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# *On Curvature and Harmonic Forms with Values in Analytic Vector Bundles*

By Kenkichi SATO

The relations between curvature and Betti numbers have been studied by K. Yano, S. Bochner and others, using Green's formula as basic tool. Especially, they proved that if a compact Riemannian manifold has negative definite Ricci curvature[1] throughout, the one-dimensional Betti number of the manifold is zero and they also extended the result to the case of the arbitrary dimensional Betti numbers [1].

The main purpose of this paper is to extend their results to vector bundle valued harmonic forms in compact Kählerian manifolds.

We know that an analytic vector bundle can be considered as a Hermitian vector bundle (that is, a reduction of the structure group to the unitary group can be defined) and that in a Hermitian vector bundle there is one and only one (1, 0)-type connection, which means the connection form is of type (1, 0) [3].

In § 1, we first consider the Whitney sum $W'\oplus P$ of the associated principal bundle $W'$ to a given vector bundle over a compact Kählerian manifold $M$ with the subbundle $P$ of its bundle of frames whose structure group is the complex general linear group. Then we can canonically define a connection in $W'\oplus P$ from the connection in $W'$ and the Kählerian connection (that is, the connection in $P$ induced from the Kählerian metric of $M$) and using this connection *basic vector fieids* can be defined which are mapped to ordinary basic vector fields in $P$ by projection of $W'\oplus P$ onto $P$. Thanks to those basic vector fields, $d''$-operator and its dual operator can be expressed in quite analogous form to those operators with respect to ordinary differential forms.

In § 2, we describe the fundamental integral formula that may be considered as Green's formula in the ordinary case.

As for curvature, the curvature in $W'\oplus P$ plays the same role in our case as the ordinary curvature does in the ordinary case. As we see in § 3, the bilinear form $R(X, Y)$ derived from the curvature plays

1) As for the definition of curvature, we follow [2]. This definition is different from that of [1] only in the signature.

the same role as Ricci curvature.

The author wishes to express his cordial thanks to Mr. H. Ozeki who has kindly read the manuscript and given him valuable criticism. Especially, the formulation and the proof of Theorem 1 are due to him; the original proof is much more complicated.

## § 1. Preliminaries

We consider a compact, complex analytic manifold $M$ of complex dimension $m$ as a base space.

### 1. Hermitian vector bundles

Let $W$ be an analytic vector bundle of dimension $n$ over $M$, that is, a bundle with $n$-dimensional complex linear space $C^n$ as standard fibre, the group $GL(n, C)$ as structure group and with complex analytic transition functions.

Let $W'$ be the associated principal bundle to $W$ and let $W_p$, $W_p'$ denote fibres over $p \in M$ of $W$ and $W'$ respectively.

We fix a reduction of the structure group to the unitary group $U(n)$, considering $W'$ to be a Hermitian vector bundle, and denote the reduced bundle by $W'(U(n))$.

By this reduction a positive definite Hermitian inner product can be defined naturally in each fibre $W_p$.

We denote these inner products by the same letter $g_1$.

By calling ordered bases of $W_p$ *frames*, $W'$ may be considered not only as the set of admissible maps of $W$, but also as the set of frames and from this point of view, $W'(U(n))$ may be considered as the set of unitary frames.

We know that in $W'$ there exists a unique connection $\Gamma_1$ such that:

(1) The connection form $\omega_1$ of $\Gamma_1$ is of type (1, 0).

(2) The Hermitian inner product is invariant under parallelism.

### 2. Kählerian connection

From now on we assume that $M$ has a Kählerian metric.

As the bundle of frames over $M$ can be reduced to $GL(m, c)$, we denote the reduced bundle by $P$ and the tangent bundle over $M$ by $T(M)$.

Since $M$ has a Kählerian metric, the structure group of $P$ can be reduced to $U(m)$. The reduced bundle will be denoted by $P(U(m))$. Thus, considering $P$ as a Hermitian vector bundle, the unique (1, 0)-type connection mentioned above coincides with the Kählerian connection. This connection we denote by $\Gamma_2$. Each tangent space $T_p$, $p \in M$ and standard fibre $V$ of $T(M)$ have an inner product and a complex structı determined by the Kählerian metric and the complex structure of $M$. The complexification $T_p^c$ of $T_p$ and $V^c$ of $V$ have a Hermitian inner product (denoted by the same letter $g_2$) as the extension of the given inner product, and are decomposed as follows:

$$T_p^c = S_p \oplus \bar{S}_p \quad \text{and} \quad V^c = U \oplus \bar{U},$$

where $S_p$ and $U$ are the sets of (1, 0)-type vectors of $T_p^c$ and $V^c$ respectively and $\bar{S}_p$, $\bar{U}$ are the sets of (0, 1)-type vectors of $T_p^c$ and $V^c$ respectively.

$S_p$ is orthogonal to $\bar{S}_p$ and $U$ to $\bar{U}$.

### 3. Whitney sum $W' \oplus P$ of $W'$ with $P$

We consider here the Whitney sum (direct sum) $\tilde{W} = W' \oplus P$ of $W'$ with $P$. If we denote by $P_p$ the fibre of $P$ over $p \in M$, we can express the fibre $\tilde{W}_p$ of $\tilde{W}$ over $p \in M$ as follows:

$$\tilde{W}_p = \{(x, y) \mid x \in W_p', y \in P_p\}.$$

We can obtain a natural reduction of the structure group, $GL(n, c) \oplus GL(m, c)$ of $\tilde{W}$ to $U(n) \oplus U(m)$, from the reduction of $W'$ to $W'(U(n))$ and of $P$ to $P(U(m))$. We denote this reduced bundle by $\tilde{W}(U(n, m))$. Then we have

$$\tilde{W}(U(n, m)) = W'(U(n)) \oplus P(U(m)).$$

Next, we define the projections $p_1$ and $p_2$ which map respectively $(x, y) \in \tilde{W}_p$ to $x \in W_p'$ and $(x, y) \in \tilde{W}_p$ to $y \in P_p$.

There is a connection $\Gamma$ in $\tilde{W}$ induced naturally from $\Gamma_1$ and $\Gamma_2$ as follows. The horizontal subspace $Q_{(x,y)}$ of the tangent space at $(x, y) \in \tilde{W}_p$ is defined as the intersection $p_1^{-1}Q_x \cap p_2^{-1}Q_y$ where $Q_x$ and $Q_y$ are the horizontal subspaces at $x \in W_p'$ and at $y \in P_p$ respectively. As for the connection form $\omega$ of $\Gamma$, we have: $\omega = p_1^*\omega_1 \oplus p_2^*\omega_2$, where $\omega_2$ is the connection form of the Kählerian connection in $P$.

The curvature form $\Omega$ can be written, using the curvature form $\Omega_1$ for $\Gamma_1$ and $\Omega_2$ for $\Gamma_2$ as follows: $\Omega = p_1^*\Omega_1 \oplus p_2^*\Omega_2$.

Now, we define *basic vector field* $B(v)$ for each $v \in V^c$ to be a horizontal vector field such that at each point $(x, y)$, it is a unique horizontal vector $B_{(x,y)}(v)$ satisfying the relation:

$$\pi B_{(x,y)}(v) = y \cdot v,$$

where $\pi$ denotes the projection of the bundle $\tilde{W}$ onto the base space $M$.

As $p_1$ and $p_2$ are complex analytic mappings, we see that $\Gamma$ is also of type (1, 0). Because of this property, $B(v)$ is of type (1, 0) or (0, 1), according as $v$ belongs to $U$ or $\bar{U}$, and $B(\bar{v})=\overline{B(v)}$. These remarks are important later.

Let $F$ be a linear space. We denote by $\bigwedge^q(F)$ the Grassmann algebra of degree $q$ over $F$.

We set $T_p(r, s)=W_p\otimes\bigwedge^r(S_p)\otimes\bigwedge^s(\bar{S}_p)$ and $V(r, s)=C^n\otimes\bigwedge^r(U)\otimes\bigwedge^s(\bar{U})$.

Then, a Hermitian inner product can be defined in $T_p(r, s)$ and $V(r, s)$, from the inner products $g_1$ and $g_2$, denoting these by the same $g$. Every point $(x, y)\in\tilde{W}_p$ may be considered to be a linear mapping of $V(r, s)$ to $T_p(r, s)$, since $x$ maps $C^n$ to $W_p$ and $y$ may be regarded as a linear mapping of $\bigwedge^r(U)\otimes\bigwedge^s(\bar{U})$ to $\bigwedge^r(S_p)\otimes\bigwedge^s(\bar{S}_p)$.

Then we can obtain the concept of parallel displacement in $\tilde{W}$ with respect to $T_p(r, s)$. And we have

**Lemma 1.** *The inner product $g$ is invariant under parallelism in the sense mentioned above.*

## § 2. Green's formula

**Theorem 1.** *Let $X$ be a compact differentiable manifold with a complete parallelisability $B_1, \cdots, B_p, A_1, \cdots, A_q$ such that $[B_i, B_j]$ are linear combinations of $A_k$ at each point, and $[A_i, B_j]$ are linear combinations of $B_k$ at each point. Then we have for an arbitrary function $f$*

$$\int B_i f d\sigma=0 \qquad i=1, \cdots, p,$$

*where $d\sigma$ is the volume element with respect to a Riemannian metric defined by requiring $B_1, \cdots, B_p, A_1, \cdots, A_q$ to be orthonormal at each point.*

Proof. It is sufficient to prove the formula for $i=1$.

Let $\omega_1, \cdots, \omega_p, \varphi_1, \cdots, \varphi_q$ be the dual base of $B_1, \cdots, B_p, A_1, \cdots, A_q$, then $d\sigma=\omega_1\wedge\cdots\wedge\omega_p\wedge\varphi_1\wedge\cdots\wedge\varphi_q$.

We set $\alpha=f\omega_2\wedge\cdots\wedge\omega_p\wedge\varphi_1\wedge\cdots\wedge\varphi_q$, then we have $d\alpha=df\wedge\omega_2\wedge\cdots\wedge\omega_p\wedge\varphi_1\wedge\cdots\wedge\varphi_q+fd(\omega_2\wedge\cdots\wedge\omega_p\wedge\varphi_1\wedge\cdots\wedge\varphi_q)$.

The first term is equal to $B_1 f d\sigma$. Since $\int d\alpha=0$ by Stokes' formula, it is sufficient to show that the second term is zero.

Indeed, we can easily see from the hypothesis of the theorem that from $\omega_k([B_i, B_j])=0$ and $\varphi_k([A_i, B_j])=0$ we have $d\omega_k(B_i, B_j)=0$ and $d\varphi_k(A_i, B_j)=0$ respectively.

We have thereby proved the theorem.

Let $\xi_i$ denote an orthonormal base of $V$ and let $A_j$ denote an orthonormal base of $V(n) \oplus U(m)$, where for simplicity we may assume that the total volume of $U(n) \oplus U(m)$ is equal to 1.

Let $A_j^*$ be the corresponding fundamental vector field, that is, the vector field determined by the one parameter group of right translations in $\tilde{W}(U(n, m))$ induced by the one parameter group corresponding to $A_j$.

Then $B(\xi_i)$ and $A_j^*$ form a complete parallelisability in $\tilde{W}(U(n, m))$. We denote by $d\sigma$ the volume element in $\tilde{W}(U(n, m))$ which is determined by that complete parallelisability.

Then we can state the following corollary which plays a fundamental role in the next section.

**Corollary.** *Let $f$ be an arbitrary function on $\tilde{W}(U(n, m))$ and let $B$ be a basic vector field. Then*

$$\int Bf d\sigma = 0 .$$

Proof. We know that the bracket of a horizontal vector field and a fundamental vector field is horizontal (Nomizu [2] Chap. II. § 4. Lemma.).

Thus we have only to prove the following

**Lemma 2.** *For arbitrary $v, v' \in V^c$, $[B(v), B(v')]$ is a vertical vector field.*

Proof. We denote by $B'(v)$, $v \in V^c$ the ordinary basic vector field in $W'(U(m))$, that is, $B_y'(v)$, $y \in W'(U(m))$, is a unique horizontal vector at $y$ which satisfies the relation $y^{-1}\pi_2 B_y'(v) = v$, where $\pi_2$ denotes the projection of $W'(U(m))$ onto $M$. (Nomizu [2] p. 49).

Let $B'(v)$ and $B'(v')$ be the ordinary basic vector fields in $W'(U(m))$ associated to $v, v' \in V^c$.

Let $I$ denote the identity map of $M$. Then we have $p_2\pi_2 = \pi I$.

Hence we see that a vector in $\tilde{W}(U(n, m))$ which is mapped by $p_2$ upon a vertical vector in $W'(U(m))$ is vertical.

Now, $B(v)$ and $B(v')$ are $p_2$-related to $B'(v)$ and $B'(v')$ respectively, hence $[B(v), B(v')]$ is $p_2$-related to $[B'(v), B'(v')]$. Since $\Gamma_2$ is a torsionless connection, $[B'(v), B'(v')]$ is a vertical vector field (Nomizu [2] Chap. III. § 2. Corollary.).

We have thereby proved the lemma.

## § 3. Harmonic $(r, s)$-type tensors in $M$ with values in $W$

We first define $(r, s)$-type tensors with values in $W$.

Assigning an element of $T_p(r, s)$ to each $p \in M$ differentiably, a tensor field with values in $W$ is obtained. We call this tensor field an $(r, s)$-type tensor field with values in $W$.

There is a one-to-one correspondence between the set of $(r, s)$-type tensor fields and the set of $V(r, s)$-valued functions in $\tilde{W}$ which satisfy the following condition[2]:

For arbitrary $a \oplus b \in GL(n, c) \oplus GL(m, c)$,

$$f(xa, yb) = (a \otimes b)^{-1} f(x, y) \tag{1}$$

where $(a \otimes b)$ is regarded as a linear automorphism of $V(r, s)$.

Let $X$ be an $(r, s)$-type tensor field and let $f_X$ be the corresponding function. Then the relation between $X$ and $f_X$ is expressed as follows:

$$X_p = (x, y) f(x, y) \quad (x, y) \in \tilde{W}_p ,$$

where $X_p$ is the value at $p$ of $X$ and $(x, y)$ is regarded as a linear transformation of $V(r, s)$ to $T_p(r, s)$.

We remark that if $f^\circ$ denotes the restriction of $f$ to $\tilde{W}(U(n, m))$, then $f^\circ$ satisfies the relation (1) in $\tilde{W}(U(n, m))$.

Conversely, given a function $f^\circ$ in $\tilde{W}(U(n, m))$ which satisfies (1) in $\tilde{W}(U(n, m))$, it can be extended uniquely to a function $f$ in $\tilde{W}$ which satisfies (1). Therefore it is sufficient to consider $f$ in $\tilde{W}(U(n, m))$. The following relations will be considered and have the meanings only in $\tilde{W}(U(n, m))$.

Let $X$ be an $(r, s)$-type tensor field. Choosing unitary bases $h_l, v_\alpha, \bar{v}_\alpha$ $l = 1, \cdots, n$, $\alpha = 1, \cdots, m$, in $C^n$, $U$ and $\bar{U}$ respectively, we can write

$$f_X = \sum f_X^{l\alpha_1 \cdots \alpha_r \bar{\beta}_1 \cdots \bar{\beta}_s} h_l \otimes (v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_r}) \otimes (\bar{v}_{\beta_1} \wedge \cdots \wedge \bar{v}_{\beta_s})$$

or in the following form:

$$f_X = \{ f_X^{l\alpha_1 \cdots \alpha_r \bar{\beta}_1 \cdots \bar{\beta}_s} \} .$$

## 1. Explicit forms of $d''$-operator and its dual operator.

We denote be $\tilde{W}_p(U(n, m))$ the fibre over $p \in M$ in $\tilde{W}(U(n, m))$.

Let $X$ and $Y$ be $(r, s)$-type tensor fields. Then $g(f_X, f_Y)$ is constant in $\tilde{W}_p(U(n, m))$. We define an inner product $(X, Y)$ by

$$(X, Y) = \int g(f_X, f_Y)\, d\sigma \tag{2}$$

---

2) C. f. Nomizu [2] Chap. III § 3. Lemma.

Noting that if we write points of $\tilde{W}_p$ as $(x, y)$ and fix $y$, then $f_X$ is complex analytic in $x$, and recalling that $B(\bar{v}_{\alpha_i})$ is of type $(0, 1)$, we can write

$$f_{d''X} = \{f_{d''X}^{l\alpha_1 \cdots \alpha_{r+1}\bar{\beta}_1 \cdots \bar{\beta}_s}\}$$
$$= \{\sum_{i=1}^{r+1} (-1)^{i+1} B(\bar{v}_{\alpha_i}) f_X^{l\alpha_1 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_{r+1}\bar{\beta}_1 \cdots \bar{\beta}_s}\}$$

For simplicity we set

$$\sum_{i=1}^{r+1} (-1)^{i+1} B(\bar{v}_{\alpha_i}) f_X^{l\alpha_1 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_{r+1}\bar{\beta}_1 \cdots \bar{\beta}_s} = a_X^{l[\alpha_1 \cdots \alpha_{r+1}]\bar{\beta}_1 \cdots \bar{\beta}_s} .$$

The suffix $X$ will be often dropped.

By virtue of (2) we can easily obtain the explicit form of dual operator $\delta$ of $d''$:

$$f_{\delta X} = \{f_{\delta X}^{l\alpha_1 \cdots \alpha_{r-1}\bar{\beta}_1 \cdots \bar{\beta}_s}\} = \{-\sum_{\alpha=1}^{m} B(v_\alpha) f_X^{l\alpha\alpha_1 \cdots \alpha_{r-1}\bar{\beta}_1 \cdots \bar{\beta}_s}\} .$$

$M$ being compact, it is well known that the necessary and sufficient condition for $X$ to be harmonic is that $d''X = 0$ and $\delta X = 0$.

We set for simplicity

$$B(v_\alpha) f^{l\alpha_1 \cdots \alpha_r\bar{\beta}_1 \cdots \bar{\beta}_s} = a_\alpha^{l\alpha_1 \cdots \alpha_r\bar{\beta}_1 \cdots \bar{\beta}_s}$$

Let $B_1$ and $B_2$ be basic vector fields. Then we have

$$[B_1, B_2] f_X = 2\Omega(B_1, B_2) f_X$$

where $\Omega$ operates on $f_X$ as a derivation[3].

## 2. The definition of bilinear form $R(X, Y)$

We adopt the following summation convention: If an index appears twice in a term, summation has to be taken on the range of the index.

We set

$$R(X, Y) = g(\overline{f_X^{l\gamma\alpha_2 \cdots \alpha_r\bar{\beta}_1 \cdots \bar{\beta}_s}} 2\Omega(B(\bar{v}_\gamma), B(v_\alpha)) f_Y, \; h_l \otimes (v_\alpha \wedge v_{\alpha_2} \wedge \cdots \wedge v_{\alpha_r}) \otimes (\bar{v}_{\beta_1} \wedge \cdots \wedge \bar{v}_{\beta_s})) .$$

Then $R$ is constant in each $\tilde{W}_p(U(n, m))$, and may be considered as a bilinear form defined at each point $p \in M$. Direct calculation shows

$$R(X, Y) = \overline{R(Y, X)} .$$

3) C. f. Nomizu [2] p. 60. Lemma.

## 3. Main theorems

We first consider the following four relations i), ii), iii) and iv).

We remark that by Lemma 1 the covariant derivative of $g$ with respect to $\Gamma$ is zero.

i) $$\begin{aligned} &B(v_\alpha)\, g(\overline{f^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(\bar v_\gamma) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \\ &= g(\overline{a_{\bar\alpha}^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(\bar v_\gamma) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \\ &\quad + g(\overline{f^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(v_\alpha)\, B(\bar v_\gamma) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \end{aligned}$$

ii) $$\begin{aligned} &g([B(\bar v_\gamma),\ B(v_\alpha)] f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r})\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \\ &= g(2\Omega(B(\bar v_\gamma),\ B(v_\alpha)) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r})\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \end{aligned}$$

From this we have

$$\begin{aligned} &g(\overline{f^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(\bar v_\gamma)\, B(v_\alpha) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \\ &\quad - g(\overline{f^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(v_\alpha)\, B(\bar v_\gamma) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \\ &= R(X,\ X) \end{aligned}$$

iii) $$\begin{aligned} &B(\bar v_\gamma)\, g(\overline{f^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(v_\alpha) f,\ h_l\otimes v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \\ &= \overline{a_\gamma^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, a_\alpha^{l\alpha\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s} \\ &\quad + g(\overline{f^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, B(\bar v_\gamma)\, B(v_\alpha) f,\ h_l\otimes(v_\alpha\wedge v_{\alpha_2}\wedge\cdots\wedge v_{\alpha_r}) \\ &\qquad\otimes(\bar v_{\beta_1}\wedge\cdots\wedge\bar v_{\beta_s})) \end{aligned}$$

iv) $$\begin{aligned} &B(\bar v_\gamma)\, g(f,\ B(\bar v_\gamma) f) \\ &= g(f,\ B(v_\gamma)\, B(\bar v_\gamma) f) + g(B(\bar v_\gamma) f,\ B(\bar v_\gamma) f) \\ &= g(f,\ B(v_\gamma)\, B(\bar v_\gamma) f) + \overline{a_{\bar\gamma}^{l\alpha_1\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, a_{\bar\gamma}^{l\alpha_1\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s} \end{aligned}$$

From i), ii), iii) and Corollary of Theorem 1 we have

$$\begin{aligned} &\int \{\overline{a_{\bar\alpha}^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, a_{\bar\gamma}^{l\alpha\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s} - R(X,\ X) \\ &\qquad - \overline{a_\gamma^{l\gamma\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}}\, a_\alpha^{l\alpha\alpha_2\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}\}\, d\sigma = 0 \end{aligned} \qquad \text{(A)}$$

But the relation

$$\overline{a_{\bar{\alpha}}^{l\gamma\alpha_2\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}}\, a_{\bar{\gamma}}^{l\alpha\alpha_2\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}$$
$$=\frac{1}{r}\,\overline{a_{\bar{\alpha}}^{l\alpha_1\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}}\, a_{\bar{\alpha}}^{l\alpha_1\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}$$
$$-\frac{1}{r(r+1)}\,\overline{a^{l[\alpha\alpha_1\cdots\alpha_r]\bar{\beta}_1\cdots\bar{\beta}_s}}\, a^{l[\alpha\alpha_1\cdots\alpha_r]\bar{\beta}_1\cdots\bar{\beta}_s}$$

holds, then from (A) we have

$$\int\Big\{\frac{1}{r}\,\overline{a_{\bar{\alpha}}^{l\alpha_1\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}}\, a_{\bar{\alpha}}^{l\alpha_1\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}$$
$$-\frac{1}{(r+1)r}\,\overline{a^{l[\alpha\alpha_1\cdots\alpha_r]\bar{\beta}_1\cdots\bar{\beta}_s}}\, a^{l[\alpha\alpha_1\cdots\alpha_r]\bar{\beta}_1\cdots\bar{\beta}_s}$$
$$-\overline{a_{\gamma}^{l\gamma\alpha_2\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}}\, a_{\alpha}^{l\alpha\alpha_2\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}-R(X,X)\Big\}\,d\sigma=0 \qquad \text{(B)}$$

From i), (B) and Corollary of Theorem 1, setting $\tilde{\Delta}=B(v_\gamma)\,B(\bar{v}_\gamma)$, we have

$$\int\Big[\frac{1}{r}\,g(f,\tilde{\Delta}f)+R(X,X)+\frac{1}{r(r+1)}\,\overline{a^{l[\alpha\alpha_1\cdots\alpha_r]\bar{\beta}_1\cdots\bar{\beta}_s}}\, a^{l[\alpha\alpha_1\cdots\alpha_r]\bar{\beta}_1\cdots\bar{\beta}_s}$$
$$+\overline{a_{\gamma}^{l\gamma\alpha_2\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}}\, a_{\alpha}^{l\alpha\alpha_2\cdots\alpha_r\bar{\beta}_1\cdots\bar{\beta}_s}\Big\}\,d\sigma=0 \qquad \text{(C)}$$

Now we can state the following

**Theorem 2.** *The necessary and sufficient condition for an* $(r, s)$*-type tensor* $X$ *with values in* $W$ *to be harmonic is that* $X$ *satisfies the following relation* :

$$g(f,\tilde{\Delta}f)+rR(X,X)=0\,.$$

Proof. Sufficiency follows from (C) and necessity can be derived by direct calculation and is omitted here.

From (B) we have

**Theorem 3.** *If* $R$ *is negative definite throughout, there is no harmonic* $(r, s)$*-type tensor with values in* $W$ *other than zero.*

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## Bibliography

[1] K. Yano and S. Bochner: Curvature and Betti numbers, Princeton, 1953.
[2] K. Nomizu: Lie groups and differential geometry, No. 2 Publications Math. Soc. Japan, 1956.
[3] S. Nakano: On complex analytic vector bundles, J. Math. Soc. Japan, **7** (1955).

*Added in proof.* I have recently become aware of a paper of S. Bochner: Curvature and Betti numbers in real and complex vector bundles. Univ. e Politec. Torino. Rend. Sem. Mat. 15 (1955–56), 225–253, where he deals with a similar problem.