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ON COMPLETE KÄHLER MANIFOLDS WITH FAST CURVATURE DECAY

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0. Introduction

We call (M, o) a Riemannian manifold with a pole iff M is a Riemannian manifold and $\exp_o: T_oM \rightarrow M$ is a global diffeomorphism. We define the radial curvatures at $x \in M$ as the sectional curvatures of all the 2-dimensional planes in T_xM which are tangent to the unique geodesic joining the pole o to x , and write $r(x)$ for the distance function from o . Suppose now our (M, o) satisfies the following conditions:

(0.1) There exist C^∞ functions $k, K: [0, \infty) \rightarrow [0, \infty)$ such that

1. $-K(r(x)) \leq$ all the radial curvatures at $x \leq k(r(x))$,

2. $\int_0^\infty sK(s)ds < \infty$,

3. $\int_0^\infty sk(s)ds \leq 1$.

In this paper, we shall prove the following theorem.

Main Theorem. *Let (M, o) be an n -dimensional complete Kähler manifold with a pole o satisfying condition (0.1) ($n \geq 2$). Moreover assume that there exists a C^∞ function $H: [0, \infty) \rightarrow [0, \infty)$ such that*

(0.2) $\int_0^\infty sH(s)ds < \infty$,

and

(0.3) $-H(r(x)) \leq$ the Ricci curvature at $x \leq H(r(x))$.

Then there exists a positive constant γ_0 depending only on $K(s)$ such that if

$$\int_0^\infty sk(s)ds < \gamma_0,$$

M is biholomorphic to C^n .

It was conjectured by Greene and Wu that if M is an n -dimensional com-

plete, simply connected Kähler manifold satisfying

$$-A(1+r(x))^{-2-\varepsilon} \leq \text{the sectional curvatures at } x \leq 0,$$

then M should be biholomorphic to C^n . This conjecture was verified by Siu and Yau (cf. [10]). In [3], Greene and Wu generalized the above result to the case:

$$-K(r(x)) \leq \text{the sectional curvatures at } x \leq 0,$$

where K is the function of (0.1) and moreover non-increasing on $[a, \infty)$ for some $a > 0$. In view of the above results and several facts of [3], it has been conjectured if a complete Kähler manifold with a pole (M, o) satisfies condition (0.1), then M should be biholomorphic to C^n . In fact, Mok, Siu, and Yau have shown (M, o) is biholomorphic to C^n in the case:

$$-A_\varepsilon(1+r(x))^{-2-\varepsilon} \leq \text{the sectional curvatures at } x \leq A_\varepsilon(1+r(x))^{-2-\varepsilon},$$

where A_ε is a sufficiently small constant depending only on ε (cf. [7]). Recently Kasue has proved Main Theorem under the stronger condition than (0.3):

$$-A(1+r(x))^{-2-\delta} \leq \text{the Ricci curvature at } x \leq A(1+r(x))^{-2-\delta},$$

where A is a constant independent of δ , then M is biholomorphic to C^n .

Combining the argument in [7], [10], and Kasue's unpublished result cited above, the proof of our Main Theorem follows from the following observation:

1. We construct a bounded solution to the equation:

$$\Delta u(x) = \theta(r(x))$$

where θ is a C^∞ function such that there exists a C^∞ function

$$h: [0, \infty) \rightarrow [0, \infty)$$

satisfying $|\theta(x)| \leq h(r(x))$ and $\int_0^\infty sh(s)ds < \infty$.

2. We construct a bounded non-vanishing holomorphic n -form on M more directly than [7].

We would like to express sincere thanks to our adviser Prof. T. Ochiai, to Dr. A. Kasue for his kindness to show us his unpublished result cited above, and to Prof. J. Kazdan who informed us Lemma 2.1 and its corollary which simplify considerably our original proof of Proposition 2.6.

1. Preliminaires

Let (M, o) be an n -dimensional complete Kähler manifold with a pole

satisfying condition (0.1) ($n \geq 2$). We recall several known results which will be used later.

Fact 1.1 (cf. [4], p. 678, Fact 2.1). *Define C^∞ functions $f(t)$, $F(t)$ by*

$$(1.2) \quad f'' + kf = 0, \quad f(0) = 0, \quad f'(0) = 1,$$

$$(1.3) \quad F'' - KF = 0, \quad F(0) = 0, \quad F'(0) = 1.$$

Then there exist constants μ and λ satisfying the following inequalities:

$$(1.4) \quad \mu \leq f'(t) \leq 1 \quad \text{and} \quad \mu t \leq f(t) \leq t,$$

$$(1.5) \quad 1 \leq F'(t) \leq \lambda \quad \text{and} \quad t \leq F(t) \leq \lambda t,$$

$$(1.6) \quad 1 - \int_0^\infty sk(s)ds \leq \mu \leq 1,$$

$$(1.7) \quad 1 \leq \lambda \leq \exp \left\{ \int_0^\infty sk(s)ds \right\}.$$

Using the results of ([4], p. 679, Lemma 2.1) and ([3], Th.C), one can obtain the following inequalities by simple computation.

Fact 1.8. *Let $f(t)$ and $F(t)$ be as in Fact 1.1. Set*

$$s(t) = \exp \left\{ \int_1^t \frac{dr}{f(r)} \right\}.$$

Then

$$(1.9) \quad (\log s)(r(x)) \text{ is plurisubharmonic on } M,$$

$$(1.10) \quad s^2(r(x)) \text{ is a } C^\infty \text{ strictly plurisubharmonic function on } M$$

and

$$2 \left(F' \frac{s^2}{f^2} \right) (r(x)) \Omega \geq L(s^2(r(x))) \geq 2 \left(f' \frac{s^2}{f^2} \right) (r(x)) \Omega,$$

$$(1.11) \quad \text{for arbitrary } p > 0,$$

$$L(\log(1+s^p)(r(x))) \geq \min. \left\{ \frac{p^2 s^p}{2(1+s^p)^2 f^2}, \frac{ps^p f'}{(1+s^p)f^2} \right\} (r(x)) \Omega,$$

$$(1.12) \quad t \leq s(t) \leq t^{1/\mu} \quad (t \geq 1)$$

and

$$t^{1/\mu} \leq s(t) \leq t \quad (0 \leq t \leq 1)$$

where Ω is the Kähler form of the given Kähler metric G and L is the Levi-form. Note that M is, in particular, a Stein manifold.

We call a differential operator on an open set U of R^{2n}

$$A = \sum_{ij} \frac{\partial}{\partial x_i} \left(a^{ij} \frac{\partial}{\partial x_j} \right), \quad a^{ij} \in C^\infty(U)$$

uniformly elliptic iff there is a positive number η (which is called as *uniform ellipticity* of A) such that for any $x \in U$ and for any tangent vector $X \in T_x(\mathbb{R}^{2n})$,

$$\eta^{-1} \sum_i X_i^2 \leq \sum_{ij} a^{ij}(x) X_i X_j \leq \eta \sum_j X_j^2.$$

Fact 1.13 (cf. [3], p. 56, Th.C and p. 80). *The exponential map*

$$\exp_o: T_o(M) \rightarrow M$$

is a quasi isometry and the real operator

$$\sqrt{|g|} \Delta = \sum_{ij} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right)$$

is uniformly elliptic with respect to the coordinates

$\exp_o: T_o(M) \rightarrow M$. Here

$$\begin{aligned} \operatorname{Re} G &= \sum_{ij} g_{ij} dx^i dx^j, \\ (g^{ij}) &\text{ is the inverse of } (g_{ij}), \\ |g| &= \det(g_{ij}). \end{aligned}$$

From now on, we call the global coordinates $\exp_o: T_o(M) \rightarrow M$ as the *natural coordinates*.

Fact 1.14 (cf. [4], p. 678, Th.1). *Let $E \rightarrow M$ be a holomorphic line bundle with a hermitian fibre metric h . Suppose the Chern form $\omega = -(\sqrt{-1}/2\pi) \partial \bar{\partial} \log h$ of the hermitian line bundle $\{E, h\}$ satisfies the condition*

$$(1.15) \quad \|\omega(x)\| \leq v(r(x)) \quad (x \in M)$$

where $v(t)$ is a non-negative C^∞ function on $[0, \infty)$ satisfying

$$(1.16) \quad \int_0^\infty sv(s) ds < \infty.$$

Then there exists a positive number v_0 such that if σ is a non-zero holomorphic section of E over M satisfying

$$(1.17) \quad \|\sigma(x)\| \leq C(1+r(x))^v$$

on M for some constant C and some $0 < v < v_0$, then σ is nowhere zero on M .

Fact 1.18 (Hörmander's $(\bar{\partial}-L^2)$ method) (cf. [2], AI-53). *Let M be a Stein manifold and $E \rightarrow M$ a holomorphic line bundle with a hermitian fibre metric h . Let φ be a plurisubharmonic function on M . Assume that there exists a positive continuous function $c(x)$ on M satisfying*

$$(1.19) \quad Ric(M) + c_1(E, h) \geq c\Omega$$

where $Ric(M)$ is the Ricci form of (M, G) and $c_1(E, h)$ is the Chern form of $\{E, h\}$. If an E -valued $C^\infty(0, p)$ -form σ on M satisfies $(p \geq 1)$

$$(1.20) \quad \bar{\partial}\sigma = 0 \quad \text{and} \quad \int_M \frac{\langle \sigma, \sigma \rangle_h}{c} e^{-\varphi} < \infty,$$

then there exists uniquely an E -valued $C^\infty(0, p-1)$ form ψ on M such that

$$(1.21) \quad \bar{\partial}\psi = \sigma \quad \text{and} \quad \int_M \langle \psi, \psi \rangle_h e^{-\varphi} \leq \int_M \frac{\langle \sigma, \sigma \rangle_h}{c} e^{-\varphi}.$$

The definition of $\langle \cdot, \cdot \rangle_h$ is as follows:

if we write locally σ as $\sigma_j e_j$ on U_j where σ_j is a $(0, p)$ -form and e_j is a local holomorphic section of E , then $\langle \sigma, \sigma \rangle_h(x)$ is defined as

$$\langle \sigma, \sigma \rangle_h(x) = \langle \sigma_j, \sigma_j \rangle(x) h(e_j, \bar{e}_j)(x)$$

where $\langle \cdot, \cdot \rangle$ is the inner-product of $(0, p)$ -form induced by the metric of M .

Fact 1.22 (cf. [6], p. 67 (7.9)). *Let A be a uniformly elliptic operator on $R^{2n}(n \geq 2)$. Then there exists the Green's function G_A of A on R^{2n} satisfying the following inequality:*

$$(1.23) \quad -\frac{c(n, \eta)^{-1}}{|x-y|^{2n-2}} \leq G_A(x, y) \leq -\frac{c(n, \eta)}{|x-y|^{2n-2}}$$

where $c(n, \eta)$ is a positive constant depending only on n and η (uniform ellipticity of A).

Fact 1.24 (Moser's submean value inequality: cf. [8], p. 462, Th.1). *Let A be a uniformly elliptic operator on $B(2R) = \{x \in R^{2n} : |x| < 2R\}$. Assume $v \in W^{1,2}(B(2R))$ satisfies*

$$(1.25) \quad \int_{B(2R)} \sum_{i,j} a^{ij} \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx \leq 0$$

for any $\phi \geq 0$ in $\mathcal{D}(B(2R))$. Then

$$(1.26) \quad \|v\|_{\infty, B(R)} \leq \frac{c(n, \eta)}{R^{2n}} \int_{B(2R)} v^2$$

where $c(n, \eta)$ is a constant as in (1.22).

Fact 1.27 (Moser's Harnack inequality: cf. [9], p. 578, Th.1). *Let V be as in (1.24) and u a positive C^∞ function defined on $B(2R)$ such that $Au = 0$ on $B(2R)$. Then*

$$(1.28) \quad \sup_{B(\bar{R})} u \leq c(n, \eta) \inf_{B(\bar{R})} u$$

where $c(n, \eta)$ is a constant as in (1.22). In particular, from (1.28), one can easily obtain the following ‘‘Liouville theorem’’: Assume A is a uniformly elliptic operator on R^{2n} . If a positive C^∞ -function u on R^{2n} satisfies $Au=0$, then u is a constant.

2. The solution of Poisson’s equation

Lemma 2.1. Let $h(|x|) \in C^\infty(R^{2n})$ depend only on $r=|x|$ for $x \in R^{2n}(n \geq 2)$, and $\Delta_0 = \sum_i \left(\frac{\partial}{\partial x_i}\right)^2$ the usual Laplacian on R^{2n} . If $h(s)$ satisfies

$$\int_0^\infty sh(s)ds < \infty,$$

then there exists a solution to the equation $\Delta_0 v = h$ such that

$$(2.2) \quad v(x) = -\frac{1}{2(n-1)\omega_{2n-1}} \int_{R^{2n}} \frac{h(|y|)}{|x-y|^{2n-2}} dy,$$

$$(2.3) \quad v(x) - v(0) = \frac{1}{2(n-1)} \int_0^{|x|} sh(s) \left[\left(1 - \frac{s}{|x|}\right)^{2n-2} \right] ds,$$

where ω_{2n-1} is the volume of the unit sphere of R^{2n} .

Proof. It is easy to see that the integral in the right hand side of (2.2) is finite. Then it is well known that $\Delta_0 v = h$. And moreover the solution to $\Delta_0 v = h$ is unique up to harmonic functions. But then both v and h depend only on r , so the equation $\Delta_0 v = h$ becomes an O.D.E.

$$h = \Delta_0 v = v_{rr} + \frac{2n-1}{r} v_r = \frac{1}{r^{2n-1}} (r^{2n-1} v_r)_r.$$

This can be integrated explicitly, just by integrating

$$r^{2n-1} v_r(r) = \int_0^r s^{2n-1} h(s) ds,$$

and we obtain

$$v_r(r) = \frac{1}{r^{2n-1}} \int_0^r s^{2n-1} h(s) ds.$$

Therefore

$$\begin{aligned} v(r) &= v(0) + \int_0^r \frac{1}{t^{2n-1}} \left[\int_0^t s^{2n-1} h(s) ds \right] dt \\ &= v(0) + \frac{1}{2(n-1)} \int_0^r h(s) s^{2n-1} \left(\frac{1}{s^{2n-2}} - \frac{1}{r^{2n-2}} \right) ds. \end{aligned}$$

Q.E.D.

From (2.3), the following is obvious.

Corollary 2.4. *If $h \geq 0$, the function $v(x)$ of (2.2) satisfies*

$$(2.5) \quad 0 \leq v(x) - v(o) \leq \frac{1}{2n-2} \int_0^{|x|} sh(s) ds.$$

Proposition 2.6. *Let θ be a C^∞ function on M and assume that there exists a C^∞ function $h: [0, \infty) \rightarrow [0, \infty)$ such that*

$$(2.7) \quad |\theta(x)| \leq h(r(x)) \text{ and } \int_0^\infty sh(s) ds < \infty.$$

Then there exists a bounded C^∞ function u on M satisfying

$$(2.8) \quad \Delta u = \theta.$$

Proof. Using the natural coordinates, set

$$A = \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right) = \sqrt{|g|} \Delta.$$

By (1.13), A is uniformly elliptic on $R^{2n} (=T_oM)$. Let G_A be the Green's function of A on R^{2n} as in (1.22). Then

$$u(x) := \int_{R^{2n}} G_A(x, y) (\sqrt{|g|} \theta)(y) dy$$

is a bounded solution of (2.8). In fact, by (1.13) and the assumption on θ , we obtain

$$|\sqrt{|g|} \theta(y)| \leq C_1 h(|y|),$$

where C_1 is a constant. By (1.23),

$$\left| \int_{R^{2n}} G_A(x, y) (\sqrt{|g|} \theta)(y) dy \right| \leq C_2 \int_{R^{2n}} \frac{h(|y|)}{|x-y|^{2n-2}} dy,$$

but from (2.1) and (2.4), the right hand side is bounded by a constant independent of x . Q.E.D.

Using a local holomorphic coordinates (z_1, \dots, z_n) , the Ricci tensor R is locally expressed as

$$R = - \sum_{\alpha\beta} \frac{\partial^2 \log |G|}{\partial z_\alpha \partial \bar{z}_\beta} dz^\alpha d\bar{z}^\beta$$

where $G = \sum_{\alpha\beta} G_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ and $|G| = \det(G_{\alpha\bar{\beta}})$. Set

$$(2.9) \quad \phi = - \sum_{\alpha\beta} G^{\alpha\bar{\beta}} \frac{\partial^2 \log |G|}{\partial z_\alpha \partial \bar{z}_\beta}$$

where $(G^{\alpha\bar{\beta}})$ is the inverse of $(G_{\alpha\bar{\beta}})$. Note that 2ϕ is the scalar curvature of $\text{Re } G$. The following is obvious from (0.3) and (2.6).

Corollary 2.10. *There exists a bounded C^∞ function u on M satisfying*

$$(2.11) \quad \Delta u = \phi .$$

So u is unique up to constants.

3. Construction of non-vanishing bounded holomorphic n -form

The following is obvious.

Lemma 3.1. *Let $E \rightarrow M$ be a hermitian vector bundle with a fibre metric h . Assume that σ is a non-zero holomorphic section of E over M such that*

$$(3.2) \quad \Delta |\sigma|_h \geq 0 \quad \text{on } B_G(R) - V$$

where $B_G(R) = \{x \in M : r(x) < R\}$, and $V = \{x \in M : |\sigma|_h(x) = 0\}$. Then

$$(3.3) \quad |\sigma|_h \in W^{1,2}(B_G(R)) ,$$

and

$$(3.4) \quad \sum_{i,j} \int_{B_G(R)} \sqrt{|g|} g^{ij} \frac{\partial |\sigma|_h}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leq 0$$

for any $\phi \geq 0$, $\phi \in \mathcal{D}(B_G(R))$ with respect to the natural coordinates.

Let v be a real valued C^∞ function on M . Then we define a fibre metric $|\cdot|_v$ on the canonical line bundle K_M of M by

$$(3.5) \quad |\cdot|_v^2 = |\cdot|_{K_M}^2 e^{-v}$$

where $|\cdot|_{K_M}$ is the fibre metric on K_M induced by G .

The next lemma immediately follows from Poincaré-Lelong's formula.

Lemma 3.6. *Let u be the function of (2.11). Then for any holomorphic n -form ξ ,*

$$(3.7) \quad \Delta \log |\xi|_{2u} = 0 \quad \text{on } M - V$$

where $V = \{x \in M : \xi(x) = 0\}$. In particular

$$(3.8) \quad \Delta |\xi|_{2u} \geq 0 \quad \text{on } M - V .$$

Proposition 3.9. *For any positive number ν , there exists a non-zero holomorphic n -form ξ on M such that*

$$(3.10) \quad |\xi|_{K_M}(x) \leq C(1+r(x))^{n(1/\mu-1)+\nu/\mu}$$

where C is a constant and μ is as in (1.1).

Proof. Let $\{U, (z_1, \dots, z_n)\}$ be a local holomorphic coordinates around o and assume $B_c(\varepsilon) \subset U$. We choose $\rho \in \mathcal{D}(R)$ satisfying

$$\begin{aligned} \rho(r) &= 1 & \text{if } |r| \leq 2^{-1}\varepsilon, \\ \rho(r) &= 0 & \text{if } |r| \geq \varepsilon, \end{aligned}$$

and

$$0 \leq \rho \leq 1.$$

Set

$$\alpha = \nu \log(1+s^2), \quad \varphi = 2n \log s$$

where s is the function in (1.8), and we choose a fibre metric $|\cdot|_\alpha$ on K_M . By direct computation using (1.1), we obtain

$$Ric(M) + c_1(K_M, |\cdot|_\alpha) \geq c\Omega,$$

where c is a positive continuous function on M .

Set

$$g(x) = \rho(r(x)) dz_1 \wedge \dots \wedge dz_n,$$

then, by (1.18), we obtain a $C^\infty(n, 0)$ -form on M with

$$(3.11) \quad \bar{\partial}\eta = \bar{\partial}g$$

and

$$(3.12) \quad \int_M |\eta|_\alpha^2 e^{-\varphi} \leq \int_M |\bar{\partial}g|_\alpha^2 \frac{e^{-\varphi}}{c} < \infty.$$

Because of the singularity of φ at o , we must have $\eta(o) = 0$. So

$$(3.13) \quad \xi = g - \eta$$

is a non-zero holomorphic n -form and satisfies

$$\xi(o) = dz_1 \wedge \dots \wedge dz_n \quad \text{and} \quad \bar{\partial}\xi = 0.$$

Since g has a compact support,

$$\int_M \frac{|\xi|_{K_M}^2}{(1+s^2)^\nu (1+s)^{2n}} < \infty.$$

In the following, C_i denotes a constant independent of ξ . Let u be the function (2.11). Then

$$\int_M \frac{|\xi|_{2u}^2}{(1+s^2)^\nu (1+s)^{2n}} \leq C_1 \int_M \frac{|\xi|_{K_M}^2}{(1+s^2)^\nu (1+s)^{2n}}.$$

From (3.1), (3.8), and (1.24),

$$\begin{aligned} \sup_{B_G(\tilde{R})} |\xi|_{2u}^2 &\leq C_2 R^{-2n} \int_{B_G(2R)} |\xi|_{2u}^2 \\ &\leq C_3 R^{-2n} (1+s(2R))^{2\nu} (1+s(2R))^{2n} \int_M \frac{|\xi|_{2u}^2}{(1+s^2)^\nu (1+s)^{2n}}. \end{aligned}$$

Here by (1.12), we obtain

$$\sup_{B_G(\tilde{R})} |\xi|_{2u}^2 \leq C_4 (1+R)^{2n(1/\mu-1)+2\nu/\mu}.$$

Because u is bounded

$$|\xi|_{K_M}(x) \leq C_5 (1+r(x))^{n(1/\mu-1)+\nu/\mu}. \tag{Q.E.D.}$$

The following lemma is proved in Moser’s paper (cf. [9]).

Lemma 3.14. *There exists a positive number δ_0 such that if u is a harmonic function on M satisfying*

$$(3.15) \quad u(x) \leq C_1(r(x)+C_2)^\delta \quad \text{for some } 0 < \delta < \delta_0$$

where C_i are constants, then

$$(3.16) \quad u = u(o) \text{ identically.}$$

Proposition 3.17. *There exists a positive number γ_1 such that if*

$$\int_0^\infty sk(s)ds < \gamma_1,$$

then there exists a holomorphic n -form ξ on M satisfying

$$0 < C^{-1} \leq |\xi|_{K_M} \leq C$$

where C is a constant.

Proof. Choose any $\gamma_1 < 1$ so that

$$n\left(\frac{1}{1-\gamma_1} - 1\right) < \nu_0$$

where ν_0 is the number of (1.14), and take $\nu > 0$ so that

$$n\left(\frac{1}{1-\gamma_1} - 1\right) + \frac{\nu}{1-\gamma_1} < \gamma_0.$$

Let ξ be the holomorphic n -form of (3.10). If $\int_0^\infty sk(s)ds < \gamma_1$, then

$$n\left(\frac{1}{\mu} - 1\right) + \frac{\nu}{\mu} < \nu_0$$

therefore ξ is nowhere zero on M by (1.14). Hence $\log|\xi|_{K_X}$ is well-defined everywhere M and

$$\Delta(\log|\xi|_{K_X}^2 - 2u) = 0 \quad \text{on } M.$$

Using the natural coordinates, set $A = \sqrt{|g|}\Delta$ and

$$v = \log|\xi|_{K_X}^2 - 2u.$$

Then A is uniformly elliptic and $Av = 0$. By (3.10)

$$v(x) \leq C\{1 + \log(1 + r(x))\}$$

where C is a constant. From (3.14), v is a constant. Since u is uniformly bounded on M , we get the conclusion. Q.E.D.

4. Construction of a biholomorphic map

Let $T^*(M)$ be the holomorphic cotangent bundle of M and w a real C^∞ function on M . We define the fibre metric $\langle \cdot, \cdot \rangle_w$ on $T^*(M)$ by

$$(4.1) \quad \langle \cdot, \cdot \rangle_w = \langle \cdot, \cdot \rangle e^{-w}$$

where $\langle \cdot, \cdot \rangle$ is the fibre metric on $T^*(M)$ induced by G .

Lemma 4.2. *There exists a bounded real C^∞ function ρ on M such that for any holomorphic 1-form σ on M ,*

$$(4.3) \quad \log\|\sigma\|_\rho \text{ is subharmonic on } M - V$$

where $\|\sigma\|_\rho^2 = \langle \sigma, \sigma \rangle_\rho$ and $V = \{x \in M : \sigma(x) = 0\}$.

Proof. From (2.6), there exists a bounded real C^∞ function ρ on M such that

$$\text{the Ricci curvature at } x \geq \Delta\rho(x).$$

Using Bochner's identity

$$\frac{1}{2}\Delta\|\sigma\|_\rho^2 = \|\nabla^p\sigma\|_\rho^2 + Ric(\sigma^\sharp, \sigma^\sharp)e^{-\rho} + \frac{1}{2}\Delta\rho\|\sigma\|_\rho^2$$

where σ^\sharp is the dual of σ and ∇^p is the covariant derivative with respect to the metric $\langle \cdot, \cdot \rangle_\rho$, we get the conclusion. Q.E.D.

We refer the proof of the following lemma to ([3], p. 43, Th.B).

Lemma 4.4. *Assume that v is a non-negative C^∞ function on M satisfying $\Delta v \geq 0$. Then*

$$(4.5) \quad \int_{B_G(o;R)} \Delta v \leq C(n, \lambda, \mu, \alpha) R^{-2} \int_{B_G(R)} v$$

where $0 < \alpha < 1$ and $C(n, \lambda, \mu, \alpha)$ is a constant depending only on n, λ, μ, α .

Proposition 4.6. *For any positive number ν , there exist holomorphic functions f_1, \dots, f_n on M such that*

$$(4.7) \quad f_i(o) = 0 \text{ and } df_i(o) = dz_i,$$

$$(4.8) \quad |f_i(x)| \leq C_1(1+r(x))^{\nu+n+1/\mu-n}$$

$$(4.9) \quad \|df_i(x)\| \leq C_2(1+r(x))^{\nu\mu+(n+1)(1/\mu-1)}$$

where C_i are constants and (z_1, \dots, z_n) are the holomorphic coordinates at o of (3.9).

Proof. Take positive numbers a, b so that $B_G(a) \subset B_G(b) \subset U$. Let χ be a C^∞ function on $[0, \infty)$ such that

$$(4.10) \quad \chi(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & t \geq b \end{cases}$$

$$\text{and } 0 \leq \chi \leq 1.$$

Set $v_i(x) = \chi(r(x))z_i$ on U , and $v_i(x) = 0$ on $M - U$.

Let ξ be the holomorphic n -form in (3.17), and set

$$(4.11) \quad \beta = \log |\xi|_{K_M}^2 \text{ and } \psi = (2n+2) \log s.$$

We define a fibre metric on the trivial line bundle $E = M \times \mathbb{C}$ by

$$(4.12) \quad h_\nu = e^{\beta - \nu \log(1+s^2)}$$

From (1.11),

$$Ric(M) + c_1(E, h_\nu) \geq c\Omega,$$

where c is a positive continuous function on M . From (1.18), there exists a C^∞ function u_i on M such that

$$(4.13) \quad \bar{\partial}u_i = \bar{\partial}v_i$$

and

$$(4.14) \quad \int_M \frac{|u_i|^2 |\xi|_{K_M}^2}{(1+s^2)^\nu s^{2n+2}} \leq \int_M \frac{\langle \bar{\partial}v_i, \bar{\partial}v_i \rangle_{h_\nu}}{c} e^{-\psi} < \infty.$$

Because of the singularity of $e^{-\psi}$ at o ,

$$u_i(o) = 0, \quad du_i(o) = 0.$$

Therefore if we set

$$f_i = v_i - u_i,$$

then

$$(4.15) \quad \bar{\partial} f_i = 0, f_i(o) = 0, \text{ and } df_i(o) = dz_i.$$

Since v_i has a compact support, (4.14) implies

$$(4.16) \quad \int_M \frac{|f_i|^2}{(1+s^2)^\nu(1+s)^{2n+2}} < \infty.$$

Using the natural coordinates, we set $A = \sqrt{|g|} \Delta$. Then from (1.13), A is uniformly elliptic.

By (3.1) and (1.24),

$$(4.17) \quad \begin{aligned} \sup_{B_{G(2^{-1}r(x))}} |f_i|^2 &\leq C_1 r(x)^{-2n} \int_{B_{G(r(x))}} |f_i|^2 \\ &\leq C_1 r(x)^{-2n} (1+s(r(x))^2)^\nu (1+s(r(x)))^{2n+2} \int_M \frac{|f_i|^2}{(1+s^2)^\nu(1+s)^{2n+2}} \\ &\quad \text{(by (4.16) and by (1.12))} \\ &\leq C_2 (1+r(x))^{2(\nu+n+1/\mu-n)} \end{aligned}$$

where C_i are constants. Let ρ be the function of (4.2). Then by (3.1) and (1.24), we obtain

$$(4.18) \quad \sup_{B_{G(2^{-1}r(x))}} \|df_i\|_\rho^2 \leq C_3 r(x)^{-2n} \int_{B_{G(r(x))}} \|df_i\|_\rho^2$$

(since ρ is bounded)

$$\leq C_4 r(x)^{-2n} \int_{B_{G(r(x))}} \|df_i\|^2 = C_4 r(x)^{-2n} \int_{B_{G(r(x))}} \square |f_i|^2$$

(observe $\Delta = 2\square$)

$$\begin{aligned} &= 2^{-1} C_4 r(x)^{-2n} \int_{B_{G(r(x))}} \Delta |f_i|^2 \\ &\leq C_5 r(x)^{-(2n+2)} \int_{B_{G(2(r(x))}} |f_i|^2 \quad \text{(from (4.4))} \\ &\leq C_6 r(x)^{-(2n+2)} \int_0^{2r(x)} (1+t)^{2(\nu+n+1)/\mu-1} dt \quad \text{(from (4.17))} \\ &\leq C_7 (1+r(x))^{2(\nu+n+1/\mu-n-1)} \end{aligned}$$

where C_i are constants. Recall that ρ is bounded, then we obtain

$$\|df_i\|(x) \leq C_8 (1+r(x))^{\nu/\mu+(n+1)(1/\mu-1)}.$$

Q.E.D.

The proofs of the following lemma and proposition are the same as those of

([7], p. 214, Lemma 3 and p. 215, Proposition).

Lemma 4.19. *There exists a positive number $\gamma_2 \leq \gamma_1$ where γ_1 is the constant of (3.17) such that if*

$$\int_0^\infty sk(s)ds < \gamma_2,$$

there exists holomorphic functions f_1, \dots, f_n on M satisfying the following conditions:

$$(4.20) \quad df_1 \wedge \dots \wedge df_n = \xi$$

where ξ is the holomorphic n -form of (3.9), and if we define holomorphic vector fields $\{X_i\}_{1 \leq i \leq n}$ on M as

$$(4.21) \quad df_i(X_j) = \delta_{ij}$$

then

$$(4.22) \quad |X_i|(x) \leq C_2 r(x)^{(n-1)(\nu/\mu + (n+1)(1/\mu - 1))} \quad 1 \leq i \leq n,$$

where C_i are constants.

We define a holomorphic map $F: M \rightarrow C^n$ by

$$F = (f_1, \dots, f_n): M \rightarrow C^n.$$

Proposition 4.23. *There exists a positive number $\gamma_0 \leq \gamma_2$ where γ_2 is the number in (4.19) such that if*

$$\int_0^\infty sk(s)ds < \gamma_0,$$

then the holomorphic map F defined above is a proper map.

Now we give the proof of our Main Theorem.

Proof of the Main Theorem. If $\int_0^\infty sk(s)ds < \gamma_0$, then from (4.20) and (4.23), F is a covering map. Since C^n is simply connected, F is biholomorphic. Q.E.D.

REMARK. Moreover if the sectional curvature of M does not change the sign, we can conclude that M is flat by using Mok-Siu-Yau's argument in ([10], p. 211).

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