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CONVERGENCE OF OPERATORS SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS

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1. Introduction and main results

Let $U \subset \mathbb{R}^d$, $d \geq 3$, U open (not necessarily bounded), and let dx denote Lebesgue measure on U . Below all functions are supposed to be real-valued. Let $a_{ij}^{(n)}, b_i^{(n)}, d_i^{(n)}, c^{(n)} \in L^1_{\text{loc}}(U; dx)$, $1 \leq i, j \leq d$, $n \in \mathbb{N} \cup \{\infty\}$ satisfying the following conditions:

(1.1) There exists $\delta \in]0, \infty[$ such that for all $n \in \mathbb{N} \cup \{\infty\}$ and dx -a.e. $x \in U$

$$\sum_{i,j=1}^d a_{ij}^{(n)}(x) \xi_i \xi_j \geq \delta \sum_{i=1}^d \xi_i^2 \text{ for all } \xi_1, \dots, \xi_d \in \mathbb{R}.$$

(1.2) There exists $M \in [0, \infty[$ such that for all $n \in \mathbb{N}$ and dx -a.e. $x \in U$

$$|a_{ij}^{(n)}(x)| \leq M, \quad 1 \leq i, j \leq d.$$

(1.3) There exist $p_{b,i}, p_{d,i}, p_c \in [d, \infty]$, $1 \leq i \leq d$, such that for all $n \in \mathbb{N} \cup \{\infty\}$

$$b_i^{(n)} \in L^{p_{b,i}}(U; dx), \quad d_i^{(n)} \in L^{p_{d,i}}(U; dx), \quad c^{(n)} \in L^{p_c/2}(U; dx).$$

Note that (1.1) is a condition only on the symmetric part of $(a_{ij})_{1 \leq i, j \leq d}$. Conditions (1.1)–(1.3) allow to construct the corresponding coercive closed forms (cf. e.g. [3, Chap. I, Sect. 2]) as follows. Let $C_0^\infty(U)$ denote the set of all infinitely differentiable functions with compact support in U . Fix $n \in \mathbb{N} \cup \{\infty\}$ and set $\partial_i := \partial/\partial x_i$, $1 \leq i \leq d$. Define

$$(1.4) \quad \mathcal{E}^{(n)}(u, v) := \sum_{i,j=1}^d \int \partial_i u \partial_j v a_{ij}^{(n)} dx + \sum_{i=1}^d \int u \partial_i v d_i^{(n)} dx \\
 + \sum_{i=1}^d \int \partial_i u v b_i^{(n)} dx + \int u v c^{(n)} dx \quad ; \quad u, v \in C_0^\infty(U).$$

For $\alpha \in]0, \infty[$ set

$$\mathcal{E}_\alpha^{(n)}(u, v) := \mathcal{E}^{(n)}(u, v) + \alpha(u, v)_{L^2(U; dx)}; u, v \in C_0^\infty(U).$$

E.g. by [4, Theorem 2.2] we know that there exists $\alpha_n \in]0, \infty[$ such that $(\mathcal{E}_{\alpha_n}^{(n)}, C_0^\infty(U))$ is closable on $L^2(U; dx)$ and its closure $(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)}))$ is a coercive closed form on $L^2(U; dx)$ in the sense of [3, Chap. I, Definition 2.4]. It is well-known (and can e.g. easily be extracted from the proof of [4, Theorem 2.2], or more precisely from the proof of the underlying [6, Theorem 1.7]) that there exist $\gamma_n \in]0, \infty[$ such that for all $u, v \in C_0^\infty(U)$

$$(1.5) \quad \left| \mathcal{E}_{\alpha_n}^{(n)}(u, v) \right| \leq \gamma_n \mathcal{E}_{\alpha_n}^{(n)}(u, u)^{1/2} \mathcal{E}_{\alpha_n}^{(n)}(v, v)^{1/2}$$

$$(1.6) \quad \gamma_n^{-1} |u|_{1,2} \leq \mathcal{E}_{\alpha_n}^{(n)}(u, u)^{1/2} \leq \gamma_n |u|_{1,2}.$$

Here $| \cdot |_{1,2}$ is the norm on the classical Sobolev space $H_0^{1,2}(U; dx)$ of order 1 in $L^2(U; dx)$, defined as the completion of $C_0^\infty(U)$ w.r.t. $| \cdot |_{1,2}$ which is given by

$$|u|_{1,2}^2 := \sum_{i=1}^d \int (\partial_i u)^2 dx + \int u^2 dx; u \in C_0^\infty(U).$$

In particular, $D(\mathcal{E}_{\alpha_n}^{(n)}) = H_0^{1,2}(U, dx)$ and (1.5), (1.6) hold for all $u \in H_0^{1,2}(U; dx)$.

REMARK 1.1. γ_n in (1.5), (1.6) only depends on α_n, δ, M and the L^p -norms of $b_i^{(n)}, d_i^{(n)}, c^{(n)}, 1 \leq i \leq d$, (cf. condition (1.3)). This can also be seen e.g. from the respective proofs in [4], [6] mentioned above. In particular, α_n and γ_n can be chosen to be independent of n , if all the L^p -norms in condition (1.3) are bounded uniformly in n .

Let $(L_{\alpha_n}, D(L_{\alpha_n})), (T_{\alpha_n, t})_{t>0}$ be the generator resp. the strongly continuous contraction semigroup associated with $(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)}))$ (cf. e.g. [3, Chap. I., Sect. 2]). Define

$$(1.7) \quad T_t^{(n)} := e^{\alpha_n t} T_{\alpha_n, t}, \quad t > 0,$$

$$(1.8) \quad L^{(n)} := L_{\alpha_n} + \alpha_n, \quad D(L^{(n)}) := D(L_{\alpha_n}).$$

Then $(L^{(n)}, D(L^{(n)}))$ generates $(T_t^{(n)})_{t>0}$ (on $L^2(U; dx)$).

REMARK 1.2.

- i) Obviously, $(L^{(n)}, D(L^{(n)}))$ and $(T_t^{(n)})_{t>0}$ are independent of the special choice of α_n .
- ii) Informally, we have for $u \in C_0^\infty(U)$ that

$$(1.9) \quad L^{(n)}u = \sum_{i,j=1}^d \partial_i(a_{ij}^{(n)} \partial_j + d_i^{(n)})u - \sum_{i=1}^d b_i^{(n)} \partial_i u - c^{(n)}u.$$

Though (1.9) is very suggestive, it is, of course, informal since $C_0^\infty(U)$ will in general not be a subset of $D(L^{(n)})$.

iii) Note that e.g. by [3, Chap. I, Theorem 2.20] $T_t^{(n)}f \in D(\mathcal{E}_{\alpha_n}^{(n)}) = H_0^{1,2}(U; dx)$ for all $f \in L^2(U; dx)$, $t > 0$.

Let $n \in \mathbb{N} \cup \{\infty\}$ and let $(G_\alpha^{(n)})_{\alpha > \alpha_n}$ be the strongly continuous resolvent associated with $(T_t^{(n)})_{t > 0}$ on $L^2(U; dx)$, i.e., for $\alpha > \alpha_n$

$$(1.10) \quad G_\alpha^{(n)}f := \int_0^\infty e^{-\alpha t} T_t^{(n)}f dt, \quad f \in L^2(U; dx),$$

(where the integral is a Bochner integral in $L^2(U; dx)$). Note that for $\alpha > \alpha_n$ and $f \in L^2(U; dx)$

$$(1.11) \quad G_\alpha^{(n)}f \in D(\mathcal{E}_{\alpha_n}^{(n)}) = H_0^{1,2}(U; dx)$$

and $\mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)}f, v) = (f, v) = \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)}f)$ for all $v \in H_0^{1,2}(U; dx)$

(cf. e.g. [3, Chap. I, Theorem 2.8] and recall (1.7)). Here for a densely defined operator $(T, D(T))$ on $L^2(U; dx)$ we denote its adjoint by $(\widehat{T}, D(\widehat{T}))$.

Consider for $1 \leq i, j \leq d$ the following conditions:

$$(1.12) \quad a_{ij}^{(n)} \xrightarrow{n \rightarrow \infty} a_{ij}^{(\infty)} =: a_{ij} \quad dx\text{-a.e. on } U.$$

$$(1.13) \quad b_i^{(n)} \xrightarrow{n \rightarrow \infty} b_i^{(\infty)} =: b_i \text{ weakly}^* \text{ in } L^{p_{b,i}}(U; dx).$$

$$(1.14) \quad d_i^{(n)} \xrightarrow{n \rightarrow \infty} d_i^{(\infty)} =: d_i \text{ weakly}^* \text{ in } L^{p_{d,i}}(U; dx).$$

$$(1.15) \quad c^{(n)} \xrightarrow{n \rightarrow \infty} c^{(\infty)} =: c \text{ weakly}^* \text{ in } L^{p_c/2}(U; dx).$$

Now we can formulate the main results of this paper.

Theorem 1.3. *Suppose that for $1 \leq i, j \leq d$ conditions (1.12), (1.13), and (1.15) are satisfied and that*

$$(1.16) \quad \left| d_i^{(n)} - d_i \right| \xrightarrow{n \rightarrow \infty} 0 \text{ weakly}^* \text{ in } L^{p_{d,i}}(U; dx), \quad \text{for all } 1 \leq i \leq d.$$

Then there exists $\alpha_0 \in]0, \infty[$ such that for all $\alpha > \alpha_0$ and all $f \in L^2(U; dx)$:

$$(i) \quad G_\alpha^{(n)}f \xrightarrow{n \rightarrow \infty} G_\alpha^{(\infty)}f =: G_\alpha f \text{ and } \widehat{G}_\alpha^{(n)}f \xrightarrow{n \rightarrow \infty} \widehat{G}_\alpha^{(\infty)}f =: \widehat{G}_\alpha f$$

weakly in $H_0^{1,2}(U; dx)$;

(ii)
$$G_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} G_\alpha f \text{ in } L^2(U; dx) ,$$

and hence for all $t > 0$

$$T_f^{(n)} f \xrightarrow{n \rightarrow \infty} T_t f \text{ in } L^2(U; dx).$$

REMARK 1.4.

- (i) We use the notion “weakly*” rather than “weakly” since $p_{b,i}, p_{d,i}, p_c$ can be equal to $+\infty$. Clearly, if we assume (1.14) then (1.16) holds if $d_i^{(n)} \xrightarrow{n \rightarrow \infty} d_i$ in dx -measure for all $1 \leq i \leq d$. Note that (1.16), of course, implies (1.14).
- (ii) Note that the last part of Theorem 1.3 (ii) is trivial, since (as is well-known and quite easy to prove) that strong convergence of strongly continuous contraction semigroups, (such as $e^{-\alpha t} T_t^{(n)} \xrightarrow{n \rightarrow \infty} e^{-\alpha t} T_t, t > 0$, in our case) is equivalent to the strong convergence of their associated resolvents. (cf. e.g. [5, Satz 1.7]).
- (iii) If conditions (1.12), (1.14), and (1.15) hold and if, in addition,

$$(1.17) \quad \left| b_i^{(n)} - b_i \right| \xrightarrow{n \rightarrow \infty} 0 \text{ weakly* in } L^{p_{b,i}}(U; dx) \quad \text{for all } 1 \leq i \leq d,$$

then by duality the assertion in part (i) of Theorem 1.3 still holds while part (ii) holds with all operators replaced by their adjoints on $L^2(U; dx)$.

By Rellich’s compact embedding theorem we get the following as an immediate consequence of Theorem 1.3 (i).

Corollary 1.5. *Suppose the U is bounded and that conditions (1.12)–(1.15) and (1.16) or (1.17) hold. Then there exists $\alpha_0 \in]0, \infty[$ such that for all $\alpha > \alpha_0, t > 0$, both $T_t^{(n)} \xrightarrow{n \rightarrow \infty} T_t$ and $G_\alpha^{(n)} \xrightarrow{n \rightarrow \infty} G_\alpha$ strongly on $L^2(U; dx)$. The same holds for their adjoints on $L^2(U; dx)$.*

As another consequence we obtain:

Corollary 1.6. *Assume that (1.12) holds and that for all $1 \leq i \leq d, b_i^{(n)} \xrightarrow{n \rightarrow \infty} b_i$ in $L^{p_{b,i}}(U; dx), d_i^{(n)} \xrightarrow{n \rightarrow \infty} d_i$ in $L^{p_{d,i}}(U; dx)$, and $c^{(n)} \rightarrow c$ in $L^{p_c/2}(U; dx)$. Then:*

- i) *There exists $\alpha_0 \in]0, \infty[$ such that for all $f \in L^2(U; dx)$ and $\alpha > \alpha_0$,*

$$G_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} G_\alpha f \text{ and } \widehat{G}_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} \widehat{G}_\alpha f \text{ in } H_0^{1,2}(U; dx).$$

- ii) *For all $t > 0$ and all $f \in L^2(U; dx)$*

$$T_t^{(n)} f \xrightarrow{n \rightarrow \infty} T_t f \text{ and } \widehat{T}_t^{(n)} f \xrightarrow{n \rightarrow \infty} \widehat{T}_t f \text{ in } H_0^{1,2}(U; dx).$$

Our proofs of all results above are purely analytic. They are presented in the next section. Theorem 1.3 extends a result by D.W. Stroock (cf. [7, Theorem II.3.13], where the case where $U = \mathbb{R}^d$, $c \equiv 0$, $d_i^{(n)} \equiv 0$, $p_{b,i} = \infty$ for all $1 \leq i \leq d$, $n \in \mathbb{N}$, was treated and the $b_i^{(n)}$, $1 \leq i \leq d$, $n \in \mathbb{N}$, were assumed to be uniformly bounded. In contrast to Stroock's our proofs are not based on heat kernel estimates. Finally, we note that we expect that by virtue of [8], [9] the results in this paper extend to the case of time-dependent coefficients (again without any uniform boundedness assumptions).

2. Proofs

Proof of Theorem 1.3. (i) For $q \in [1, \infty]$ let $\| \cdot \|_q$ denote the usual norm in $L^q(U; dx)$. By the conditions and Remark 1.1, α_n and γ_n can be chosen to be independent of n , i.e., $\gamma_n =: \gamma_0 > 0$ and $\alpha := \alpha_0 > 0$ for all $n \in \mathbb{N} \cup \{\infty\}$, say. In particular, for all $\alpha > \alpha_0$

$$\sup_n \left\| \alpha \widehat{G}_\alpha^{(n)} \right\| =: C_\alpha < \infty$$

where $\| \cdot \|$ denotes operator norm on $L^2(U; dx)$. Hence by (1.11)

$$(2.1) \quad \mathcal{E}_\alpha^{(n)}(\widehat{G}_\alpha^{(n)} f, \widehat{G}_\alpha^{(n)} f) = (f, \widehat{G}_\alpha^{(n)} f) \leq \alpha^{-1} C_\alpha \|f\|_2^2.$$

Fix $f \in L^2(U; dx)$, $\alpha > \alpha_0$. Since $\gamma_n = \gamma_0$ for all $n \in \mathbb{N}$, (1.6) and (2.1) imply that

$$(2.2) \quad \sup_n \left| \widehat{G}_\alpha^{(n)} f \right|_{1,2} =: C < \infty.$$

Then by the Banach–Alaoglu theorem there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ and $\widehat{G}f \in H_0^{1,2}(U; dx)$ such that

$$\widehat{G}_\alpha^{(n_k)} f \xrightarrow[k \rightarrow \infty]{} \widehat{G}f \text{ weakly in } H_0^{1,2}(U; dx).$$

So, it remains to show that $\widehat{G}f = \widehat{G}_\alpha f$. For simplicity of notation we replace $(n_k)_{k \in \mathbb{N}}$ again by $(n)_{n \in \mathbb{N}}$ and, since $(\widehat{G}_\alpha^{(n)} f)_{n \in \mathbb{N}}$ converges (strongly) in $L^2(V; dx)$ for every open ball V in U by Rellich's theorem, we may also assume that

$$(2.3) \quad \widehat{G}_\alpha^{(n)} f \longrightarrow \widehat{G}f \quad dx\text{-a.e..}$$

CLAIM 1. Let $v \in C_0^\infty(U)$. Then

$$\lim_{n \rightarrow \infty} [\mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f)] = 0.$$

Suppose Claim 1 has been proven. Then by the weak convergence of $(\widehat{G}_\alpha^{(n)} f)_{n \in \mathbb{N}}$ in $H_0^{1,2}(U; dx)$ and (1.5), (1.6) it follows that

$$\begin{aligned} \mathcal{E}_\alpha(v, \widehat{G}f) &= \lim_{n \rightarrow \infty} \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f) = \lim_{n \rightarrow \infty} \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f) = (v, f) \\ &= \mathcal{E}_\alpha(v, \widehat{G}_\alpha f) \end{aligned}$$

for all $v \in C_0^\infty(U)$, hence $\widehat{G}_\alpha f = \widehat{G}f$ and the proof is complete.

To prove Claim 1 note that for $n \in \mathbb{N}$

$$\begin{aligned} (2.4) \quad \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f) &= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i v \partial_j \widehat{G}_\alpha^{(n)} f \, dx + \sum_{i=1}^d \int (d_i - d_i^{(n)}) v \partial_i \widehat{G}_\alpha^{(n)} f \, dx \\ &\quad + \sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i v \widehat{G}_\alpha^{(n)} f \, dx + \int (c - c^{(n)}) v \widehat{G}_\alpha^{(n)} f \, dx. \end{aligned}$$

By the Cauchy–Schwarz inequality and (2.2) the first summand converges to zero as $n \rightarrow \infty$ because of Lebesgue’s dominated convergence theorem.

Let us recall that by Sobolev’s Lemma if $\lambda := (2^{2/3}(d - 1))/((d - 2)d^{1/2})$, then for all $u \in C_0^\infty(U)$

$$(2.5) \quad \|u\|_{\frac{2d}{d-2}} \leq \lambda \left(\int |\nabla u|_{\mathbb{R}^d}^2 \, dx \right)^{1/2}$$

(cf. e.g. [2, Theorem 1.7.1]). For $K := \text{supp } v$ (2.2) and (2.5) imply that $\{\widehat{G}_\alpha^{(n)} f \mid n \in \mathbb{N}\}$ is uniformly $d/(d - 2)$ –integrable on K w.r.t. dx . Hence by (2.3)

$$(2.6) \quad \widehat{G}_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} \widehat{G}f \text{ in } L^{\frac{d}{d-2}}(K; dx).$$

Since for all $i \in \{1, \dots, d\}$

$$\sup_n \left\| b_i^{(n)} \right\|_{L^{d/2}(K; dx)} < \infty$$

and

$$\sup_n \left\| c^{(n)} \right\|_{L^{d/2}(K; dx)} < \infty$$

(because $p_{d,i} \geq d > d/2$ and $p_c \geq d/2$), it follows that both $b_i^{(n)} \xrightarrow{n \rightarrow \infty} b_i$ and $c^{(n)} \xrightarrow{n \rightarrow \infty} c$ weakly* in $L^{d/2}(K; dx)$. Hence (2.6) implies that both the third and fourth summand on the right hand side of (2.4) converge to zero. To prove that the

same holds for the second, fix $i \in \{1, \dots, d\}$ and note that

$$\begin{aligned} & \left| \int (d_i - d_i^{(n)}) v \partial_i \widehat{G}_\alpha^{(n)} f \, dx \right| \\ & \leq \left(\int (d_i - d_i^{(n)})^2 v^2 \, dx \right)^{1/2} \left(\int (\partial_i \widehat{G}_\alpha^{(n)} f)^2 \, dx \right)^{1/2}. \end{aligned}$$

Hence by (2.2) it is sufficient to realize that by the Cauchy–Schwarz inequality (applied to the measure $|d_i - d_i^{(n)}| v^2 \, dx$)

$$\int (d_i - d_i^{(n)})^2 v^2 \, dx \leq \left(\int |d_i - d_i^{(n)}| v^2 \, dx \right)^{1/2} \left(\int |d_i - d_i^{(n)}|^3 v^2 \, dx \right)^{1/2},$$

and to recall that by (1.16) $|d_i - d_i^{(n)}| \xrightarrow[n \rightarrow \infty]{} 0$ weakly* in $L^{p_{d,i}}(U; dx)$ and thus, because $p_{d,i} \geq d \geq 3$, and $\text{supp} v$ is compact,

$$\sup_n \int |d_i - d_i^{(n)}|^3 v^2 \, dx < \infty.$$

Now Claim 1 is proved. To show that also $G_\alpha^{(n)} f \xrightarrow[n \rightarrow \infty]{} G_\alpha f$ weakly in $H_0^{1,2}(U; dx)$ we note that by (1.11) for all $n \in \mathbb{N}$

$$\mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f, G_\alpha^{(n)} f) = (f, G_\alpha^{(n)} f) \leq \alpha^{-1} C_\alpha \|f\|_2^2.$$

So, as above

$$\sup_n \|G_\alpha^{(n)} f\|_{1,2} =: C < \infty$$

and

$$G_\alpha^{(n_k)} f \xrightarrow[k \rightarrow \infty]{} Gf \text{ weakly in } H_0^{1,2}(U; dx), \text{ hence weakly in } L^2(U; dx)$$

for some subsequence $(n_k)_{k \in \mathbb{N}}$ and some $Gf \in H_0^{1,2}(U; dx)$. Again we only have to show that $Gf = G_\alpha f$. But we know that $\widehat{G}_\alpha^{(n)} f \xrightarrow[n \rightarrow \infty]{} \widehat{G}_\alpha f$ weakly in $L^2(U; dx)$, hence $G_\alpha^{(n)} f \xrightarrow[n \rightarrow \infty]{} G_\alpha f$ weakly in $L^2(U; dx)$, so

$$G_\alpha f = Gf,$$

and the proof of assertion (i) is complete.

(ii) By Remark 1.4 (i) it suffices to prove the first statement.

CLAIM 2. Let $f_n \in L^2(U; dx)$, $n \in \mathbb{N}$, such that $f_n \xrightarrow{n \rightarrow \infty} 0$ weakly in $L^2(U; dx)$. Then

$$\widehat{G}_\alpha^{(n)} f_n \xrightarrow{n \rightarrow \infty} 0 \text{ weakly in } H_0^{1,2}(U; dx).$$

Suppose Claim 2 has been proven, then for $\alpha > \alpha_0$ (where α_0 is as in assertion (i)), $f \in L^2(U; dx)$ and all $n \in \mathbb{N}$

$$\left\| G_\alpha^{(n)} f \right\|_2^2 = \int G_\alpha f G_\alpha^{(n)} f dx + \int \widehat{G}_\alpha^{(n)} (G_\alpha^{(n)} f - G_\alpha f) f dx.$$

By part (i) the first summand converges to $\|G_\alpha f\|_2^2$ while by Claim 2 the second summand converges to zero. Using part (i) again we conclude that

$$G_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} G_\alpha f \text{ in } L^2(U; dx).$$

To prove the claim, by (1.6) and Remark 1.1 as well as (2.2) it suffices to show that

$$\lim_{n \rightarrow \infty} \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) = 0 \text{ for all } v \in C_0^\infty(U).$$

So, let $v \in C_0^\infty(U)$, then by (1.11)

$$\mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) = (v, f_n) + \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f_n).$$

So, it remains to be shown that

$$\lim_{n \rightarrow \infty} (\mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f_n)) = 0.$$

But

$$\begin{aligned} & \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f_n) \\ &= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i v \partial_j \widehat{G}_\alpha^{(n)} f_n dx + \sum_{i=1}^d \int (d_i - d_i^{(n)}) v \partial_i \widehat{G}_\alpha^{(n)} f_n dx \\ & \quad + \sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i v \widehat{G}_\alpha^{(n)} f_n dx + \int (c - c^{(n)}) v \widehat{G}_\alpha^{(n)} f_n dx. \end{aligned}$$

By (2.1) $(\widehat{G}_\alpha^{(n)} f_n)_{n \in \mathbb{N}}$ is bounded in $H_0^{1,2}(U; dx)$, hence by exactly the same arguments as in the proof of Claim 1 (with $K := \text{supp} v$) we obtain that

$$\widehat{G}_\alpha^{(n)} f_n \xrightarrow{n \rightarrow \infty} h \text{ in } L^{\frac{d}{d-2}}(K; dx)$$

for some $h \in H_0^{1,2}(U; dx)$. Now also the rest of the proof of Claim 2 is entirely analogous to that of Claim 1.

Thus the proof of assertion (ii) is complete. □

Proof of Corollary 1.6. (i) Let $\alpha_0 \in]0, \infty[$ be as in Theorem 1.3. Fix $\alpha > \alpha_0$ and $f \in L^2(U; dx)$. Then by Remark 1.1 (cf. the beginning of the proof for Theorem 1.3) it suffices to prove

$$(2.7) \quad \lim_{n \rightarrow \infty} \mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f - G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) = 0,$$

since by duality the same then holds for $\widehat{G}_\alpha f, \widehat{G}_\alpha^{(n)} f, n \in \mathbb{N}$. But by applying (1.11) twice we have for all $n \in \mathbb{N}$

$$\begin{aligned} & \mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f - G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) \\ &= \mathcal{E}_\alpha(G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) - \mathcal{E}_\alpha^{(n)}(G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) \\ &= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i G_\alpha f \partial_j (G_\alpha^{(n)} f - G_\alpha f) dx \\ & \quad + \sum_{i=1}^d \int (d_i - d_i^{(n)}) G_\alpha f \partial_i (G_\alpha^{(n)} f - G_\alpha f) dx \\ & \quad + \sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i G_\alpha f (G_\alpha^{(n)} f - G_\alpha f) dx \\ & \quad + \int (c - c^{(n)}) G_\alpha f (G_\alpha^{(n)} f - G_\alpha f) dx. \end{aligned}$$

Since by Theorem 1.3, $G_\alpha^{(n)} f \rightharpoonup_\infty G_\alpha f$ weakly in $H_0^{1,2}(U; dx)$, it is clear that the first summand converges to zero as $n \rightarrow \infty$. To see that the same is true for the others we only have to realize that after applying Hölder’s inequality we have to deal with integrals of type

$$I_n := \int g_n^2 u_n^2 dx, \quad n \in \mathbb{N},$$

where $g_n \rightarrow 0$ in $L^p(U; dx), p \in [d, \infty[$, $u_n \in H_0^{1,2}(U; dx)$ such that $\sup_n \|u_n\|_{1,2} < \infty$. But using Hölder’s inequality and (2.5) we obtain that

$$\begin{aligned} I_n &\leq \left(\int u_n^2 dx \right)^{\frac{p-d}{p}} \left(\int g_n^{2p/d} u_n^2 dx \right)^{d/p} \\ &\leq \|u_n\|_2^{\frac{2(p-d)}{p}} \|g_n\|_p^2 \lambda^{2d/p} \|u_n\|_{1,2}^{2d/p}, \end{aligned}$$

hence $I_n \xrightarrow{n \rightarrow \infty} 0$ and the proof of assertion (i) is complete.

(ii) E.g. by [1, Theorem 3.4 (iii)], (1.6) and Remark 1.1 it follows that $(T_t^{(n)})_{n \in \mathbb{N}}$ is a strongly continuous semigroup on $H_0^{1,2}(U; dx)$ and that $(G_\alpha^{(n)})_{\alpha > \alpha_0}$ is the associated resolvent. Hence assertion (ii) follows by Remark 1.4 (ii). □

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