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On the Quotient Semi-Group of a Noncommutative Semi-Group

By Kentaro MURATA

In this short note we remark, following K. Asano¹⁾, that a non-commutative semi-group g with a certain condition can be embedded into the quotient semi-group G . The necessary and sufficient condition for the existence of the quotient semi-group is the same as the case of a ring. Moreover if g is a ring, we can define the addition in G in a natural manner and G is just the quotient ring of g .

Definition 1. An element λ in a semi-group g is called *regular*, if the following two conditions are satisfied: 1) $a\lambda = b\lambda$ ($a, b \in g$) implies $a = b$ and 2) $\lambda a = \lambda b$ ($a, b \in g$) implies $a = b$.

If g has the unit, the elements having their inverse elements in g are obviously regular.

In the following we assume that a semi-group g has regular elements. It is clear that all regular elements in g form a sub-semi-group g^* of g .

Definition 2. Let m be a sub-semi-group of g^* . If a semi-group G which contains g satisfies the next three conditions, we call G a *left quotient semi-group* of g by m .

- (1) G has a unit 1.
- (2) Every element α in m has an inverse α^{-1} in G .
- (3) For every x in G , there exists α in m such that αx is contained in g .

In particular if $m = g^*$, we call G a left quotient semi-group of g . According to Definition 2, every element s in G is clearly expressible in the form $s = \alpha^{-1}a$, where $\alpha \in m$ and $a \in g$. If g has a left (or right) unit e , then $e = 1$.²⁾

Lemma 1. If for every a in g and every α in m there exist α' in m and a' in g such that $\alpha'a = a'\alpha$ then, for any n elements $\lambda_i \in m$ ($i = 1, \dots, n$) there exist n elements $c_i \in g$ ($i = 1, \dots, n$) satisfying the following condition:

¹⁾ K. Asano, Arithmetische Idealtheorie in nichtkommutativen Ringen, Japan. Journ. Math. **16** (1939); Über die Quotientenbildung von Schieferringen, Journ. Math. Soc. Japan **1** (1949).

²⁾ $e = e1 = e\lambda\lambda^{-1} = \lambda\lambda^{-1} = 1$ ($e = \lambda e = \lambda^{-1}\lambda e = \lambda^{-1}\lambda = 1$).

$$c_1 \lambda_1 = \dots = c_n \lambda_n = \gamma \in m.$$

Proof. According to the assumption, there exist a_1 in g and α_2 in m such that $a_1 \lambda_1 = \alpha_2 \lambda_2$. Next, for $\alpha_2 \lambda_2$ and λ_3 , there exist a_2 in g and α_3 in m such that $a_2 \lambda_2 = \alpha_3 \lambda_3$. Next, for $\alpha_3 \lambda_3$ and λ_4 , there exist a_3 in g and α_4 in m such that $a_3 \lambda_3 = \alpha_4 \lambda_4$. By induction we complete our proof.

Lemma 2. If, under the same assumption as in lemma 1, there exists one element pair x_0, y_0 in g such that $x_0 \alpha = y_0 \beta \in m$ ($\alpha, \beta \in m$) and $x_0 a = y_0 b$ ($a, b \in g$), then every pair x, y in g satisfying the condition $x \alpha = y \beta \in m$ satisfies $x a = y b$.

Proof. Putting $\theta = x_0 \alpha = y_0 \beta$ and $\varphi = x \alpha = y \beta$, we take c and δ in g and m respectively as $c \theta = \delta \varphi \in m$. Then $c \theta = c x_0 \alpha = c y_0 \beta = \delta \varphi = \delta x \alpha = \delta y \beta$, $c x_0 = \delta x$, $c y_0 = \delta y$, hence $\delta x a = c x_0 a = c y_0 b = \delta y b$, that is, $x a = y b$.

Theorem 1. In order that there exists a left quotient semi-group of g by m , it is necessary and sufficient that for every $a \in g$ and every $\lambda \in m$ there exist $a' \in g$ and $\lambda' \in m$ satisfying $\lambda' a = a' \lambda$. And such a left quotient semi-group is uniquely determined by m and g apart from its isomorphism.

Proof. Let G be a left quotient semi-group of g by m . Then for $a \lambda^{-1}$ ($a \in g, \lambda \in m$) in G , there exists $\lambda' \in m$ such that $\lambda' \cdot a \lambda^{-1} = a' \in g$, namely $\lambda' a = a' \lambda$. Hence the condition is necessary. We show now the condition is sufficient. First, we assume g has the unit 1. Let G be the set of all symbols (α, a) , $\alpha \in m, a \in g$. We can introduce the equality of the elements in G as follows: (α, a) is equal to (β, b) if and only if $x a = y b$ for every x and y satisfying $x \alpha = y \beta \in m, x, y \in g$. Then, according to Lemma 2, in order that $(\alpha, a) = (\beta, b)$ it is sufficient that there exists at least one element pair x, y in g satisfying $x a = y b$ and $x \alpha = y \beta \in m$. As we can then readily prove, the above-defined equality fulfils the equivalence relation. In particular $(\lambda \alpha, \lambda a) = (\alpha, a)$ ($\lambda \in m$). Now we define the multiplication of the elements in G as follows:

$$(\alpha, a) (\beta, b) = (\beta' \alpha, a' b), \beta' a = a' \beta, \beta' \in m, a' \in g.$$

The product is independent of the choice of $a' \in g$ and $\beta' \in m$. For if $\beta'' a = a'' \beta, \beta'' \in m, a'' \in g$, then taking u and δ such that $u \beta' = \delta \beta'' \in m$ ($u \in g, \delta \in m$), we get $u \beta' a = \delta \beta'' a \in m$, and

$$u a' \beta = u \beta' a = \delta \beta'' a = \delta a'' \beta.$$

Hence $u a' = \delta a'', u a' b = \delta a'' b$, that is, $(\beta' \alpha, a' b) = (\beta'' \alpha, a'' b)$. In

particular, if the product of a and β is commutative, then $(\alpha, a)(\beta, b) = (\beta\alpha, ab)$. Further, if $(\alpha, a) = (\alpha_1, a_1)$ and $(\beta, b) = (\beta_1, b_1)$ then $(\alpha, a)(\beta, b) = (\alpha_1, a_1)(\beta_1, b_1)$. For if we choose c, c_1, a' in g and β' in m satisfying $\gamma = c\beta = c_1\beta_1 \in m$, $\beta'a = a'\gamma$, then $(\alpha, a)(\beta, b) = (\beta'\alpha, a'cb) = (\beta'\alpha, a'c_1b_1) = (\alpha, a)(\beta_1, b_1)$. Similarly $(\alpha, a)(\beta_1, b_1) = (\alpha_1, a_1)(\beta_1, b_1)$. One can easily verify the associative law of the multiplication introduced above. Hence G is a semi-group. The mapping $a \rightarrow (1, a)$ gives an isomorphism of g into G . Identifying a and $(1, a)$, we can see that G contains g , $1 = (1, 1)$ is the unit of G , every element $\alpha = (1, \alpha)$ in m has an inverse $(\alpha, 1)$ in G , and

$$\alpha(\alpha, a) = (1, \alpha)(\alpha, a) = (\alpha, \alpha a) = (1, a) = a.$$

Hence G is a left quotient semi-group of g by m . Secondly, if g has no unit, then we add it to g , and denote the new by g' . Then there exists a left quotient semi-group G of g' by m . And it is easily seen that G is a left quotient semi-group of g by m . Finally let G' be an arbitrary left quotient semi-group of g by m , then every element in G' is expressible as the form $\alpha^{-1}a$ where α is in m and a in g . The mapping $\alpha^{-1}a \rightarrow (\alpha, a)$ gives an isomorphism of G' onto G , q. e. d.

Corollary 1. In order that there exists a left quotient semi-group of a semi-group g , it is necessary and sufficient that for every regular element λ and every element a in g there exist a' in g and a regular element λ' satisfying $a'\lambda = \lambda'a$.

Corollary 2. Let all elements in a semi-group g be regular. In order that there exists a left quotient group of g , it is necessary and sufficient that for any two elements α, β in g there exist another two elements α', β' in g satisfying $\alpha'\alpha = \beta'\beta$.

Lemma 3. Let G be a left quotient semi-group of g by m . Let further $x_i (i = 1, \dots, n)$ be any n elements in g , then we can choose an element $\gamma \in m$ such that $\gamma x_i \in g (i = 1, \dots, n)$.

Proof. There exist α_i in m such that $\alpha_i x_i \in g (i = 1, \dots, n)$. And if we take, according to Lemma 1, $\gamma = c_1 \alpha_1 = \dots = c_n \alpha_n \in m$, then $\gamma x_i \in g (i = 1, \dots, n)$.

Now, we shall consider a quotient system of some algebraic system. Let ν be a semi-group with regular elements, and m a sub-semi-group of ν consisting of regular elements. Let further ν has besides the multiplication, binary operations denoted by \circ , which satisfy the following conditions:

[1] If a and $b (a, b \in \nu)$ are composable, (with respect to \circ) then ca and cb are composable, also ac and bc are composable. And $ca \circ cb = c(a \circ b)$, $ac \circ bc = (a \circ b)c$.

[2] If, for some $\gamma \in \mathfrak{m}$, γa and γb ($a, b \in \mathfrak{o}$) are composable, then a and b are composable, also if $a\gamma$ and $b\gamma$ are composable then so are a and b .

For example, let \mathfrak{o} be a (noncommutative or commutative) ring, and \circ the addition, then the above-mentioned conditions are of course fulfilled.

Theorem 2. Let S be a left quotient semi-group of \mathfrak{o} by \mathfrak{m} . If there exists an element $\lambda \in \mathfrak{m}$, such that λx and λy ($x, y \in S$) are contained in \mathfrak{o} and composable, we define the composition of x and y by

$$(*) \quad x \circ y = \lambda^{-1}(\lambda x \circ \lambda y).$$

Then S becomes an algebraic system of the same kind of \mathfrak{o} .

Proof. If x and y (where $x, y \in S$) are composable in the sense of the above-defined (*), then $x \circ y$ is uniquely determined independently of the choice of λ in \mathfrak{m} . For if μx and μy (where $\mu x \in \mathfrak{o}$, $\mu y \in \mathfrak{o}$, $\mu \in \mathfrak{m}$) are composable, then, by taking η and c satisfying $\eta \mu = c \lambda = \gamma \in \mathfrak{m}$ ($\eta \in \mathfrak{m}$, $c \in \mathfrak{o}$), we get

$$\begin{aligned} \gamma \lambda^{-1}(\lambda x \circ \lambda y) &= c \lambda \lambda^{-1}(\lambda x \circ \lambda y) = c(\lambda x \circ \lambda y) \\ &= c \lambda x \circ c \lambda y = \eta \mu x \circ \eta \mu y = \eta(\mu x \circ \mu y) = \gamma \mu^{-1}(\mu x \circ \mu y). \end{aligned}$$

Namely $\lambda^{-1}(\lambda x \circ \lambda y) = \mu^{-1}(\mu x \circ \mu y)$.

Next, if x and y in S are composable, then so are zx and zy ($z \in S$) and $z x \circ z y = z(x \circ y)$. By the hypothesis, there exists λ in \mathfrak{m} such that $x \circ y = \lambda^{-1}(\lambda x \circ \lambda y)$. If we choose α, τ and a such that $\alpha z \in \mathfrak{o}$, $\alpha \cdot z x \in \mathfrak{o}$, $\alpha \cdot z y \in \mathfrak{o}$ ($\alpha \in \mathfrak{m}$) and $\tau(\alpha z) = a \lambda$ ($\tau \in \mathfrak{m}$, $a \in \mathfrak{o}$), then $\sigma z x = a \lambda x$, $\sigma z y = a \lambda y$ ($\sigma = \tau \alpha$). Since $a \lambda x = \sigma z x$ and $a \lambda y = \sigma z y$ are composable in \mathfrak{o} , $z x$ and $z y$ are composable in S , and

$$\begin{aligned} z(x \circ y) &= z \lambda^{-1}(\lambda x \circ \lambda y) = \sigma^{-1} a(\lambda x \circ \lambda y) \\ &= \sigma^{-1}(a \lambda x \circ a \lambda y) = \sigma^{-1}(\sigma z x \circ \sigma z y) = z x \circ z y, \quad (\sigma \in \mathfrak{m}). \end{aligned}$$

If x and y in S are composable, then xz and yz are composable, and $(x \circ y)z = xz \circ yz$. By the hypothesis, there exists $\lambda \in \mathfrak{m}$ such that $x \circ y = \lambda^{-1}(\lambda x \circ \lambda y)$. If we take μ and α such that $\mu z \in \mathfrak{o}$ ($\mu \in \mathfrak{m}$), $\alpha \lambda x \mu^{-1} \in \mathfrak{o}$ and $\alpha \lambda y \mu^{-1} \in \mathfrak{o}$ ($\alpha \in \mathfrak{m}$), then $x \circ y = \sigma^{-1}(\sigma x \circ \sigma y)$, $\sigma = \alpha \lambda$. According to the latter part of the condition [2], $\sigma x \mu^{-1}$ and $\sigma y \mu^{-1}$ are composable in \mathfrak{o} . Hence

$$\begin{aligned} (x \circ y)z &= \sigma^{-1}(\sigma x \circ \sigma y)z = \sigma^{-1}(\sigma x \mu^{-1} \mu \circ \sigma y \mu^{-1} \mu)z \\ &= \sigma^{-1}(\sigma x \mu^{-1} \circ \sigma y \mu^{-1}) \mu z = \sigma^{-1}(\sigma x \mu^{-1} \mu z \circ \sigma y \mu^{-1} \mu z) \\ &= \sigma^{-1}(\sigma x z \circ \sigma y z) = xz \circ yz. \end{aligned}$$

Every element in m has an inverse in S . Then condition [2] is therefore easily obtained from the condition [1]. Thus our theorem is proved.

Let \mathfrak{o} be a noncommutative ring containing non-nilfactors. The non-nilfactor is clearly a regular element in \mathfrak{o} in the sense of Definition 1. Hence we have

*Corollary.*³⁾ *Let m be a sub-semi-group consisting of non-nilfactors in a given ring \mathfrak{o} . In order that there exists a left quotient ring S of \mathfrak{o} by m , it is necessary and sufficient that, for every $\alpha \in m$ and every $a \in \mathfrak{o}$, there exist $a' \in \mathfrak{o}$ and $\alpha' \in m$ satisfying $a' \alpha = \alpha' a$.*

Putting the word *right* in place of *left* in the above-mentioned argument, we can argue similarly as above. But the existence of a left quotient semi-group and a right one are independent, and if there exist both, then they are the same⁴⁾.

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^{3), 4)} K. Asano, 1. c.

