q-1)/4+k+1, (q-1)/4)$ or $(k+1, (q-1)/2)$. Let $P$ be a subgroup of $C_N(z)$ of order $\sqrt{q}$. Then $F(P)=\{\gamma\}$ and $P$ is semi-regular on $\Omega-M$. If $|\Delta_2|=(q-1)/4$, then $q/5=(q-1)/4=-(q-5)/4$ and $q/5=(q-1)/4+k+1$. From this, $q=5^4$, $k=3$, $|\Delta_1|=10$ and $|\Delta_2|=6$. Since $(C_N(z))^2=S_3$, $X^a=S_3$ and so $|X|\geq 3^2$. As $X$ acts on $\Delta_1$ and $|\Delta_1|=1\mod 3$, $|G_a|\geq |X_a|\geq 3^2$, contrary to $N^a=PSL(2,25)$. If $|\Delta_2|=(q-1)/2$, $q/5=(q-1)/2=-(q-3)/2$, so $q=3^2$, $k=1$, $N^a=D_3$ and $\Delta_1=\{\alpha, \beta\}$. Hence $C_N(z)$ fixes $\alpha$ and $\beta$, so that $PGL(2,3)=C_N(z)=\Omega-M$. If $|\Delta_2|=(q-1)/2$, then $q/5=\frac{q}{2}(q+2\epsilon-2k-2)$. Hence $(q-\epsilon+2k+2)(q+\epsilon+2k-2)(k+1-\epsilon)$.

Proof. Set $S=\{(\gamma, u)|\gamma\in F(u), u\in z\}$, where $z$ is an involution in $N^a$. We now count the number of elements of $S$ in two ways. Since $N^a=N^\alpha \cap N^h$, $F(z)=\{\gamma|z\in N^a\}$ and hence $C_G(z)$ is transitive on $F(z)$ by Lemma 2.1. Therefore $|S|=|\Omega||z^G|=|z^G||F(z)|$. Since $r=1$, $|\Omega|=1+|N^a|: N^a:=kq(q+\epsilon)/2+1$ and by Lemma 2.8 $|F(z)|=(q-\epsilon)/2+k+1$. Since $G_\alpha\supseteq N^a$, $z^G\alpha$ is contained in $N^a$ and so $|G_\alpha|: C_{G_\alpha}(z)=|N^a|: C_{N^\alpha}(z)=q(q+\epsilon)/2$. Hence $(q-\epsilon)/2+k+1|(kq(q+\epsilon)+2(q+\epsilon))$. On the other hand, $|F(z)|=|C_G(z)|_2|C_{G_\alpha}(z)|_2\leq |G_\alpha|_2$ and $|G_\alpha|_2=|G_\alpha|_2=|\Omega|_2$ because $|G_\alpha|: C_{G_\alpha}(z)=q(q+\epsilon)/2=1\mod 2$. Hence $|q-\epsilon+2k+2|\leq |kq(q+\epsilon)+2|$. Since $kq(q+\epsilon)+2=(kq+2k(q-1))\leq q(q-2k+2)+2((k+2k-\epsilon)(k-1)+k+1)$ and $q(q+\epsilon)+2(2k-2)(q-\epsilon+2k+2)+2(k+2-\epsilon)(k+1-\epsilon)$, we have (3.10).

(3.11) Suppose $|Y|\geq 3$. Then one of the following holds.
(i) $N^a=N^\alpha \cap N^h=D_4$.  
(ii) $N^a=N^\alpha \cap N^h=D_4$. and $N_G(Y)^{F(\gamma)}$ has a regular normal subgroup.

Proof. Suppose false. Then, by (3.5), (3.8) and Lemma 2.9, $N_G(Y)^{F(\gamma)}=R(3)$ or there exists a prime $p_1\geq 5$ such that $N_G(Y)^{F(\gamma)}\geq PSL(2,p_1)$ and $V/Y\cong Z_{p_1}$. By (i) of (3.1) and (3.9), $F(N^a_0)=\{\alpha, \beta\}$. On the other hand, $(N^a_0)^{F(\gamma)}=N^a_0/Y\cong Z_2$. Hence $N_G(Y)^{F(\gamma)}\neq R(3)$ and $C_G(Y)^{F(\gamma)}\neq R(3)$.  

By (i) of (3.4) and Lemma 2.7, we have \( C_{G_{a}}(Y) = V \langle f_i \rangle \), where \( f_i \) is a field automorphism of \( N^* \). Let \( t \) be the order of \( f_i \), \( n = tm \) and let \( p^m \equiv \varepsilon_i \equiv 1 \mod 4 \). Clearly \( C_{G_{a}}(Y)^{F(Y)} \supseteq V^{F(Y)} \simeq \mathbb{Z}_{n_1} \) and \( |C_{G_{a}}(Y)^{F(Y)}| \mid t \), so that \( (p_i - 1)/2 \mid t \).

First we assume that \( t \) is even and set \( t = 2t_i \). Then \( Y \leq C_{N^*}(f_i) = PGL(2, p^m) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \) is a cyclic subgroup of \( C_{N^*}(f_i) \) of order \( p^m - \varepsilon_1 \) and \( (p^m - 1)/2(p^m - \varepsilon_1) = p_1 \). Put \( s = \sum_{i=1}^{t_i} (p^m)^i \). Then \( (p^m + \varepsilon_1)s/2 = p_1 \), so that we have either (i) \( t_i - 1 = 1 \) and \( p_1 = (p^m + \varepsilon_1)/2 \) or (ii) \( t_i \geq 2 \), \( p^m = 3 \) and \( p_1 = s \). In the case (i), \( 2 \leq (p_i - 1)/2 = (p^m + \varepsilon_1 - 2)/4 \mid 2t_i = 2 \). Hence \( (p_1, q) = (5, 3^4) \) or \((4, 11^3)\). Let \( z \) be as in (3.7). As mentioned in the proof of (3.10), \( |F(z)| = (q - 1)/2 + k + 1 \), \( |\Omega| = k(q + 1)/2 + 1 \) and \( C_G(z) \) is transitive on \( F(z) \). If \( q = 3^4 \), then \( |F(z)| = 46 \) and \( |\Omega| = 2 \cdot 19 \cdot 23 \). Hence \( |C_G(z)| = |F(z)| |C_G(z)N^*/N^*| = 46 \cdot 2^t \cdot 80 = 2^{t+1} \cdot 5 \cdot 23 \) with \( 0 \leq i \leq 3 \). Let \( P \) be a Sylow 23-subgroup of \( C_G(z) \) and \( Q \) a Sylow 5-subgroup of \( C_G(z) \). Since \( 11 \not| \Omega \), \( P \) is a subgroup of \( N_r \) for some \( r \in \Omega \) and \( F(P) = \{ y \} \). Hence \( ye \Lambda/7 \), contrary to \( C_{N^*}(z) = D_{150} \). In the case (ii), we have \( (p_i - 1)/2 = \sum_{i=1}^{t_i - 1} 9^i/2 \mid t = 2t_i \). From this, \( 9^{t_i - 1} \leq 4t_i \), hence \( t_i = 1 \), a contradiction.

Assume \( t \) is odd. Then \( Y \leq C_{N^*}(f_i) = PGL(2, p^m) \) by Lemma 2.6 (viii). As \( |V/Y| = p_1 \) and \( p_1 \) is a prime, \( Y \simeq Z_{(p^m - t_i)}(q - \varepsilon)/(p^m - \varepsilon_1) = p_1 \). Hence \( \sum_{i=1}^{t_i - 1} (p^m)^i (\varepsilon_i)^{t_i - 1} = p_1 \) and \( (p_i - 1)/2 = \sum_{i=1}^{t_i - 1} (p^m)^i (\varepsilon_i)^{t_i - 1} - 1)/2 \mid t \). In particular \( 2t \geq (p^m)^{t_i - 1} - (p^m)^{t_i - 2} = (p^m)^{t_i - 2} \geq 2(p^m)^{t - 2} \). From this \( t = 3 \), \( m = 1 \), \( p_1 = 7 \) and \( q = 3^4 \), so that \( N_r^* \simeq Z_2 \times Z_2 \), a contradiction.

(3.12) (i) of (3.11) does not occur.

Proof. Let \( G^a \) be a minimal counterexample to (3.12) and \( M \) a minimal normal subgroup of \( G \). By the hypothesis, \( G \) has no regular normal subgroup and hence \( M_{\pm 1} \). As \( M_{\pm 1} \) is a normal subgroup of \( G_{a} \), by (i) of (3.4), \( M_{a} \) contains \( N^* \). By (3.9), \( r = 1 \), hence \( M \) is doubly transitive on \( \Omega \). Therefore \( G = M \) and \( G \) is a nonabelian simple group.

Since \( N_r^* = D_4 \cdot k = 1 \) and \( q - \varepsilon + 4 \mid (4 - \varepsilon)(2 - \varepsilon) + 1 \mid (4 - \varepsilon)(2 - \varepsilon) \) by (3.10). Hence we have \( q = 7, 9, 11, 19, 27 \) or 43.

Let \( x \) be an element of \( N_r^* \). If \( |x| > 2 \), by Lemma 2.8, \( |F(x)| = 1 + |N_r^*| \times 1/|N_r^*| = 2 \) and if \( |x| = 2 \), similarly we have \( |F(x)| = (q - \varepsilon)/2 + 2 \). Assume \( q = 9 \) and let \( d \) be an involution in \( G_{a} - N^* \) such that \( \langle d \rangle N^* \) is isomorphic to \( PGL(2, p^m) \).
We may assume \( d \in G_{ab} \). Since \( \langle d \rangle N^a \) is transitive on \( \Omega - \{ \alpha \} \), by Lemmas 2.3 and 2.6 (vii), (ix), \(|F(d)| = 2(q-1)(q+1)/2(q+1)+1 = (q+1)/2\), while \(|F(x)| = (q+1)/2 + 2\) for \( x \in I(N^a) \). Hence \( d \) is an odd permutation, contrary to the simplicity of \( G \). Thus \( G_a = N^a \) if \( q \neq 9, 27 \) and \(|G_a/N^a| = 1, 3\) if \( q = 27 \).

If \( q = 9 \), \(|\Omega| = 1 + 9 \cdot 10/2 = 2 \cdot 3 \cdot 5 \) with \( 0 \leq i \leq 2 \). Let \( P \) be a Sylow 23-subgroup of \( G \). Since \( \text{Aut}(Z_{23}) = 3 \cdot |N_c(P)| = 2 \cdot |PSL(2, 9)| = 2 \cdot 3^3 \cdot 5 \) with \( 0 < \alpha < 2 \), let \( \gamma \) be a Sylow 7-subgroup of \( G \). Since \( \gamma \) is a 7-subgroup of \( G \), \( |\gamma| = 2^8 \cdot 3^2 \cdot 5 \) for some \( a \) with \( 0 < a < 6 \). By a Sylow's theorem, \( 2^8 \cdot 3^2 \cdot 5 = 2^6 \cdot 3^2 \cdot 5 = 1 \pmod{23} \), a contradiction.

If \( q = 27, |\Omega| = 1 + 27 \cdot 26/2 = 2 \cdot 5 \cdot 11 \) and \(|G_a| = 2^2 \cdot 3^4 \cdot 7 \cdot 13 \) with \( 0 < a < 1 \). Let \( P \) be a Sylow 11-subgroup of \( G \). Since \( \text{Aut}(Z_{23}) = 3 \cdot |N_c(P)| = 2 \cdot |PSL(2, 9)| = 2 \cdot 3^3 \cdot 5 \cdot 13 \) with \( 0 < a < 11 \), and \(|G| = 2^5 \cdot 3^5 \cdot 7 \cdot 11 \).

We now argue that \( \langle \gamma \rangle N^a \sim D_{24} \). Let \( R \) be the Sylow 3-subgroup of \( N^a_R \). If \( t \) centralizes \( R \), \( R \) acts on \( F(t) \) and so \( F(R) \subseteq F(t) \) as \(|F(t)| = 8 \) and \(|F(R)| = 2 \). Hence \( \alpha \sim \alpha \), contrary to the choice of \( t \). Therefore \( t \) inverts \( R \) and \( \langle t \rangle N^a \) is isomorphic to \( Z_2 \times D_{12} \) or \( D_{24} \). Suppose \( \langle t \rangle N^a \sim Z_2 \times D_{12} \). Then \( \langle t \rangle N^a \) contains fifteen involutions and so we can take \( u \in I(\langle t \rangle N^a) \) satisfying \(|F(u)| = 0 \) and \( \langle t \rangle N^a \sim \langle u \rangle \times N^a \). As \(|F(u)| = 0 \), \(|F(u')| = |\Omega|/2 = 28 \). By Lemma 2.3, 28 = \(|C_u(u)| \times |\langle u \rangle N^a \cap u^c|/24 \) and hence \(|C_u(u)| = 2^3 \cdot 7 \) or \( 2^5 \cdot 3 \cdot 7 \). Since \( \langle u \rangle N^a = N_c(R) \), we have \(|C_u(u) : C_u(u) \cap N_c(R)| = 2 \cdot 7 \) or \( 2^2 \cdot 7 \). By a Sylow's theorem, \(|C_u(u) : C_u(u) \cap N_c(R)| = 2^2 \cdot 7 \), so that \(|C_u(u)| = 2^5 \cdot 3 \cdot 7 \). Let \( Q \) be a Sylow 7-subgroup of \( C_u(u) \). Then \(|C_u(u) \cap N_c(Q)| = 2^5 \cdot 3 \cdot 7 \) or \( 2^2 \cdot 7 \) by a Sylow's theorem. Hence \( 2^2 \cdot 7 \mid |N_c(Q)| \). Since \( \text{Aut}(Z_7) = Z_2 \times Z_3 \),
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Let \( U \) be a Sylow 2-subgroup of \( N_\alpha \) and set \( L = N_G(U) \). It follows from (3.3) and Lemma 2.6 (iv) that \( L \cap N^* = A_4 \), \( L^{(U)} = A_4 \) and \( |L| = 2^3 \cdot 3 \). Let \( T, \langle \rho \rangle \) be Sylow 2- and 3-subgroup of \( L \), respectively. Obviously \( L \supset \Gamma \) and \( C_{\langle \rho \rangle}(x) = 1 \).

On the other hand \( T > \Gamma \) and \( \langle \rho \rangle \supset N^* = D_{24} \) and so \( T = Z_2 \times Z_2 \) because \( C_{\langle \rho \rangle}(x) = 1 \).

By Theorem 5.4.5 of [2], \( T \) is dihedral or semi-dihedral. Hence \( N_G(T)/C_{\langle \rho \rangle}(T) \) is (Aut\( T \)) is a 2-group, so that \( C_{\langle \rho \rangle}(x) = 1 \), a contradiction.

Let \( p_1 \) be an odd prime such that \( p_1 \mid |\Omega| \) and \( p_1 \), \( |G_a| \) and let \( P \) be a Sylow \( p_1 \)-subgroup of \( G \). Clearly \( P \) is semi-regular on \( \Omega \) and so any element in \( C_G(P) \) has at least \( p_1 \) fixed points. If \( x \) is an element of \( N_\alpha \) and its order is at least three, then \( F(x) = |F(Y)| = 4 \) by Lemma 2.8. Since \( |N_\alpha| = (q - \varepsilon)/3 \), we have \( |\Omega| = 1 + 3(q + \varepsilon)/2 \).

If \( q = 5^2 \), then \( |\Omega| = 2^4 \cdot 61 \) and \( |G_a| = 2^{4+i} \cdot 3 \cdot 5^2 \cdot 13 \) (\( 0 \leq i \leq 2 \)). Let \( P \) be a Sylow 61-subgroup of \( G \). Then \( P \cong Z_{61} \). As mentioned above, \( 5, 13 \) and \( |C_G(P)| \) and so \( 5^2, 13 \) and \( |N_G(P)| \). Hence \( |G| = N_G(P) = 2^a \cdot 3^{i+1} \cdot 13 \), where \( 0 \leq a \leq 10 \) and \( 0 \leq b, c \leq 1 \). But we can easily verify \( |G| : N_G(P) \mid 1 \equiv 1 \) (mod 61), contrary to a Sylow's theorem.

If \( q = 7^2 \), then \( |\Omega| = 2^4 \cdot 919 \) and \( |G_a| = 2^{7+i} \cdot 3 \cdot 5^2 \cdot 7^2 \) (\( 0 \leq i \leq 2 \)). Let \( P \) be a Sylow 919-subgroup of \( G \). By the similar argument as above, we obtain \( 5, 7 \) and \( |N_G(P)| \). Hence \( |G : N_G(P)| \equiv 1 \) (mod 61), a contradiction.

If \( q = 11^2 \), then \( |\Omega| = 2^4 \cdot 173 \) and \( |G_a| = 2^{7+i} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \) (\( 0 \leq i \leq 2 \)). Let \( P \) be a Sylow 173-subgroup of \( G \). Similarly we have \( 3, 5, 11, 61 \) and \( |N_G(P)| \) and so \( |G : N_G(P)| = 2^4 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \equiv -5 \cdot 2^2 \) (mod 173), where \( 0 \leq a \leq 12 \). Hence \( |G : N_G(P)| \equiv 1 \) (mod 61), a contradiction.

If \( q = 59 \), then \( |\Omega| = 2^4 \cdot 17 \cdot 151 \) and \( |G_a| = 2^{7+i} \cdot 3 \cdot 5 \cdot 29 \cdot 59 \) (\( 0 \leq i \leq 1 \)). Let \( P \) be a Sylow 17-subgroup of \( G \). Similarly we have \( 3, 5, 29, 59 \) and \( |N_G(P)| \) and so \( |G : N_G(P)| = 2^4 \cdot 3 \cdot 5 \cdot 29 \cdot 59 \cdot 151^4 \equiv 10 \cdot 2^2 \) or \( 12 \cdot 2^2 \) (mod 17), where \( 0 \leq a \leq 4 \) and \( 0 \leq b \leq 1 \). From this, we have a contradiction.

If \( q = 71 \), then \( |\Omega| = 2^5 \cdot 233 \) and \( |G_a| = 2^{7+i} \cdot 3 \cdot 5 \cdot 7 \cdot 71 \) (\( 0 \leq i \leq 1 \)). Let \( P \) be
a Sylow 233-subgroup of $G$. Since $3,5,7,71 | N_{G}(P)$, $|G|: N_{G}(P)| = 2^{a} \cdot 3^{b} \cdot 5^{c} \cdot 7^{d} \cdot 71^{e}$ (mod 233), where $0 \leq a, b, c, d, e \leq 9$. Similarly we get a contradiction.

We now consider the case $|Y| < 3$. By (ii) of (3.5), $N_{\beta}^{a} = Z_{2} \times Z_{2}$ or $N_{\beta}^{a} = D_{8}$ and $N_{\beta}^{a} \cap N_{\beta} = 0 \leq Z_{2} \times Z_{2}$.

(3.14) The case that $N_{\beta}^{a} = Z_{2} \times Z_{2}$ does not occur.

Proof. Set $\Delta = F(N_{\beta}^{a})$. Then $|\Delta| = 3r+1$ and $\Delta = F(N_{\beta}^{a}N_{\beta}^{a})$ by (ii) of (3.1) and Corollary B1 of [7]. Since $|N_{\beta}^{a}| = 4$, we have $q = p^a = 3,(mod 8)$ and so $n$ is odd. Hence $|G_{a}/N_{\beta}^{a}| = 2$ and $N_{\beta}^{a} \cap N_{\beta}^{a} = N_{\beta}^{a}N_{\beta}^{a}N_{\beta}^{a}N_{\beta}^{a} = 1$ or $Z_{2}$ by (3.2). Suppose $N_{\beta}^{a}N_{\beta}^{a}$ is a Sylow 2-subgroup of $G_{a}$, hence $G_{a}(N_{\beta}^{a}N_{\beta}^{a})$ is doubly transitive by a Witt's theorem. Since $N_{\beta}^{a}N_{\beta}^{a} = D_{8}$ and $|\Delta|$ is even, $G_{a}(N_{\beta}^{a}N_{\beta}^{a})$ is also doubly transitive. Let $g$ be an element of $G_{a}(N_{\beta}^{a}N_{\beta}^{a})$ such that $\alpha' = \beta$ and $\beta' = \alpha$. Then $N_{\beta}^{a} = g^{-1}N_{\beta}^{a}g = N_{\beta}^{a}$ and hence $N_{\beta}^{a} = N_{\beta}^{a} \cap N_{\beta}$, a contradiction. Thus $N_{\beta}^{a} = N_{\beta}^{a} \cap N_{\beta} = Z_{2} \times Z_{2}$.

Let $z$ be an involution in $N_{\beta}^{a}$ and $t \in zG$ an involution such that $\alpha' = \beta$. Set $\Gamma = \{\gamma, \delta \mid \gamma, \delta \in \Omega, \gamma \neq \delta\}$. We consider the action of the element $z$ on $\Gamma$.

By the similar argument as in the proof of (3.12), $|F(z)| = |F(z)| - 1)/2 + (|\Omega| - |F(z)|)/2 = |C_{\Omega}(z)| = |C_{\Omega}(z)| \cdot \langle z \rangle \cdot |C_{\Omega}(z)| \cdot |C_{\Omega}(z)| \cdot |C_{\Omega}(z)|$. Since $N_{\beta}^{a} = N_{\beta}^{a} \cap N_{\beta}^{a}$ by Lemma 2.6 (ii), $z_{G} \cap N_{\beta} = z_{G}^{a}$ and so $|C_{\Omega}(z)| = |F(z)| \cdot |C_{\Omega}(z)| \cdot |C_{\Omega}(z)| \cdot |C_{\Omega}(z)|$, so that $|G_{a}| = |\Omega| = 1 (mod |F(z)|)$. Since $|G_{a}|/N_{\beta}^{a} = |G_{a}/N_{\beta}^{a} \cap N_{\beta}^{a}| = 2$, we have $|G_{a}| |8n$. Clearly $|\Omega| = 1 + g(q - \epsilon) (q + \epsilon) r/8$ and by Lemma 2.8 (i), $|F(z)| = 1 + 3 (q - \epsilon) r/4$. Hence $1 + 3(q - \epsilon) r/4 < 8n(1 + g(q - \epsilon) (q + \epsilon) r/8)$. Put $n = rs$. Then $3r - 3er + 4 (4s + g(q - \epsilon) (q + \epsilon) r) 3r = 864 r^{2} + 4s (3pq) (3pq - 3er) (3er + 3er)$. Hence $3r - 3er + 4 | 864 r^{2} + 4s (3pq - 3er) (3er + 3er)$. Since $3q - 3er < 864 r^{2}$, $n = 1$ or $(n, p) = (3, 5, 3, -1)$, while none of these satisfy (3.14). Therefore $m = (2q^{2} + (2q + 9)q - 9r)/(3q - 3q + 4)$. It follows that $(q, m) = (19, 27/2), (37, 28), (83, 449/8)$ or $(149, 411/4)$. Since $m$ is an integer, we have $(q, m) = (37, 28)$. But $m \leq |\langle t \rangle| G_{a} | \leq 16$, a contradiction. Thus (3.14)
holds.

(3.15) The case that \( N_\alpha^* = D_6 \) and \( N_\alpha^* \cap N_\beta = Z_2 \times Z_2 \) does not occur.

Proof. Let \( \Delta, L \) and \( K \) be as defined in (3.6). By (3.6), there exists an element \( x \) in \( L_\alpha \) such that its order is odd and \( \langle x^\alpha \rangle \) is regular on \( \Delta - \{ \alpha \} \).

Since \( (L_\alpha)' \leq N_\alpha^* \) by (3.6) and \( N_\alpha^* = D_6 \), \( x \) stabilizes a normal series \( N_\alpha^* \unlhd N_\alpha^* \unlhd N_\alpha^* \). Hence \( x \) centralizes \( N_\alpha^* \) by Theorem 5.3.2 of [2] and so \( x^{-1}N_\alpha^* = N_\alpha^* \). Put \( \gamma = \beta^2 \).

If \( r = 1 \), then \( \beta = \gamma \), so that \( N_\alpha^* = N_\alpha^* \). From this, \( N_\alpha^* = N_\alpha^* \).

By the doubly transitivity of \( G \), \( N_\alpha^* = N_\alpha^* \), hence \( N_\alpha^* = N_\alpha^* \cap N_\beta \), a contradiction. Therefore \( r = 1 \) and \( \Delta = \{ \alpha, \beta \} \).

Set \( \langle \delta \rangle = Z(N_\alpha^*) \), \( \Delta_1 = \alpha^{C_\alpha(\delta)} \) and let \( \{ \Delta_1, \Delta_2, \ldots, \Delta_j \} \) be the set of \( C_\alpha(\delta) \)-orbits on \( F(\delta) \).

Since \( L \unlhd N_\alpha^* \cap N_\beta \) and by (3.2), \( N_\alpha^* \cap N_\beta < 1 \), \( z \) is contained in \( N_\alpha^* \cap N_\beta \). Hence, by Lemma 2.1, \( \beta \in \Delta_1 \) and \( k \) is at least two. By Lemma 2.8, \( |F(\delta)| = 1 + (q - \varepsilon)|N_\alpha^*| = 1 + 5(q - \varepsilon)/8 \).

Clearly \( |C_\alpha(\delta)| = (q - \varepsilon)/8 \) and so \( |\Delta_1| \geq 1 + (q - \varepsilon)/8 \). If \( \gamma \in F(\delta) - \Delta_1 \), then \( C_\alpha(\delta) \unlhd Z_2 \times Z_2 \), for otherwise \( \langle \delta \rangle = Z(N_\alpha^*) \nleq N_\alpha^* \cap N_\beta \) and by Lemma 2.1 \( \gamma \in \Delta_1 \), a contradiction. Hence one of the following holds.

(i) \( k = 3 \) and \( |\Delta_1| = 1 + (q - \varepsilon)/8 \), \( |\Delta_2| = 1 + (q - \delta)/4 \).

(ii) \( k = 2 \) and \( |\Delta_1| = 1 + (q - \varepsilon)/8 \), \( |\Delta_2| = (q - \varepsilon)/2 \).

(iii) \( k = 2 \) and \( |\Delta_1| = 1 + 3(q - \varepsilon)/8 \), \( |\Delta_2| = (q - \varepsilon)/4 \).

Let \( \gamma \in F(\delta) - \Delta_1 \). Then, \( z \in G_\gamma \cap N_\gamma \) and so \( C_\alpha(\gamma) \cong D_{q+,r} \) or \( PGL(2, \sqrt{q}) \) by Lemma 2.6 (vii), (viii), (ix).

If \( C_\alpha(\gamma) \cong D_{q+,r} \), then \( (q + \varepsilon)/2 \mid |\Delta_1| \) and so \( q = 7 \) and (ii) occurs. But \( (q + \varepsilon)/2 = 3 \mid |\Delta_2| - 1 - 1 = 1 \), a contradiction. If \( C_\alpha(\gamma) \cong PGL(2, \sqrt{q}) \), then (i) does not occur because \( \sqrt{q} \not\mid q - \varepsilon \). Hence \( \sqrt{q} \mid |\Delta_1| \) and \( \sqrt{q} \mid |\Delta_2| - 1 \). From this, \( q = 25 \) and (iii) occurs. In this case, we have \( |\Delta_1| = 10 \), so that an element of \( C_\alpha(\delta) \) of order 3 is contained in \( N_\delta^* \) for some \( \delta \in \Delta_1 \), contrary to \( N_\delta^* = N_\beta^* = D_6 \).

4. Case (II)

In this section we assume that \( N_\alpha^* \cong PGL(2, p^m) \), where \( n = 2mk \) and \( k \) is odd. Since \( n \) is even, \( q = p^m \equiv 1 \pmod{4} \). We set \( p^m \equiv \varepsilon \equiv \{ \pm 1 \} \pmod{4} \). In section 7 we shall consider the case that \( N_{\alpha}^* = C_4 \). Therefore we assume \( (p, m) = (3, 1) \) in this section.

(4.1) The following hold.

(i) \( N_\beta^*/N_\alpha^* \cap N_\beta^* = 1 \) or \( Z_2 \) and \( N_\alpha^* \cap N_\beta^* \cong (N_\alpha^*)' \cong PSL(2, p^m) \).

(ii) If \( (p, m) = (5, 1) \), there exists a cyclic subgroup \( Y \) of \( (N_\alpha^*)' \) such that \( N_{\alpha}^*(Y) = D_{q+,r} \) and \( N_0(Y) = D_{q+,r} \) is doubly transitive.

Proof. As \( N_\alpha^* \cong N_\alpha^* \cap N_\beta^* \), either \( N_\beta^*/N_\alpha^* \cap N_\beta^* \leq Z_2 \) or \( N_\alpha^* \cap N_\beta^* = 1 \). If \( N_\alpha^* \cap N_\beta^* = 1 \), by Lemma 2.2 and 2.6 (vi), \( N_\beta^* = N_\alpha^* \cap N_\beta^* \cong N_\beta^* N_\beta^*/N_\beta^* = Z_2 \times Z_2 \), a
Contradiction. Therefore \( N^\alpha/\alpha \cap N^\beta = 1 \) or \( N^\alpha \cap N^\beta \geq (N^\alpha)^{\alpha'} = PSL(2, p^m) \).

Now we assume that \((p, m) \neq (3,1), (5,1)\) and let \( z \) be an involution in \((N^\beta)^{\alpha'}\). Then \( C_{N^\beta}(z) \cong D_{2(p^m - 1)} \) by Lemma 2.6 (vii). Suppose \( C_{N^\beta}(z) \) is not a 2-subgroup and put \( Y = 0(C_{N^\beta}(z)) \). Then, if \( Y^z \leq G_{ab} \) for some \( g \in G \), we have \( Y^z \leq N^\alpha \) and \( Y^z \leq N^\beta \), where \( \gamma = \alpha^z \) and \( \delta = \beta^z \). By (i) \( Y^z \leq N^\alpha \cap N^\beta \) and so \( Y^z = Y^\alpha \) for some \( h \in N^\alpha \cap N^\beta \). Thus \( N_G(Y)^{\alpha'} \) is doubly transitive. Assume that \( C_{N^\beta}(z) \) is a 2-subgroup and set \( C_{N^\beta}(z) = \langle u, v \mid u^2 = v^{-1}, v^2 = 1 \rangle \). We may assume that \( v \in (N^\beta)^{\alpha'} \) and \( \langle u, v \rangle \) is a Sylow 2-subgroup of \((N^\beta)^{\alpha'}\). Since \( p^m \neq 3,5 \), the order of \( u^2 \) is at least four. On the other hand there is no element of order \( |u^2| \) in \( \langle u, v \rangle \). Hence any element of order \( |u^2| \) which is contained in \( N^\beta \) is necessarily an element of \( N^\alpha \). By the similar argument as above, \( N_G(Y)^{\alpha'} \) is doubly transitive.

(4.2) Let notations be as in (4.1). Suppose \((p, m) \neq (3,1), (5,1)\) and set \( \Delta = F(Y) \) and \( X = N_G(Y) \). Then \( |\Delta| = rs(p^m + \varepsilon)/2 + 1 \), where \( s = \sum_{i=0}^{k-1} p^{2mi} \), \( C_G(N^\alpha) = 1 \) and one of the following holds.

\begin{enumerate}
  \item \( X^\alpha \leq AGL(1, 2^c) \) for some integer \( c \).
  \item \( X^\alpha = PSL(2, p_1) \) or \( PGL(2, p_1) \), \( r = 1 \) and \( 2p_1 = p^m + \varepsilon \).
\end{enumerate}

Proof. By Lemma 2.8 (ii), \( |\Delta| = 1 + |N^\alpha \cap X|/r |N^\alpha \cap X| = 1 + (p^{2m} - 1)/r(2p^m - \varepsilon) = (p^m + \varepsilon)/2 + 1 \). By (4.1) and Lemma 2.9, we have (i), (ii) or \( X^\alpha = R(3) \).

Assume that \( X^\alpha = R(3) \). Then \( rs(p^m + \varepsilon)/2 + 1 = 28 \), hence \( k = 1 \) and \( r(p^m + \varepsilon)/2 = 27 \). Since \( r \) is odd and \( r \) is odd, we have \( r = m = 1 \) and \( q = 3^2 \). But a Sylow 3-subgroup of \( X^\alpha \) is cyclic because \( N^\alpha \cap X \cong D_{p+2} \), and \( X^\alpha/X \cap N^\alpha = X^\alpha N^\alpha/N^\alpha \leq Z_2 \times Z_2 \), a contradiction. Thus (i) or (ii) holds.

(4.3) (i) of (4.2) does not occur.

Proof. Let notations be as in (4.2). Suppose \( X^\alpha \leq AGL(1, 2^c) \) and put \( W = C_{N^\beta}(Y) \). Then \( Y \leq W \cong Z_{p^m - 1} \). Since \( C_{N^\alpha}(Y) \) is cyclic, \( W \) is a characteristic subgroup of \( C_{N^\alpha}(Y) \) and \( W \) is a normal subgroup of \( X^\alpha \). Hence \( \leq X^\alpha \) and \( X \cap N^\beta = 1 \) or \( Z_2 \). By Lemmas 2.4 and 2.6, \( F(X \cap N^\beta) = 1 + |X \cap N^\beta| |N^\beta| : X \cap N^\beta | \times r |N^\beta| = l + r \). Since \( 1 + r < |\Delta| \), \( (X \cap N^\beta)^{\alpha'} = Z_2 \) and hence \( (1 + r)^{2} = rs(p^m + \varepsilon)/2 + 1 \) by Lemma 2.5. From this, \( r = s(p^m + \varepsilon)/2 - 2mk \) and so \( p^{2m}(k - 1) + mk \leq 2 \). Hence \( m = k = r = 1 \) and \( q = 7^2 \).

Let \( R \) be a Sylow 3-subgroup of \( N^\beta \). Since \( N^\beta \cong PGL(2, 7) \), we have \( R \cong Z_3 \). By Lemmas 2.4 and 2.6, \( |F(R)| = 1 + (7^2 - 1) |N^\beta| : N^\beta \cap N^\beta / |N^\beta| = 4 \). Hence \( N_G(R)^{F(R)} = \Delta_4 \) or \( S_4 \). But is a Sylow 3-subgroup of \( N_G(R)^{F(R)} \) because \( N^\alpha = PSL(2, 7^2) \), contrary to \( N_G(R)^{F(R)} = \Delta_4 \) or \( S_4 \).

(4.4) (ii) of (4.2) does not occur.
Proof. Let notations be as in (4.2). Suppose $X^\Delta \trianglelefteq PSL(2,p_1)$. By the similar argument as in (4.3), $C_{N_\beta}(Y) \subseteq X_\Delta$ and so $C_{N_\alpha}(Y) = Z_{p_1}$, and $N_{N_\alpha}(Y)^{\Delta} = D_{2p_1}$. Hence $|X^\Delta| = |(p_1 - 1)/2|$. Since $X^\Delta \trianglelefteq PSL(2,p_1)$, $p_1(p_1 - 1)/2 | |(X^\Delta)|$, hence $p_1 - 1 | 8n$. As $k = 1$ and $2p_1 = p^\alpha + \epsilon$, we have $p^\alpha + \epsilon - 2 | 32m$. From this, $(p,m,p_1) = (11,1,5), (3,2,5)$ or $(3,3,13)$.

Let $R$ be a cyclic subgroup of $N_\alpha^\beta$ such that $R = Z((p^\alpha + \epsilon)/2)$. By Lemma 2.6, $N_\alpha(R)^{F(R)}$ is doubly transitive and by Lemma 2.8 (ii), $|F(R)| = 1 + |N_\alpha(R)| - |N_{N_\alpha}(R)^{(R)}| = 42$ and $N_\alpha(R)^{F(R)} = Z_6$. Since $|N_{N_\alpha}(R): N_{N_\alpha}^\beta(R)| = 6, N_{N_\alpha}(R)^{F(R)} = N_{\alpha}(R)^{F(R)}$. Hence $N_{N_\alpha}(R)/K = Z_6$, where $K = (N_{N_\alpha}(R))^{F(R)}$. But $N_{N_\alpha}(R)/(N_{N_\alpha}(R))^* = Z_2 \times Z_2$, a contradiction.

If $(p,m,p_1) = (3,2,5)$, $|N_\alpha(R)| = 5$ and so by [9], $|F(R)| = 15$. Since $|N_{N_\alpha}(R): N_{N_\alpha}^\beta(R)| = 4, N_{N_\alpha}(R)^{\alpha} = Z_4$, contrary to $N_{N_\alpha}(R)/(N_{N_\alpha}(R))^* = Z_2 \times Z_2$.

If $(p,m,p_1) = (3,3,13)$, $|F(R)| = 15$. By [9], $N_{\alpha}(R)^{F(R)}$ is not solvable, a contradiction.

(4.5) $p^\alpha = 5.$

Proof. Assume that $p^\alpha = 5$. Then $n = 2k$ with $k$ odd and $N_\alpha^\beta = PGL(2,5)$ $\cong S_5$. First we argue that $N_\alpha^\beta = N_\alpha \cap N_\beta$. Suppose false. Then $C_\alpha(N_\beta^\alpha) = 1$ by Lemma 2.2, and $N_\alpha^\beta/N_\alpha \cap N_\beta = Z_2$ by (4.1). Since $N_\alpha^\beta \cap N_\beta = N_\alpha^\beta/N_\alpha \cap N_\beta = Z_2$ and the outer automorphism group of $S_5$ is trivial, we have $Z(N_\alpha^\beta N_\beta^\alpha) = Z_2$.

Let $\omega$ be the involution of $Z(N_\alpha^\beta N_\beta^\alpha)$ and let $\omega \in I(N_\alpha^\beta) - I(N_\alpha)$. Since $C_{N_\alpha}(\omega)$ is doubly transitive of order 2 and $C_{N_\alpha}(\omega) \cong PGL(2,5)$, $\omega$ acts on $N_\alpha^\beta$ as a field automorphism of order 2 and $C_{N_\alpha}(\omega) = Z_2$. By Lemma 2.8 $|F(\omega)| = 1 + r(5^2 - 1)/24$. Let $P$ be a Sylow 5-subgroup of $C_{N_\alpha}(\omega)$. Then $|P| = 5^s$ and $|\gamma| = 5^k - 1$ for each $\gamma \in N_\alpha^\beta$. Since $P$ acts on $F(\omega) - \{\alpha\}$, we have $5^{s-1}r(5^k - 1)/24$, so that $k = 1$ and $|F(\omega)| = 6 = r/k$. Hence $C_{N_\alpha}(\omega)^{F(\omega)} = Z_2$, so $C_{N_\alpha}(\omega)^{F(\omega)} = Z_2$. But clearly $\omega \in N_\alpha^\beta \cap N_\beta$ by Lemma 2.1, a contradiction. Therefore (a) holds.

By Lemma 2.5, $(r + 1)^2 = 3rs + 1$ and so $r = 3s - 2/k$. Hence $k = r = 1$ and $G_\alpha/N_\alpha^\beta \leq Z_2 \times Z_2$. Let $z$ be an involution in $N_\beta$. Then $|F(z)| = 1 + 24 \cdot 25/120 = 6$. Some Doubly Transitive Permutation Groups
by Lemma 2.8 and $|\Omega| = 1 + |N^a| = 66$ as $r = 1$. By the similar argument as in the proof of (3.12), $|F(z)||F(z)|-1)/2 + (|\Omega| - |F(z)|)/2 = |C_G(z)||z^G \cap \langle t \rangle G_{ab}|/|\langle t \rangle G_{ab}|$, where $t$ is an involution such that $\alpha^t = \beta$. Hence $|z^G \cap \langle t \rangle G_{ab}| = 15|G_{ab}|/|C_G(z)|$. Set $H = \langle t \rangle G_{ab}$ and let $R$ be a Sylow 3-subgroup of $N^a_{ab}$. By Lemma 2.8, $|F(R)| = 1 + 24 \cdot 10/120 = 3$. Set $F(R) = \{\alpha, \beta, \gamma\}$. On the other hand, as $N^a_{ab} = S_5$ and Out$(S_5) = 1$, we have $H = Z(H) \times N^a_{ab}$ and $|Z(H)| = 2, 4$ or $H = C_H(N^a_{ab}) \times N^a_{ab}$ and $Z(G_{ab}) = Z_2 \times Z_2$, contrary to Lemma 2.6 (ix). In the former case, we have $|Z(H)| = 2$. For otherwise $Z(H) < G$ and $Z(H) \cap G = 1$, so letting $u \in Z(H) \cap Z^G$, we have $|R| = 3 |F(u)| = 1 = 5$, a contradiction. Therefore $Z(H) = Z_2$ and so $|z^G \cap H| = 25 + 25 = 50$, while $|z^G \cap H| = 15|G_{ab}|/|C_G(z)| = 15 \cdot 120/24 = 75$, a contradiction.

5. Case (III)

In this section we assume that $N^a_{ab} = PSL(2, p^m)$, where $n = mk$ and $k$ is odd. Set $p^m \equiv \varepsilon \equiv \{\pm 1\}$ (mod 4). Then $q \equiv \varepsilon$ (mod 4) as $k$ is odd. In section 6 we shall consider the case that $N^a_{ab} = A_4$, so we assume $(p, m) \neq (3, 1)$ in this section. From this $N^a_{ab}$ is a nonabelian simple group and so $N^a_{ab} = N^a_{ab} \cap N^a_{ab}$ or $N^a_{ab} \cap N^a_{ab} = 1$. If $N^a_{ab} \cap N^a_{ab} = 1$, then $C_G(N^a_{ab}) = 1$ by Lemma 2.2 and $N^a_{ab} = N^a_{ab} \cap N^a_{ab} \cap N^a_{ab} = N^a_{ab} \cap N^a_{ab} = 1$. Set $Y$ be a cyclic subgroup of $C_N^a(z) = D_{p^m}$ of index 2. Since $C_G(a)(z)$ is doubly transitive, we have $F(Y) = F(z)$. By Lemma 2.9, $C_G(z)^F(z) = 1$ and one of the following holds.

(a) $C_G(z)^F(z) = ATL(1, 2^r)$.
(b) $C_G(z)^F(z) \geq PSL(2, p_1)^{p_1 \geq 5}$, $r = 1$ and $|N^a_{ab}(z)| = p_1$.
(c) $C_G(z)^F(z) = R(3)$.

Let $Y$ be a cyclic subgroup of $C_N^a(z) = D_{p^m}$ of index 2. Since $C_G(a)(z)$ is doubly transitive, we have $F(Y) = F(z)$. By the similar argument as in (3.1), $N^a_{ab} \cap N(C_N^a(z)) = C_N^a(z)$ or $N^a_{ab} \cap N(C_N^a(z)) = A_4$. Hence by Lemmas 2.3 and 2.4, $|F(C_N^a(z))| = 1 + |C_N^a(z)| |N^a_{ab}(z)| |N^a_{ab}(z)| |N^a_{ab}(z)| |N^a_{ab}(z)| = 1 + |A_4| |N^a_{ab}(z)| |r| |N^a_{ab}(z)| |r| |N^a_{ab}(z)| |r| |N^a_{ab}(z)|$. Therefore $F(C_N^a(z)) = r + 1$ or $3r + 1$. From this $C_N^a(z)^F(z) \geq Z_2$. In the case (a), $(r + 1)^2 = 1 + (p^m - \varepsilon)r/(p^m - \varepsilon)$ by Lemma 2.5 and hence $r = (p^m - \varepsilon)/(p^m - \varepsilon) - 2/|mk|$. Since $(p^m - \varepsilon)/(p^m - \varepsilon) \geq ((p^m)^k + 1)/(p^m + 1) = \sum_{i=0}^{k} (-1)^i/m(i(p^m - p^m + 1)) < k$, we have $p^m(k - 1)/(p^m - p^m + 1) \leq mk$, hence $((p^m)^k - 1)/k(m(p^m - p^m + 1)) < 1$. Thus $k = 3$, $m = 1$ and $p = 3$, contrary to $(p, m) = (3, 1)$.

In the case (b), $r = 1$, $p_1 = (p^m - \varepsilon)/(p^m - \varepsilon)$, $p_1(p_1 - 1)/2$ and $s \mid 4mkp_1$, where $s$ is the order of $C_G(a)(z)^F(z)$. Hence $p_1 - 1/s$ is.


\[ \geq \left( \frac{p^n+1}{p^n-1} \right) - 1 = \sum_{k=0}^{\infty} \left( -\frac{p^n}{p^n-1} \right)^k \geq \frac{p^{m(k-2)}(p^n-1)}{2k \leq 4m(p^n-1)} \leq 1 \text{ because } p^n \neq 3. \] Hence \( k = 3 \) and \( p^n = 5 \), so that \( p^m = 30 \). If \( m = 3 \), we have a contradiction.

In the case (c), \( r+l=4 \) and \( 1+(p^n-\varepsilon)(p^n-\varepsilon)=28 \) and so \( r=3 \) and \( (p^n-\varepsilon)(p^n-\varepsilon)=9 \). Hence \( 9 \geq \frac{p^{m+1}+p^{m}}{2} \geq p^m-1 \geq 1 \) because \( p^n \neq 3 \).

**6. Case (IV)**

In this section we assume that \( N^a = A_4 \) and \( q = 3, 5 \text{ (mod 8)} \). If \( N^a \cap N^b = 1 \), by Lemma 2.2, \( C_G(N^a) = 1 \) and so \( N^a / N^b \cong N^a / N^b \cap N^b \leq Z_2 \times Z_2 \). Hence \( N^a / N^b \cap N^b = 1 \text{ or } Z_2 \), so that \( z^G \cap G_{ab} = z^G \cap N^a = z^G \) for an involution \( z \in N^a \).

Therefore \( C_G(z)^{F(z)} \) is doubly transitive. By Lemma 2.9, \( C_G(N^a) = 1 \) and one of the following holds.

- (a) \( C_G(z)^{F(z)} \cong A_4 \) for some integer \( c \geq 1 \).
- (b) \( C_G(z)^{F(z)} \cong PSL(2, p) \) \( (p \geq 5) \), \( r = 1 \) and \( |C_{N^a}(z) : C_{N^a}(z)| = p^m \).
- (c) \( C_G(z)^{F(z)} \cong R(3) \).

Let \( T \) be a Sylow 2-subgroup of \( N^a \). Then \( z \in T \) and by Lemmas 2.3 and 2.4, \( |F(T)| = 1+|N_{N^a}(T)| |r| |N^a| = r+1 \). By Lemma 2.8 (i), \( |F(z)| = (q-\varepsilon)r/4+1 \).

Hence \( F(z)^{F(z)} \cong Z_2 \) if \( q = 5 \). If \( q = 5 \), as \( PSL(2,5) \cong PSL(2,4) \), from Lemma 2.5, we have \( r = 3 \text{ or } 5 \).

**In the case (a), \( r+l=4 \text{ and } 1+(p^n-\varepsilon)(p^n-\varepsilon)=28 \text{ and so } r=3 \text{ and } (p^n-\varepsilon)(p^n-\varepsilon)=9.**

Hence \( 9 \geq \frac{p^{m+1}+p^{m}}{2} \geq p^m-1 \) so that \( p^n = 3 \), a contradiction.

**In the case (b), \( p_1(p_1-1)/2 \text{ and } s^2 \geq 8n(q-\varepsilon) \geq 4n^2 \).**

where \( s \) is the order of \( C_{G_a}(z)^{F(z)} \). Hence \( p_1 \geq 8n \). Since \( p_1 = (q-\varepsilon)/4 \), \( p^n \geq 4^3 \geq 32n \) and so we have \( q = 11, 13, 19, 27 \) or 37. If \( q = 27 \), by Lemma 2.6, \( C_{G_a}(z) = D_{2q-4} \) or \( D_{2(q-4)} \) and so \( C_{G_{ab}}(z)^{F(z)} \cong Z_2 \). Hence \( (p_1-1)/2 = 2 \). From this \( q = 19 \). Let \( R \) be a Sylow 3-subgroup of \( G_{ab} \). By the similar argument as in the case (a), \( N_G(R)^{F(R)} \) is doubly transitive and \( |F(R)| = 1+18 \frac{3}{2} = 7 \). Hence \( |G| = 7 \). On the other hand \( |G| = 7 \), \( C_{G_a}(z) = (1+|N^a : N^b|)|G_a| = (1+18 \cdot 19 \cdot 20/2 \cdot 12) \cdot 2^4 \cdot 18 \cdot 19 \cdot 20/2 = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 13 \cdot 19 \) with \( 0 \leq i \leq 1 \), a contradiction. If \( q = 27 \), then \( |C_G(z)| = |F(z)| \times |C_{G_a}(z)| = 8 \times |G_a| \) while \( |\Omega| = 1+|N^a : N^b| = 1+26 \cdot 27 \cdot 28/2 \cdot 12 = 820 = 2^3 \cdot 5 \cdot 41 \) and so \( |G| = 4 |G_a| \). Therefore \( |C_G(z)| = |G_a| \), a contradiction.

**In the case (c), \( r+l=4 \text{ and } 1+(q-\varepsilon)r/4=28.**

Hence \( r = 3 \) and \( q = 37, \)
contrary to \( r | n \).

7. Case (V)

In this section we assume that \( N^*_a = S_4 \) and \( q = 7,9 \pmod{16} \). We note that \( 4 \nmid n \).

First we argue that \( N^*_a = N^* \cap N^b \). Suppose \( N^*_a \cap N^b \). Then \( C_o(N^a) = 1 \) by Lemma 2.2. Since \( N^*_a / N^a \cap N^b \Rightarrow N^*_a / N^b \approx Z_2 \times Z_2 \), we have \( N^* \cap N^b \approx A_4 \) and \( N^*_b / N^a \cap N^b \approx Z_2 \). So \( N^*_a / N^b \approx N^*_b / N^b \cap N^b \approx Z_2 \). Hence as \( \text{Out}(S_4) = 1 \), \( Z(N^*_a / N^b) \approx Z_2 \). Set \( \langle t \rangle = Z(N^*_b / N^b) \) and let \( t \in I(N^*_b) - I(N^a) \). Since \( C_N(t) \supseteq N^*_b = S_4 \) and \( \langle t \rangle N^a = N^*_b \), by Lemma 2.6, we have \( C_{N^a}(t) = PGL(2, \sqrt{q}) \) and \( |F(t)| = 1 + 3(q - \varepsilon) r / 8 \) by Lemma 2.8.

Let \( P \) be a Sylow \( p \)-subgroup of \( C_{N^a}(t) \). Then \( |P| = \sqrt{q} \). If \( p = 3 \), \( P \) acts semi-regularly on \( F(t) - \{ \alpha \} \) and so \( \sqrt{q} | 3(q - \varepsilon) r / 8 \). Therefore \( \sqrt{q} | r \) and so \( 5^e > n^2 \) for any positive integer \( n \). This is a contradiction. If \( p = 3 \), \( P : \sqrt{q} = \sqrt{3(q - \varepsilon) r / 8} \). Hence \( \sqrt{q} | 3(q - \varepsilon) r / 8 \) and so \( q | 81 r^2 \). In particular, \( 3^e = q | 81 n^2 \). From this, \( n \leq 7 \). Since \( q = 3^e \equiv 7 \) or \( 9 \pmod{16} \), we have \( q = 3^2 \) or \( 3^6 \). If \( q = 3^6 \), \( |\Omega| = 1 + |N^a : N^*_a| = 1 + 8 \cdot 9 \cdot 10 / 2 \cdot 24 = 16 \), a contradiction by [9] if \( q = 3^6 \), \( |F(t)| = 1 + 273r \) and \( |F(t) - \{ \alpha \}| \geq |C_{N^a}(t)| \geq |PGL(2, 3^3)| / 8 = 2457 \) contrary to \( r | 3 \). Thus \( N^*_a = N^* \cap N^b \).

Let \( V \) be a cyclic subgroup of \( N^*_a \) of order 4 and let \( U \) be a Sylow 2-subgroup of \( N^*_a \) containing \( V \). Then \( U = N_{G_a}(V) \), \( |F(V)| = 1 + (q - \varepsilon) r / 8 \) by Lemma 2.8 and \( |F(U)| = 1 + 8 \cdot 3r / 24 = r + 1 \) by Lemmas 2.3 and 2.4. If \( q = 7,9 \), then \( |F(U)| < |F(V)| \) and hence \( U^{F(V)} \approx Z_2 \). Suppose \( q = 7 \) or \( 9 \). Then \( r = 1 \) as \( r | n \). Hence \( |\Omega| = 1 + |N^a : N^*_a| = 8 \) or 16. By [10], we have a contradiction. Therefore \( U^{F(V)} \approx Z_2 \).

Suppose \( V^e \leq G_{a^b} \) for some \( g \in G \) and set \( \gamma = \alpha^g \). Then \( V^e \leq g^{-1} N^a g \cap G_{a^b} \leq N^a \cap G_{a^b} \leq N^a \cap N^b = N^a \). Since \( N^*_a = S_4 \), \( V^e = V^h \) for some \( h \in N^*_a \). Hence \( C_o(V^{F(V)}) \) is doubly transitive. By Lemma 2.9, \( C_o(N^a) = 1 \) and one of the following holds.

(a) \( N_c(V^{F(V)}) \leq AGL(1, 2^7) \).
(b) \( N_c(V^{F(V)}) \geq PSL(2, p_1) \), \( p_1 = (q - \varepsilon) / 8 \geq 5 \).
(c) \( N_c(V^{F(V)}) = R(3) \).

In the case (a), \( r + 1 = 1 + (q - \varepsilon) r / 8 \) by Lemma 2.5 and so \( r = (q - \varepsilon - 16) / 8 \) and \( r | n \). From this \( q = 23 \) or 25 and \( r = 1 \). Since \( |\Omega| = 1 + |N^a : N^*_a| = 2 \cdot 127 \) or \( 2 \cdot 163 \), we have \( |G/2| = 2 |G_a/2| \) while \( |N_c(V)||2| = |F(V)||2| |G_a(V)||2| = 4 |G_a/2| \), contrary to \( |C_o(V)||2| = |G| \).

In the case (b), \( p_1 (p_1 - 1) / 2 \) and \( s \mid 2n(q - \varepsilon) / 4 = 4np_1 \), where \( s \) is the order of \( N_{G_a}(V^{F(V)}) \). Hence \( p_1 | 8n \). From this, \( p^e - \varepsilon - 8 \mid 64n \) and so \( q = 23, \, 41, \, 71 \) or 73. Since \( p_1 \) is a prime and \( p_1 = (q - \varepsilon) / 8 \geq 5 \), \( q = 23, \, 71, \) or 73. Therefore \( q = 41 \) and \( |\Omega| = 1 + |N^a : N^*_a| = 1 + 40 \cdot 41 \cdot 42 / 2 \cdot 24 = 2^2 \cdot 359 \), so that \( |G/2| = 4 |G_a/2| \).
Since $N_\beta^*=N^* \cap N^\beta$, $C_c(z)^{F(z)}$ is transitive by Lemma 2.1. On the other hand $|F(z)|=1+40 \cdot 9/24=16$ by Lemma 2.8 (i) and so $|C_c(z)|=16|C_{G_\alpha}(z)|=16|G_\alpha|$, contrary to $|C_c(z)|||G|$.

In the case (c), $r+1=4$ and $1+(q-\varepsilon)r/8=28$. Hence $r=3$ and $q=71$ or 73, contrary to $r \mid n$.

8. Case (VI)

In this section we assume that $N_\beta^*=A_5$ and $q \equiv 3, 5 \pmod{8}$. In particular, $n$ is odd. If $N_\beta^*=N^* \cap N^\beta$, then $N^* \cap N^\beta=1$, $C_c(N^*)=1$ and so $N_\beta^*=N_\beta^*/N^\beta/N^\beta \leq \text{Out}(N^\beta)=Z \times Z$, a contradiction. Hence $N_\beta^*=N^* \cap N^\beta$. Let $z$ be an involution in $N_\beta^*$ and $T$ a Sylow 2-subgroup of $N_\beta^*$ containing $z$. Then, by Lemma 2.8 $|F(z)|=1+(q-\varepsilon)15r/60=1+(q-\varepsilon)r/4$ and by Lemmas 2.3 and 2.4 $|F(T)|=1+12 \cdot 5r/60=1+r$. Since $N_\beta^*=N^* \cap N^\beta$, $z^G \cap G_\alpha^*=z^G \cap N_\beta^*=z^N_\beta$ and so $C_c(z)^{F(z)}$ is doubly transitive. By Lemma 2.9, $C_c(N^*)=1$ and one of the following holds.

(a) $C_c(z)^{F(z)}=\text{Alt}(1,2)$.

(b) $C_c(z)^{F(z)}=\text{PSL}(2, p_1)$, $p_1=(q-\varepsilon)/4 \geq 5$.

(c) $C_c(z)^{F(z)}=R(3)$.

In the case (a), by Lemma 2.5, $(q-\varepsilon)/4=1$ or $(r+1)/2=1+(q-\varepsilon)r/4$. Hence $q=5$ or $r=(q-\varepsilon-8)/4 \mid n$. If $q=5$, then $N_\beta^*=N^*$, a contradiction. Therefore $p^*-\varepsilon-8 \mid 4n$ and so $n=1$ and $q=11$ or 13. If $q=13$, we have $5 \not\mid |G_\alpha|$, a contradiction. Hence $q=11$ and $|\Omega|=1+|N^*: N_\beta^*=1+10 \cdot 11 \cdot 12/2 \cdot 60=12$. By [9], $C_\alpha \simeq M_{11}$, $|\Omega|=12$ and so (iii) of our theorem holds.

In the case (b), we have $p_1(p_1-1)/2 \mid s$ and $s \mid 2n(q-\varepsilon)/2=4np_1$, where $s$ is the order of $C_c_{G_\alpha}(z)^{F(z)}$. Hence $p_1=18n$ and so $p^*-\varepsilon=4 \mid 32n$. From this $q=19, 27$ or 37. Since $5 \not\mid |G_\alpha|$, $q=27, 37$. Hence $q=19$ and $|\Omega|=1+|N^*: N_\beta^*=1+18 \cdot 19 \cdot 20/2 \cdot 60=2 \cdot 29$. Since $G_\alpha=PGL(2, 19)$ or $PGL(2, 19)$, $|G|=|\Omega||G_\alpha|=2 \cdot 29 \cdot 2 \cdot 18 \cdot 19 \cdot 20/2=2^{3+i} \cdot 3^3 \cdot 5 \cdot 19 \cdot 29$ with $0 \leq i \leq 1$. Let $P$ be a Sylow 29-subgroup of $G$. Then $P$ is semi-regular on $\Omega$ and 3, 5, 19 $\not\mid |N_c(P)|$ because $N_c(P)/C_c(P) \leq Z_4 \times Z_7$. Hence $|G: N_c(P)|=2^i \cdot 3^3 \cdot 5 \cdot 19$ with $0 \leq j \leq 4$, while $2^i \cdot 3^3 \cdot 5 \cdot 19 \equiv 1 \pmod{29}$ for any $j$ with $0 \leq j \leq 4$, contrary to a Sylow's theorem.

If $C_c(z)^{F(z)}=R(3), r+1=4$ and $1+(q-\varepsilon)r/4=28$ and hence $r=3, q=37$, contrary to $r \mid n$.

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References


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