

Title	Convergence of operators semigroups generated by elliptic operators
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Citation	Osaka Journal of Mathematics. 34(4) P.923-P.932
Issue Date	1997
Text Version	publisher
URL	<a href="https://doi.org/10.18910/8026">https://doi.org/10.18910/8026</a>
DOI	10.18910/8026
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## CONVERGENCE OF OPERATORS SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS

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(Received October 22, 1996)

### 1. Introduction and main results

Let  $U \subset \mathbb{R}^d$ ,  $d \geq 3$ ,  $U$  open (not necessarily bounded), and let  $dx$  denote Lebesgue measure on  $U$ . Below all functions are supposed to be real-valued. Let  $a_{ij}^{(n)}$ ,  $b_i^{(n)}$ ,  $d_i^{(n)}$ ,  $c^{(n)} \in L^1_{\text{loc}}(U; dx)$ ,  $1 \leq i, j \leq d$ ,  $n \in \mathbb{N} \cup \{\infty\}$  satisfying the following conditions:

(1.1) There exists  $\delta \in ]0, \infty[$  such that for all  $n \in \mathbb{N} \cup \{\infty\}$  and  $dx$ -a.e.  $x \in U$

$$\sum_{i,j=1}^d a_{ij}^{(n)}(x) \xi_i \xi_j \geq \delta \sum_{i=1}^d \xi_i^2 \text{ for all } \xi_1, \dots, \xi_d \in \mathbb{R}.$$

(1.2) There exists  $M \in [0, \infty[$  such that for all  $n \in \mathbb{N}$  and  $dx$ -a.e.  $x \in U$

$$|a_{ij}^{(n)}(x)| \leq M, \quad 1 \leq i, j \leq d.$$

(1.3) There exist  $p_{b,i}$ ,  $p_{d,i}$ ,  $p_c \in [d, \infty]$ ,  $1 \leq i \leq d$ , such that for all  $n \in \mathbb{N} \cup \{\infty\}$

$$b_i^{(n)} \in L^{p_{b,i}}(U; dx), \quad d_i^{(n)} \in L^{p_{d,i}}(U; dx), \quad c^{(n)} \in L^{p_c/2}(U; dx).$$

Note that (1.1) is a condition only on the symmetric part of  $(a_{ij})_{1 \leq i, j \leq d}$ . Conditions (1.1)–(1.3) allow to construct the corresponding coercive closed forms (cf. e.g. [3, Chap. I, Sect. 2]) as follows. Let  $C_0^\infty(U)$  denote the set of all infinitely differentiable functions with compact support in  $U$ . Fix  $n \in \mathbb{N} \cup \{\infty\}$  and set  $\partial_i := \partial/\partial x_i$ ,  $1 \leq i \leq d$ . Define

$$(1.4) \quad \mathcal{E}^{(n)}(u, v) := \sum_{i,j=1}^d \int \partial_i u \partial_j v a_{ij}^{(n)} dx + \sum_{i=1}^d \int u \partial_i v d_i^{(n)} dx \\
 + \sum_{i=1}^d \int \partial_i u v b_i^{(n)} dx + \int u v c^{(n)} dx \quad ; \quad u, v \in C_0^\infty(U).$$

For  $\alpha \in ]0, \infty[$  set

$$\mathcal{E}_\alpha^{(n)}(u, v) := \mathcal{E}^{(n)}(u, v) + \alpha(u, v)_{L^2(U; dx)}; u, v \in C_0^\infty(U).$$

E.g. by [4, Theorem 2.2] we know that there exists  $\alpha_n \in ]0, \infty[$  such that  $(\mathcal{E}_{\alpha_n}^{(n)}, C_0^\infty(U))$  is closable on  $L^2(U; dx)$  and its closure  $(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)}))$  is a coercive closed form on  $L^2(U; dx)$  in the sense of [3, Chap. I, Definition 2.4]. It is well-known (and can e.g. easily be extracted from the proof of [4, Theorem 2.2], or more precisely from the proof of the underlying [6, Theorem 1.7]) that there exist  $\gamma_n \in ]0, \infty[$  such that for all  $u, v \in C_0^\infty(U)$

$$(1.5) \quad \left| \mathcal{E}_{\alpha_n}^{(n)}(u, v) \right| \leq \gamma_n \mathcal{E}_{\alpha_n}^{(n)}(u, u)^{1/2} \mathcal{E}_{\alpha_n}^{(n)}(v, v)^{1/2}$$

$$(1.6) \quad \gamma_n^{-1} |u|_{1,2} \leq \mathcal{E}_{\alpha_n}^{(n)}(u, u)^{1/2} \leq \gamma_n |u|_{1,2}.$$

Here  $| \cdot |_{1,2}$  is the norm on the classical Sobolev space  $H_0^{1,2}(U; dx)$  of order 1 in  $L^2(U; dx)$ , defined as the completion of  $C_0^\infty(U)$  w.r.t.  $| \cdot |_{1,2}$  which is given by

$$|u|_{1,2}^2 := \sum_{i=1}^d \int (\partial_i u)^2 dx + \int u^2 dx; u \in C_0^\infty(U).$$

In particular,  $D(\mathcal{E}_{\alpha_n}^{(n)}) = H_0^{1,2}(U, dx)$  and (1.5), (1.6) hold for all  $u \in H_0^{1,2}(U; dx)$ .

**REMARK 1.1.**  $\gamma_n$  in (1.5), (1.6) only depends on  $\alpha_n, \delta, M$  and the  $L^p$ -norms of  $b_i^{(n)}, d_i^{(n)}, c^{(n)}, 1 \leq i \leq d$ , (cf. condition (1.3)). This can also be seen e.g. from the respective proofs in [4], [6] mentioned above. In particular,  $\alpha_n$  and  $\gamma_n$  can be chosen to be independent of  $n$ , if all the  $L^p$ -norms in condition (1.3) are bounded uniformly in  $n$ .

Let  $(L_{\alpha_n}, D(L_{\alpha_n})), (T_{\alpha_n, t})_{t>0}$  be the generator resp. the strongly continuous contraction semigroup associated with  $(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)}))$  (cf. e.g. [3, Chap. I., Sect. 2]). Define

$$(1.7) \quad T_t^{(n)} := e^{\alpha_n t} T_{\alpha_n, t}, \quad t > 0,$$

$$(1.8) \quad L^{(n)} := L_{\alpha_n} + \alpha_n, \quad D(L^{(n)}) := D(L_{\alpha_n}).$$

Then  $(L^{(n)}, D(L^{(n)}))$  generates  $(T_t^{(n)})_{t>0}$  (on  $L^2(U; dx)$ ).

**REMARK 1.2.**

- i) Obviously,  $(L^{(n)}, D(L^{(n)}))$  and  $(T_t^{(n)})_{t>0}$  are independent of the special choice of  $\alpha_n$ .
- ii) Informally, we have for  $u \in C_0^\infty(U)$  that

$$(1.9) \quad L^{(n)}u = \sum_{i,j=1}^d \partial_i(a_{ij}^{(n)} \partial_j + d_i^{(n)})u - \sum_{i=1}^d b_i^{(n)} \partial_i u - c^{(n)}u.$$

Though (1.9) is very suggestive, it is, of course, informal since  $C_0^\infty(U)$  will in general not be a subset of  $D(L^{(n)})$ .

iii) Note that e.g. by [3, Chap. I, Theorem 2.20]  $T_t^{(n)}f \in D(\mathcal{E}_{\alpha_n}^{(n)}) = H_0^{1,2}(U; dx)$  for all  $f \in L^2(U; dx)$ ,  $t > 0$ .

Let  $n \in \mathbb{N} \cup \{\infty\}$  and let  $(G_\alpha^{(n)})_{\alpha > \alpha_n}$  be the strongly continuous resolvent associated with  $(T_t^{(n)})_{t > 0}$  on  $L^2(U; dx)$ , i.e., for  $\alpha > \alpha_n$

$$(1.10) \quad G_\alpha^{(n)}f := \int_0^\infty e^{-\alpha t} T_t^{(n)}f dt, \quad f \in L^2(U; dx),$$

(where the integral is a Bochner integral in  $L^2(U; dx)$ ). Note that for  $\alpha > \alpha_n$  and  $f \in L^2(U; dx)$

$$(1.11) \quad G_\alpha^{(n)}f \in D(\mathcal{E}_{\alpha_n}^{(n)}) = H_0^{1,2}(U; dx)$$

and  $\mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)}f, v) = (f, v) = \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)}f)$  for all  $v \in H_0^{1,2}(U; dx)$

(cf. e.g. [3, Chap. I, Theorem 2.8] and recall (1.7)). Here for a densely defined operator  $(T, D(T))$  on  $L^2(U; dx)$  we denote its adjoint by  $(\widehat{T}, D(\widehat{T}))$ .

Consider for  $1 \leq i, j \leq d$  the following conditions:

$$(1.12) \quad a_{ij}^{(n)} \xrightarrow{n \rightarrow \infty} a_{ij}^{(\infty)} =: a_{ij} \quad dx\text{-a.e. on } U.$$

$$(1.13) \quad b_i^{(n)} \xrightarrow{n \rightarrow \infty} b_i^{(\infty)} =: b_i \text{ weakly}^* \text{ in } L^{p_{b,i}}(U; dx).$$

$$(1.14) \quad d_i^{(n)} \xrightarrow{n \rightarrow \infty} d_i^{(\infty)} =: d_i \text{ weakly}^* \text{ in } L^{p_{d,i}}(U; dx).$$

$$(1.15) \quad c^{(n)} \xrightarrow{n \rightarrow \infty} c^{(\infty)} =: c \text{ weakly}^* \text{ in } L^{p_c/2}(U; dx).$$

Now we can formulate the main results of this paper.

**Theorem 1.3.** *Suppose that for  $1 \leq i, j \leq d$  conditions (1.12), (1.13), and (1.15) are satisfied and that*

$$(1.16) \quad \left| d_i^{(n)} - d_i \right| \xrightarrow{n \rightarrow \infty} 0 \text{ weakly}^* \text{ in } L^{p_{d,i}}(U; dx), \quad \text{for all } 1 \leq i \leq d.$$

*Then there exists  $\alpha_0 \in ]0, \infty[$  such that for all  $\alpha > \alpha_0$  and all  $f \in L^2(U; dx)$  :*

$$(i) \quad G_\alpha^{(n)}f \xrightarrow{n \rightarrow \infty} G_\alpha^{(\infty)}f =: G_\alpha f \text{ and } \widehat{G}_\alpha^{(n)}f \xrightarrow{n \rightarrow \infty} \widehat{G}_\alpha^{(\infty)}f =: \widehat{G}_\alpha f$$

weakly in  $H_0^{1,2}(U; dx)$ ;

(ii) 
$$G_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} G_\alpha f \text{ in } L^2(U; dx) ,$$

and hence for all  $t > 0$

$$T_f^{(n)} f \xrightarrow{n \rightarrow \infty} T_t f \text{ in } L^2(U; dx).$$

REMARK 1.4.

- (i) We use the notion “weakly\*” rather than “weakly” since  $p_{b,i}, p_{d,i}, p_c$  can be equal to  $+\infty$ . Clearly, if we assume (1.14) then (1.16) holds if  $d_i^{(n)} \xrightarrow{n \rightarrow \infty} d_i$  in  $dx$ -measure for all  $1 \leq i \leq d$ . Note that (1.16), of course, implies (1.14).
- (ii) Note that the last part of Theorem 1.3 (ii) is trivial, since (as is well-known and quite easy to prove) that strong convergence of strongly continuous contraction semigroups, (such as  $e^{-\alpha t} T_t^{(n)} \xrightarrow{n \rightarrow \infty} e^{-\alpha t} T_t, t > 0$ , in our case) is equivalent to the strong convergence of their associated resolvents. (cf. e.g. [5, Satz 1.7]).
- (iii) If conditions (1.12), (1.14), and (1.15) hold and if, in addition,

$$(1.17) \quad \left| b_i^{(n)} - b_i \right| \xrightarrow{n \rightarrow \infty} 0 \text{ weakly* in } L^{p_{b,i}}(U; dx) \quad \text{for all } 1 \leq i \leq d,$$

then by duality the assertion in part (i) of Theorem 1.3 still holds while part (ii) holds with all operators replaced by their adjoints on  $L^2(U; dx)$ .

By Rellich’s compact embedding theorem we get the following as an immediate consequence of Theorem 1.3 (i).

**Corollary 1.5.** *Suppose the  $U$  is bounded and that conditions (1.12)–(1.15) and (1.16) or (1.17) hold. Then there exists  $\alpha_0 \in ]0, \infty[$  such that for all  $\alpha > \alpha_0, t > 0$ , both  $T_t^{(n)} \xrightarrow{n \rightarrow \infty} T_t$  and  $G_\alpha^{(n)} \xrightarrow{n \rightarrow \infty} G_\alpha$  strongly on  $L^2(U; dx)$ . The same holds for their adjoints on  $L^2(U; dx)$ .*

As another consequence we obtain:

**Corollary 1.6.** *Assume that (1.12) holds and that for all  $1 \leq i \leq d, b_i^{(n)} \xrightarrow{n \rightarrow \infty} b_i$  in  $L^{p_{b,i}}(U; dx), d_i^{(n)} \xrightarrow{n \rightarrow \infty} d_i$  in  $L^{p_{d,i}}(U; dx)$ , and  $c^{(n)} \rightarrow c$  in  $L^{p_c/2}(U; dx)$ . Then:*

- i) *There exists  $\alpha_0 \in ]0, \infty[$  such that for all  $f \in L^2(U; dx)$  and  $\alpha > \alpha_0$ ,*

$$G_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} G_\alpha f \text{ and } \widehat{G}_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} \widehat{G}_\alpha f \text{ in } H_0^{1,2}(U; dx).$$

- ii) *For all  $t > 0$  and all  $f \in L^2(U; dx)$*

$$T_t^{(n)} f \xrightarrow{n \rightarrow \infty} T_t f \text{ and } \widehat{T}_t^{(n)} f \xrightarrow{n \rightarrow \infty} \widehat{T}_t f \text{ in } H_0^{1,2}(U; dx).$$

Our proofs of all results above are purely analytic. They are presented in the next section. Theorem 1.3 extends a result by D.W. Stroock (cf. [7, Theorem II.3.13], where the case where  $U = \mathbb{R}^d$ ,  $c \equiv 0$ ,  $d_i^{(n)} \equiv 0$ ,  $p_{b,i} = \infty$  for all  $1 \leq i \leq d$ ,  $n \in \mathbb{N}$ , was treated and the  $b_i^{(n)}$ ,  $1 \leq i \leq d$ ,  $n \in \mathbb{N}$ , were assumed to be uniformly bounded. In contrast to Stroock's our proofs are not based on heat kernel estimates. Finally, we note that we expect that by virtue of [8], [9] the results in this paper extend to the case of time-dependent coefficients (again without any uniform boundedness assumptions).

**2. Proofs**

Proof of Theorem 1.3. (i) For  $q \in [1, \infty]$  let  $\| \cdot \|_q$  denote the usual norm in  $L^q(U; dx)$ . By the conditions and Remark 1.1,  $\alpha_n$  and  $\gamma_n$  can be chosen to be independent of  $n$ , i.e.,  $\gamma_n =: \gamma_0 > 0$  and  $\alpha := \alpha_0 > 0$  for all  $n \in \mathbb{N} \cup \{\infty\}$ , say. In particular, for all  $\alpha > \alpha_0$

$$\sup_n \left\| \alpha \widehat{G}_\alpha^{(n)} \right\| =: C_\alpha < \infty$$

where  $\| \cdot \|$  denotes operator norm on  $L^2(U; dx)$ . Hence by (1.11)

$$(2.1) \quad \mathcal{E}_\alpha^{(n)}(\widehat{G}_\alpha^{(n)} f, \widehat{G}_\alpha^{(n)} f) = (f, \widehat{G}_\alpha^{(n)} f) \leq \alpha^{-1} C_\alpha \|f\|_2^2.$$

Fix  $f \in L^2(U; dx)$ ,  $\alpha > \alpha_0$ . Since  $\gamma_n = \gamma_0$  for all  $n \in \mathbb{N}$ , (1.6) and (2.1) imply that

$$(2.2) \quad \sup_n \left| \widehat{G}_\alpha^{(n)} f \right|_{1,2} =: C < \infty.$$

Then by the Banach–Alaoglu theorem there exists a subsequence  $(n_k)_{k \in \mathbb{N}}$  and  $\widehat{G}f \in H_0^{1,2}(U; dx)$  such that

$$\widehat{G}_\alpha^{(n_k)} f \xrightarrow{k \rightarrow \infty} \widehat{G}f \text{ weakly in } H_0^{1,2}(U; dx).$$

So, it remains to show that  $\widehat{G}f = \widehat{G}_\alpha f$ . For simplicity of notation we replace  $(n_k)_{k \in \mathbb{N}}$  again by  $(n)_{n \in \mathbb{N}}$  and, since  $(\widehat{G}_\alpha^{(n)} f)_{n \in \mathbb{N}}$  converges (strongly) in  $L^2(V; dx)$  for every open ball  $V$  in  $U$  by Rellich's theorem, we may also assume that

$$(2.3) \quad \widehat{G}_\alpha^{(n)} f \longrightarrow \widehat{G}f \quad dx\text{-a.e..}$$

CLAIM 1. Let  $v \in C_0^\infty(U)$ . Then

$$\lim_{n \rightarrow \infty} [\mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f)] = 0.$$

Suppose Claim 1 has been proven. Then by the weak convergence of  $(\widehat{G}_\alpha^{(n)} f)_{n \in \mathbb{N}}$  in  $H_0^{1,2}(U; dx)$  and (1.5), (1.6) it follows that

$$\begin{aligned} \mathcal{E}_\alpha(v, \widehat{G}f) &= \lim_{n \rightarrow \infty} \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f) = \lim_{n \rightarrow \infty} \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f) = (v, f) \\ &= \mathcal{E}_\alpha(v, \widehat{G}_\alpha f) \end{aligned}$$

for all  $v \in C_0^\infty(U)$ , hence  $\widehat{G}_\alpha f = \widehat{G}f$  and the proof is complete.

To prove Claim 1 note that for  $n \in \mathbb{N}$

$$\begin{aligned} (2.4) \quad \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f) &= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i v \partial_j \widehat{G}_\alpha^{(n)} f \, dx + \sum_{i=1}^d \int (d_i - d_i^{(n)}) v \partial_i \widehat{G}_\alpha^{(n)} f \, dx \\ &\quad + \sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i v \widehat{G}_\alpha^{(n)} f \, dx + \int (c - c^{(n)}) v \widehat{G}_\alpha^{(n)} f \, dx. \end{aligned}$$

By the Cauchy–Schwarz inequality and (2.2) the first summand converges to zero as  $n \rightarrow \infty$  because of Lebesgue’s dominated convergence theorem.

Let us recall that by Sobolev’s Lemma if  $\lambda := (2^{2/3}(d - 1))/((d - 2)d^{1/2})$ , then for all  $u \in C_0^\infty(U)$

$$(2.5) \quad \|u\|_{\frac{2d}{d-2}} \leq \lambda \left( \int |\nabla u|_{\mathbb{R}^d}^2 \, dx \right)^{1/2}$$

(cf. e.g. [2, Theorem 1.7.1]). For  $K := \text{supp } v$  (2.2) and (2.5) imply that  $\{\widehat{G}_\alpha^{(n)} f \mid n \in \mathbb{N}\}$  is uniformly  $d/(d - 2)$ –integrable on  $K$  w.r.t.  $dx$ . Hence by (2.3)

$$(2.6) \quad \widehat{G}_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} \widehat{G}f \text{ in } L^{\frac{d}{d-2}}(K; dx).$$

Since for all  $i \in \{1, \dots, d\}$

$$\sup_n \left\| b_i^{(n)} \right\|_{L^{d/2}(K; dx)} < \infty$$

and

$$\sup_n \left\| c^{(n)} \right\|_{L^{d/2}(K; dx)} < \infty$$

(because  $p_{d,i} \geq d > d/2$  and  $p_c \geq d/2$ ), it follows that both  $b_i^{(n)} \xrightarrow{n \rightarrow \infty} b_i$  and  $c^{(n)} \xrightarrow{n \rightarrow \infty} c$  weakly\* in  $L^{d/2}(K; dx)$ . Hence (2.6) implies that both the third and fourth summand on the right hand side of (2.4) converge to zero. To prove that the

same holds for the second, fix  $i \in \{1, \dots, d\}$  and note that

$$\begin{aligned} & \left| \int (d_i - d_i^{(n)}) v \partial_i \widehat{G}_\alpha^{(n)} f \, dx \right| \\ & \leq \left( \int (d_i - d_i^{(n)})^2 v^2 \, dx \right)^{1/2} \left( \int (\partial_i \widehat{G}_\alpha^{(n)} f)^2 \, dx \right)^{1/2}. \end{aligned}$$

Hence by (2.2) it is sufficient to realize that by the Cauchy–Schwarz inequality (applied to the measure  $|d_i - d_i^{(n)}| v^2 \, dx$ )

$$\int (d_i - d_i^{(n)})^2 v^2 \, dx \leq \left( \int |d_i - d_i^{(n)}| v^2 \, dx \right)^{1/2} \left( \int |d_i - d_i^{(n)}|^3 v^2 \, dx \right)^{1/2},$$

and to recall that by (1.16)  $|d_i - d_i^{(n)}| \xrightarrow[n \rightarrow \infty]{} 0$  weakly\* in  $L^{p_{d,i}}(U; dx)$  and thus, because  $p_{d,i} \geq d \geq 3$ , and  $\text{supp} v$  is compact,

$$\sup_n \int |d_i - d_i^{(n)}|^3 v^2 \, dx < \infty.$$

Now Claim 1 is proved. To show that also  $G_\alpha^{(n)} f \xrightarrow[n \rightarrow \infty]{} G_\alpha f$  weakly in  $H_0^{1,2}(U; dx)$  we note that by (1.11) for all  $n \in \mathbb{N}$

$$\mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f, G_\alpha^{(n)} f) = (f, G_\alpha^{(n)} f) \leq \alpha^{-1} C_\alpha \|f\|_2^2.$$

So, as above

$$\sup_n \|G_\alpha^{(n)} f\|_{1,2} =: C < \infty$$

and

$$G_\alpha^{(n_k)} f \xrightarrow[k \rightarrow \infty]{} Gf \text{ weakly in } H_0^{1,2}(U; dx), \text{ hence weakly in } L^2(U; dx)$$

for some subsequence  $(n_k)_{k \in \mathbb{N}}$  and some  $Gf \in H_0^{1,2}(U; dx)$ . Again we only have to show that  $Gf = G_\alpha f$ . But we know that  $\widehat{G}_\alpha^{(n)} f \xrightarrow[n \rightarrow \infty]{} \widehat{G}_\alpha f$  weakly in  $L^2(U; dx)$ , hence  $G_\alpha^{(n)} f \xrightarrow[n \rightarrow \infty]{} G_\alpha f$  weakly in  $L^2(U; dx)$ , so

$$G_\alpha f = Gf,$$

and the proof of assertion (i) is complete.

(ii) By Remark 1.4 (i) it suffices to prove the first statement.



CLAIM 2. Let  $f_n \in L^2(U; dx)$ ,  $n \in \mathbb{N}$ , such that  $f_n \xrightarrow{n \rightarrow \infty} 0$  weakly in  $L^2(U; dx)$ . Then

$$\widehat{G}_\alpha^{(n)} f_n \xrightarrow{n \rightarrow \infty} 0 \text{ weakly in } H_0^{1,2}(U; dx).$$

Suppose Claim 2 has been proven, then for  $\alpha > \alpha_0$  (where  $\alpha_0$  is as in assertion (i)),  $f \in L^2(U; dx)$  and all  $n \in \mathbb{N}$

$$\left\| G_\alpha^{(n)} f \right\|_2^2 = \int G_\alpha f G_\alpha^{(n)} f dx + \int \widehat{G}_\alpha^{(n)} (G_\alpha^{(n)} f - G_\alpha f) f dx.$$

By part (i) the first summand converges to  $\|G_\alpha f\|_2^2$  while by Claim 2 the second summand converges to zero. Using part (i) again we conclude that

$$G_\alpha^{(n)} f \xrightarrow{n \rightarrow \infty} G_\alpha f \text{ in } L^2(U; dx).$$

To prove the claim, by (1.6) and Remark 1.1 as well as (2.2) it suffices to show that

$$\lim_{n \rightarrow \infty} \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) = 0 \text{ for all } v \in C_0^\infty(U).$$

So, let  $v \in C_0^\infty(U)$ , then by (1.11)

$$\mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) = (v, f_n) + \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f_n).$$

So, it remains to be shown that

$$\lim_{n \rightarrow \infty} (\mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f_n)) = 0.$$

But

$$\begin{aligned} & \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) - \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)} f_n) \\ &= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i v \partial_j \widehat{G}_\alpha^{(n)} f_n dx + \sum_{i=1}^d \int (d_i - d_i^{(n)}) v \partial_i \widehat{G}_\alpha^{(n)} f_n dx \\ & \quad + \sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i v \widehat{G}_\alpha^{(n)} f_n dx + \int (c - c^{(n)}) v \widehat{G}_\alpha^{(n)} f_n dx. \end{aligned}$$

By (2.1)  $(\widehat{G}_\alpha^{(n)} f_n)_{n \in \mathbb{N}}$  is bounded in  $H_0^{1,2}(U; dx)$ , hence by exactly the same arguments as in the proof of Claim 1 (with  $K := \text{supp} v$ ) we obtain that

$$\widehat{G}_\alpha^{(n)} f_n \xrightarrow{n \rightarrow \infty} h \text{ in } L^{\frac{d}{d-2}}(K; dx)$$

for some  $h \in H_0^{1,2}(U; dx)$ . Now also the rest of the proof of Claim 2 is entirely analogous to that of Claim 1.

Thus the proof of assertion (ii) is complete. □

**Proof of Corollary 1.6.** (i) Let  $\alpha_0 \in ]0, \infty[$  be as in Theorem 1.3. Fix  $\alpha > \alpha_0$  and  $f \in L^2(U; dx)$ . Then by Remark 1.1 (cf. the beginning of the proof for Theorem 1.3) it suffices to prove

$$(2.7) \quad \lim_{n \rightarrow \infty} \mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f - G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) = 0,$$

since by duality the same then holds for  $\widehat{G}_\alpha f, \widehat{G}_\alpha^{(n)} f, n \in \mathbb{N}$ . But by applying (1.11) twice we have for all  $n \in \mathbb{N}$

$$\begin{aligned} & \mathcal{E}_\alpha^{(n)}(G_\alpha^{(n)} f - G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) \\ &= \mathcal{E}_\alpha(G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) - \mathcal{E}_\alpha^{(n)}(G_\alpha f, G_\alpha^{(n)} f - G_\alpha f) \\ &= \sum_{i,j=1}^d \int (a_{ij} - a_{ij}^{(n)}) \partial_i G_\alpha f \partial_j (G_\alpha^{(n)} f - G_\alpha f) dx \\ & \quad + \sum_{i=1}^d \int (d_i - d_i^{(n)}) G_\alpha f \partial_i (G_\alpha^{(n)} f - G_\alpha f) dx \\ & \quad + \sum_{i=1}^d \int (b_i - b_i^{(n)}) \partial_i G_\alpha f (G_\alpha^{(n)} f - G_\alpha f) dx \\ & \quad + \int (c - c^{(n)}) G_\alpha f (G_\alpha^{(n)} f - G_\alpha f) dx. \end{aligned}$$

Since by Theorem 1.3,  $G_\alpha^{(n)} f \rightharpoonup_\infty G_\alpha f$  weakly in  $H_0^{1,2}(U; dx)$ , it is clear that the first summand converges to zero as  $n \rightarrow \infty$ . To see that the same is true for the others we only have to realize that after applying Hölder’s inequality we have to deal with integrals of type

$$I_n := \int g_n^2 u_n^2 dx, \quad n \in \mathbb{N},$$

where  $g_n \rightarrow 0$  in  $L^p(U; dx), p \in [d, \infty[, u_n \in H_0^{1,2}(U; dx)$  such that  $\sup_n \|u_n\|_{1,2} < \infty$ . But using Hölder’s inequality and (2.5) we obtain that

$$\begin{aligned} I_n &\leq \left( \int u_n^2 dx \right)^{\frac{p-d}{p}} \left( \int g_n^{2p/d} u_n^2 dx \right)^{d/p} \\ &\leq \|u_n\|_2^{\frac{2(p-d)}{p}} \|g_n\|_p^2 \lambda^{2d/p} \|u_n\|_{1,2}^{2d/p}, \end{aligned}$$

hence  $I_n \xrightarrow{n \rightarrow \infty} 0$  and the proof of assertion (i) is complete.

(ii) E.g. by [1, Theorem 3.4 (iii)], (1.6) and Remark 1.1 it follows that  $(T_t^{(n)})_{n \in \mathbb{N}}$  is a strongly continuous semigroup on  $H_0^{1,2}(U; dx)$  and that  $(G_\alpha^{(n)})_{\alpha > \alpha_0}$  is the associated resolvent. Hence assertion (ii) follows by Remark 1.4 (ii). □

ACKNOWLEDGEMENT. We would like to thank S. Albeverio for fruitful discussions on this paper. Financial support of the German Science Foundation through SFB 343 (Bielefeld) and the EC–Science Project SC1\*CT92-0784 is gratefully acknowledged. The second named author is partially supported by VISTA, a research cooperation between the Norwegian Academy of Science and Statoil.

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