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CONVERGENCE OF OPERATORS
SEMIGROUPS GENERATED
BY ELLIPTIC OPERATORS

MICHAEL RÖCKNER and TU-SHENG ZHANG

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1. Introduction and main results

Let $U \subset \mathbb{R}^d$, $d \geq 3$, $U$ open (not necessarily bounded), and let $dx$ denote Lebesgue measure on $U$. Below all functions are supposed to be real-valued. Let $a_{ij}^{(n)}$, $b_i^{(n)}$, $d_i^{(n)}$, $c^{(n)} \in L^1_{\text{loc}}(U; dx)$, $1 \leq i, j \leq d$, $n \in \mathbb{N} \cup \{\infty\}$ satisfying the following conditions:

(1.1) There exists $\delta \in ]0, \infty[$ such that for all $n \in \mathbb{N} \cup \{\infty\}$ and $dx$-a.e. $x \in U$

\[ \sum_{i,j=1}^{d} a_{ij}^{(n)}(x)\xi_i\xi_j \geq \delta \sum_{i=1}^{d} \xi_i^2 \quad \text{for all } \xi_1, \ldots, \xi_d \in \mathbb{R}. \]

(1.2) There exists $M \in ]0, \infty[$ such that for all $n \in \mathbb{N}$ and $dx$-a.e. $x \in U$

\[ |a_{ij}^{(n)}(x)| \leq M, \quad 1 \leq i, j \leq d. \]

(1.3) There exist $p_b, p_d, p_c \in [d, \infty)$, $1 \leq i \leq d$, such that for all $n \in \mathbb{N} \cup \{\infty\}$

\[ b_i^{(n)} \in L^{p_b,i}(U; dx), \quad d_i^{(n)} \in L^{p_d,i}(U; dx), \quad c^{(n)} \in L^{p_c/2}(U; dx). \]

Note that (1.1) is a condition only on the symmetric part of $(a_{ij})_{1 \leq i, j \leq d}$. Conditions (1.1)–(1.3) allow to construct the corresponding coercive closed forms (cf. e.g. [3, Chap. I, Sect. 2]) as follows. Let $C_0^\infty(U)$ denote the set of all infinitely differentiable functions with compact support in $U$. Fix $n \in \mathbb{N} \cup \{\infty\}$ and set $\partial_i := \partial/\partial x_i$, $1 \leq i \leq d$. Define

\[ \mathcal{E}^{(n)}(u, v) := \sum_{i,j=1}^{d} \int \partial_i u \partial_j v a_{ij}^{(n)}(x) \, dx + \sum_{i=1}^{d} \int u \partial_i v d_i^{(n)} \, dx \]

\[ + \sum_{i=1}^{d} \int \partial_i uv b_i^{(n)} \, dx + \int uv c^{(n)} \, dx; \quad u, v \in C_0^\infty(U). \]
For \( \alpha \in ]0, \infty[ \) set
\[
\mathcal{E}^{(n)}_{\alpha}(u, v) := \mathcal{E}^{(n)}(u, v) + \alpha(u, v)_{L^2(U; dx)}; u, v \in C^\infty_0(U).
\]

E.g. by [4, Theorem 2.2] we know that there exists \( \alpha_n \in ]0, \infty[ \) such that \( (\mathcal{E}^{(n)}_{\alpha_n}, C^\infty_0(U)) \) is closable on \( L^2(U; dx) \) and its closure \( (\mathcal{E}^{(n)}_{\alpha_n}, D(\mathcal{E}^{(n)}_{\alpha_n})) \) is a coercive closed form on \( L^2(U; dx) \) in the sense of [3, Chap. I, Definition 2.4]. It is well-known (and can e.g. easily be extracted from the proof of [4, Theorem 2.2], or more precisely from the proof of the underlying [6, Theorem 1.7]) that there exist \( \gamma_n \in ]0, \infty[ \) such that for all \( u, v \in C^\infty_0(U) \)
\[
|\mathcal{E}^{(n)}_{\alpha_n}(u, v)| \leq \gamma_n \mathcal{E}^{(n)}_{\alpha_n}(u, u)^{1/2} \mathcal{E}^{(n)}_{\alpha_n}(v, v)^{1/2}
\]
\[
\gamma_n^{-1}|u|_{1,2} \leq \mathcal{E}^{(n)}_{\alpha_n}(u, u)^{1/2} \leq \gamma_n|u|_{1,2}.
\]
Here \( | \cdot |_{1,2} \) is the norm on the classical Sobolev space \( H^{1,2}_0(U; dx) \) of order 1 in \( L^2(U; dx) \), defined as the completion of \( C^\infty_0(U) \) w.r.t. \( | \cdot |_{1,2} \) which is given by
\[
|u|_{1,2}^2 := \sum_{i=1}^d \int (\partial_i u)^2 dx + \int u^2 dx; u \in C^\infty_0(U).
\]

In particular, \( D(\mathcal{E}^{(n)}_{\alpha_n}) = H^{1,2}_0(U, dx) \) and (1.5), (1.6) hold for all \( u \in H^{1,2}_0(U; dx) \).

**Remark 1.1.** \( \gamma_n \) in (1.5), (1.6) only depends on \( \alpha_n, \delta, M \) and the \( L^p \)-norms of \( b^{(n)}_i, d^{(n)}_i, c^{(n)}_i, 1 \leq i \leq d \), (cf. condition (1.3)). This can also be seen e.g. from the respective proofs in [4], [6] mentioned above. In particular, \( \alpha_n \) and \( \gamma_n \) can be chosen to be independent of \( n \), if all the \( L^p \)-norms in condition (1.3) are bounded uniformly in \( n \).

Let \( (L_{\alpha_n}, D(L_{\alpha_n})), (T_{\alpha_n,t})_{t>0} \) be the generator resp. the strongly continuous contraction semigroup associated with \( (\mathcal{E}^{(n)}_{\alpha_n}, D(\mathcal{E}^{(n)}_{\alpha_n})) \) (cf. e.g. [3, Chap. I., Sect. 2]). Define
\[
T^{(n)}_t := e^{\alpha_n t}T_{\alpha_n, t}, \quad t > 0,
\]
\[
L^{(n)} := L_{\alpha_n} + \alpha_n, \quad D(L^{(n)}) := D(L_{\alpha_n}).
\]
Then \( (L^{(n)}, D(L^{(n)})) \) generates \( (T^{(n)}_t)_{t>0} \) (on \( L^2(U; dx) \)).

**Remark 1.2.**

i) Obviously, \( (L^{(n)}, D(L^{(n)})) \) and \( (T^{(n)}_t)_{t>0} \) are independent of the special choice of \( \alpha_n \).

ii) Informally, we have for \( u \in C^\infty_0(U) \) that
\[ L^{(n)}u = \sum_{i,j=1}^{d} \partial_i (a_{ij}^{(n)} \partial_j + a_{ij}^{(n)})u - \sum_{i=1}^{d} b_i^{(n)} \partial_i u - c^{(n)}u. \]

Though (1.9) is very suggestive, it is, of course, informal since \( C_0^\infty(U) \) will in general not be a subset of \( D(L^{(n)}) \).

iii) Note that e.g. by [3, Chap. I, Theorem 2.20] \( T_t^{(n)}f \in D(\mathcal{E}_\alpha^{(n)}) = H_0^{1,2}(U; dx) \) for all \( f \in L^2(U; dx), t > 0 \).

Let \( n \in \mathbb{N} \cup \{\infty\} \) and let \( (G_\alpha^{(n)})_{\alpha > \alpha_n} \) be the strongly continuous resolvent associated with \( (T_t^{(n)})_{t>0} \) on \( L^2(U; dx) \), i.e., for \( \alpha > \alpha_n \)

\[ G_\alpha^{(n)} f := \int_0^\infty e^{-\alpha t} T_t^{(n)} f dt, \quad f \in L^2(U; dx), \]

(where the integral is a Bochner integral in \( L^2(U; dx) \)). Note that for \( \alpha > \alpha_n \) and \( f \in L^2(U; dx) \)

\[ G_\alpha^{(n)} f \in D(\mathcal{E}_\alpha^{(n)}) = H_0^{1,2}(U; dx) \]

and \( \mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f, v) = (f, v) = \mathcal{E}_\alpha^{(n)}(v, \tilde{G}_\alpha^{(n)} f) \) for all \( v \in H_0^{1,2}(U; dx) \)

(cf. e.g. [3, Chap. I, Theorem 2.8] and recall (1.7)). Here for a densely defined operator \((T, D(T))\) on \( L^2(U; dx) \) we denote its adjoint by \((T^*, D(T^*))\).

Consider for \( 1 \leq i, j \leq d \) the following conditions:

\[ a_{ij}^{(n)} \xrightarrow{n \to \infty} a_{ij}^{(\infty)} =: a_{ij} \quad dx\text{-a.e. on } U. \]
\[ b_i^{(n)} \xrightarrow{n \to \infty} b_i^{(\infty)} =: b_i \text{ weakly* in } L^{p_a,i}(U; dx). \]
\[ d_i^{(n)} \xrightarrow{n \to \infty} d_i^{(\infty)} =: d_i \text{ weakly* in } L^{p_d,i}(U; dx). \]
\[ c^{(n)} \xrightarrow{n \to \infty} c^{(\infty)} =: c \text{ weakly* in } L^{p_c/2}(U; dx). \]

Now we can formulate the main results of this paper.

**Theorem 1.3.** Suppose that for \( 1 \leq i, j \leq d \) conditions (1.12), (1.13), and (1.15) are satisfied and that

\[ \left| d_i^{(n)} - d_i \right| \xrightarrow{n \to \infty} 0 \text{ weakly* in } L^{p_d,i}(U; dx), \quad \text{for all } 1 \leq i \leq d. \]

Then there exists \( \alpha_0 \in ]0, \infty[ \) such that for all \( \alpha > \alpha_0 \) and all \( f \in L^2(U; dx) \):

(i) \( G_\alpha^{(n)} f \xrightarrow{n \to \infty} G_\alpha^{(\infty)} f =: G_\alpha f \) and \( \tilde{G}_\alpha^{(n)} f \xrightarrow{n \to \infty} \tilde{G}_\alpha^{(\infty)} f =: \tilde{G}_\alpha f \)
weakly in $H_0^{1,2}(U;dx)$;

(ii) \[ G^{(n)}_\alpha f \to \infty G_\alpha f \text{ in } L^2(U;dx), \]

and hence for all $t > 0$

\[ T^{(n)}_f f \to \infty T_t f \text{ in } L^2(U;dx). \]

**Remark 1.4.**

(i) We use the notion "weakly*" rather than "weakly" since $p_{bi}, p_{di}, p_c$ can be equal to $\pm \infty$. Clearly, if we assume (1.14) then (1.16) holds if $d^{(n)}_i \to \infty d_i$ in $dx$-measure for all $1 \leq i \leq d$. Note that (1.16), of course, implies (1.14).

(ii) Note that the last part of Theorem 1.3 (ii) is trivial, since (as is well-known and quite easy to prove) that strong convergence of strongly continuous contraction semigroups, (such as $e^{-at}T^{(n)}_t \to \infty e^{-at}T_t$, $t > 0$, in our case) is equivalent to the strong convergence of their associated resolvents. (cf. e.g. [5, Satz 1.7]).

(iii) If conditions (1.12), (1.14), and (1.15) hold and if, in addition,

\[ |b^{(n)}_i - b_i| \to \infty 0 \text{ weakly* in } L^{p_i}(U;dx) \text{ for all } 1 \leq i \leq d, \]

then by duality the assertion in part (i) of Theorem 1.3 still holds while part (ii) holds with all operators replaced by their adjoints on $L^2(U;dx)$.

By Rellich's compact embedding theorem we get the following as an immediate consequence of Theorem 1.3 (i).

**Corollary 1.5.** Suppose the $U$ is bounded and that conditions (1.12)–(1.15) and (1.16) or (1.17) hold. Then there exists $\alpha_0 \in ]0, \infty[$ such that for all $\alpha > \alpha_0$, $t > 0$, both $T^{(n)}_t \to \infty T_t$ and $G^{(n)}_\alpha \to \infty G_\alpha$ strongly on $L^2(U;dx)$. The same holds for their adjoints on $L^2(U;dx)$.

As another consequence we obtain:

**Corollary 1.6.** Assume that (1.12) holds and that for all $1 \leq i \leq d$, $b^{(n)}_i \to \infty b_i$ in $L^{p_i}(U;dx)$, $d^{(n)}_i \to \infty d_i$ in $L^{p_d}(U;dx)$, and $c^{(n)} \to c$ in $L^{p_c/2}(U;dx)$. Then:

i) There exists $\alpha_0 \in ]0, \infty[$ such that for all $f \in L^2(U;dx)$ and $\alpha > \alpha_0$,

\[ G^{(n)}_\alpha f \to \infty G_\alpha f \text{ and } \hat{G}^{(n)}_\alpha f \to \infty \hat{G}_\alpha f \text{ in } H_0^{1,2}(U;dx). \]

ii) For all $t > 0$ and all $f \in L^2(U;dx)$

\[ T^{(n)}_t f \to \infty T_t f \text{ and } \hat{T}^{(n)}_t f \to \infty \hat{T}_t f \text{ in } H_0^{1,2}(U;dx). \]
Our proofs of all results above are purely analytic. They are presented in the next section. Theorem 1.3 extends a result by D.W. Stroock (cf. [7, Theorem II.3.13], where the case where \( U = \mathbb{R}^d, c \equiv 0, d^{(n)}_i \equiv 0, p_{bi} = \infty \) for all \( 1 \leq i \leq d, n \in \mathbb{N} \), was treated and the \( b_i^{(n)}, 1 \leq i \leq d, n \in \mathbb{N} \), were assumed to be uniformly bounded. In contrast to Stroock's our proofs are not based on heat kernel estimates. Finally, we note that we expect that by virtue of [8], [9] the results in this paper extend to the case of time-dependent coefficients (again without any uniform boundedness assumptions).

2. Proofs

Proof of Theorem 1.3. (i) For \( q \in [1, \infty] \) let \( \| \|_q \) denote the usual norm in \( L^q(U; dx) \). By the conditions and Remark 1.1, \( \alpha_n \) and \( \gamma_n \) can be chosen to be independent of \( n \), i.e., \( \gamma_n =: \gamma_0 > 0 \) and \( \alpha := \alpha_0 > 0 \) for all \( n \in \mathbb{N} \cup \{ \infty \} \), say. In particular, for all \( \alpha > \alpha_0 \)

\[
\sup_n \| \alpha \widetilde{G}_\alpha^{(n)} \| =: C_\alpha < \infty
\]

where \( \| \| \) denotes operator norm on \( L^2(U; dx) \). Hence by (1.11)

\[
(2.1) \quad \mathcal{E}_\alpha^{(n)}(\widetilde{G}_\alpha^{(n)} f, \widetilde{G}_\alpha^{(n)} f) = (f, \widetilde{G}_\alpha^{(n)} f) \leq \alpha^{-1} C_\alpha \| f \|_2^2.
\]

Fix \( f \in L^2(U; dx), \alpha > \alpha_0 \). Since \( \gamma_n = \gamma_0 \) for all \( n \in \mathbb{N} \), (1.6) and (2.1) imply that

\[
(2.2) \quad \sup_n \left| \widetilde{G}_\alpha^{(n)} f \right|_{1,2} =: C < \infty.
\]

Then by the Banach–Alaoglu theorem there exists a subsequence \( (n_k)_{k \in \mathbb{N}} \) and \( \tilde{G} f \in H^{1,2}_0(U; dx) \) such that

\[
\tilde{G}_\alpha^{(n_k)} f \rightharpoonup \tilde{G} f \quad \text{weakly in } H^{1,2}_0(U; dx).
\]

So, it remains to show that \( \tilde{G} f = \tilde{G}_\alpha f \). For simplicity of notation we replace \( (n_k)_{k \in \mathbb{N}} \) again by \( (n)_{n \in \mathbb{N}} \) and, since \( (\tilde{G}_\alpha^{(n)} f)_{n \in \mathbb{N}} \) converges (strongly) in \( L^2(V; dx) \) for every open ball \( V \) in \( U \) by Rellich's theorem, we may also assume that

\[
(2.3) \quad \tilde{G}_\alpha^{(n)} f \longrightarrow \tilde{G} f \quad dx-a.e..
\]

Claim 1. Let \( v \in C_0^\infty(U) \). Then

\[
\lim_{n \to \infty} \left[ \mathcal{E}_\alpha(v, \tilde{G}_\alpha^{(n)} f) - \mathcal{E}_\alpha^{(n)}(v, \tilde{G}_\alpha^{(n)} f) \right] = 0.
\]
Suppose Claim 1 has been proven. Then by the weak convergence of \((\tilde{G}_{\alpha}^{(n)} f)_{n \in \mathbb{N}}\) in \(H^{1,2}_0(U; dx)\) and (1.5), (1.6) it follows that
\[
\mathcal{E}_\alpha(v, \tilde{G}_f) = \lim_{n \to \infty} \mathcal{E}_\alpha(v, \tilde{G}_f^{(n)}) = \lim_{n \to \infty} \mathcal{E}_\alpha^{(n)}(v, \tilde{G}_f^{(n)}) = (v, f)
\]
for all \(v \in C_0^\infty(U)\), hence \(\tilde{G}_f = \tilde{G}_f\) and the proof is complete.

To prove Claim 1 note that for \(n \in \mathbb{N}\)
\[
\mathcal{E}_\alpha(v, \tilde{G}_f^{(n)}) - \mathcal{E}_\alpha^{(n)}(v, \tilde{G}_f^{(n)})
= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i v \partial_j \tilde{G}_f^{(n)} f dx
+ \sum_{i=1}^d \int (d_i - d_i^{(n)}) v \partial_i \tilde{G}_f^{(n)} f dx
\]
\[
\sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i v \tilde{G}_f^{(n)} f dx
\]
\[
\int (c - c^{(n)}) v \tilde{G}_f^{(n)} f dx.
\]

By the Cauchy–Schwarz inequality and (2.2) the first summand converges to zero as \(n \to \infty\) because of Lebesgue's dominated convergence theorem.

Let us recall that by Sobolev's Lemma if \(\lambda := (2^{2/3}(d - 1))/((d - 2)d^{1/2})\), then for all \(u \in C_0^\infty(U)\)
\[
\|u\|_{2d/2} \leq \lambda \left( \int |\nabla u|^2 dx \right)^{1/2}
\]
(cf. e.g. [2, Theorem 1.7.1]). For \(K := \text{supp } v (2.2)\) and (2.5) imply that \(\{\tilde{G}_f^{(n)} f \mid n \in \mathbb{N}\}\) is uniformly \(d/(d - 2)\)--integrable on \(K\) w.r.t. \(dx\). Hence by (2.3)
\[
\tilde{G}_f^{(n)} f \xrightarrow{n \to \infty} \tilde{G}_f \text{ in } L^{2d/2}(K; dx).
\]
Since for all \(i \in \{1, \ldots, d\}\)
\[
\sup_n \left\| b_i^{(n)} \right\|_{L^{d/2}(K; dx)} < \infty
\]
and
\[
\sup_n \left\| c^{(n)} \right\|_{L^{d/2}(K; dx)} < \infty
\]
(because \(p_{d,i} \geq d > d/2\) and \(p_c \geq d/2\)), it follows that both \(b_i^{(n)} \xrightarrow{n \to \infty} b_i\) and \(c^{(n)} \xrightarrow{n \to \infty} c\) weakly* in \(L^{d/2}(K; dx)\). Hence (2.6) implies that both the third and fourth summand on the right hand side of (2.4) converge to zero. To prove that the
same holds for the second, fix $i \in \{1, \ldots, d\}$ and note that

$$\left| \int (d_i - d_i^{(n)}) v \bar{G}_\alpha^{(n)} f \, dx \right| \leq \left( \int (d_i - d_i^{(n)})^2 v^2 \, dx \right)^{1/2} \left( \int (\partial_i \bar{G}_\alpha^{(n)} f)^2 \, dx \right)^{1/2}.$$

Hence by (2.2) it is sufficient to realize that by the Cauchy–Schwarz inequality (applied to the measure $|d_i - d_i^{(n)}| v^2 \, dx$)

$$\int (d_i - d_i^{(n)})^2 v^2 \, dx \leq \left( \int |d_i - d_i^{(n)}| v^2 \, dx \right)^{1/2} \left( \int |d_i - d_i^{(n)}|^3 v^2 \, dx \right)^{1/2},$$

and to recall that by (1.16) $|d_i - d_i^{(n)}|_{n \to \infty} 0$ weakly* in $L^{p_d, 1}(U; dx)$ and thus, because $p_{d,i} \geq d \geq 3$, and suppv is compact,

$$\sup_n \int |d_i - d_i^{(n)}|^3 v^2 \, dx < \infty.$$

Now Claim 1 is proved. To show that also $G_\alpha^{(n)} f_n \rightharpoonup_{L^2} G_\alpha f$ weakly in $H_0^{1,2}(U; dx)$ we note that by (1.11) for all $n \in \mathbb{N}$

$$E_\alpha^{(n)}(G_\alpha^{(n)} f, G_\alpha^{(n)} f) = (f, G_\alpha^{(n)} f) \leq \alpha^{-1} C_\alpha \|f\|_2^2.$$

So, as above

$$\sup_n \left| G_\alpha^{(n)} f \right|_{1,2} =: C < \infty$$

and

$$G_\alpha^{(n_k)} f_{k \to \infty} \rightharpoonup_{L^2} G f$$

for some subsequence $(n_k)_{k \in \mathbb{N}}$ and some $Gf \in H_0^{1,2}(U; dx)$. Again we only have to show that $Gf = G_\alpha f$. But we know that $\bar{G}_\alpha^{(n)} f_n \rightharpoonup \bar{G}_\alpha f$ weakly in $L^2(U; dx)$, hence $G_\alpha^{(n)} f_n \rightharpoonup \infty G_\alpha f$ weakly in $L^2(U; dx)$, so

$$G_\alpha f = Gf,$$

and the proof of assertion (i) is complete.

(ii) By Remark 1.4 (i) it suffices to prove the first statement.
Claim 2. Let \( f_n \in L^2(U; dx), n \in \mathbb{N}, \) such that \( f_n \rightharpoonup 0 \) weakly in \( L^2(U; dx). \) Then

\[ \hat{G}_\alpha^{(n)} f_n \rightharpoonup 0 \quad \text{weakly in } H^{1,2}_0(U; dx). \]

Suppose Claim 2 has been proven, then for \( \alpha > \alpha_0 \) (where \( \alpha_0 \) is as in assertion (i)), \( f \in L^2(U; dx) \) and all \( n \in \mathbb{N} \)

\[ \left\| G_\alpha^{(n)} f \right\|_2^2 = \int G_\alpha f G_\alpha^{(n)} f \, dx + \int \hat{G}_\alpha^{(n)} (G_\alpha^{(n)} f - G_\alpha f) f \, dx. \]

By part (i) the first summand converges to \( \|G_\alpha f\|_2^2 \) while by Claim 2 the second summand converges to zero. Using part (i) again we conclude that

\[ G_\alpha^{(n)} f_n \rightharpoonup \infty G_\alpha f \text{ in } L^2(U; dx). \]

To prove the claim, by (1.6) and Remark 1.1 as well as (2.2) it suffices to show that

\[ \lim_{n \to \infty} \mathcal{E}_\alpha(v, \hat{G}_\alpha^{(n)} f_n) = 0 \text{ for all } v \in C_0^\infty(U). \]

So, let \( v \in C_0^\infty(U), \) then by (1.11)

\[ \mathcal{E}_\alpha(v, \hat{G}_\alpha^{(n)} f_n) = (v, f_n) + \mathcal{E}_\alpha(v, \hat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \hat{G}_\alpha^{(n)} f_n). \]

So, it remains to be shown that

\[ \lim_{n \to \infty} (\mathcal{E}_\alpha(v, \hat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \hat{G}_\alpha^{(n)} f_n)) = 0. \]

But

\[ \mathcal{E}_\alpha(v, \hat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \hat{G}_\alpha^{(n)} f_n) \]

\[ = \sum_{i,j=1}^d (a_{ij} - a^{(n)}_{ij}) \partial_i v \partial_j \hat{G}_\alpha^{(n)} f_n dx + \sum_{i=1}^d (d_i - d^{(n)}_i) \nu \partial_i \hat{G}_\alpha^{(n)} f_n dx \]

\[ + \sum_{i=1}^d (b_i - b^{(n)}_i) \nu \partial_i \hat{G}_\alpha^{(n)} f_n dx + \int (c - c^{(n)}) \nu \hat{G}_\alpha^{(n)} f_n dx. \]

By (2.1) \( (\hat{G}_\alpha^{(n)} f_n)_{n \in \mathbb{N}} \) is bounded in \( H^{1,2}_0(U; dx), \) hence by exactly the same arguments as in the proof of Claim 1 (with \( K := \text{supp} v \)) we obtain that

\[ \hat{G}_\alpha^{(n)} f_n \rightharpoonup \infty h \quad \text{in } L^2(K; dx) \]

for some \( h \in H^{1,2}_0(U; dx). \) Now also the rest of the proof of Claim 2 is entirely analogous to that of Claim 1.
Thus the proof of assertion (ii) is complete.

Proof of Corollary 1.6. (i) Let $\alpha_0 \in [0, \infty]$ be as in Theorem 1.3. Fix $\alpha > \alpha_0$ and $f \in L^2(U; dx)$. Then by Remark 1.1 (cf. the beginning of the proof for Theorem 1.3) it suffices to prove

\begin{equation}
\lim_{n \to \infty} E^{(n)}(G^{(n)}_\alpha f - G_\alpha f, G^{(n)}_\alpha f - G_\alpha f) = 0,
\end{equation}

since by duality the same then holds for $\hat{G}_\alpha f, \hat{G}^{(n)}_\alpha f, n \in \mathbb{N}$. But by applying (1.11) twice we have for all $n \in \mathbb{N}$

\begin{align*}
E^{(n)}(G^{(n)}_\alpha f - G_\alpha f, G^{(n)}_\alpha f - G_\alpha f) &= E_\alpha(G^nf, G^{(n)}_\alpha f - G_\alpha f) - E^{(n)}(G^nf, G^{(n)}_\alpha f - G_\alpha f) \\
&= \sum_{i,j=1}^d \int (a_{ij} - a^{(n)}_{ij}) \partial_i G^nf \partial_j(G^{(n)}_\alpha f - G_\alpha f) dx \\
&\quad + \sum_{i=1}^d \int (d_i - d^{(n)}_i) G^nf \partial_i(G^{(n)}_\alpha f - G_\alpha f) dx \\
&\quad + \sum_{i=1}^d \int (b_i - b^{(n)}_i) \partial_i G_\alpha f(G^{(n)}_\alpha f - G_\alpha f) dx \\
&\quad + \int (c - c^{(n)}) G_\alpha f(G^{(n)}_\alpha f - G_\alpha f) dx.
\end{align*}

Since by Theorem 1.3, $G^{(n)}_\alpha f \rightharpoonup G_\alpha f$ weakly in $H^{1,2}_0(U; dx)$, it is clear that the first summand converges to zero as $n \to \infty$. To see that the same is true for the others we only have to realize that after applying Hölder’s inequality we have to deal with integrals of type

$$I_n := \int g_n^2 u_n^2 dx, \quad n \in \mathbb{N},$$

where $g_n \to 0$ in $L^p(U; dx)$, $p \in [d, \infty]$, $u_n \in H^{1,2}_0(U; dx)$ such that $\sup_n |u_n|_{1,2} < \infty$. But using Hölder’s inequality and (2.5) we obtain that

$$I_n \leq \left( \int u_n^2 dx \right)^{\frac{p-d}{p}} \left( \int g_n^{2p/d} u_n^2 dx \right)^{d/p} \leq \|u_n\|_2^{2(p-d)/p} \|g_n\|_p^{2d/p} \|u_n\|_{1,2}^{2d/p},$$

hence $I_n \rightharpoonup 0$ and the proof of assertion (i) is complete.

(ii) E.g. by [1, Theorem 3.4 (iii)], (1.6) and Remark 1.1 it follows that $(T^{(n)}_t)_{n \in \mathbb{N}}$ is a strongly continuous semigroup on $H^{1,2}_0(U; dx)$ and that $(G^{(n)}_\alpha)_{\alpha > \alpha_0}$ is the associated resolvent. Hence assertion (ii) follows by Remark 1.4 (ii).
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