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Title	On F-projective stable stems
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Citation	Osaka Journal of Mathematics. 1979, 16(2), p. 505-528
Version Type	VoR
URL	https://doi.org/10.18910/8046
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Osaka J. Math.
16 (1979), 505–528

ON F-PROJECTIVE STABLE STEMS

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(Received May 22, 1978)

In this note we study F-projective stable stems in dimension n with $7 \leqq n \leqq 22$, where F denotes the complex ($F=C$) or quaternionic ($F=H$) number field. D. Randall [9] determined them in dimension $\leqq 6$.

We use the notations and terminologies defined in the previous paper [8] or the book of Toda [11] without any reference.

1. Definitions and results

Given a pointed space X and a positive integer m, we define

$$\pi_m^{SF}(X) = \begin{cases} \text{image of } p_n^*\colon \{FP_n, X\} \to \{S^{nd-1}, X\} & \text{if } m = nd-1 \\ 0 \quad \text{if } m \not\equiv -1 \bmod (d)\,. \end{cases}$$

An element of $\pi_m^{SF}(X)$ is said to be *F-projective.* In this note we only consider the case of X being the spheres. Remark that π_{nd-1}^{SF} (S^l) is a subgroup of G_{nd-l-1}. We say that the m-stem G_m is *fully F-projective* if there exist integers l and n with $m=nd-l-1$ and $\pi_{nd-1}^{SF}(S^l)=G_m$.

Given a positive integer m, we consider the following problems.

$(Q.1)_m$ Compute $\pi_{nd-1}^{SF}(S^l)$ for each n and l with $m=nd-l-1$.

$(Q.2)_m$ What elements of G_m are F-projective?

$(Q.3)_m$ Is G_m fully F-projective?

Of course answers of $(Q.1)_m$ solve $(Q.2)_m$ and $(Q.3)_m$. Our main results are tabled as follows. Here 0 means that the problem is completely solved but no signed place not completely solved yet*). Details are given in (1.6) and §2.

In what follows in this section we prove some general results. Since p_n^H is the composition of p_{2n}^C and the canonical map $CP_{2n} \to HP_n$, we have

*) Recently in his dissertation, R.E. Snow has determined the C-projectivity of the 2-components for the stems less than or equal to 15.

m \ F	$(Q.1)_m$ H	$(Q.1)_m$ C	$(Q.2)_m$ H	$(Q.2)_m$ C	$(Q.3)_m$ H	$(Q.3)_m$ C
7	0	0	0	0	no	no
8	0		0	0	no	yes
9	0	0	0	0	no	yes
10	0		0	0	no	yes
11	0	0	0	0	yes	yes
13	0	0	0	0	yes	yes
15				0	no	yes
17					no	
21				0		yes
22			0	0	yes	yes

Proposition 1.1. *$\pi^{SH}_{4n-1}(S^l)$ is contained in $\pi^{SC}_{4n-1}(S^l)$ for any l and n.*

We have also

Proposition 1.2. *If $a \in G_m$ or $b \in G_n$ is F-projective, then $ab \in G_{m+n}$ is F-projective.*

Proposition 1.3. *If $0 \leqq j < d$, $\pi^{SF}_{(n+k)d-1}(S^{nd-j})$ is equal to the image of $p_{n+k,k}{}^*\colon \{FP_{n+k,k}, S^{nd-j}\} \to \{S^{(n+k)d-1}, S^{nd-j}\}$.*

These can be proved easily so we omit the details.
In [7] we proved the following.

Proposition 1.4. *$\pi^{SF}_{(n+k)d-1}(S^{nd})$ contains a cyclic subgroup of the order* den$[F\{n, k\}\alpha_F(n, k)]$.

Recall that $FP_{n+k,k}$ can be identified with the Thom space $(FP_k)^{n\xi_k}$ [3]. Let $M_k(F)$ be the order of ξ_k in the J-group $J(FP_k)$, which was determined by Adams-Walker [2] and Sigrist-Suter [10]. Then we have

Proposition 1.5. *If $m \equiv n \bmod (M_{k+1}(F))$, then*

$$\pi^{SF}_{(m+k)d-1}(S^{md-j}) = \pi^{SF}_{(n+k)d-1}(S^{nd-j})$$

for $0 \leqq j < d$.

Proof. For a vector bundle τ, $S(\tau)$ and $D(\tau)$ denote the associated sphere and disk bundle respectively. Without any loss of generality we may assume $m > n$. By assumption there exists an integer l and a fibre homotopy equivalence [3]

$$f' \colon S((m-n)\xi_{k+1} \oplus \underline{l}) \to S(\underline{(m-n)d+l})$$

where $\underline{j}$ denotes the real j-dimensional trivial vector bundle over FP_{k+1}. Naturally we can extend f' to a fibre homotopy equivalence

$$D((m-n)\xi_{k+1} \oplus \underline{l}) \to D(\underline{(m-n)d+l})$$

and to a fibre homotopy equivalence

$$f'' \colon (D(m\xi_{k+1} \oplus \underline{l}), S(m\xi_{k+1} \oplus \underline{l})) \to (D(n\xi_{k+1} \oplus (\underline{(m-n)d+l}),\ S(n\xi_{k+1} \oplus (\underline{(m-n)d+l})).$$

Hence we have a homotopy equivalence

$$\begin{aligned} f''' \colon E^l FP_{m+k+1,k+1} &= (FP_{k+1})^{m\xi_{k+1} \oplus \underline{l}} \\ &\to (FP_{k+1})^{n\xi_{k+1} \oplus (\underline{(m-n)d+l})} = E^{(m-n)d+l} FP_{n+k+1,k+1} \end{aligned}$$

where E denotes the reduced suspension. Consider the following diagram in which the horizontal sequences are the natural cofibrations.

$$\begin{array}{ccccccc} E^l S^{(m+k)d-1} & \xrightarrow{E^l p_{m+k\ k}} & E^l FP_{m+k,k} & \overset{i}{\subset} & E^l FP_{m+k+1,k+1} & \xrightarrow{q} & E^{l+1} S^{(m+k)d-1} \\ & & & & \Big\downarrow f''' & & \\ E^{(m-n)d+l} S^{(n+k)d-1} & \xrightarrow{E^{(m-n)d+l} p_{n+k,k}} & E^{(m-n)d+l} FP_{n+k,k} & \overset{i}{\subset} & E^{(m-n)d+l} FP_{n+k+1,k+1} & \xrightarrow{q} & E^{(m-n)d+l+1} S^{(n+k)d-1}. \end{array}$$

By cellular approximation we may assume that there exists

$$f \colon E^l FP_{m+k,k} \to E^{(m-n)d+l} FP_{n+k,k}$$

with $i \circ f = f''' \circ i$ and so there exists

$$h \colon E^{l+1} S^{(m+k)d-1} \to E^{(m-n)d+l+1} S^{(n+k)d-1}$$

with $h \circ q = q \circ f'''$. In the stable category f is clearly an equivalence and so h is an equivalence, too. Therefore in the stable category we have the following commutative square in which the vertical stable maps are equivalences.

$$\begin{tikzcd}
S^{(m+k)d-1} \arrow[r, "p_{m+k,k}"] \arrow[d, "h"] & FP_{m+k,k} \arrow[d, "f"] \\
E^{(m-n)d}S^{(n+k)d-1} \arrow[r, "E^{(m-n)d}p_{n+k,k}"] & E^{(m-n)d}FP_{n+k,k} .
\end{tikzcd}$$

This and (1.3) complete the proof.

We prove a negative result.

Theorem 1.6. *Let μ_k $(k\geqq 0)$ denote the Adams element in G_{8k+1} [1]. Then μ_k is not H-projective.*

Proof. Consider a commutative diagram in which f and f' are stable maps

$$\begin{tikzcd}
S^{4n+8k-1} \arrow[r, "p_{n+2k,2k}"] \arrow[d, "="] & HP_{n+2k,2k} \arrow[r, phantom, "\subset"] \arrow[d, "f"] & HP_{n+2k+1,2k+1} \arrow[d, "f'"] \\
S^{4n+8k-1} \arrow[r, "f\circ p_{n+2k,2k}"] & S^{4n-2} \arrow[r] & C(f\circ p_{n+2k,2k}) .
\end{tikzcd}$$

Apply $\tilde{K}$ to this diagram; since $\tilde{K}(X)=0$ if X is a finite complex with cells of only odd dimensions, we have the following commutative diagram

$$\begin{tikzcd}
0 & \tilde{K}(HP_{n+2k,2k}) \arrow[l] & \tilde{K}(HP_{n+2k+1,2k+1}) \arrow[l] & \tilde{K}(S^{4n+8k}) \arrow[l] & 0 \arrow[l] \\
0 & \tilde{K}(S^{4n-2}) \arrow[l] \arrow[u, "f^*"] & \tilde{K}(C(f\circ p_{n+2k,k})) \arrow[l] \arrow[u, "f'^*"] & \tilde{K}(S^{4n+8k}) \arrow[l, "\pi^*"'] \arrow[u, "="] & 0 \arrow[l]
\end{tikzcd}$$

Let $a\in\tilde{K}(C(f\circ p_{n+1k,2k}))$ be an element which maps to the generator $g_C^{2n-1}\in \tilde{K}(S^{4n-2})$, and $b\in\tilde{K}(C(f\circ p_{n+2k,2k}))$ be the generator of the image of π^* with $f'^*(b)=z^{n+2k}$. Then a and b generate $\tilde{K}(C(f\circ p_{n+2k,2k}))$. We have

$$\psi^2(a) = 2^{2n-1}a+\lambda b$$

for some integer λ, and

$$e_C(f\circ p_{n+2k,2k}) = \lambda/(2^{2n+4k}-2^{2n-1}) .$$

Put $f'^*(a)=\sum_{i=0}^{2k} a_i z^{n+i}$. Then

$$\psi^2(f'^*(a)) = \sum_i a_i(z^2+4z)^{n+i} = \sum_{i,j} a_i\binom{n+i}{j-i}4^{n+2i-j}z^{n+j} ,$$

$$\psi^2(f'^*(a)) = f'^*(\psi^2(a)) = 2^{2n-1}\sum_{i=0}^{2k} a_i z^{n+i}+\lambda z^{n+2k} .$$

Comparing the coefficients of z^{n+2k}, we have

$$\lambda=\sum_{i=0}^{2k-1} a_i\binom{n+i}{2k-i}4^{n+2i-2k}+(2^{2n+4k}-2^{2n-1})a_{2k}$$

and so

$$e_C(f\circ p_{n+2k,2k})=\sum_{i=0}^{2k-1} a_i\binom{n+i}{2k-i}4^{n+2i-2k}/(2^{2n+4k}-2^{2n-1}).$$

On the other hand

$$\begin{aligned}0&=f^*(ch(g_C^{2n-1}))=ch(f^*(g_C^{2n-1}))=\sum_{i=0}^{2k-1} a_i(ch(z))^{n+i}\\&=\sum_{i=0}^{2k-1} a_i(\phi_H(t))^{n+i}.\end{aligned}$$

Since $\phi_H(t)=t+$higher terms, we have

$$a_0=a_1=\cdots=a_{2k-1}=0$$

and then

$$e_C(f\circ p_{n+2k,2k})=0.$$

Since μ_k has non-trivial e_C-invariant, the conclusion follows.

Since $\mu_0=\eta$, μ_0 is C-projective. We shall prove that μ_1 is C-projective (2.9).

2. Computations

From now on, we work in the stable category of pointed spaces and stable maps between them with exceptions in (2.3), (ii) of (2.4), (2.5) and (2.7).

Concerning with F-projective 7-stems we have

Theorem 2.1. (i) $\pi_{4n+7}^{SH}(S^{4n})\cong Z/\mathrm{den}[H\{n,2\}\alpha_H(n,2)]$.
(ii) $\pi_{2n+7}^{SC}(S^{2n})\cong Z/\mathrm{den}[C\{n,4\}\alpha_C(n,4)]$.

Proof. Given $f\in\{HP_{n+2,2},S^{4n}\}$, we have

$$e_C(f\circ p_{n+2,2})=-\deg(f)\alpha_H(n,2)$$

from Theorem 1.1 of [7]. Since $e_C\colon G_7\to Z/2^4\cdot3\cdot5$ is an isomorphism, the conclusion (i) follows. By the same methods (ii) follows too.

By an easy calculation we have

$$\mathrm{den}[H\{n,2\}\alpha_H(n,2)]\,|\,2^2\cdot3\cdot5$$

and these are equal when for example $n=4$, and

$$\mathrm{den}[C\{n,4\}\alpha_C(n,4)]\,|\,2^3\cdot3\cdot5$$

and these are equal when for example $n=13$. Thus, since $G_7=Z_{2^4}\{\sigma\}\oplus Z_{15}$, we have

Corollary 2.2. *$2\sigma\in G_7$ is not H-projective but C-projective, and σ is not C-projective.*

Recall that $g_4=p_2^H\colon S^7\to S^4$ denotes the Hopf map. Let $g_n=E^{n-4}g_4\in\pi_{n+3}(S^n)$ for $n>4$. Then we have

Lemma 2.3. $g_5=\nu_5+\alpha_1(5)$.

We have also

Lemma 2.4. (i) *$\langle\eta, m\nu, n\nu\rangle=\langle\eta, mg_\infty, ng_\infty\rangle\supset\frac{1}{2}mn\langle\eta, 2\nu, \nu\rangle$ for any integers m and n with $mn\equiv 0$ mod(2).*

(ii) *$\{\eta_5, \nu_6, 2\nu_9\}_1=\{\eta_5, mg_6, 2ng_9\}_1=\varepsilon_5$ for any odd integers m and n.*

Proof. We have

$$\begin{aligned}
&\langle\eta, mg_\infty, ng_\infty\rangle=\langle\eta, m\nu, ng_\infty\rangle+\langle\eta, m\alpha_1, ng_\infty\rangle \qquad \text{by (3.8) of [11]},\\
&\langle\eta, m\nu, ng_\infty\rangle\subset\langle\eta, m\nu, n\nu\rangle+\langle\eta, m\nu, n\alpha_1\rangle \qquad \text{by (3.8) of ibid.},\\
&\langle\eta, m\nu, n\alpha_1\rangle=\langle\eta, m\nu, 16n\alpha_1\rangle \qquad \text{since } 3\alpha_1=0\\
&\qquad\subset\langle\eta, 16m\nu, n\alpha_1\rangle \qquad \text{by (3.5) of ibid.},\\
&\qquad\equiv 0 \qquad \text{since } 8\nu=0,
\end{aligned}$$

and so

$$\langle\eta, m\nu, ng_\infty\rangle\subset\langle\eta, m\nu, n\nu\rangle$$

but their indeterminacies are equal to ηG_7, hence

$$\begin{aligned}
\langle\eta, m\nu, ng_\infty\rangle&=\langle\eta, m\nu, n\nu\rangle\\
&\supset\frac{1}{2}mn\langle\eta, 2\nu, \nu\rangle \qquad \text{by (3.5) and (3.8) of [11].}
\end{aligned}$$

We have also

$$\langle\eta, m\alpha_1, ng_\infty\rangle=\langle\eta, 4m\alpha_1, ng_\infty\rangle\supset\langle 4\eta, m\alpha_1, ng_\infty\rangle\equiv 0$$

and so

$$\langle\eta, m\alpha_1, ng_\infty\rangle\equiv 0$$

and then

$$\langle\eta, mg_\infty, ng_\infty\rangle=\langle\eta, m\nu, n\nu\rangle\supset\frac{1}{2}mn\langle\eta, 2\nu, \nu\rangle.$$

Thus the conclusion (i) follows.

By the proof of (6.1) of [11]

$$E^2\varepsilon_3=\varepsilon_5=\{\eta_5, \nu_6, 2\nu_9\}_1.$$

Given $a\in\pi_{11}(S^8)$ and $b\in\pi_8(S^5)$ with $b\circ a=0$, we consider the Toda bracket

$$\{\eta_5, E^1b, E^1a\}_1 \in \pi_{13}(S^5)/(\pi_{10}(S^5)E^2a+\eta_5E^1\pi_{12}(S^5)).$$

By Toda [11] it is easy to see that $\eta_5E^1\pi_{12}(S^5)=\pi_{10}(S^5)E^2a=0$. Hence $\{\eta_5, E^1b, E^1a\}_1$ consists of a single element. Then by the same methods as the proof of (i) we have

$$\{\eta_5, \nu_6, 2\nu_9\}_1 = \{\eta_5, m\nu_6, 2n\nu_9\}_1 = \{\eta_5, mg_6, 2ng_9\}_1$$

for any odd integers m and n. Thus the conclusion (ii) follows.

We have

Lemma 2.5. (i) *$i^*\colon \{HP_{n+2,2}, S^{4n-1}\} \to \{S^{4n}, S^{4n-1}\}$ is an isomorphism.*

(ii) *$i^*\colon \{HP_{n+2,2}, S^{4n-2}\} \to \{S^{4n}, S^{4n-2}\}$ is an isomorphism if n is odd.*

(iii) *If n is even, we have a split exact sequence:*

$$0 \to \{S^{4n+4}, S^{4n-2}\} \xrightarrow{q^*} \{HP_{n+2,2}, S^{4n-2}\} \xrightarrow{i^*} \{S^{4n}, S^{4n-2}\} \to 0.$$

Proof. Considering the Puppe exact sequence associated with the cofibration $S^{4n+3} \to HP_{n+1,1} \subset HP_{n+2,2}$, we obtain (i), since $G_4=G_5=0$. Recall that

$$p_{n+1,1}=ng_{4n}\colon S^{4n+3} \to HP_{n+1,1}=S^{4n}$$

from [5] (or see (1.14) of [8]). We have the following exact sequence:

$$\begin{array}{ccccc} \{S^{4n+1}, S^{4n-2}\} & \xrightarrow{p_{n+1,1}{}^*} & \{S^{4n+4}, S^{4n-2}\} & \xrightarrow{q^*} & \{HP_{n+2,2}, S^{4n-2}\} \\ =Z_{24}\{g_\infty\} & & =Z_2\{\nu^2\} & & \\ & \xrightarrow{i^*} & \{S^{4n}, S^{4n-2}\} & \to & \{S^{4n+3}, S^{4n-2}\} \\ & & & & =0 \end{array}$$

Since $p_{n+1,1}{}^*(g_\infty)=ng_\infty^2=n\nu^2$, $p_{n+1,1}{}^*$ is epimorphic and i^* is isomorphic if n is odd. Thus the conclusion (ii) follows. If n is even, $p_{n+1,1}{}^*=0$ and we obtain the short exact sequence in (iii). Hence $\{HP_{n+2,2}, S^{4n-2}\} \cong Z_4$ or $Z_2 \oplus Z_2$. Suppose that $\{HP_{n+2,2}, S^{4n-2}\} \cong Z_4$. Then $q^*(\nu^2)$ is divisible by 2. Hence $p_{n+2,2}{}^*(q^*(\nu^2))=0$ since $2G_9=0$. But $q\circ p_{n+2,2}=p_{n+2,1}=(n+1)g_{4n+4}$, therefore $p_{n+2,2}{}^*(q^*(\nu^2))=(n+1)\nu^3 \neq 0$. This is a contradiction. Thus $\{HP_{n+2,2}, S^{4n-2}\} \cong Z_2 \oplus Z_2$. This completes the proof.

Recall that $KO^*(HP_n)=KO^*[\xi]/(\xi^n)$. Using the complexification $c\colon KO^* \to K^*$ we can easily prove the following. Details are omitted.

Lemma 2.6. $\psi^3(\xi)=3^4\xi+3^3y_1\xi^2+3^2y_2\xi^3$.

Now we determine H-projective 8 and 9-stems. Recall that $G_8=Z_2\{\bar\nu\} \oplus Z_2\{\varepsilon\}$ and $G_9=Z_2\{\nu^3\} \oplus Z_2\{\eta\varepsilon\} \oplus Z_2\{\mu\}$ with the relations $\eta\sigma=\bar\nu+\varepsilon$ and $\eta\bar\nu=\nu^3$. We have

Theorem 2.7. *The groups* $\pi^{SH}_{4n+7}(S^{4n-j})$ $(j=1, 2)$ *are given by the following table.*

$n \bmod (4)$	$\pi^{SH}_{4n+7}(S^{4n-1})$	$\pi^{SH}_{4n+7}(S^{4n-2})$
1	$Z_2\{\varepsilon\}$	$Z_2\{\eta\varepsilon\}$
2	$Z_2\{\bar{\nu}\}$	$Z_2\{\nu^3\}$
3	$Z_2\{\eta\sigma\}$	$Z_2\{\eta^2\sigma\}$
0	0	$Z_2\{\nu^3\}$

Proof. By (i) of (2.5), $\{HP_{n+2,2}, S^{4n-1}\}\cong Z_2$. Let f be a generator of it. Then $\pi^{SH}_{4n+7}(S^{4n-1})$ is a subgroup of G_8 generated by $f\circ p_{n+2,2}$ and we have the following commutative diagram

$$\begin{tikzcd}
S^{4n+6} \arrow[dr, "(n+1)g_\infty"] & & & S^{4n+7} \arrow[d, "p_{n+2,2}"'] \arrow[dr, "p_{n+2,1}"] & \\
 & S^{4n+3} \arrow[r, "p_{n+1,1}"] & S^{4n} \arrow[r, phantom, "\subset"] \arrow[dr, "\eta"'] & HP_{n+2,2} \arrow[r] \arrow[d, "f"] & S^{4n+4} \\
 & & & S^{4n-1} &
\end{tikzcd}$$

Since $p_{n+1,1}=ng_{4n}$ and $p_{n+2,1}=(n+1)g_{4n+4}$, we have

$$f\circ p_{n+2,2}\in\langle\eta,\ ng_\infty,\ (n+1)g_\infty\rangle\,.$$

By (i) of (2.4) this Toda bracket contains $\frac{1}{2}n(n+1)\langle\eta,\ 2\nu,\ \nu\rangle$. Hence

$$\begin{aligned}
\langle\eta,\ ng_\infty,\ (n+1)g_\infty\rangle &= \begin{cases} \eta\circ G_7 & \text{if } n\equiv 0 \text{ or } 3 \bmod(4) \\ \langle\eta,\ 2\nu,\ \nu\rangle & \text{if } n\equiv 1 \text{ or } 2 \bmod(4) \end{cases} \\
&= \begin{cases} \{0,\ \eta\sigma\} & \text{if } n\equiv 0 \text{ or } 3 \bmod(4) \\ \{\varepsilon,\ \bar{\nu}\} & \text{if } n\equiv 1 \text{ or } 2 \bmod(4) \end{cases}
\end{aligned}$$

Hence

(*) $f\circ p_{n+2,2}=0$ or $\eta\sigma$ if $n\equiv 0$ or $3 \bmod(4)$, and ε or $\bar{\nu}$ if $n\equiv 1$ or $2 \bmod(4)$.

Suppose that $n\equiv 1 \bmod(4)$. By (ii) of (2.4), $\varepsilon_5=\{\eta_5,\ ng_6,\ (n+1)g_9\}_1$. Consider the following diagram:

$$
\begin{tikzcd}
S^{12} \arrow[dr, "2g_9"] & & & S^{13} \arrow[d, "E^2p_3"] \arrow[dr, "2g_{10}"] & \\
& S^9 \arrow[r, "E^2g_4"] & S^6 \arrow[dr, "\eta_5"'] \arrow[r, phantom, "\subset"] & E^2HP_3 \arrow[r] & S^{10} \\
& & & S^5 &
\end{tikzcd}
$$

Then we have

$$\varepsilon_5 \in \text{Image of } (E^2p_3)^*: [E^2HP_3, S^5] \to \pi_{13}(S^5)$$

and $\varepsilon \in \pi_{11}^{SH}(S^3)$, and then

$$\pi_{11}^{SH}(S^3) = Z_2\{\varepsilon\} .$$

If $n \geqq 2$, we have

$$\varepsilon_{4n-1} = E^{4n-6}\{\eta_5, ng_6, (n+1)g_9\}_1 \in \{\eta_{4n-1}, ng_{4n}, (n+1)g_{4n+3}\}_{4n-5}$$

by Proposition (1.3) of [11]. Since the Toda bracket in the right hand is a coset of $\pi_{4n+4}(S^{4n-1})(n+1)g_{4n+4} + \eta_{4n-1}E^{4n-5}\pi_{12}(S^5) = 0$, we have

$$\varepsilon_{4n-1} = \{\eta_{4n-1}, ng_{4n}, (n+1)g_{4n+3}\}_{4n-5} .$$

Since $[HP_{n+2,2}, S^{4n-1}] \cong \{HP_{n+2,2}, S^{4n-1}\}$, f is representable by an unstable map, we denote it by the same letter f. Then

$$\varepsilon_{4n-1} = f \circ p_{n+2,2} .$$

Thus $\pi_{4n+7}^{SH}(S^{4n-1}) = Z_2\{\varepsilon\}$ if $n \equiv 1 \bmod (4)$. From (ii) of (2.5), $\{HP_{n+2,2}, S^{4n-2}\} = \eta\{HP_{n+2,2}, S^{4n-1}\} \cong Z_2$ if n is odd. Hence $\pi_{4n+7}^{SH}(S^{4n-2}) = Z_2\{\eta\varepsilon\}$ if $n \equiv 1 \bmod (4)$.

We use the Adams d_R- and e_R-invariants [1]. Let $e_1 \in KO^{-1}$ be the generator, and put $e_9 = g_R e_1 \in KO^{-9}$. For $f \in \{HP_{n+2,2}, S^{4n-1}\}$ we have the commutative diagram:

$$
\begin{tikzcd}
S^{4n+7} \arrow[r, "p_{n+2,2}"] \arrow[d, "="] & HP_{n+2,2} \arrow[d, "f"] \arrow[r, phantom, "\subset"] & HP_{n+3,3} \arrow[d, "f'"] \\
S^{4n+7} \arrow[r, "f \circ p_{n+2,2}"] & S^{4n-1} \arrow[r, "j"] & C(f \circ p_{n+2,2}) .
\end{tikzcd}
$$

Apply $\widetilde{KO}^{-4n-9}$ to this diagram, then we have the following commutative diagram in which the horizontal sequences are exact:

$$
\begin{tikzcd}
0 & \widetilde{KO}^{-4n-9}(HP_{n+2,2}) \arrow[l] & \widetilde{KO}^{-4n-9}(HP_{n+3,3}) \arrow[l] & \widetilde{KO}^{-4n-9}(S^{4n+8}) \arrow[l] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-4n-9}(S^{4n-1}) \arrow[l] \arrow[u, "f^*"] & \widetilde{KO}^{-4n-9}(C(f \circ p_{n+2,2})) \arrow[l, "j^*"'] \arrow[u, "f'^*"] & \widetilde{KO}^{-4n-9}(S^{4n+8}) \arrow[l] \arrow[u, "="] & 0 . \arrow[l]
\end{tikzcd}
$$

Let $a\in\widetilde{KO}^{-4n-9}(C(f\circ p_{n+2,2}))$ be an element which maps to a generator of $\widetilde{KO}^{-4n-9}(S^{4n-1})\cong Z$, and $b\in\widetilde{KO}^{-4n-9}(C(f\circ p_{n+2,2}))$ be the element which is the image of the generator of $\widetilde{KO}^{-4n-9}(S^{4n+8})\cong Z_2$. Since $\widetilde{KO}^{-4n-9}(HP_{n+2,2})=Z_2\{e_9\xi^n\}$ and $\widetilde{KO}^{-4n-9}(HP_{n+3,3})=Z_2\{e_9\xi^n\}\oplus Z_2\{e_1\xi^{n+2}\}$ we have

$$f'^*(a)=xe_9\xi^n+ye_1\xi^{n+2}$$

for some $x, y\in Z_2$. We have also

$$\psi^3(a)=3^{4n+4}a+\lambda b$$

for some $\lambda\in Z_2$, and

$$e_R(f\circ p_{n+2,2})=\lambda\,.$$

We have

$$\begin{aligned} f'^*(\psi^3(a)) &= f'^*(3^{4n+4}a+\lambda b)=3^{4n+4}f'^*(a)+\lambda f'^*(b)\\ &= 3^{4n+4}xe_9\xi^n+(3^{4n+4}y+\lambda)e_1\xi^{n+2}\,,\end{aligned}$$

and

$$\begin{aligned} f'^*(\psi^3(a)) &= \psi^3(f'^*(a))=\psi^3(xe_9\xi^n+ye_1\xi^{n+2})\\ &= x\psi^3(e_9)\psi^3(\xi^n)+y\psi^3(e_1)\psi^3(\xi^{n+2})\\ &= x3^4e_9(3^{4n}\xi^n+3^{4n-1}ny_1\xi^{n+1}+3^{4n-2}ny_2\xi^{n+2})\\ &\qquad\qquad +ye_13^{4(n+2)}\xi^{n+2} \qquad \text{by (2.6)}\\ &= x3^{4n+4}e_9\xi^n+(x3^{4n+2}n+y3^{4n+8})e_1\xi^{n+2} \quad \text{since } e_9y_1=0\\ &\qquad\qquad\qquad\qquad \text{and } e_9y_2=e_1\,.\end{aligned}$$

Comparing the coefficients of $e_1\xi^{n+2}$, we have

$$\lambda=nx \quad (\text{in } Z_2)\,.$$

On the other hand the following triangle is commutative by (i) of (2.5).

$$\begin{tikzcd} S^{4n}=HP_{n+1,1} \arrow[r, "i", hook] \arrow[dr, "\eta"'] & HP_{n+2,2} \arrow[d, "f"] \\ & S^{4n-1}\,. \end{tikzcd}$$

Hence we have the commutative triagle

$$\begin{tikzcd} \widetilde{KO}^{-4n-9}(S^{4n}) & \widetilde{KO}^{-4n-9}(HP_{n+2,2}) \arrow[l, "i^*"'] \\ & \widetilde{KO}^{-4n-9}(S^{4n-1}) \arrow[u, "f^*"'] \arrow[ul, "\eta^*"] \end{tikzcd}$$

and $i^*f^*j^*(a)=xe_9\xi^n$ where $j^*(a)$ is the generator of $\widetilde{KO}^{-4n-9}(S^{4n-1})\cong Z$. Since $\eta^*=d_R(\eta)\neq 0$, we have $x\neq 0$ and so

$$e_R(f\circ p_{n+2,2})=n\,.$$

Since $e_R(\eta\sigma)\neq 0$ [1], by (*) we know that $\pi^{SH}_{4n+7}(S^{4n-1})=Z_2\{\eta\sigma\}$ if $n\equiv 3 \bmod (4)$, or 0 if $n\equiv 0 \bmod (4)$. Then $\pi^{SH}_{4n+7}(S^{4n-2})=Z_2\{\eta^2\sigma\}$ if $n\equiv 3 \bmod (4)$ from (2.5).

Suppose that n is even. By the fact $e_C(\nu^3)=e_C(\eta\varepsilon)=0$ and the proof of (1.6), we see that

$$Z_2\{\nu^3\}\subset\pi^{SH}_{4n+7}(S^{4n-2})\subset Z_2\{\nu^3\}\oplus Z_2\{\eta\varepsilon\}\,.$$

If $\pi^{SH}_{4n+7}(S^{4n-2})=Z_2\{\nu^3\}\oplus Z_2\{\eta\varepsilon\}$, $\pi^{SH}_{4n+7}(S^{4n-2})$ contains the J-image $\eta^2\sigma=\nu^3+\eta\varepsilon$, that is, there exists $h\in\{HP_{n+2,2}, S^{4n-2}\}$ with $h\circ p_{n+2,2}=\eta^2\sigma$. Using $\widetilde{KO}^{-4n-10}$ and the same methods as above we have

$$e_R(h\circ p_{n+2,2})=nx=0$$

for some $x\in Z_2$, but this is a contradiction since $e_R(\eta^2\sigma)\neq 0$ [1]. Therefore

$$(**)\qquad \pi^{SH}_{4n+7}(S^{4n-2})=Z_2\{\nu^3\}\ \text{if } n \text{ is even.}$$

Next suppose that $n\equiv 2 \bmod (4)$. By (*), $\pi^{SH}_{4n+7}(S^{4n-1})=Z_2\{\bar{\nu}\}$ or $Z_2\{\varepsilon\}$. If $\pi^{SH}_{4n+7}(S^{4n-1})=Z_2\{\varepsilon\}$, $\pi^{SH}_{4n+7}(S^{4n-2})$ contains $\eta\varepsilon$. This contradicts to (**). Thus $\pi^{SH}_{4n+7}(S^{4n-1})=Z_2\{\bar{\nu}\}$ and the proof is completed.

Concerning with C-projective 8-stems we prove

Theorem 2.8. *$\pi^{SC}_{2n+7}(S^{2n-1})$ is equal to*
(i) G_8 *if* $n\equiv 2$ *or* $4 \bmod (8)$,
(ii) 0 *if n is odd,*
(iii) $Z_2\{\eta\sigma\}$ *or* G_8 *if* $n\equiv 0$ *or* $6 \bmod (8)$.

Proof. Suppose that n is even. Since $q_3\circ p_{n+4,4}=p_{n+4,1}=\eta$ from (i) of (1.13) of [8], $\pi^{SC}_{2n+7}(S^{2n-1})$ contains $\sigma\circ q_3\circ p_{n+4,4}=\sigma\eta$. Then by (1.1) and (2.7), $\pi^{SC}_{2n+7}(S^{2n-1})=G_8$ if $n\equiv 2$ or $4 \bmod (8)$.

Next suppose that n is odd. Put $n=2m+1$. Consider the following Puppe exact sequences:

$$\{S^{4m+3}, S^{4m+1}\}\xrightarrow{(Ep_{2m+2,1})^*}\{S^{4m+4}, S^{4m+1}\}\xrightarrow{q^*}\{CP_{2m+3,2}, S^{4m+1}\}$$
$$\to\{S^{4m+2}, S^{4m+1}\}\xrightarrow{p_{2m+2,1}{}^*}\{S^{4m+3}, S^{4m+1}\}\,,$$

$$\{S^{4m+6}, S^{4m+1}\}\xrightarrow{q^*}\{CP_{2m+4,3}, S^{4m+1}\}\xrightarrow{i^*}\{CP_{2m+3,2}, S^{4m+1}\}$$
$$\to\{S^{4m+5}, S^{4m+1}\}\,.$$

Since $p_{2m+2,1}=\eta$ and $\eta^3=12g_\infty$, $\{CP_{2m+4,3}, S^{4m+1}\}\cong Z_{12}$. Let $a\in\{CP_{2m+4,3}, S^{4m+1}\}$

be an element with $i^*(a)=q^*(g_\infty)$. Then a is a generator. Let $f\in\{CP_{2m+5,4}$ $S^{4m+1}\}$ be an element. Then $f|_{CP_{2m+4,3}}=xa$ for some integer x. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & S^{4m+9} \arrow[dl, "s"'] \arrow[d, "p_{2m+5,4}"] \\
CP_{2m+3,2} \arrow[r, hook, "i"] \arrow[d, "q"] & CP_{2m+4,3} \arrow[r, hook] \arrow[d, "q"] \arrow[ddr, "xa"'] & CP_{2m+5,4} \arrow[dd, "f"] \\
CP_{2m+3,1} \arrow[d, "="] & CP_{2m+4,2} \arrow[d, "\pi"] & \\
S^{4m+4} \arrow[r, equal] & HP_{m+2,1} \arrow[r, dashed, "xg_\infty"'] & S^{4m+1}
\end{tikzcd}$$

where the fact $p_{2m+5,1}=0$ assures the existence of s. We have

$$xg_\infty\circ\pi\circ q\circ i = xg_\infty\circ q = xa\circ i\,.$$

Since i^* is monomorphic in the above Puppe sequence, we have

$$xg_\infty\circ\pi\circ q = xa\,.$$

Then

$$f\circ p_{2m+5,4} = xa\circ s = xg_\infty\circ\pi\circ q\circ s = xg_\infty\circ 0 = 0$$

since $\pi\circ q\circ s\in G_5=0$. This completes the proof.

Concerning with C-projective 9-stems we prove

Theorem 2.9. *$\pi_{2n+9}^{SC}(S^{2n})$ is equal to*

(i) G_9	*if* $n\equiv 5$, 7 mod(8), 3,9 mod(16), *or* 17 mod(32),
(ii) $Z_2\{\eta^2\sigma\}\oplus Z_2\{\eta\varepsilon\}$	*if* $n\equiv 11$ mod(16) *or* 1 mod(32),
(iii) $Z_2\{\nu^3\}$	*if* $n\equiv 0$ mod(4),
(iv) 0	*if* $n\equiv 2$ mod(4).

Proof. By (1.1) of [7]

$$e_C(f\circ p_{n+5,5}) = -\deg(f)\alpha_C(n, 5)$$

for $f\in\{CP_{n+5,5}, S^{2n}\}$. Hence $\pi_{2n+9}^{SC}(S^{2n})$ contains μ if and only if $\nu_2(C\{n, 5\}\times\alpha_C(n, 5))=-1$, since $e_C(\mu)=\dfrac{1}{2}$ and $e_C(\nu^3)=e_C(\eta\varepsilon)=e_C(\eta^2\sigma)=0$. By (1.16) and (3.1) of [8] and an elementary analysis, we have

$$\nu_2(C\{n,5\})=\begin{cases} 4 \text{ if } n\equiv 4,\ 5,\ 6 \text{ or } 7 \bmod (2^3) \\ 3 \text{ if } n\equiv 3 \bmod (2^3)\ 8,\ 9 \text{ or } 10 \bmod (2^4) \\ 2 \text{ if } n\equiv 1,\ 2 \bmod (2^4) \text{ or } 16 \bmod (2^5) \\ 1 \text{ if } n\equiv 32 \bmod (2^6) \\ 0 \text{ if } n\equiv 0 \bmod (2^6)\,, \end{cases}$$

$$\nu_2(\alpha_C(n,5))=\begin{cases} -5 \text{ if } n\equiv 5 \text{ or } 7 \bmod (2^3) \\ -4 \text{ if } n\equiv 6 \bmod (2^3),\ 3 \text{ or } 9 \bmod (2^4) \\ -3 \text{ if } n\equiv 10 \bmod (2^4),\ 11 \text{ or } 17 \bmod (2^5) \\ -2 \text{ if } n\equiv 4,\ 8 \bmod (2^4),\ 18 \bmod (2^5),\ 27 \text{ or } 33 \bmod (2^6) \\ -1 \text{ if } n\equiv 16,\ 28 \bmod (2^5),\ 2 \bmod (2^6) \text{ or } 59 \bmod (2^7) \\ \geqq 0 \text{ if } n\equiv 0,\ 12 \bmod (2^5),\ 1,\ 34 \bmod (2^6) \text{ or } 123 \bmod (2^7)\,. \end{cases}$$

Hence $\pi_{2n+9}^{SC}(S^{2n})$ contains μ if and only if $n\equiv 5$, $7 \bmod (2^3)$, 3, $9 \bmod (2^4)$, or $17 \bmod (2^5)$.

If n is odd, $q_4\circ p_{n+5,5}=p_{n+5,1}=\eta$ and $\pi_{2n+9}^{SC}(S^{2n})$ contains $\{S^{2n+8}, S^{2n}\}\circ q_4\circ p_{n+5,5}$ $=G_8\circ\eta=Z_2\{\eta^2\sigma\}\oplus Z_2\{\eta\varepsilon\}$. Thus the conclusions (i) and (ii) follow.

Next consider the case of n being even. First we show that $\pi_{2n+9}^{SC}(S^{2n})$ does not contain J-image $\eta^2\sigma=\nu^3+\eta\varepsilon$. Consider a commutative diagram:

$$\begin{tikzcd} S^{2n+9} \arrow[r, "p_{n+5,5}"] \arrow[d, "="] & CP_{n+5,5} \arrow[d, "f"] & \subset & CP_{n+6,6} \arrow[d, "f'"] \\ S^{2n+9} \arrow[r, "f\circ p_{n+5,5}"] & S^{2n} \arrow[rr] & & C(f\circ p_{n+5,5}) \end{tikzcd}$$

We apply $\widetilde{KO}$ if $n\equiv 0 \bmod (4)$ or $\widetilde{KO}^{-4}$ if $n\equiv 2 \bmod (4)$ to this diagram. The methods for $n\equiv 0 \bmod (4)$ and $n\equiv 2 \bmod (4)$ are quite similar to a part of the proof of (2.7), so we sketch the proof only for $n\equiv 0 \bmod (4)$. Put $n=4m$. We have the following commutative diagram:

$$\begin{tikzcd} 0 & \widetilde{KO}(CP_{4m+5,5}) \arrow[l] & \widetilde{KO}(CP_{4m+6,6}) \arrow[l] & \widetilde{KO}(S^{8m+10}) \arrow[l] & 0 \arrow[l] \\ 0 & \widetilde{KO}(S^{8m}) \arrow[l] \arrow[u, "f^*"] & \widetilde{KO}(C(f\circ p_{4m+5,5})) \arrow[l] \arrow[u, "f'^*"] & \widetilde{KO}(S^{8m+10}) \arrow[l] \arrow[u, "="] & 0\,. \arrow[l] \end{tikzcd}$$

Let a and b be elements of $\widetilde{KO}(C(f\circ p_{4m+5,5}))$ such that a maps to a generator of $\widetilde{KO}(S^{8m})\cong Z$ and b is the image of the generator of $\widetilde{KO}(S^{8m+10})\cong Z_2$. Then

$$\psi^3(a)=3^{4m}a+\lambda b$$

for some $\lambda\in Z_2$, and

$$e_R(f \circ p_{4m+5,5}) = \lambda\,.$$

Since $\widetilde{KO}(CP_{4m+6,6}) = Z\{z_0^{2m}, z_0^{2m+1}, z_0^{2m+2}\} \oplus Z_2\{z_0^{2m+3}\}$ [4], we may put $f'^*(a)=\sum_{i=0}^{3} d_i z_0^{2n+i}$ for some integers d_i $(0 \leqq i \leqq 2)$ and $d_3 \in Z_2$. Analysing the equation $f'^*(\psi^3(a)) = \psi^3(f'^*(a))$, we know that $\lambda=0$. Hence J-image $\eta^2\sigma$ is not contained in $\pi_{2n+9}^{SC}(S^{2n})$, since $e_R(\eta^2\sigma) \neq 0$ [1]. Therefore $\pi_{2n+9}^{SC}(S^{2n})=0$, $Z_2\{\nu^3\}$ or $Z_2\{\eta\varepsilon\}$ if n is even.

Second we show (iii). Cnsider the following diagram in which the triangle is commutative by (1.15) of [8].

$$\begin{tikzcd}
 & & & S^{2n+9} \arrow[dl, "\left(\frac{n}{2}+3\right)g_\infty"'] \arrow[d, "p_{n+5,2}"] \\
S^{2n+7} \arrow[r, "p_{n+4,1}"] & S^{2n+6} \arrow[r, phantom, "="] & CP_{n+4,1} \arrow[dr, "\nu^2"'] & CP_{n+5,2} \arrow[l, hook', "i"'] \\
 & & & S^{2n}
\end{tikzcd}$$

Since $p_{n+4,1}=\eta$, $\nu^2 p_{n+4,1}=0$ and there exists $h \in \{CP_{n+5,2}, S^{2n}\}$ with $h \circ i = \nu^2$. Then $h \circ p_{n+5,2} = \nu^2 \circ \left(\frac{1}{2}n+3\right) g_\infty = \nu^3$ if $n \equiv 0 \bmod (4)$ or 0 if $n \equiv 2 \bmod (4)$. Thus $\pi_{2n+9}^{SC}(S^{2n}) = Z_2\{\nu^3\}$ if $n \equiv 0 \bmod (4)$, and the conclusion (iii) follows.

Third we show (iv). Suppose that $n \equiv 2 \bmod (4)$. Consider the following diagram in which the two horizontal and one vertical sequences are parts of suitable Puppe exact sequences.

$$\begin{tikzcd}
 & & \{S^{2n+4}, S^{2n}\} = 0 \arrow[d] & \\
\{S^{2n+6}, S^{2n}\} \arrow[r, "q^*"] & \{CP_{n+4,4}, S^{2n}\} \arrow[r, "i^*"] & \{CP_{n+3,3}, S^{2n}\} \arrow[r, "p_{n+3,3}{}^*"] \arrow[d, "i'^*"] & \{S^{2n+5}, S^{2n}\} = 0 \\
 & \{S^{2n+2}, S^{2n}\} = Z_2\{\eta^2\} \arrow[r, "q_1{}^*"] \arrow[dr, "p_{n+2,1}{}^*"'] & \{CP_{n+2,2}, S^{2n}\} \arrow[r, "i''^*"] \arrow[d, "p_{n+2,2}{}^*"] & \{S^{2n}, S^{2n}\} \\
 & & \{S^{2n+3}, S^{2n}\} = Z_{24}\{g_\infty\} &
\end{tikzcd}$$

Since $p_{n+2,1}=\eta$ and $\eta^3=12g_\infty \neq 0$, $p_{n+2,1}{}^*$ is monomorphic and the image of $q_1{}^*$ is not contained in the image of i'^*, and so $\{CP_{n+3,3}, S^{2n}\} \cong Z$ and i^* is isomorphic on a free subgroup. Then we can choose $h \in \{CP_{n+4,4}, S^{2n}\}$ which is a generator of a free part and satisfies $i''^*i'^*i^*(h)=\deg(h)=C\{n,4\}$. Let $s\in\{S^{2n+9}, CP_{n+4,4}\}$ be an element with $p_{n+5,5}=i_1\circ s$. Let f be any element of $\{CP_{n+5,5}, S^{2n}\}$. Then $f\circ i_1=(\deg(f)/C\{n, 4\})h+e\circ q$ for some $e\in\{S^{2n+6}, S^{2n}\}$ and

$$f\circ p_{n+5,5}=f\circ i_1\circ s=(\deg(f)/C\{n, 4\})h\circ s+e\circ q\circ s\,.$$

Since $q\circ s=\left(\frac{1}{2}n+3\right)g_\infty$ or $\left(\frac{1}{2}n+15\right)g_\infty$ from (1.15) of [8], $q\circ s$ is divisible by 2, and then

$$f\circ p_{n+5,5}=(\deg(f)/C\{n, 4\})h\circ s$$

for $\{S^{2n+6}, S^{2n}\}\cong Z_2$. By (1.16) and (3.1) of [8], we know easily that

$$C\{n, 4\}=24/(n, 24)=2^2\cdot 3\Big/\left(\frac{1}{2}n, 3\right),$$

$$\nu_2(C\{n, 5\})=\begin{cases} 4 \text{ if } n\equiv 6 \bmod(8) \\ 3 \text{ if } n\equiv 10 \bmod(16) \\ 2 \text{ if } n\equiv 2 \bmod(16)\,. \end{cases}$$

Hence if $n\equiv 6 \bmod(8)$ or $10 \bmod(16)$, $C\{n, 5\}/C\{n, 4\}\equiv 0 \bmod(2)$ and $f\circ p_{n+5,5}=0$ since $\deg(f)$ is a multiple of $C\{n, 5\}$. Thus the conclusion (iv) follows if $n\equiv 6 \bmod(8)$ or $10 \bmod(16)$. In case of $n\equiv 2 \bmod(16)$, we constructed the following commutative diagram in the proof of (v) of (3.1) in [8] and found that $q_1\circ s_3$ is divisible by 2.

$$\begin{tikzcd}
 & & & S^{2n+9} \arrow[d, "p_{n+5,5}"] \arrow[dl, "s_1"'] \arrow[ddl, bend right, "s_2"'] \arrow[ddlll, bend right, "s_3"'] \\
 & & CP_{n+4,4} \arrow[r, phantom, "\subset"] \arrow[d, "3"] & CP_{n+5,5} \arrow[d, "3"] \\
CP_{n+2,2} = S^{2n} \vee S^{2n+2} \arrow[r, phantom, "\subset"] & CP_{n+3,3} \arrow[r, phantom, "\subset"] & CP_{n+4,4} \arrow[r, phantom, "\subset"] & CP_{n+5,5}
\end{tikzcd}$$

Choose $u\in\{CP_{n+2,2}, S^{2n}\}$ with $\deg(u)=1$. Then $f|_{CP_{n+2,2}}=\deg(f)u+e\circ q_1$ for some $e\in\{S^{2n+2}, S^{2n}\}$, and

$$\begin{aligned} f\circ p_{n+5,5} &= 3f\circ p_{n+5,5}, \qquad \text{since } 2G_9=0 \\ &= f|_{CP_{n+2,2}}\circ s_3 \\ &= \deg(f)u\circ s_3+e\circ q_1\circ s_3 \\ &= \deg(f)u\circ s_3, \qquad \text{since } e\in G_2=Z_2 \text{ and } 2\,|\,q_1\circ s_3\,. \end{aligned}$$

By (1.16) and (3.1) of [8]

$$\nu_2(C\{n, 5\})\geqq 1$$

hence $\deg(f)\equiv 0 \bmod (2)$ and

$$f\circ p_{n+5,5}=0$$

since $u\circ s_3\in G_9$ and $2G_9=0$. Thus $\pi_{2n+9}^{SC}(S^{2n})=0$ if $n\equiv 2 \bmod (16)$ and the proof is completed.

We determine H-projective 10-stems. Recall that $G_{10}=Z_2\{\eta\mu\}\oplus Z_3\{\beta_1\}$.

Theorem 2.10. *$\pi_{4n+7}^{SH}(S^{4n-3})=Z_3\{\beta_1\}$ if $n\equiv 1 \bmod (3)$ or 0 if $n\not\equiv 1 \bmod (3)$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
S^{4n+6} \arrow[dr, "(n+1)g_\infty"] & & & S^{4n+7} \arrow[d, "p_{n+2,2}"'] \arrow[dr, "p_{n+2,1}"] & \\
& S^{4n+3} \arrow[r, "p_{n+1,1}"] & S^{4n} \arrow[r, hook, "i"] & HP_{n+2,2} \arrow[r] & S^{4n+4}
\end{tikzcd}$$

Given $f\in\{HP_{n+2,2}, S^{4n-3}\}$, we have $f\circ i=mg_\infty$ for some integer m with $mn\equiv 0 \bmod (2)$, since $p_{n+1,1}=ng_\infty$ and $0=f\circ i\circ p_{n+1,1}=mn\nu^2$. By definition of Toda bracket we have

$$f\circ p_{n+2,2}\in\langle f\circ i, p_{n+1,1}, (n+1)g_\infty\rangle.$$

Since all Toda brackets which appear in this proof have zero indeterminacies from a similar method as the proof of (i) of (2.4), we have

$$\begin{aligned}\langle f\circ i, p_{n+1,1}, (n+1)g_\infty\rangle &= \langle mg_\infty, ng_\infty, (n+1)g_\infty\rangle\\ &= \frac{1}{2}mn(n+1)\langle\nu, 2\nu, \nu\rangle+mn(n+1)\langle\alpha_1,\alpha_1,\alpha_1\rangle.\end{aligned}$$

But

$$\begin{aligned}\langle\nu, 2\nu, \nu\rangle &= -\langle 2\nu, \nu, 2\nu\rangle && \text{by (3.10) of [11]}\\ &= -\langle\nu, 4\nu, \nu\rangle && \text{by (3.5) of ibid.}\\ &= -2\langle\nu, 2\nu, \nu\rangle && \text{by (3.8) of ibid.}\\ &= 0\end{aligned}$$

and

$$\langle\alpha_1, \alpha_1, \alpha_1\rangle=\beta_1 \quad \text{by p. 180 of ibid.}$$

and then

$$f\circ p_{n+2,2}=mn(n+1)\beta_1.$$

Conversely for any m with $mn\equiv 0 \bmod (2)$ there exists $f\in\{HP_{n+2,2}, S^{4n-3}\}$ with $f\circ i=mg_\infty$. Thus the conclusion follows.

We prove

Theorem 2.11. $\pi^{SC}_{2n+9}(S^{2n-1})$ *is equal to*

(i)	G_{10}	*if* $n\equiv 1 \bmod(6)$,
(ii)	$Z_2\{\eta\mu\}$	*if* $n\equiv 3 \bmod(6)$,
(iii)	$Z_3\{\beta_1\}$	*if* $n\equiv 4 \bmod(6)$,
(iv)	0	*if* $n\equiv 0 \bmod(6)$,
(v)	0 *or* $Z_3\{\beta_1\}$	*if* $n\equiv 2 \bmod(6)$,
(vi)	$Z_2\{\eta\mu\}$ *or* G_{10}	*if* $n\equiv 5 \bmod(6)$.

Proof. First we suppose that n is odd. Since $q_4\circ p_{n+5,5}=p_{n+5,1}=\eta$, $\pi^{SC}_{2n+9}(S^{2n-1})$ contains $\mu\circ q_4\circ p_{n+5,5}=\mu\eta$ and (vi) follows, (i) also follows from (1.1) and (2.10). Given $f\in\{CP_{n+5,5}, S^{2n-1}\}$, we have

$$0=f|_{CP_{n+1,1}}\circ p_{n+1,1}=f|_{CP_{n+1,1}}\circ\eta$$

so $f|_{CP_{n+1,1}}=0$ and

$$\pi^{SC}_{2n+9}(S^{2n-1}) = \text{image of } p_{n+5,4}{}^*:\ \{CP_{n+5,4}, S^{2n-1}\}\to\{S^{2n+9}, S^{2n-1}\}\,.$$

In case of $n\equiv 3$ mod(6) we construct a commutative diagram:

$$\begin{tikzcd}
 & & & & S^{2n+9} \arrow[d, "p_{n+5,4}"] \arrow[ddl, "s_1"'] \arrow[dddll, "s_2"'] \arrow[dddllll, "s_3"'] \\
 & & & & CP_{n+5,4} \arrow[d, "2"] \\
 & & & CP_{n+4,3} \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d, "4"] & CP_{n+5,4} \arrow[d, "4"] \\
CP_{n+2,1} \arrow[r, phantom, "="] & S^{2n+2} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{n+3,2} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{n+4,3} \arrow[r, phantom, "\subset"] & CP_{n+5,4}
\end{tikzcd}$$

Since $q_3\circ p_{n+5,4}=p_{n+5,1}=\eta$, $q_3\circ 2p_{n+5,4}=0$ and there exists s_1 with $i\circ s_1=2p_{n+5,4}$. By (1.15) of [8] $q_2\circ s_1=(n+3)g_\infty$. Then $4q_2\circ s_1=0$ and there exists s_2 with $i\circ s_2=4s_1$. Since $q_1\circ s_2\in G_5=0$, there exists s_3 with $i\circ s_3=s_2$. Thus the construction of the above diagram is completed. Given $f\in\{CP_{n+5,4}, S^{2n-1}\}$, we have

$$\begin{aligned} 8f\circ p_{n+5\ 4} &= f|_{CP_{n+2,1}}\circ s_3 \\ &= 0,\ \text{since } G_3\circ G_7=0 \end{aligned}$$

so $\pi^{SC}_{2n+9}(S^{2n-1})$ does not contain $Z_3\{\beta_1\}$ and hence (ii) follows.

Next we suppose that n is even. If $\pi^{SC}_{2n+9}(S^{2n-1})$ contains $\eta\mu$, that is, there exists $f\in\{CP_{n+5}, S^{2n-1}\}$ with $f\circ p_{n+5}=\eta\mu$, we have the following commutative triangle

$$\begin{tikzcd}
& \widetilde{KO}^{-6n-9}(S^{2n-1}) \arrow[dl, "f^*"'] \arrow[dr, "d_R(\eta\mu)"] & \\
\widetilde{KO}^{-6n-9}(CP_{n+5}) \arrow[rr, "p_{n+5}^*"'] & & \widetilde{KO}^{-6n-9}(S^{2n+9})
\end{tikzcd}$$

But $6n+9\equiv 1 \bmod (8)$ (if $n\equiv 0 \bmod (4)$) or $5 \bmod (8)$ (if $n\equiv 2 \bmod (4)$) and hence $\widetilde{KO}^{-6n-9}(CP_{n+5})=0$ by Theorem 2 of Fujii [4] and

$$d_R(\eta\mu)=p_{n+5}{}^*f^*=0\,.$$

This is a contradiction since $d_R(\eta\mu)\neq 0$ [1]. Thus $\pi_{2n+9}^{SC}(S^{2n-1})$ does not contain $\eta\mu$. Hence (v) follows.

In case of $n\equiv 0 \bmod (6)$, we obtain the following commutative diagram by the methods used in the proof of (3.1) of [8].

$$\begin{tikzcd}
& & & & S^{2n+9} \arrow[d, "p_{n+5,5}"] \arrow[dl, "s_1"'] \arrow[ddll, bend right, "s_2"'] \arrow[ddlll, bend right, "s_3"'] \arrow[dddllll, bend right, "s_4"'] \\
& & & CP_{n+4,4} \arrow[r, phantom, "\subset"] \arrow[d, "2^3"] & CP_{n+5,5} \arrow[d, "2^3"] \\
& CP_{n+2,2} \arrow[r, phantom, "\subset"] \arrow[d, "2^5\cdot 5"] & CP_{n+3,3} \arrow[r, phantom, "\subset"] & CP_{n+4,4} \arrow[r, phantom, "\subset"] & CP_{n+5,5} \arrow[d, "2^5\cdot 5"] \\
CP_{n+1,1}=S^{2n} \arrow[r, phantom, "\subset"] & CP_{n+2,2} \arrow[rrr, phantom, "\subset"] & & & CP_{n+5,5}
\end{tikzcd}$$

Given $f\in\{CP_{n+5,5},\ S^{2n-1}\}$, we have

$$2^8\cdot 5f\circ p_{n+5,5}=f\,|_{CP_{n+1,1}}\circ s_4\in G_1\circ G_9=Z_2\,.$$

Thus $\pi_{2n+9}^{SC}(S^{2n-1})$ does not contain $Z_3\{\beta_1\}$. Hence (iv) follows.

In case of $n\equiv 4 \bmod (6)$, we construct the following commutative diagram which implies (iii) since $h\circ g\circ p_{n+5,4}\in\langle\alpha_1,\alpha_1,\alpha_1\rangle=\beta_1$.

$$\begin{tikzcd}
& & S^{2n+9} \arrow[d, "p_{n+5,4}"] \arrow[ddr, "\alpha_1"] & & \\
& CP_{n+4,3} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{n+5,4} \arrow[d, dashed, "g"] \arrow[dr, dashed, "f"] & & \\
S^{2n+5} \arrow[r, "\alpha_1"] & S^{2n+2} \arrow[r] \arrow[dr, "\alpha_1"'] & C(\alpha_1) \arrow[r, "q"] \arrow[d, dashed, "h"] & S^{2n+6} \arrow[r, "\alpha_1"] & S^{2n+3} \\
& & S^{2n-1} & &
\end{tikzcd}$$

$\alpha_1^2=0$ assures the existence of h. By Theorem 2.6 of Randall [9], there exists f with $f\circ p_{n+5,4}=\alpha_1$. Consider the Puppe exact sequence

$$\cdots\to\{CP_{n+2,1}, S^{2n+2}\}\xrightarrow{p^*_{n+2,1}}\{S^{2n+3}, S^{2n+2}\}\to\{CP_{n+3,2}, S^{2n+3}\}\to \{CP_{n+2,1}, S^{2n+3}\}=0\to\cdots$$

Since $p_{n+2,1}=(n+1)\eta=\eta$, the above $p^*_{n+2,1}$ is an epimorphism, hence $\{CP_{n+3,2}, S^{2n+3}\}=0$. Considering the suitable Puppe sequences, we know easily that i^*: $\{CP_{n+5,4}, S^{2n+3}\}\to\{CP_{n+4,3}, S^{2n+3}\}$ and q_2^*: $\{S^{2n+6}, S^{2n+6}\}\to\{CP_{n+4,3}, S^{2n+6}\}$ are isomorphisms. Consider the Puppe exact sequence

$$\cdots\to\{CP_{n+3,2}, S^{2n+2}\}\xrightarrow{p^*_{n+3,2}}\{S^{2n+5}, S^{2n+2}\}\xrightarrow{q_2^*}\{CP_{n+4,3}, S^{2n+3}\} \to\{CP_{n+3,2}, S^{2n+3}\}=0\to\cdots$$

Then we have the following diagram

$$\begin{tikzcd}
 & \{S^{2n+6}, S^{2n+6}\} \arrow[r, "q_2^*", "\cong"'] \arrow[d, "\alpha_{1*}"] & \{CP_{n+4,3}, S^{2n+6}\} \arrow[d, "\alpha_{1*}"] & \\
0 \arrow[r] \;\pi^{SC}_{2n+5}(S^{2n+2}) \arrow[r] & \{S^{2n+6}, S^{2n+3}\} \arrow[r, "q_2^*"] & \{CP_{n+4,3}, S^{2n+3}\} \arrow[r] & 0 \\
 & & \{CP_{n+5,4}, S^{2n+3}\} \arrow[u, "\cong", "i^*"'] &
\end{tikzcd}$$

By Theorem 2.6 of [9], $\alpha_1\in\pi^{SC}_{2n+5}(S^{2n+2})$. Hence the image of α_{1*} in the left hand side is contained in $\pi^{SC}_{2n+5}(S^{2n+2})$, and the image of α_{1*} in the right hand side is zero. Therefore $i^*(\alpha_1\circ f)=\alpha_{1*}(f\circ i)=0$ and $\alpha_1\circ f=0$. Thus there exists g with $q\circ g=f$.

This completes the proof.

We determine F-projective 11-stem. Given $f\in\{HP_{n+3,3}, S^{4n}\}$ we have

$$e'_R(f\circ p_{n+3,3})=-\frac{1}{2}\deg(f)\alpha_H(n, 3)$$

by (1.5) of [8]. Since e'_R: $G_{11}\to Z_{504}$ is an isomorphism, we have

Theorem 2.12. $\pi^{SH}_{4n+11}(S^{4n})\cong Z/\mathrm{den}\left[\frac{1}{2}H\{n, 3\}\alpha_H(n, 3)\right]$.

We have also

Theorem 2.13. *$\pi^{SC}_{2n+11}(S^{2n})$ is isomorphic to*

(i) $Z/2\,\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$ *if* $n\equiv 0 \bmod (2)$, $5,7 \bmod (8)$, $11 \bmod (16)$, 1 *or* $3 \bmod (32)$,

(ii) $Z/\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$ *if* $n\equiv 9 \bmod (16)$, 17 *or* 19 $\bmod (32)$.

Proof. Let $u(n)$ be the order of the cyclic group $\pi^{SC}_{2n+11}(S^{2n})$. Given $f\in \{CP_{n+6,6}, S^{2n}\}$, we have

$$e'_R(f\circ p_{n+6,6}) = \frac{1}{2}a_6(f) - \frac{1}{2}\deg(f)\alpha_C(n, 6)$$

for some integer $a_6(f)$ by (1.5) of [8]. Choose f_0 with $\deg(f_0)=C\{n, 6\}$. Then

$$u(n) = \mathrm{den}\left[\frac{1}{2}a_6(f_0) - \frac{1}{2}C\{n, 6\}\alpha_C(n, 6)\right]$$

for $e'_R\colon G_{11}\to Z_{504}$ is an isomorphism. Then it is easy to see that $u(n)$ is equal to $\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$ or $2\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$, and equal to $2\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$ if $\nu_2(\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)])\geqq 1$. By (1.16) and (3.1) of [8], $\nu_2(\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)])\geqq 1$ if and only if $n\equiv 7 \bmod (8)$, $11 \bmod (16)$ or $n\equiv 0 \bmod (2)$ and $n\not\equiv 4 \bmod (8)$, $50 \bmod (64)$ and $0 \bmod (128)$. First suppose that $n\equiv 4 \bmod (8)$, $50 \bmod (64)$ or $0 \bmod (128)$. Since $q_5\circ p_{n+6,6}=p_{n+6,1}=\eta$, $\pi^{SC}_{2n+11}(S^{2n})$ contains $\mu\eta\circ q_5\circ p_{n+6,6}=\mu\eta^2=4\zeta$ and hence $u(n)$ is even and in fact $u(n)=2\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$. Thus $u(n)=2\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)]$ if n is even. Next consider the case of n being odd. By (1.16), (3.1), (iii) of (1.4) of [8] and an easy calculation, we check that $a_6(f_0)\equiv 0 \bmod (2)$ (if $n\equiv 3 \bmod (4)$ or $33 \bmod (64)$) or $1 \bmod (2)$ (if $n\equiv 5 \bmod (8)$, $9 \bmod (16)$, $17 \bmod (32)$ or $1 \bmod (64)$). Then by also an easy calculation $u(n)$ is determined as the forms given in Theorem. The proof is completed.

It is easily seen from (1.16) and (2.1) of [8] that $\mathrm{den}\left[\frac{1}{2}H\{3, 3\}\alpha_H(3, 3)\right]=504$, and hence G_{11} is fully H-projective and fully C-projective by (1.1). Thus we have

Corollary 2.14. *G_{11} is fully H- and C-projective.*

Concerning with F-projective 12-stems, we have no problems, since $G_{12}=0$. Recall that $G_{13}=Z_3\{\beta_1\alpha_1\}$. We have

Theorem 2.15. *$\pi^{SH}_{4n+11}(S^{4n-2})$ is equal to*
(i) G_{13} *if* $n\equiv 0$ *or* $2 \bmod (3)$,
(ii) 0 *if* $n\equiv 1 \bmod (3)$.

Proof. Since $q_2\circ p_{n+3,3}=p_{n+3,1}=(n+2)g_\infty$ from (2.10) of [5] (or see (1.14) of [8]), $\pi^{SH}_{4n+11}(S^{4n-2})$ contains $\beta_1\circ q_2\circ p_{n+3,3}=(n+2)\beta_1\alpha_1$. Thus the conclusion (i) follows. Suppose that $n\equiv 1 \bmod (3)$. Then $8q_2\circ p_{n+3,3}=0$ and there exists $s\in\{S^{4n+11}, HP_{n+2,2}\}$ with $i_1\circ s=8p_{n+3,3}$. Given $f\in\{HP_{n+3,3}, S^{4n-2}\}$ we have

$$f\circ p_{n+3,3} = 16f\circ p_{n+3,3} = 2f\circ i_1\circ s\,.$$

But $2\{HP_{n+2,2}, S^{4n-2}\}=0$ by (2.5). Thus $2f\circ i_1\circ s=0$ and the conclusion (ii) follows.

We have also

Theorem 2.16. *$\pi_{2n+13}^{SC}(S^{2n})$ is equal to*
(i) G_{13} *if* $n\equiv 0$ *or* 2 mod(3),
(ii) 0 *if* $n\equiv 1$ mod(3).

Proof. By Randall [9, Theorems 2.5, 2.6], $\alpha_1\in\pi_{2n+13}^{SC}(S^{2n+10})$ if and only if $n\equiv 0$ or 2 mod(3). Then (i) follows from (1.2). In case of $n\equiv 1$ mod(6), (ii) was proved in the proof of (vii) of [8]. By the same methods we can prove (ii) in case of $n\equiv 4$ mod(6). We omit the details.

Concerning with F-projective 14-stems, we prove the following. Recall that $G_{14}=Z_2\{\sigma^2\}\oplus Z_2\{\kappa\}$.

Theorem 2.17. $\pi_{4n+11}^{SH}(S^{4n-3})=Z_2\{\sigma^2\}$ *if* $n\equiv 6$ mod(8).

Proof. Suppose that $n\equiv 6$ mod(8). Since $q_1\circ p_{n+3,2}=p_{n+3,1}=(n+2)g_\infty$, $3q_1\circ p_{n+3,2}=0$ and there exists $s\in\{S^{4n+11}, HP_{n+2,1}\}$ with $i_1\circ s=3p_{n+3,2}$. Since $\sigma\circ p_{n+2,1}=(n+1)\sigma\circ g_\infty=(n+1)\sigma\nu=0$, there exists $f\in\{HP_{n+3,2}, S^{4n-3}\}$ with $f\circ i_1=\sigma$. Put $n=8m+6$. Then by (ii) of (1.13) of [8], we have

$$e_C(s)=(8m+7)(20m+17)/2^4\cdot 3\cdot 5\,.$$

Hence $\sharp s\equiv 0$ mod(2^4) and

$$\begin{aligned} f\circ p_{n+3,2} &= f\circ 3p_{n+3,2}, \text{ since } 2G_{14}=0\\ &= \sigma s\\ &= \sigma^2\,. \end{aligned}$$

Thus $\pi_{4n+11}^{SH}(S^{4n-3})$ contains σ^2. By the following Theorem (2.18), $\eta\circ\pi_{4n+11}^{SH}(S^{4n-3})$ (which is a subgroup of $\pi_{4n+11}^{SH}(S^{4n-4})$) does not contain $\eta\kappa$ and hence $\pi_{4n+11}^{SH}(S^{4n-3})$ does not contain κ. This completes the proof.

Recall that $G_{15}=Z_2\{\eta\kappa\}\oplus Z_{2^5}\{\rho\}\oplus Z_{15}$ and there is a split exact sequence

$$0\to Z_2\{\eta\kappa\}\to G_{15}\xrightarrow{e_C} Z/2^5\cdot 3\cdot 5\to 0\,.$$

We have

Theorem 2.18. *$\pi_{4n+15}^{SH}(S^{4n})$ is isomorphic to*
(i) $Z_2\{\eta\kappa\}\oplus Z/v(n)$ *if* $n\equiv 0$ *or* 3 mod(4),
(ii) $Z/v(n)$ *if* $n\equiv 5$ mod(8),
(iii) $Z_2\{\eta\kappa\}\oplus Z/v(n)$ *or* $Z/v(n)$ *if* $n\equiv 2$ mod(4) *or* 1 mod(8),
and $\pi_{4n+15}^{SH}(S^{4n})$ does not contain $\eta\kappa$ if $n\equiv 5$ mod(8), where $v(n)=\mathrm{den}[H\{n, 4\}\times\alpha_H(n, 4)]$.

Proof. The conclusions (i), (ii) and (iii) follow from (1.2) of [8], because

$\eta\kappa\in\pi^{SH}_{4n+15}(S^{4n})$ if $n\equiv 0$ or $3 \bmod (4)$ from (2.2) of [8]. Next consider the case of $n\equiv 5 \bmod (8)$. Since $q_3\circ p_{n+4,4}=(n+3)g_\infty$, $3q_3\circ p_{n+4,4}=0$ and there exists $s\in\{S^{4n+15}, HP_{n+3,3}\}$ with $i_1\circ s=3p_{n+4,4}$. Let $a\in\{HP_{n+3,3}, S^{4n}\}$ be an element with $\deg(a)=H\{n, 3\}$. Then a generates a free part of $\{HP_{n+3,3}, S^{4n}\}$ which is of rank 1. Given $f\in\{HP_{n+4,4}, S^{4n}\}$, we have

$$f\circ i_1=(\deg(f)/H\{n, 3\})a+e\circ q_2$$

for some $e\in\{HP_{n+3,1}, S^{4n}\}=G_8$ and

$$\begin{aligned}3f\circ p_{n+4,4} &= f\circ i_1\circ s\\ &= (\deg(f)/H\{n, 3\})a\circ s+e\circ q_2\circ s\\ &= (\deg(f)/H\{n, 3\})a\circ s, \text{ since } G_8\circ G_7=0\,.\end{aligned}$$

But by (1.16) and (2.1) of [8], $\nu_2(H\{n, 3\})=3$ and $\nu_2(H\{n, 4\})=6$. Thus $\deg(f)/H\{n, 3\}\equiv 0 \bmod (8)$ since $\deg(f)$ is a multiple of $H\{n, 4\}$. Suppose that $\pi^{SH}_{4n+11}(S^{4n})$ contains $\eta\kappa+x$ for some x which is orthogonal to $Z_2\{\eta\kappa\}$, then $\eta\kappa+x=f\circ p_{n+4,4}$ for some $f\in\{HP_{n+4,4}, S^{4n}\}$. Then

$$\eta\kappa+3x=3f\circ p_{n+4,4}=(\deg(f)/H\{n, 3)\}a\circ s$$

and hence $\eta\kappa+3x$ is divisible by 8. This is a contradiction, for $\#(\eta\kappa)=2$. Thus $\pi^{SH}_{4n+11}(S^{4n})$ does not contain $\eta\kappa+x$ for any $x\in G_{15}$ which is orthogonal to $Z_2\{\eta\kappa\}$. This completes the proof.

By (1.16) and (2.1) of [8] we have easily that $\nu_2(v(n))\leqq 4$, and $\nu_2(v(n))=4$ if and only if $n\equiv 25 \bmod (32)$. Hence we have

Corollary 2.19. *$\rho\in G_{15}$ is not H-projective but 2ρ or $2\rho+\eta\kappa$ is H-projective.*

By (1.1), (2.18) and the above split exact sequence we have

Theorem 2.20. *$\pi^{SC}_{2n+15}(S^{2n})$ is isomorphic to*
(i) $Z_2\{\eta\kappa\}\oplus Z/w(n)$ *if n is even,*
(ii) $Z_2\{\eta\kappa\}\oplus Z/w(n)$ *or* $Z/w(n)$ *if n is odd,*
where $w(n)=\mathrm{den}[C\{n, 8\}\alpha_C(n, 8)]$.

By (1.16) and (3.1) of [8] we have that $\nu_2(w(n))=5$ if and only if $n\equiv 50 \bmod (64)$, and in case of $n\equiv 2 \bmod (4)$, we have that $\nu_3(w(n))=1$ if and only if $n\equiv 14$, 22, 26, 34 mod(36), 10, 38, 46, 74 mod(108), 82 or 190 mod(324), and $\nu_5(w(n))=1$ if and only if $n\equiv 2$, 14, 18 mod(20), 10, 30, 70 or 90 mod(100). Hence we have

Corollary 2.21. *G_{15} is fully C-projective and the smallest n for which $\pi^{SC}_{2n+15}(S^{2n})=G_{15}$ is 178.*

Recall that $G_{17}=Z_2\{\eta\eta^*\}\oplus Z_2\{\nu\kappa\}\oplus Z_2\{\eta^2\rho\}\oplus Z_2\{\bar{\mu}\}$. We have

Proposition 2.22. *$\bar{\mu}$ and the Adams element $\mu_2 \in G_{17}$ are not contained in $\pi^{SC}_{2n+17}(S^{2n})$ if $n \not\equiv 3 \bmod (2^7)$.*

Proof. Since $e_C(\bar{\mu}) = e_C(\mu_2) = \frac{1}{2}$ from (12.13) of [1], it will suffice to show that $\nu_2(C\{n, 9\}\alpha_C(n, 9)) \geqq 0$ if $n \not\equiv 3 \bmod (2^7)$. Indeed by (1.16) and (3.1) of [8] we have

$$C\{n, 9\}/(C\{n, 8\}\operatorname{den}[C\{n, 8\}\alpha_C(n, 8)]) = \begin{cases} 1 \text{ or } 2 \text{ if } n \equiv 3 \bmod (2^7) \text{ or } 1 \bmod (2^9) \\ 1 \qquad \text{otherwise} \end{cases}$$

and an calculation shows that if $n \not\equiv 3 \bmod (2^7)$ and $1 \bmod (2^9)$ we have $\nu_2(C\{n, 9\}\alpha_C(n, 9)) \geqq 0$, and if $n \equiv 1 \bmod (2^9)$ we have $\nu_2(C\{n, 8\}\alpha_C(n, 9)) \geqq 0$ and hence $\nu_2(C\{n, 9\}\alpha_C(n, 9)) \geqq 0$, and the conclusion follows.

By Randall [9, Theorems 2.5, 2.6] we know that $\nu \in \pi^{SC}_{2n+17}(S^{2n+14})$ if and only if $n \equiv 3 \bmod (4)$. And by (i) of (1.13) of [8], $p_{n+9,1} = (n+8)\eta = n\eta$, and so $\eta \in \pi^{SC}_{2n+17}(S^{2n+16})$ if and only if n is odd. Thus if $n \equiv 3 \bmod (4)$, $\pi^{SC}_{2n+17}(S^{2n})$ contains $\nu\kappa$, $\eta\eta^*$ and $\eta^2\rho$. Hence we have

Corollary 2.23. *If $n \equiv 3 \bmod (4)$, then $\pi^{SC}_{2n+17}(S^{2n})$ contains $Z_2\{\eta\eta^*\} \oplus Z_2\{\nu\kappa\} \oplus Z_2\{\eta^2\rho\}$.*

Recall that there exists a split exact sequence [1]

$$0 \to Z_2 \to G_{19} \xrightarrow{e'_R} Z_{264} \to 0 .$$

By (1.5) of [8] we have

Proposition 2.24. *$\pi^{SH}_{4n+19}(S^{4n})$ contains a cyclic subgroup of the order $\operatorname{den}\left[\frac{1}{2}H\{n, 5\}\alpha_H(n, 5)\right]$.*

Take $f \in \{CP_{n+10,10}, S^{2n}\}$ with $\deg(f) = C\{n, 10\}$. From (1.5) of [8]

$$e'_R(f \circ p_{n+10,10}) = \frac{1}{2}a_{10} - \frac{1}{2}C\{n, 10\}\alpha_C(n, 10)$$

for some integer a_{10}, and so $\pi^{SC}_{2n+19}(S^{2n})$ contains a cyclic subgroup of the order $\operatorname{den}\left[\frac{1}{2}a_{10} - \frac{1}{2}C\{n, 10\}\alpha_C(n, 10)\right]$. Even if we can not determine $a_{10} \bmod (2)$, we have $\operatorname{den}\left[\frac{1}{2}a_{10} - \frac{1}{2}C\{n, 10\}\alpha_C(n, 10)\right] = \operatorname{den}\left[\frac{1}{2}C\{n, 10\}\alpha_C(n, 10)\right]$ when

$$(*) \qquad \nu_2(C\{n, 10\}\alpha_C(n, 10)) \leqq -1 .$$

For example if $n \equiv 10, 12, 14 \bmod (2^4)$, $18, 20, 22 \bmod (2^5)$, $6, 34, 36 \bmod (2^6)$ or

102 mod(2^7), then $C\{n, 10\}=C\{n, 7\}$ den$[C\{n, 7\}\alpha_C(n, 8)]$ by (3.1) of [8] and (*) is satisfied. This follows from elementary but routine calculation using (1.16) of [8]. Hence we have

Proposition 2.25. *If* $n\equiv 10, 12, 14$ mod(2^4), 18, 20, 22 mod(2^5), 6, 34, 36 mod(2^6) *or* 102 mod(2^7), *then* $\pi^{SC}_{2n+19}(S^{2n})$ *contains a cyclic subgroup of the order* den$\left[\frac{1}{2}C\{n, 7\}\cdot \text{den}[C\{n, 7\}\alpha_C(n, 8)]\cdot\alpha_C(n, 10)\right]$.

Recall that $G_{21}=Z_2\{\eta\bar{\kappa}\}\oplus Z_2\{\sigma^3\}$ from [6]. By (1.2) and (2.17) we have

Proposition 2.26. *If* $n\equiv 4$ mod(8), *then* $\pi^{SH}_{4n+19}(S^{4n-2})$ *contains* σ^3.

Since $p^C_{m,1}=(m-1)\eta$, by (2.26) we have

Proposition 2.27. *If* $n\equiv 7$ mod(16), *then* $\pi^{SC}_{2n+21}(S^{2n})=G_{21}$.

Recall that $G_{22}=Z_2\{\varepsilon\kappa\}\oplus Z_2\{\nu\bar{\sigma}\}$ from [6]. Since $p^H_{m,1}=(m-1)g_\infty$, by (1.2) and (2.7) we have

Proposition 2.28. $\pi^{SF}_{4n+19}(S^{4n-3})$ *is equal to* G_{22} *if* $n\equiv 3$ mod(4), *and contains* $Z_2\{\varepsilon\kappa\}$ *if* $n\equiv 2$ mod(4) *or* $Z_2\{\nu\bar{\sigma}\}$ *if n is odd.*

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