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ON IRREDUCIBLE UNITARY REPRESENTATIONS OF SOME SPECIAL LINEAR GROUPS OF THE SECOND ORDER, II

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1. Introduction

In this Part II of series of the present work, we discuss the construction of irreducible representations of binary modular congruence group mod p^λ. The method is analogous with that of Part I [6] where we discussed the construction of discrete series of $SL(2, \boldsymbol{K})$, where $\boldsymbol{K}$ is a non-discrete locally compact field.

There exists a homomorphism of $SL(2, \boldsymbol{Z})$ into the symplectic group associated with $\mathbf{Z}/(p^\lambda)\times \boldsymbol{Z}/(p^\lambda)$ and the kernel of this homomorphism is the principal congruence group mod p^λ. So we have a homomorphic imbedding of the modular congruence group into the symplectic group associated with $\boldsymbol{Z}/(p^\lambda)\times \boldsymbol{Z}/(p^\lambda)$. A. Weil [7] constructed a natural projective unitary representation of the symplectic group associated with a locally compact abelian group G on $L^2(G)$. If we take $G=\boldsymbol{Z}/(p^\lambda)\times \boldsymbol{Z}/(p^\lambda)$ and restrict the projective representation to the modular congruence group, we can show that it is a representation in the ordinary sense. The representation thus obtained conicide with the one constructed by H. D. Kloosterman [3] who used the transformation formula of theta functions. The decomposition into invariant irreducible subspaces and the calculation of their traces were performed in detail in [3] and they give the greater part (in fact, for the case $\lambda=1$, all) of irreducible representations.

If we take $G=\boldsymbol{Z}/(p^\lambda)\times \boldsymbol{Z}/(p^{\lambda-1})$ and apply the construction described above, we also have a new representation of the modular congruence group. The complete decomposition into irreducible representations is not undertaken in this paper, and we only show for the special case $\lambda=2$ that all irreducible representations absent in H. D. Kloosterman's work are obtained as invariant subspaces of this representation.

The traces of irreducible representations of the modular congruence group mod p were calculated by G. F. Frobenius. E. Hecke, in connection with his study of the general theory of modular functions, raised the problem of determining all irreducible representations and their traces of the modular congruence group mod p^λ. The first contributions to this problem were published almost simultaneously by H. Rohrbach [5] and H. W. Praetorius [4], both of whom claculated the traces for the special case $\lambda=2$. The general problem was attacked by H. D. Kloosterman as mentioned above.

This Part II is almost independent of Part I and follows directly from §2 of Part I where we summarized the results in Chapter I of [7], In §2 of this paper, we collect definitions and state the principle of our construction. The reconstruction of the representation obtained by H. D. Kloosterman and the construction of a new one are done in §3 and §4 respectively. Preliminary results for the decomposition of the latter into invariant subspaces are contained in §5. In Appendix we consider the special case $\lambda=2$ and calculate the traces of representations on some invariant subspaces. Comparing it with the results in [5], we see that they are irreducible and fill up representations absent in H.D. Kloosterman's work.

Professor H. Yoshizawa informed the author that J. A. Shalika had obtained analogous and, in some points, more explicit results by a different method.

2. Definitions and the principle of the construction

Let us fix an odd prime number p and a natural number λ. For $\alpha\in\boldsymbol{Z}$, $p^n\|\alpha$ implies that the highest power of p which divide α is p^n. For $\alpha\in\boldsymbol{Z}$ such that $\alpha\not\equiv 0\,(p)$, α^{-1} is an integer which satisfies $\alpha\cdot\alpha^{-1}\equiv 1(p^\lambda)$. For $u=(u_1, u_2)\in\boldsymbol{Z}\times\boldsymbol{Z}$, we say $u\equiv 0\,(p^n)$ (or $p^n\,|\,u$) if $p^n\,|\,u_1$ and $p^n\,|\,u_2$. $p^n\|u$ implies that $u\equiv 0(p^n)$ and $u\not\equiv 0(p^{n+1})$. u mod p^n are understood in the same way.

Put $\Gamma=SL(2, \boldsymbol{Z})$ and let us denote with $\Gamma(p^\lambda)$ the principal congruence subgroup mod p^λ:

$$\Gamma(p^\lambda)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma;\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}(p^\lambda)\right\}.$$

$\Gamma(p^\lambda)$ is a normal subgroup of Γ and we call $G(p^\lambda)=\Gamma/\Gamma(p^\lambda)$ the modular congruence group mod p^λ. For $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma$, let l be the integer such that $p^l\|\gamma$ and put $\gamma=p^l\gamma_0$.

We shall apply the general theory of Chapter I in [7] (or see our Part I, §2), taking $G=\boldsymbol{Z}/(p^\lambda)\times\boldsymbol{Z}/(p^\lambda)$ in §3 and $G=\boldsymbol{Z}/(p^\lambda)\times\boldsymbol{Z}/(p^{\lambda-1})$ in §4. They are self-dual and an explicit identification of G^* with G is given separately in §3 and §4.

For $\alpha\in\boldsymbol{Z}$, define homomorphism α of G by $u\alpha=(\alpha u_1, \alpha u_2)$. This establishes a homomorphism of Γ into $Sp(G)$ and the kernel of this homomorphism is $\Gamma(p^\lambda)$. So $G(p^\lambda)$ is imbedded homormophically into $Sp(G)$, so into $B_0(G)$. The image of $g\in G(p^\lambda)$ in $B_0(G)$ by above imbedding is simply denoted

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Added in proof.

Professor M. Kuga informed the author that J. A. Shalika, in 1965, had stated the connection of these problems with the work of A. Weil.

with g.

It is known that the natural homomorphism π_0 of $\boldsymbol{B}_0(G)$ (a group of unitary operators in $L^2(G)$) to $B_0(G)$ is surjective and its kernel is the group of constant multiples of the identity. Let us fix a mapping $\boldsymbol{r}$ from $B_0(G)$ to $\boldsymbol{B}_0(G)$ such that $\pi_0\circ\boldsymbol{r}$ is the identity. If s, s' and s'' in $\boldsymbol{B}_0(G)$ satisfy $ss'=s''$, then there esists a constant $c(s, s')$ such that $\boldsymbol{r}(s)\boldsymbol{r}(s')=c(s, s')\boldsymbol{r}(s'')$.

Let $\mathfrak{H}=L^2(G)$ and V and V' be operators on $\mathfrak{H}$. We shall mean with the notation $V\Phi(u)\sim V'\Phi(u)$ ($\Phi\in\mathfrak{H}$, $u\in G$) that there exists a non-zero constant C such that $V=CV'$. We mostly use this notation as

$$V\Phi(u)\sim\sum_{v\in G}K(u, v)\Phi(v)\,,$$

where V' is defined by $V'\Phi(u)=\sum_{v\in G}K(u, v)\Phi(v)$.

Now consider the sum

$$F_\sigma(n)=\sum_{x \bmod p^n} e^{2\pi i\sigma\frac{x^2}{p^n}}$$

where σ is an integer such that $\sigma\not\equiv 0\ (p)$. $F_\sigma(1)$ is ordinary Gaussian sum and $F_\sigma(1)=p^{1/2}\left(\frac{\sigma}{p}\right)\varepsilon_0$, where $\left(\frac{\sigma}{p}\right)$ is the Legendre symbol and $\varepsilon_0=1$ or i according as $\left(\frac{-1}{p}\right)=1$ or $\left(\frac{-1}{p}\right)=-1$. It is known

$$(1)\qquad F_\sigma(2n)=p^n\,,\quad F_\sigma(2n+1)=p^{n+1/2}\left(\frac{\sigma}{p}\right)\varepsilon_0$$

(see [2, pp. 227–228]).

3. Reconstruction of the representation of $G(p^\lambda)$ obtained by H.D. Kloosterman

Take $G=\boldsymbol{Z}/(p^\lambda)\times\boldsymbol{Z}/(p^\lambda)$. Let Δ be an integer which is without square factor and $\Delta\not\equiv 0\ (p)$. For $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$, $v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in G$, put

$$\langle u, v\rangle=e\left[\frac{{}^t u Q v}{p^\lambda}\right]\quad (e[x]=e^{2\pi i x})\,,$$

where $Q=\begin{pmatrix}1&0\\0&\Delta\end{pmatrix}$. $\langle\ ,\ \rangle$ defines a selfduality of G.

In this case, for $\gamma\not\equiv 0\,(p)$ and $\Phi\in\mathfrak{H}$,

$$(2)\qquad \boldsymbol{r}\begin{pmatrix}1&\beta\\0&1\end{pmatrix}\Phi(u)\sim\Phi(u)e\left[\frac{2^{-1}\beta}{p^\lambda}Q[u]\right]\quad (Q[u]={}^t uQu)$$

and

$$(3)\qquad \boldsymbol{r}\begin{pmatrix}0&-\gamma^{-1}\\\gamma&0\end{pmatrix}\Phi(u)\sim\sum_{v\in G}\Phi(v)\langle -u\gamma^{-1}, v\rangle.$$

For $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G(p^\lambda)$ with $\gamma\not\equiv 0(p)$, by the identity

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}1 & \alpha\gamma^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -\gamma^{-1}\\ \gamma & 0\end{pmatrix}\begin{pmatrix}1 & \delta\gamma^{-1}\\ 0 & 1\end{pmatrix}$$

and formulas (2) and (3), we have

$$(4)\qquad r(g)\Phi(u)\sim\sum_{v\in G}e\left[\frac{2^{-1}\alpha\gamma^{-1}Q[u]+2^{-1}\delta\gamma^{-1}Q[u]-\gamma^{-1\,t}uQv}{p^\lambda}\right]\Phi(v)\,.$$

Now let $\gamma\equiv 0(p)$ and $\gamma\equiv p^l\gamma_0$, $\gamma_0\not\equiv 0(p)$. $\alpha\not\equiv 0(p)$ in this case and $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=s\begin{pmatrix}\gamma & \delta\\ -\alpha & -\beta\end{pmatrix}\left(s=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)$.

So

$$r(g)\Phi(u)\sim r(s)r\begin{pmatrix}\gamma & \delta\\ -\alpha & -\beta\end{pmatrix}\Phi(u)$$
$$\sim\sum_{v\in G}e\left[-\frac{{}^tuQv}{p^\lambda}\right]\sum_{w\in G}e\left[\frac{-2^{-1}\gamma\alpha^{-1}Q[v]+2^{-1}\beta\alpha^{-1}Q[w]+\alpha^{-1\,t}vQw}{p^\lambda}\right]\Phi(w)\,.$$

Let us evaluate the summation over v. Put

$$\varphi=\sum_{v\in G}e\left[-\frac{{}^tuQv}{p^\lambda}\right]\cdot e\left[\frac{-2^{-1}\gamma\alpha^{-1}Q[v]+\alpha^{-1\,t}vQw}{p^\lambda}\right].$$

Then

$$\varphi=\sum_{v\in G}e\left[-\frac{\alpha^{-1}}{p^\lambda}\{2^{-1}\gamma Q[v]+{}^t(u\alpha-w)Qv\}\right].$$

So we have

$$|\varphi|^2=\sum_{v,v'\in G}e\left[-\frac{\alpha^{-1}}{p^\lambda}\{2^{-1}\gamma(Q[v']-Q[v])+{}^t(u\alpha-w)Q(v'-v)\}\right].$$
$$=\sum_{v,t\in G}e\left[-\frac{\alpha^{-1}}{p^\lambda}\{2^{-1}\gamma Q[t]+\gamma{}^tvQt+{}^t(u\alpha-w)Qt\}\right].$$

Summation over v is 0 unless $p^{\lambda-l}\,|\,t$. Therefore

$$|\varphi|^2=p^{2\lambda}\sum_{t\in G,\,p^{\lambda-l}|t}e\left[-\frac{\alpha^{-1\,t}(u\alpha-w)Qt}{p^\lambda}\right].$$

So φ is 0 unless $p^l\,|\,u\alpha-w$. Now let $p^l\,|\,u\alpha-w$ and put $u\alpha-w=ap^l$, then

$$\varphi=\sum_{v\in G}e\left[-\frac{\alpha^{-1}}{p^{\lambda-l}}\{2^{-1}\gamma_0Q[v]+{}^taQv\}\right]$$
$$=\sum_{u\in G}e\left[-\frac{\alpha^{-1}}{p^{\lambda-l}}\{2^{-1}\gamma_0Q[u+\gamma_0^{-1}a]-2^{-1}\gamma_0^{-1}Q[a]\}\right]$$
$$=e\left[\frac{2^{-1}\alpha^{-1}\gamma_0^{-1}}{p^{\lambda-l}}Q[a]\right]\sum_{v\in G}e\left[-\frac{2^{-1}\alpha^{-1}\gamma_0}{p^{\lambda-l}}Q[v]\right].$$

So we have

$$r(g)\Phi(u) \sim \sum_{w\in G, p^l | u\alpha - w} e\left[\frac{2^{-1}\alpha^{-1}\beta}{p^\lambda} Q[w]\right] e\left[\frac{2^{-1}\alpha^{-1}\gamma_0^{-1}}{p^{\lambda+l}} Q[u\alpha - w]\right]\Phi(w),$$

or

$$(5) \qquad r(g)\Phi(u) \sim \sum_{w\in G} k(g\,|\,u, w)\Phi(w),$$

where

$$(6) \quad k(g\,|\,u, v) = e\left[\frac{2^{-1}\alpha\gamma_0^{-1}Q[u] + 2^{-1}\delta\gamma_0^{-1}Q[v] - \gamma_0^{-1}\,{}^t uQv}{p^{\lambda+l}}\right], \quad \text{if} \quad p^l\,|\,u\alpha - v,$$

$$= 0, \qquad \text{otherwise.}$$

We have assumed that $\gamma \equiv 0(p)$ i.e. $l \geqslant 1$, however (5), (6) are valid for all $g \in G(P^\lambda)$.

Now let $gg' = g''$, where

$$g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad g' = \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} \quad \text{and} \quad g'' = \begin{pmatrix}\alpha'' & \beta''\\ \gamma'' & \delta''\end{pmatrix}$$

with $\gamma = p^l\gamma_0$, $\gamma' = p^{l'}\gamma_0'$ and $\gamma'' = p^{l''}\gamma_0''$ ($\gamma_0 \not\equiv 0(p)$ etc.). There exists a constant $c = c(g, g')$ such that

$$\sum_{v\in G} k(g\,|\,u, v)k(g'\,|\,v, w) = ck(g''\,|\,u, w).$$

Putting $u = w = 0$ in this identity, we have

$$c = \sum_{v\in G, p^l|v, p^{l'}|v\alpha'} e\left[\frac{2^{-1}\delta\gamma_0^{-1}}{p^{\lambda+l}} Q[v]\right] e\left[\frac{2^{-1}\alpha'\gamma_0'^{-1}}{p^{\lambda+l'}} Q[v]\right]$$

$$= \sum_{v\in G, p^l|v, p^{l'}|v\alpha'} e\left[\frac{2^{-1}\gamma''\gamma_0^{-1}\gamma_0'^{-1}}{p^{\lambda+l+l'}} Q[v]\right].$$

Assume $l \geqslant l'$, then

$$c = \sum_{v\in G, p^l|v} e\left[\frac{2^{-1}\gamma_0''\gamma_0^{-1}\gamma_0'^{-1}}{p^{\lambda+l+l'-l''}} Q[v]\right]$$

$$= \sum_{v \bmod p^{\lambda-l}} e\left[\frac{2^{-1}\gamma_0''\gamma_0^{-1}\gamma_0'^{-1}}{p^{\lambda-l+l'-l''}} Q[v]\right].$$

Let first $k = \lambda - l + l' - l'' > 0$. Writing $v = v' + v''p^k$, where v' and v'' run through a complete system of residues mod p^k and $p^{l''-l'}$ respectively. Then

$$c = p^{2(l''-l')} \sum_{v' \bmod p^k} e\left[\frac{2^{-1}\gamma_0''\gamma_0^{-1}\gamma_0'^{-1}}{p^k} Q[v']\right].$$

By (1),

$$(7) \qquad c = p^{\lambda-l-l'+l''}\left(\frac{-\Delta}{p}\right)^{\lambda-l-l'+l''}.$$

Next, let $\lambda \leqslant l+l''-l'$. In this case $c=p^{2(\lambda-l)}$. $l=l'$ implies $l''=\lambda$. If $1\leqslant l'<l$, then by $\gamma\alpha'+\delta\gamma'=\gamma''$, $p^{l}\|\gamma\alpha'$ and $p^{l'}\|\delta\gamma'$, we have $l''=l'$ and $l=\lambda$. Remaining case $0=l'<l$ (recall we have assumed $l'\leqslant l$) can be discussed analogously and implies $l''=0$, $l=\lambda$. So (7) is valid even in the case $\lambda\leqslant l+l''-l'$.

Put

$$K(g\,|\,u, v) = p^{-\lambda+l}\left(\frac{-\Delta}{p}\right)^{\lambda-l} k(g\,|\,u, v),$$

and define operator $T(g)$ by

$$T(g)\Phi(u) = \sum_{v\in G} K(g\,|\,u, v)\Phi(v),$$

Then we have $T(g)T(g')=T(gg')$ if $l\geqslant l'$, in particular, $T(g)T(g^{-1})=I$. So $T(g)T(g')=T(gg')$ without restriction $l\geqslant l'$. The obtained representation is unitary because $T(g^{-1})=T(g)^*$, which can be verified directly.

4. Construction of a new representation of $G(p^\lambda)$

Put $G=\boldsymbol{Z}/(p^\lambda)\times\boldsymbol{Z}/(p^{\lambda-1})$ $(\lambda\geqslant 2)$. Let Δ be an integer without square factor such that $\Delta=p\Delta'$, $\Delta'\not\equiv 0(p)$ and σ be an integer such that $\sigma\not\equiv 0(p)$. For $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$, $v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in G$, put

$$\langle u, v\rangle = e_\sigma\left[\frac{2^t uQv}{p^\lambda}\right] \quad (e_\sigma[x]=e^{2\pi i\sigma x}),$$

where $Q=\begin{pmatrix}1 & 0\\ 0 & \Delta\end{pmatrix}$. G is self-dual with respect to $\langle\, ,\rangle$.

In this case, for $\alpha, \gamma\not\equiv 0(p)$ and $\Phi\in\mathfrak{H}=L^2(G)$,

$$r\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\Phi(u)\sim\Phi(u\alpha),$$

$$r\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\Phi(u)\sim\Phi(u)e_\sigma\left[\frac{\beta}{p^\lambda}Q[u]\right] \quad (Q[u]={}^t uQu)$$

and

$$r\begin{pmatrix}0 & -\gamma^{-1}\\ \gamma & 0\end{pmatrix}\Phi(u)\sim\sum_{v\in G}\Phi(v)\langle -u\gamma^{-1}, v\rangle.$$

For $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G(p^\lambda)$ with $\gamma\not\equiv 0(p)$, we have

$$r(g)\Phi(u)\sim\sum_{v\in G} e_\sigma\left[\frac{\alpha\gamma^{-1}Q[u]+\delta\gamma^{-1}Q[v]-2\gamma^{-1}\,{}^t uQv}{p^\lambda}\right]\Phi(v).$$

Now let $\gamma\equiv 0(p)$ and $\gamma=p^l\gamma_0$, $\gamma_0\not\equiv 0(p)$ with $1\leqslant l\leqslant\lambda-1$. We have

$$r(g)\Phi(u) \sim r(s)r\begin{pmatrix}\gamma & \delta \\ -\alpha & -\beta\end{pmatrix}\Phi(u)$$

$$\sim \sum_{v\in G} e_\sigma\left[-\frac{2{}^tuQv}{p^\lambda}\right]\sum_{w\in G} e_\sigma\left[\frac{-\gamma\alpha^{-1}Q[u]+\beta\alpha^{-1}Q[w]+2\alpha^{-1}\,{}^tvQw}{p^\lambda}\right]\Phi(w)\,.$$

Let us evaluate the summation over v. Put

$$\varphi = \sum_{v\in G} e_\sigma\left[-\frac{2{}^tuQv}{p^\lambda}\right]e_\sigma\left[\frac{-\gamma\alpha^{-1}Q[v]+2\alpha^{-1}\,{}^tvQw}{p^\lambda}\right]$$

Then

$$\varphi = \sum_{v\in G} e_\sigma\left[-\frac{\alpha^{-1}}{p^\lambda}\{\gamma Q[v]+2\,{}^t(u\alpha-w)Qv\}\right].$$

So

$$|\varphi|^2 = \sum_{v,v'\in G} e_\sigma\left[-\frac{\alpha^{-1}}{p^\lambda}\{\gamma(Q[v']-Q[v])+2\,{}^t(u\alpha-w)Q(v'-v)\}\right]$$

$$= \sum_{v,t\in G} e_\sigma\left[-\frac{\alpha^{-1}}{p^\lambda}\{\gamma Q[t]+2\gamma\,{}^tvQt+2\,{}^t(u\alpha-w)\,Qt\}\right].$$

Summation over v is 0 unless $p^{\lambda-l}\,|\,t_1$ and $p^{\lambda-l-1}\,|\,t_2$. Therefore

$$|\varphi|^2 = p^{2\lambda-1}\sum_{t\in G,\,p^{\lambda-l}|t_1,\,p^{\lambda-l-1}|t_2} e_\sigma\left[-\frac{2\alpha^{-1}\,{}^t(u\alpha-w)Qt}{p^\lambda}\right]$$

So φ is 0 unless $p^l\,|\,u\alpha-w$. Now let $p^l\,|\,u\alpha-w$ and put $u\alpha-w=ap^l$, then

$$\varphi = \sum_{v\in G} e_\sigma\left[-\frac{\alpha^{-1}}{p^{\lambda-l}}\{\gamma_0 Q[v]+2\,{}^taQv\}\right]$$

$$= e_\sigma\left[\frac{\alpha^{-1}\gamma_0^{-1}}{p^{\lambda-l}}\,Q[a]\right]\sum_{v\in G} e_\sigma\left[-\frac{\alpha^{-1}\gamma_0}{p^{\lambda-l}}\,Q[v]\right]$$

So we have

$$r(g)\Phi(u) \sim \sum_{w\in G,\,p^l|u\alpha-w} e_\sigma\left[\frac{\alpha^{-1}\beta}{p^\lambda}\,Q[w]\right]e_\sigma\left[\frac{\alpha^{-1}\gamma_0^{-1}}{p^{\lambda+l}}\,Q[u\alpha-w]\right]\Phi(w)\,,$$

or

$$(8)\qquad r(g)\Phi(u) \sim \sum_{w\in G} k(g\,|\,u,w)\Phi(w)\,,$$

where

$$(9)\qquad k(g\,|\,u,v) = e_\sigma\left[\frac{\alpha\gamma_0^{-1}Q[u]+\delta\gamma_0^{-1}Q[v]-2\gamma_0^{-1}\,{}^tuQv}{p^{\lambda+l}}\right],\quad \text{if}\quad p^l\,|\,u\alpha-v\,,$$

$$= 0\,,\qquad\qquad \text{otherwise.}$$

We have assumed $\gamma\equiv 0(p)$ i.e. $l\geqslant 1$, however (8), (9) are valid for $g\in G(p^\lambda)$ with $l\leqslant\lambda-1$. If $l=\lambda$, then

$$r(g)\Phi(u) \sim \sum_{v \in G} k(g \mid u, v)\Phi(v) ,$$

where

$$\begin{aligned} k(g \mid u, v) &= e_\sigma\left[\frac{\alpha\beta}{p^\lambda} Q[u]\right], \quad \text{if} \quad p^\lambda \mid \alpha u_1 - v_1 , \quad p^{\lambda-1} \mid \alpha u_2 - v_2 \\ &= 0 , \qquad \text{otherwise,} \end{aligned}$$

which can be shown directly.

Now let $gg' = g''$, where

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad g' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \quad \text{and} \quad g'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$$

with $\gamma = p^l \gamma_0$, $\gamma' = p^l \gamma_0'$ and $\gamma'' = p^{l''} \gamma_0''$ ($\gamma_0 \not\equiv 0(p)$ etc.). There exists a constant $c = c(g, g')$ such that

$$\sum_{v \in G} k(g \mid u, v) k(g' \mid v, w) = c k(g'' \mid u, w) .$$

Assuming $l' \leqslant l \leqslant \lambda - 1$, put $u = w = 0$ in this identity. We have

$$\begin{aligned} c &= \sum_{v \in G, p^l \mid v} e_\sigma\left[\frac{\delta\gamma_0^{-1}}{p^{\lambda+l}} Q[v]\right] e_\sigma\left[\frac{\alpha'\gamma_0'^{-1}}{p^{\lambda+l'}} Q[v]\right] \\ &= \sum_{v \in G, p^l \mid v} e_\sigma\left[\frac{\gamma''\gamma_0^{-1}\gamma_0'^{-1}}{p^{\lambda+l+l'}} Q[v]\right] \\ &= \sum_{v_1 \bmod p^{\lambda-l},\, v_2 \bmod p^{\lambda-l-1}} e_\sigma\left[\frac{\gamma_0''\gamma_0^{-1}\gamma_0'^{-1}}{p^{\lambda-l+l'-l''}} Q[v]\right] . \end{aligned}$$

Let first $k = \lambda - l + l' - l'' > 0$. Then

$$c = p^{2(l''-l')} \sum_{v_1' \bmod p^k,\, v_2' \bmod p^{k-1}} e_\sigma\left[\frac{\gamma_0''\gamma_0^{-1}\gamma_0'^{-1}}{p^k} Q[v']\right] .$$

Using (1), we have

$$c = {}^{\lambda-l-l'+l''-1/2} \left(\frac{\Delta'}{p}\right)^{\lambda-l+l'-l''-1} \left(\frac{\gamma_0\gamma_0'\gamma_0''\sigma}{p}\right) \varepsilon_0 .$$

Next, let $\lambda \leqslant l + l'' - l'$. This occurs only if $l = l'$ and $l'' = \lambda$ and $c = p^{2(\lambda-l)-1}$ in this case.

Put

$$\begin{aligned} K(g \mid u, v) &= p^{-\lambda+l+1/2} \left(\frac{\Delta'}{p}\right)^{\lambda-l-1} \left(\frac{\gamma_0\sigma}{p}\right) \varepsilon_0^{-1} k(g \mid u, v) , \quad \text{if} \quad l \leqslant \lambda - 1 , \\ &= \left(\frac{\alpha}{p}\right) k(g \mid u, v) , \quad \text{if} \quad l = \lambda , \end{aligned}$$

and define operator $T(g)$ by

$$T(g)\Phi(u)=\sum_{v\in G}K(g\,|\,u,v)\Phi(v)\,.$$

$T(g)$ is a unitary representation of $G(p^\lambda)$

5. Preliminary results for the decomposition of the representation in §4 into invariant subspaces.

5.1. Automorphism of Q.

Let us consider the set of all matrices

$$V=\begin{pmatrix}x_1 & -\Delta x_2\\ x_2 & x_1\end{pmatrix}$$

with x_1, x_2 satisfying

$$x_1^2+\Delta x_2^2\equiv 1\,(p^\lambda)\,. \tag{10}$$

We introduce in this set following equivalence relation:

$\begin{pmatrix}x_1 & -\Delta x_2\\ x_2 & x_1\end{pmatrix}$ and $\begin{pmatrix}y_1 & -\Delta y_2\\ y_2 & y_1\end{pmatrix}$ are equivalent if an only if $x_1\equiv y_1(p^\lambda)$ and $x_2\equiv y_2(p^\lambda)$. Then it form a group $\mathfrak{G}$ of order $2p^\lambda$ with ordinary multiplication rule of matrices. $V\in\mathfrak{G}$ induces an automorphism of G defined by $G\ni a=\begin{pmatrix}a_1\\ a_2\end{pmatrix}\to V\begin{pmatrix}a_1\\ a_2\end{pmatrix}$. It is shown that

$${}^tVQV\equiv Q(p^\lambda)\qquad\text{for}\quad V\in\mathfrak{G}\,. \tag{11}$$

5.2. Stationary subgroups. Let us determine the stationary subgroup of $\mathfrak{G}$ at $a=(a_1, a_2)\in G$ i.e. the elements of $\mathfrak{G}$ which statisfy

$$\begin{cases} x_1a_1-\Delta x_2a_2=a_1(p^\lambda)\\ x_2a_1+\ \ x_1a_2=a_2(p^{\lambda-1})\,.\end{cases} \tag{12}$$

Put for $k\geqslant 1$ $S_k=\{a=(a_1, a_2)\in G;\, p^{\lambda-k}\|a\}$. For $a\in S_k$, if we write $a=p^{\lambda-k}a^0$, (12) reduces to

$$\begin{cases}(x_1-1)a_1^0-\Delta x_2a_2^0\equiv 0(p^k)\\ x_2a_1^0+(x_1-1)a_2^0\equiv 0(p^{k-1})\,.\end{cases} \tag{12$'$}$$

x_1, x_2 with (10) satisfy (12)$'$ if and only if

$$x_1\equiv 1(p^k)\,,\quad x_2\equiv 0(p^{k-1})\,, \tag{13}$$

which is verified by considering the case $a_1^0\not\equiv 0(p)$ and $a_1^0\equiv 0(p)$ separately. We will denote with $\mathfrak{S}_k$ the subgroup of elements of $\mathfrak{G}$ which satisfy (13).

The order of the group $\mathfrak{G}/\mathfrak{S}_k$ is $2p^\lambda/p^{\lambda-k+1}=2p^{k-1}$. Number of elements of G which are contained in S_k is $p^{2k-3}(p-1)$ if $k\geqslant 2$ and $p-1$ if $k=1$. So numbers of $\mathfrak{G}/\mathfrak{S}_k$-transitive parts of S_k are $2^{-1}p^{k-2}(p^2-1)$ if $k\geqslant 2$ and $2^{-1}(p-1)$ if $k=1$.

$\mathfrak{G}_1/\mathfrak{G}_\lambda$ is isomorphic to $\boldsymbol{Z}/(p^{\lambda-1})$. For the case $\lambda=2$, explicit form of the isomorphism is found in Appendix.

In general, the isomorphism is established by the aid of the theory of p-adic exponential function (see for instance [1, pp. 177–179]) with the additional assumption that $p>3$.

5.3. A quadratic number field. Let d' be an integer without square factors such that $d'\equiv-\Delta'$, $d'\equiv 2(4)$ (see [3, p. 377]) and put $d=pd'$, then d is square free and $d\equiv-\Delta(p^{2\lambda})$, $d\equiv 2(4)$. Consider the quadratic number fileld $Q(w)$, where $W=\sqrt{d}$. By natural homomorphism from integers of $Q(w)$ to G defined by $a=a_1+wa_2\to(a_1, a_2)$, G can be identified with the residue classes of integrs of $Q(w)$ with respect to the following equivalence relation: $a_1=a+wa_2$ and $b_1=b+wb_2$ are equivalent if and only if $a_1\equiv b_1(p^\lambda)$ and $a_2\equiv b_2(p^{\lambda-1})$. The equivalence class containing $a=a_1+wa_2$ is also denoted with a. We write $a\equiv b(p^l)$, if $a_1\equiv b_1(p^l)$ and $a_2\equiv b_2(p^l)$ for $l\leqslant\lambda-1$.

The transformation of G induced by $V=\begin{pmatrix} x_1 & -\Delta x_2 \\ x_2 & x_1 \end{pmatrix}\in\mathfrak{G}$ is written as $a\to\varepsilon a(\varepsilon=x_1+wx_2)$ by the above identification. Thus $\mathfrak{G}$ is identified with multiplicative group of all integers $\varepsilon=x_1+wx_2$ in $Q(w)$, x_1, x_2 satisfying (10) and determined mod p^λ.

5.4. Invariant subspaces corresponding to the primitive charactors. Let $\chi(\varepsilon)$ be a character of $\mathfrak{G}/\mathfrak{S}_\lambda$ such that its restriction to $\mathfrak{S}_{\lambda-1}/\mathfrak{S}_\lambda$ is not trivial. We call such character a primitive character. Now let us consider the subspace $\mathfrak{H}_\chi$ of $\mathfrak{H}$ consisting of elements Φ which satisfy $\Phi(\varepsilon u)=\chi(\varepsilon)\Phi(u)$ for all $\varepsilon\in\mathfrak{G}$. $\mathfrak{H}_\chi$ is invariant subspaces and let $T_\chi(g)=T(g)\,|\,\mathfrak{H}_\chi$. If $\Phi\in\mathfrak{H}_\chi$, then $\Phi(u)=0$ unless $u\in S_\lambda$. Let θ be a system of representatives of the $\mathfrak{G}/\mathfrak{S}_\chi$-transitive parts of S_λ. Then for $\Phi\in\mathfrak{H}_\chi$,

$$\begin{aligned} T(g)\Phi(u) &= \sum_{v\in G} K(g\,|\,u, v)\Phi(v) \\ &= \sum_{v\in\theta}\ \sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda} K(g\,|\,u, \varepsilon v)\Phi(\varepsilon v) \\ &= \sum_{v\in\theta}\Big[\sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda} K(g\,|\,u, \varepsilon v)\chi(\varepsilon)\Big]\Phi(v)\,. \end{aligned}$$

Therefore

$$\begin{aligned} T_rT_\chi(g) &= \sum_{a\in\theta}\ \sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda} K(g\,|\,a, \varepsilon a)\chi(\varepsilon) \\ &= \frac{1}{2p^{\lambda-1}}\sum_{a\in S_\lambda}\ \sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda} K(g\,|\,a, \varepsilon a)\chi(\varepsilon)\,. \end{aligned}$$

Let us write this formula more explicitly. First let $l\leqslant\lambda-1$, then

$$(14)\quad \frac{1}{c}T_rT_\chi(g)=\sum_{a\in S_\lambda}\sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda,\ \varepsilon a\equiv a\alpha(p^l)} e_\sigma\left[\frac{\gamma_0^{-1}(\alpha+\delta-\varepsilon-\bar{\varepsilon})N(a)}{p^{\lambda+l}}\right]\chi(\varepsilon)\,,$$

where

$$c=\frac{1}{2p^{\lambda-1}}p^{-\lambda+l+1/2}\left(\frac{\Delta'}{p}\right)^{\lambda-l-1}\left(\frac{\gamma_0\sigma}{p}\right)\varepsilon_0^{-1}\,.$$

When $l=0$, the congruence $\varepsilon a\equiv a\alpha(p^l)$ is no restriction on ε. Now let $l\geqslant 1$ and let us only consider g with $\alpha\equiv 1(p^l)$. If we put $\varepsilon=x_1+wx_2$, then $\varepsilon a\equiv a(p^l)$ is equivalent to

$$(15)\quad \begin{cases}(x_1-1)a_1-\Delta x_2a_2\equiv 0(p^l)\\ x_2a_1+(x_1-1)a_2\equiv 0(p^l)\,.\end{cases}$$

If $a_1\not\equiv 0(p)$, then ε satisfies (15) if and only if $\varepsilon\in\mathfrak{G}_{l+1}$. If $a_1\equiv 0(p)$ (in this case $a_2\not\equiv 0(p)$), then ε satisfies (15) if and only if $\varepsilon\in\mathfrak{G}_l$. So (14) reduces to

$$(14)'\quad \frac{1}{c}T_rT_\chi(g)=\sum_{a\in S_\lambda}\sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda} e_\sigma\left[\frac{\gamma_0^{-1}(\alpha+\delta-\varepsilon-\bar{\varepsilon})N(a)}{p^{\lambda+l}}\right]\chi(\varepsilon)\,,\quad \text{if}\quad l=0\,,$$

$$=\sum_{a\in S_\lambda,\ a_1\not\equiv 0(p)}\ \sum_{\varepsilon\in\mathfrak{S}_{l+1}/\mathfrak{S}_\lambda} e_\sigma\left[\frac{\gamma_0^{-1}(\alpha+\delta-\varepsilon-\bar{\varepsilon})N(a)}{p^{\lambda+l}}\right]\chi(\varepsilon)$$

$$+\sum_{a\in S_\lambda,\ a_1\equiv 0(p)}\ \sum_{\varepsilon\in\mathfrak{S}_l/\mathfrak{S}_\lambda} e_\sigma\left[\frac{\gamma_0^{-1}(\alpha+\delta-\varepsilon-\bar{\varepsilon})N(a)}{p^{\lambda+l}}\right]\chi(\varepsilon)\,,$$

$$\text{if}\quad 1\leqslant l\leqslant\lambda-1\,.$$

Next let $l=\lambda$, then

$$T_rT_\chi(g)=\frac{1}{2p^{\lambda-1}}\left(\frac{\alpha}{p}\right)\sum_{a\in S_\lambda}\sum_{\varepsilon\in\mathfrak{G}/\mathfrak{S}_\lambda,\ \varepsilon a=a\alpha} e_\sigma\left[\frac{\alpha\beta N(a)}{p^\lambda}\right]\chi(\varepsilon)\,.$$

If we put $\varepsilon=x_1+wx_2$, $\varepsilon a=a\alpha$ is equivalent to

$$\begin{cases}(x_1-\alpha)a_1-\Delta x_2a_2\equiv 0(p^\lambda)\\ x_2a_1+(x_1-\alpha)a_2\equiv 0(p^{\lambda-1})\,.\end{cases}$$

If $a_1\not\equiv 0(p)$, then

$$(x_1-\alpha)(a_1^2+\Delta a_2^2)\equiv 0(p^\lambda)\,.$$

So $x_1-\alpha\equiv 0(p^\lambda)$ and $x_2\equiv 0(p^{\lambda-1})$. It is necessary for the existence of such ε that $\alpha\equiv\pm 1(p^\lambda)$. If $\alpha\equiv\pm 1(p^\lambda)$, $\varepsilon a=a\alpha$ if and only if $\pm\varepsilon\in\mathfrak{S}_\lambda$. Now if $a_1\equiv 0(p)$, then

$$x_2(a_1^2+\Delta a_2^2)\equiv 0(p^\lambda)\,.$$

So $x_2\equiv 0(p^{\lambda-1})$ and $x_1-\alpha\equiv 0(p^{\lambda-1})$. It is necessary for the existence of such ε that $\alpha\equiv\pm 1(p^{\lambda-1})$. If $\alpha\pm\equiv 1(p^{\lambda-1})$, $\varepsilon a=a\alpha$ if and only if $\pm\varepsilon\in\mathfrak{S}_\lambda$. We have thus obtained

$$(16)\quad T_rT_\chi(g)=\frac{1}{2p^{\lambda-1}}\chi(\pm 1)\left(\frac{\alpha}{p}\right)\sum_{a\in S_\lambda}e_\sigma\left[\frac{\beta N(a)}{p^\lambda}\right],\quad \text{if}\quad \alpha\equiv\pm 1(p^\lambda),$$
$$=\frac{1}{2p^{\lambda-1}}\chi(\pm 1)\left(\frac{\alpha}{p}\right)\sum_{a\in S_\lambda,\, a_1\equiv 0(p)}e_\sigma\left[\frac{\beta N(a)}{p^\lambda}\right],\quad \text{if}\quad \alpha\equiv\pm 1(p^{\lambda-1}),$$
$$=0,\qquad \text{otherwise.}$$

Appendix Discussion of the case λ=2.

In this appendix we calculate traces of representations T_χ in § 5.4. explicitly for the case $\lambda=2$ and see that they are irreducible and together with irreducible representations constructed by H.D. Kloosterman [3] exhaust all irreducible representations of $G(p^2)$. For calculation of traces we use the representative of conjugate classes in $G(p^2)$ introduced in [5]. Note that if $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, then $g'=sgs^{-1}=\begin{pmatrix}\delta & -\gamma\\ -\beta & \alpha\end{pmatrix}\left(s=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)$ and $T_rT_\chi(g)=T_rT_\chi(g')$. We write H instead of σ and use the notation $\sum'_{x \bmod p}$ instead of $\sum_{x \bmod p,\, x\not\equiv 0(p)}$.

Let first $l=2$. If $\alpha\equiv\pm 1(p^2)$, then we have the following results by (16):

$$T_rT_\chi(g)=\frac{1}{2}(\pm 1)^f\left(-1+p^{1/2}\left(\frac{\beta\Delta' H}{p}\right)\varepsilon_0\right),\quad \text{if}\quad \beta\not\equiv 0(p),$$
$$=\frac{1}{2}(\pm 1)^f\left(-1+p^{3/2}\left(\frac{\beta_0 H}{p}\right)\varepsilon_0\right),\quad \text{if}\quad \beta=p\beta_0,\quad \beta_0\not\equiv 0(p),$$
$$=\frac{1}{2}(\pm 1)^f(p^2-1),\qquad \text{if}\quad \beta\equiv 0(p^2),$$

where $(-1)^f=\chi(-1)\left(\frac{-1}{p}\right)$. Traces corresponding to the representatives E, F, A, B, P and Q are obtained.

Trace of $D^\lambda(l=2$ and $p\|\alpha-1)$ is also calculated by (16) and is equal to $\frac{1}{2}(p-1)$.

Next, let $l=1$. $\eta\to\varepsilon(\eta)=1-2^{-1}\Delta\eta^2+w\eta$ establishes the isomorphism between $\boldsymbol{Z}/(p)$ and $\mathfrak{S}_1/\mathfrak{S}_2$, so the primitive charcater χ is written as $\chi(\varepsilon(\eta))=e\left[\frac{K\eta}{p}\right]$ $K\not\equiv 0(p)$). By (14)′,

$$\frac{1}{c}\,T_rT_\chi(g)=A+B\quad\left(c=\frac{1}{2p}p^{-1/2}\left(\frac{\gamma_0 H}{p}\right)\varepsilon_0^{-1}\right),$$

where

$$A=\sum_{a\in S_2,\, a_1\not\equiv 0(p)}e_H\left(\frac{\gamma_0^{-1}(\alpha+\delta-2)N(a)}{p^3}\right)$$

and

$$B=\sum_{a\in S_2,\, a_1\equiv 0(p)}\ \sum_{\eta \bmod p}e\left[\frac{K\eta}{p}\right]e_H\left[\frac{\gamma_0^{-1}(\alpha+\delta-2+\Delta\eta^2)N(a)}{p^3}\right]$$

We have

$$B=\sum_{a_1,a_2 \bmod p,\ a_2\not\equiv 0(p)}\ \sum_{\eta \bmod p} e\left[\frac{K\eta}{p}\right]e_H\left[\frac{\gamma_0^{-1}(\alpha+\delta-2+\Delta\eta^2)(a_1^2p+\Delta' a_2^2)}{p^3}\right]$$

$$=\sum_{a_1 \bmod p} e_H\left[\frac{\gamma_0^{-1}(\alpha+\delta-2)a_1^2}{p}\right]\sum_{a_2 \bmod p}{}'\ \sum_{\eta \bmod p} e\left[\frac{K\eta}{p}\right]e_H\left[\frac{\gamma_0^{-1}\Delta'(\alpha+\delta-2+\Delta\eta^2)a_2^2}{p^2}\right]$$

$$=\sum_{a_1 \bmod p} e_H\left[\frac{\gamma_0^{-1}(\alpha+\delta-2)a_1^2}{p}\right]p^{1/2}\left(\frac{\gamma_0 H}{p}\right)\varepsilon_0\sum_{a_2 \bmod p}{}' e_H\left(\frac{\gamma_0^{-1}\Delta'(\alpha+\delta-2)a_2^2}{p^2}\right)$$

$$\times e\left[-\frac{\sigma\gamma_0 a_2^{-2}}{p}\right],$$

where $\sigma=2^{-2}K^2H^{-1}\Delta'^{-2}$. If $\alpha+\delta-2=\tau p$ ($\tau\not\equiv 0(p)$), we have $A=0$ and

$$B=p^{3/2}\left(\frac{\gamma_0 H}{p}\right)\varepsilon_0\sum_{a_2 \bmod p}{}' e\left[\frac{H\tau\gamma_0^{-1}\Delta' a_2^2-\sigma\gamma_0 a_2^{-2}}{p}\right]$$

We have , for example,

$$T_rT_\chi(P^{(\tau)})=\frac{1}{2}\sum_{a_2 \bmod p}{}' e\left[\frac{\rho a_2^2+\sigma\tau a_2^{-2}}{p}\right],$$

where $\rho=-H\Delta'$. If $\alpha+\delta-2=\tau p^2$ ($\tau\not\equiv 0(p)$),

$$T_rT_\chi(g)=\frac{1}{2}\left(-1+p\left(\frac{\tau}{p}\right)\right).$$

Finally, let $l=0$. Additional assumption $(\alpha+\delta)^2-4\not\equiv 0(p)$ implies that $T_rT_\chi(g)=0$. The results are as following table, where n is an integer such that $n\not\equiv 0(p)$ and $\left(\frac{n}{p}\right)=-1$.

Representative U	$T_rT_\chi(U)$
$E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\frac{1}{2}(p^2-1)$
$F=-E$	$\frac{1}{2}(-1)^f(p^2-1)$
$A=\begin{pmatrix}1&0\\p&1\end{pmatrix}$	$\frac{1}{2}\left(-1+p^{3/2}\left(\frac{-\sigma}{p}\right)\varepsilon_0\right)$
$B=\begin{pmatrix}1&0\\np&1\end{pmatrix}$	$\frac{1}{2}\left(-1-p^{3/2}\left(\frac{-\sigma}{p}\right)\varepsilon_0\right)$
$C^\mu=\begin{pmatrix}1&\mu p\\n\mu p&1\end{pmatrix}$ $\mu=1,2,\cdots,\frac{p-1}{2}$	$-\frac{1}{2}(p+1)$
$D^\lambda=\begin{pmatrix}1+\lambda p&0\\0&1-\lambda p\end{pmatrix}$ $\lambda=1,2,\cdots,\frac{p-1}{2}$	$\frac{1}{2}(p-1)$
$P^{(\tau)}=\begin{pmatrix}1&\tau p\\1&1+\tau p\end{pmatrix}$ $\tau=0,1,\cdots,p-1$	$\frac{1}{2}\sum_{x \bmod p}{}' e\left[\frac{\rho x^2+\sigma\tau x^{-2}}{p}\right]$
$Q^{(\tau)}=\begin{pmatrix}1&\tau n^{-1}p\\n&1+\tau p\end{pmatrix}$ $\tau=0,1,\cdots,p-1$	$\frac{1}{2}\sum_{x \bmod p}{}' e\left[\frac{\rho n x^2+\sigma\tau n^{-1}x^{-2}}{p}\right]$
$G^{(t)}=\begin{pmatrix}1&t\\1&1+t\end{pmatrix}$ $t=1,2,\cdots,p^2-1$, $t^2+4t\not\equiv 0\ (p)$	0

So they coincide with characters $\chi_2^{(\rho,\sigma)}(G)$ in [5]. There it is shown that they are irreducible and that there exist $4(p-1)$ different characters obtained, for example, for $\rho=1, 2, \cdots, p-1$; $\left(\frac{\rho}{p}\right)=\pm 1$; $f=0, 1$. Corresponding $T_\chi(g)$ are exactly those irreducible representations absent in the construction of [3].

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