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ON THE BOUNDARY BEHAVIOR OF HARMONIC MORPHISMS AT THE BOUNDARY OF COMPACTIFICATIONS OF MARTIN TYPE

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Introduction

The boundary behavior of analytic functions on the unit disc has been an interesting material of the complex analysis. Among various investigations on this problem some results such as the theorem of Fatou-Plessner and the theorem of Riesz-Frostman-Nevanlinna were generalized in a far extensive context, that is, for analytic mappings of Riemann surfaces and their relevant compactifications. These were found in the book of Constantinescu-Cornea [1].

Later [2], they considered harmonic morphisms between Brelot's harmonic spaces under Wiener's compactification and revealed that above mentioned theorems are based essentially on the potential theoretic character. By using relations between the Wiener and the Martin compactifications of Brelot's harmonic spaces with additional assumptions, the author obtained informations on the boundary behavior of harmonic morphisms at the Martin boundary [5], [6].

Under the framework of Constantinescu-Cornea [3], K. Oja [10] generalized theorems of Fatou-Plessner and of Riesz. Recently [7], in order to discuss the Naïm theory [9] in the context of harmonic spaces of Constantinescu-Cornea the author defined the compactifications of Martin type. Here we give a supplementary remark on mappings of type *Bl* and translate the informations on the Wiener boundary into those on the Martin type compactification following the spirit of [6]. It is still possible to discuss the problem without any compactification as in [11], [12].

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1. *Preliminaries.*

Let X be a $\mathcal{P}$-harmonic space in the sense of Constantinescu-Cornea [3]. We assume that X has a countable base and 1 is superharmonic on X. Further, we assume that X has a compactification of Martin type X^* [7], i.e., (1) X^* is metrizable and resolutive, (2) there is a finite continuous function $K(x, z)$ defined on $X\times\Delta(\Delta=X^*\backslash X)$ such that $k_z: x\rightarrow k(x, z)$ is non-negative and harmonic on X for every $z\in\Delta$, (3) there is a non-negative Borel measure μ on Δ and a set Δ_1 corresponding to a certain class of minimal harmonic functions k_z such that $\mu(\Delta\backslash\Delta_1)=0$ and $\mu(T)=0$ if T is negligible, (4) for every $u\in\mathrm{HB}(X)$ there is a resolutive function f such that $u(x)=\mathrm{H}_f(x)=\int k(x, z)f(z)d\mu(z)$ for every $x\in X$. Then, X has the Wiener compactification X^W which is resolutive [4] and X^* is a quotient of X^W, i.e., there is a continuous surjection $\pi: X^W\rightarrow X^*$ satisfying $\pi(x)=x$ for every $x\in X$.

Let X' be a second harmonic space such that X' has a countable base and 1 is superharmonic on X'. We suppose further that X' is *connected* and *elliptic* if $X'\notin\mathcal{P}$, i.e., if X' is not $\mathcal{P}$-harmonic. For a compactification X'^* of X' we make the following assumption:

Case (i) $X'^*=X'$ if X' is compact,

Case (ii) X'^* is an arbitrary metrizable compactification if X' is non-compact and $X'\notin\mathcal{P}$,

Case (iii) X'^* is an arbitrary metrizable and resolutive compactification if $X'\in\mathcal{P}$.

Remarks. (1) The case (iii) occurs since 1 is harmonizable [4], and X' is non-compact if $X'\in\mathcal{P}$.

(2) If $X'\notin\mathcal{P}$, then X' is completely degenerated, i.e., 0 is the only one potential on X' and all non-negative superharmonic functions are harmonic and they are proportional ([3], Excercise 3.1.10).

(3) If $F'\subset X'$ is closed and non-polar then $X'\backslash F'\in\mathcal{P}$ ([3], Exicercise 6.2.5).

Proposition 1.

Let $f'\in\mathcal{C}(X'^*)$, F' be a compact non-polar subset of X' with non-empty interior and $G'=X'\backslash F'$. Then the restriction of f' to G' is harmonizable on G'.

Proof. *Case (i)*: G' is relatively compact and is resolutive since, in virtue of the above Remark (3), G' is an open subset of a $\mathcal{P}$-harmonic space ([3], Th. 2.4.2). Hence by the same argument as in [8], 2.1.4, $\mathrm{h}_{f'}^{G',G'}=\mathrm{H}_{f'}^{G',X'}$. In the following we refer to [8] without mentioning any possible modification.

Case (ii): We show first that $\mathrm{h}_1^{G',X'}=0$. This function $\mathrm{h}_1^{G',X'}$ is associated with 0 on $\partial G'$ ([8], 2.1.5). Let p_0' be an Evans function on G' of some potential $p'>0$ on G'. Then, $\liminf\,[p_0'-\mathrm{h}_1^{G',X'}]\geq 0$ on $\partial G'$ ([8] 1.2.14). The function

$$v'=\begin{cases}\inf\,(p_0'-\mathrm{h}_1^{G',X'},\ 0) & \text{on } G'\\ 0 & \text{on } F'\end{cases}$$

is superharmonic on X'. Since, by Remark (2), the non-negative superharmonic function $1+v'$ on X' is harmonic and is proportional to 1, we have $v'=0$, which implies $\mathrm{h}_1^{G',X'}=0$, since $p_0'-\mathrm{h}_1^{G',X'}\geq 0$. Considering f'^+ and f'^- separately, we obtain $\mathrm{h}_{f'}^{G',X'}=0$. Thus, $\bar{\mathrm{h}}_{f'}^{G',G'}=\bar{\mathrm{h}}_{f'}^{G',X'}+\mathrm{H}_{f'}^{G',X'}=\mathrm{H}_{f'}^{G',X'}$ ([8], 2.1.4) and $\underline{\mathrm{h}}_{f'}^{G',G'}=-\bar{\mathrm{h}}_{(-f')}^{G',G'}=-\mathrm{H}_{(-f')}^{G',X'}=\mathrm{H}_{f'}^{G',X'}$.

Case (*iii*): Since f' is harmonizable on X' ([8], 3.2.9) it is harmonizable on G' ([8], 2.2.3), q.e.d..

Let φ: $X\rightarrow X'$ be a *harmonic morphism* [10], i.e., φ: $X\rightarrow X'$ is continuous and for every open subset U' of X' with $\varphi^{-1}(U')\neq\emptyset$ and for every hyperharmonic function u' on U', $u'\circ\varphi$ is hyperharmonic on $\varphi^{-1}(U')$.

φ is called *locally polarly non-constant* [10] if there is no open subset U of X such that $\varphi(U)$ is a singleton and polar.

φ is *of type Bl at* $x'\in X'$ if there is an open neighborhood U' of x' such that $\varphi^{-1}(U')=\emptyset$ or $\mathrm{h}_1^{\varphi^{-1}(U'),X}=0$. A harmonic morphism which is of type *Bl* at each point of X' is called *of type Bl.*

2. The boundary behavior of harmonic morphisms at Wiener boundary

We consider the Wiener compactification X^W of X. For the definition and properties of X^W, one may refer to [4]. The following results are obtained, except for the theorem on mappings of type *Bl*, by K. Oja and we quote them with brief proof.

Let φ: $X\rightarrow X'$ be a harmonic morphism. We define, for $\tilde{x}\in\Delta^W=X^W\backslash X$, the cluster set

$$\varphi^*(\tilde{x})=\cap\{\overline{\varphi(U^*\cap X)};\ U^* \textit{ is a neighborhood of } \tilde{x} \textit{ in } X^W\},$$

where the closure is taken in the compact space X'^*.

Theorem of Fatou-Plessner.
$\varphi^*(\tilde{x})$ is either X'^* or a singleton, and only the latter case occurs if $X'\in\mathcal{P}$.

For the case (i) and the case (ii), in view of Prop. 1, we may use a result of K. Oja, [10], Lemma 3.1, and in the case (iii) $f'\circ\varphi$ is a Wiener function on X for every Wiener function f' on X' ([10], Lemma 4.7), and every Wiener function on X can be extended continuously to X^W.

Putting

$$\tilde{P}=\{\tilde{x}\in\Delta^W;\ \varphi^*(\tilde{x})=X'^*\},$$
$$\tilde{F}=\{\tilde{x}\in\Delta^W;\ \varphi^*(x) \text{ is a singleton}\}$$

we can rewrite above theorem as

$$\Delta^W=\tilde{P}\cup\tilde{F} \text{ and } \Delta^W=\tilde{F} \text{ provided } X'\in\mathcal{P}.$$

Further, we have $\Delta^W \backslash \Gamma^W \subset \tilde{F}$, where Γ^W is the harmonic boundary of X^W ([10], Cor. 6.14).

In the same way as [2], Th. 6.3, it is shown that $\tilde{P}$ is an open and closed subset of Γ^W.

We call a set $A' \subset X'^*$ is *polar in* X'^* if $A' \cap X'$ is polar and if for any open subset $G' \subset X'$ with $G' \in \mathcal{P}$ and for every $a'_0 \in G'$ there is a non-negative hyperharmonic function u' on G' such that $u'(a'_0) < +\infty$ and

$$\lim_{\substack{a' \to x' \\ a' \in G'}} u'(a') = +\infty \quad \text{for all} \quad x' \in \overline{G'} \cap A' \cap \Delta' .$$

Theorem of Riesz.

Let φ be locally polarly non-constant, and $\tilde{A}$ be a subset of Δ^W. Then, if $A' = \bigcup_{\tilde{x} \in \tilde{A}} \varphi^*(\tilde{x})$ is polar in X'^* then $\bar{\mathrm{H}}^W_{1_{\tilde{A}}} = 0$.

Proof. Let K'_1 and K'_2 be disjoint compact subsets of X' with non-empty interiors and $G'_i = X' \backslash K'_i$ $(i=1, 2)$. Then, there is a non-negative hyperharmonic functions u'_i on G'_i such that u'_i is finite on a dense subset of G'_i and $\lim_{\substack{a' \to x' \\ a' \in G'}} u'_i(a') = +\infty$ for every $x' \in \bar{G}'_i \cap A'$ ([10], Lemma 6.9, Th. 6.11 and [3], Excercise 3.1.11). Putting $u_i = u'_i \circ \varphi$, $G_i = \varphi^{-1}(G'_i)$ and $\tilde{A}_i = \{\tilde{x} \in \Delta^W \backslash \overline{(X \backslash G_i)};\ \lim_{\substack{a \to \tilde{x} \\ a \in G_i}} u_i(a) = +\infty\}$ we have $\tilde{A} \subset \tilde{A}_1 \cup \tilde{A}_2$ and $\bar{\mathrm{H}}^W_{1_{\tilde{A}_i}} = 0$. The last assertion is derived if we prove the following:

(1) $\hat{\mathrm{R}}^{X \backslash G_i}_u = u$, *where* $u = \bar{\mathrm{H}}^W_{1_{\tilde{A}_i}}$,

and

(2) $u \le \hat{\mathrm{R}}^{X \backslash G_i}_{\mathrm{H}^W_f} = q$ *and q is a potential on X, where* $f = 1_{\Delta^W \backslash \overline{(X \backslash G_i)}}$.

Let v_1 be a non-negative hyperharmonic function on X and $v_1 \ge 1$ on $X \backslash G_i$. The function

$$v_2 = \begin{cases} 1 & \text{on } X \backslash G_i \\ \inf(1, v_1 + \varepsilon u_i) & \text{on } G_i \end{cases}$$

satisfies $v_2 \ge u$ for every $\varepsilon > 0$. In particular, $v_1 + \varepsilon u_i \ge u$ on G_i. Since $P = \{x \in G_i : u_i(x) = \infty\} = \varphi^{-1}(\{x' \in G'_i : u'_i(x') = \infty\})$ is polar, $v_1 \ge u$ on G_i. Hence $v_1 \ge u$, for $v_1 \ge 1 \ge u$ on $X \backslash G_i$. Now, let v be a non-negative hyperharmonic function on X satisfying $v \ge u$ on $X \backslash G_i$. Then $v_1 + v - u$ is non-negative and hyperhamonic on X and $v_1 + v - u \ge 1$ on $X \backslash G_i$, which means $v_1 + v - u \ge \mathrm{R}^{X \backslash G_i}_1$ and therefore $\mathrm{R}^{X \backslash G_i}_1 + \mathrm{R}^{X \backslash G_i}_u - u \ge \mathrm{R}^{X \backslash G_i}_1$. Thus we have $\mathrm{R}^{X \backslash G_i}_u \ge u$, which completes the proof of (1).

To prove (2), let $\mathfrak{v}$ be a non-negative hyperharmonic function on X satisfying $\underline{\lim}\, \mathfrak{v} \ge \alpha > 1$ at every point of $\Delta^W \cap \overline{(X \backslash G_i)}$. Then, there exists an open subset V^* of X^W such that $V^* \supset \Delta^W \cap \overline{(X \backslash G_i)}$ and $\mathfrak{v} \ge 1$ on $V = V^* \cap X$. Letting

$K=(X\setminus G_i)\setminus V$, we have $\hat{\mathrm{R}}_1^K+v\geq\hat{\mathrm{R}}_1^{X/G_i}\geq\hat{\mathrm{R}}_{\mathrm{H}_f^W}^{X\setminus G_i}=q\geq\mathrm{h}_q^{X,X}$, hence $\hat{\mathrm{R}}_1^K+\alpha\mathrm{H}_g^W\geq\mathrm{h}_q^{X,X}$, where $g=1_{\Delta^W\cap\overline{(X\setminus G_i)}}$. Since K is compact and α (>1) is arbitrary, we have $\mathrm{h}_q^{X,X}\leq\mathrm{H}_q^W$, therefore $\mathrm{h}_q^{X,X}\leq\mathrm{H}_{\min(f,g)}^W=0$, i.e., q is a potential.

The following results on mappings of type *Bl* are obtained quite in the same way as in [2].

Proposition 2.

$$\{\varphi^*(\tilde{x});\ \tilde{x}\in\Gamma^W\setminus\tilde{P}\}\cap X' = \{x'\in X';\ \varphi \textit{ is not of type Bl at } x'\}.$$

Proof. Let $\tilde{x}\in\Gamma^W\setminus\tilde{P}$ and $x'=\varphi^*(\tilde{x})\in X'$. For any open neighborhood U' of x', since $U=\varphi^{-1}(U')\neq\emptyset$, we have to show that $\bar{\mathrm{h}}_1^{U,X}\neq0$. Suppose for a moment that $\bar{\mathrm{h}}_1^{U,X}=0$. If $(\bar{U}\setminus U)\cap X\neq\emptyset$, $1=\bar{\mathrm{h}}_1^{U,X}+\mathrm{H}_1^{U,X}$ implies $1=\mathrm{H}_1^{U,X}=\mathrm{R}_1^{X\setminus U}$ on U, and hence $\mathrm{R}_1^{X\setminus U}=1$ in X. Select a neighborhood V^* of $\tilde{x}$ such that $\bar{V}^*\cap\overline{X\setminus U}=\emptyset$. This is possible since $\tilde{x}\notin\overline{X\setminus U}$. Then $p=\inf\{\hat{\mathrm{R}}_1^{V^*\cap X}, \hat{\mathrm{R}}_1^{X\setminus U}\}$ is a potential ([8], 3.2.23) and $0=\liminf_{\tilde{x}} p=\liminf_{\tilde{x}}\hat{\mathrm{R}}_1^{V^*\cap X}=1$ which is a contradiction. If $(\bar{U}\setminus U)\cap X=\emptyset$, we have $1=\bar{\mathrm{h}}_1^{U,X}$ and this contradicts our assumption.

Next, we prove that φ is of type *Bl* at each point $x'\notin\varphi^*(\Gamma^W\setminus\tilde{P})\cap X'$. Since $\varphi^*(\Gamma^W\setminus\tilde{P})\cap X'$ is a closed subset of X', we can find an open neighborhood U' of x' such that $\bar{U}'\cap\varphi^*(\Gamma^W\setminus\tilde{P})=\emptyset$. We may suppose that U' is relatively compact and $U'\neq X'$. Then we have $\Gamma^W\subset\overline{X\setminus\varphi^{-1}(\bar{U}')}$ and hence $\mathrm{R}_1^{X\setminus\varphi^{-1}(U')}=1$, which shows as above that $\bar{\mathrm{h}}_1^{\varphi^{-1}(U'),X}=0$.

Theorem (*Characterization of mappings of type Bl*)

(1) *When $X'\notin P$, φ is of type Bl if and only if $\Gamma^W=\tilde{P}$.*

(2) *When $X'\in P$, φ is of type Bl if and only if $\varphi^*(\Gamma^W)\cap X'=\emptyset$.*

Proof. (1): If φ is of type *Bl*, by Prop. 2, $\varphi^*(\Gamma^W\setminus\tilde{P})\cap X'=\emptyset$, which implies $\Gamma^W=\tilde{P}$ whenever X' is compact. When X' is non-compact, we can deduce that $\Gamma^W\setminus\tilde{P}$ is of harmonic measure zero and hence $\Gamma^W\setminus\tilde{P}=\emptyset$ since $\tilde{P}$ is open and clised in Γ^W. The converse is trivial.

(2): In this case, $\tilde{P}=\emptyset$ and φ is extended continuously on X^W. Hence the assertion is an immediate consequence of Prop. 2.

REMARK. If $X'\in\mathcal{P}$ and φ is of type *Bl* then we can show $\varphi^*(\Gamma^W)=\Gamma'$ ([2] Cor. 6.2, [8] 3.1.7).

3. The fine boundary behavior of harmonic morphisms

Let φ be a harmonic morphism between X and X' and X^* be a compactification of Martin type. In [7], we have shown that each point $z\in\Delta_1$ has the fine filter $\mathcal{G}_z=\{E\subset X;\ \mathrm{R}_{k_z}^{X\setminus E}\neq k_z\}$ and the set

$$\Delta_2 = \{z\in\Delta_1;\ \mathcal{G}_z \text{ converges to } z\}$$

contains μ-almost all boundary points, i.e., $\mu(\Delta\backslash\Delta_2)=0$. For $z\in\Delta_2$ we define

$$\hat{\phi}(z) = \cap \{\overline{\varphi(E)};\ E \in \mathcal{G}_z\}\ ,$$

where the closure is taken in X'^*. Let

$$\begin{aligned}\hat{P} &= \{z\in\Delta_2;\ \hat{\phi}(z)=X'^*\}\,,\\ \hat{F} &= \{z\in\Delta_2;\ \hat{\phi}(z) \text{ is a singleton}\}\,.\end{aligned}$$

Then as in [5], [6] we can prove the following key theorem which reveals the relation between $\hat{\phi}$ and φ^*:

Key Theorem.
Let

$$N = (\Delta\backslash\Delta_2) \cup [\Delta_2\backslash(\hat{F} \cup \hat{P})]$$

and

$$\tilde{N} = (\Delta^W\backslash\Gamma^W) \cup [\pi^{-1}(\Delta\backslash\hat{P}) \cap \tilde{P}] \cup (\tilde{F}\backslash\tilde{F}_1)\,,$$

where $\tilde{F}_1=\{\tilde{x}\in\tilde{F};\ \pi(\tilde{x})\in\Delta_2,\ \varphi^*(\tilde{x})=\hat{\phi}(\pi(\tilde{x}))\}$.
Then, $\mu(N)=0$ *and* $\tilde{N}$ *is of harmonic measure zero and*

$$\hat{\phi}(z) = \varphi^*(\tilde{x}) \quad \textit{if} \quad z\in\Delta\backslash N \quad \textit{and} \quad \tilde{x}\in(\pi^{-1}(z)\cap\Gamma^W)\backslash\tilde{N}\,.$$

The above theorem makes it possible for us to translate informations about φ^* into those of $\hat{\phi}$.

Theorem of Fatou-Plessner.

$$\hat{\phi}(z) = X'^* \textit{ or a sinleton for } \mu\textit{-almost all } z\,.$$

This is immediate from the definition of N in the key thoerm.

Theorem of Riesz.
Suppose φ *is locally polarly non-constant and let* A *be a* μ*-measurable subset of* Δ_2. *If* $\bigcup_{z\in A}\hat{\phi}(z)$ *is polar in* X'^**, then* A *is polar.*

In fact, using notations in the key theorem and putting

$$\tilde{A} = [\pi^{-1}(A\backslash N) \cap \Gamma^W]\backslash\tilde{N}$$

we have

$$\bigcup_{\tilde{x}\in\tilde{A}} \varphi^*(\tilde{x}) \subset \bigcup_{z\in A}\hat{\phi}(z)\,,$$

which implies that $\tilde{A}$ is of harmonic measure zero, i.e., $\bar{\mathrm{H}}^W_{1_{\tilde{A}}}=0$. Since $\pi^{-1}(A)\cap\Gamma^W\subset\tilde{A}\cup\pi^{-1}(N)\cup\tilde{N}$, we have $\mathrm{H}^W_{1_{\pi^{-1}(A)}}=0$, and hence $\mathrm{H}_{1_A}=\mathrm{H}^W_{1_{\pi^{-1}(A)}}=0$. Thus A is polar ([10], Cor. 6.6).

Theorem on the boundary chracterization of mappings of type *Bl*.

(1) *When* $X' \notin \mathcal{P}$, φ *is of type Bl if and only* if $\mu(\hat{F})=0$.

(2) *When* $X' \in \mathcal{P}$, φ *is of type Bl if and only if* $\phi(z) \cap X' = \emptyset$ *for* μ*-almost all* z.

Proof. (1): If φ is of type Bl, then $\tilde{P}=\Gamma^W$ so that $\mu(\hat{F})=0$, by the key theorem. Conversely, if φ is not of type Bl, then $\Gamma^W \cap \tilde{F} \neq \emptyset$. Since $\Gamma^W \cap \tilde{F}$ is an open subset of Γ^W, it follows that it has a positive harmonic measure at some point of X. Hence $\mu(\pi(\Gamma^W \cap \tilde{F}))>0$, which implies $\mu(\tilde{F})>0$ by the key theorem.

(2): Since $\tilde{P}=\emptyset$ in this case $\mu(\Delta \backslash \pi(\tilde{F}_1))=0$ by the key theorem. Since $\phi(z) \in \varphi^*(\Gamma^W)$ for $z \in \pi(\tilde{F}_1)$, it follows that $\varphi^*(\Gamma^W) \cap X' = \emptyset$ is equivalent to $\phi(z) \cap X' = \phi$ for μ-almost all z (cf. the proof of [6], Th. 5.3).

REMARK. It is readily seen that if $X' \in \mathcal{P}$ and if φ is of type Bl then $\Gamma' \backslash \bigcup_{z \in \hat{F}} \phi(z)$ is of harmonic measure zero.

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