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BOUNDED EXPONENTIAL SUMS

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1. Introduction

T. Kamae has asked (personal communication) whether it is possible to find a sequence (a_k) of ± 1 's such that the sums

$$\sum_{k=m}^{m+n} a_k e^{-ik\theta}$$

stay bounded (for all integers m and n with $n \geq 0$) for all $\theta \in [-\pi, \pi)$ (with the bound possibly depending on θ). We show that there is no such sequence. In fact, the only such real-valued sequences must be "essentially zero" in a sense explained below.

This conclusion is reached by adopting a dynamical viewpoint, applying the Spectral Theorem, and showing that every nonzero element of L^2 must have nonzero mean power at some frequency. This latter observation is equivalent to the triviality of the intersection of all the spaces of "twisted coboundaries" for a unitary operator.

2. Results

Suppose that $a = (a_k) \in \mathbf{R}^{\mathbf{Z}}$ is a doubly infinite sequence with the property that

$$\left| \sum_{k=m}^{m+n} a_k e^{-ik\theta} \right| \leq c(\theta) < \infty \quad \text{for all } m \in \mathbf{Z}, \text{ all } n \geq 0, \text{ and all } \theta \in [-\pi, \pi).$$

Taking $n = \theta = 0$, we see that a is bounded and so takes values in a compact interval I . Let X denote the closure of the orbit of a under the shift transformation σ in the compact metric space $I^{\mathbf{Z}}$. Let μ be a shift-invariant Borel probability measure on X .

Given $x \in X$ and a block $B = b_0 \cdots b_n$ which appears in x , we can find a block $D = d_0 \cdots d_n$ in a such that $|b_i - d_i| < 1/(n+1)$ for $i = 0, \dots, n$. Consequently

$$\left| \sum_{k=0}^n b_k e^{-ik\theta} \right| \leq c(\theta) + 1 \quad \text{for all } \theta.$$

If $Tg = g \circ \sigma$ for $g \in L^2(X, \mu)$ and $f(x) = \pi_0 x = x_0$ for $x \in X$, we have then that

$$\left\| \sum_{k=0}^n T^k f e^{-ik\theta} \right\|_2 \leq c(\theta) + 1 \quad \text{for all } \theta \text{ and all } n \geq 0.$$

We will see that this is impossible unless $f=0$ in L^2 . Since this cannot happen for a sequence a which assumes only finitely many values, all nonzero, the original question will be settled. For general sequences, the conclusion is that boundedness against all θ is possible only if projection onto the central coordinate is 0 a.e. with respect to every invariant measure on the orbit closure X of the sequence; that is, the only invariant probability measure on X is concentrated on the fixed point 0^∞ . In this case we say that the sequence (a_k) is *essentially zero*.

Theorem. *Let H be a Hilbert space and $T: H \rightarrow H$ a unitary operator. For each $n=1, 2, \dots$, $\theta \in [-\pi, \pi)$, and $f \in H$ let*

$$S_n^\theta f = \sum_{k=0}^{n-1} e^{-ik\theta} T^k f.$$

If $\sup_n \|S_n^\theta f\| < \infty$ for all θ , then $f=0$.

Proof. Applying the Spectral Theorem with common notations and conventions, we may write

$$\begin{aligned} Tf &= \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda) f, \\ S_n^\theta f &= \int_{-\pi}^{\pi} \sum_{k=0}^{n-1} e^{ik(\lambda-\theta)} dE(\lambda) f = \int_{-\pi}^{\pi} \frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}} dE(\lambda) f, \end{aligned}$$

and

$$\|S_n^\theta f\|^2 = \int_{-\pi}^{\pi} \left| \frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}} \right|^2 d\|Ef\|^2(\lambda).$$

The following Lemma will show that such expressions cannot stay bounded for any positive measure (such as $\nu = \|E(\cdot)f\|^2$ if $f \neq 0$ a.e.), thereby completing the proof.

Lemma. *There is a constant $C > 0$ such that if ν is a positive measure on $[-\pi, \pi)$, n is a positive integer, $\varepsilon > 0$, and*

$$A_n(\theta) = \frac{1}{n} \int_{-\pi}^{\pi} \left| \frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}} \right|^2 d\nu(\lambda),$$

then $\nu\{\theta \in [-\pi, \pi): A_n(\theta) < \varepsilon\} < \frac{\varepsilon}{C}$.

Proof. Let C_1 and C_2 be positive constants such that $|\alpha| < \pi$ implies that $C_1|\alpha| \leq |1-e^{i\alpha}| \leq C_2|\alpha|$. Then

$$A_n(\theta) \geq \frac{1}{n} \int_{\theta-(\pi/n)}^{\theta+(\pi/n)} \left| \frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}} \right|^2 d\nu(\lambda) \geq \left[\frac{C_1}{C_2} \right]^2 n \nu \left(\theta - \frac{\pi}{n}, \theta + \frac{\pi}{n} \right).$$

Let $\varepsilon > 0$ and $n > 0$, and let $\delta = \delta(\varepsilon) = \nu\{\theta: A_n(\theta) < \varepsilon\}$. Suppose that $\delta > 0$, since otherwise we are finished. Choose a compact set $K \subset \{\theta: A_n(\theta) < \varepsilon\}$ with $\nu(K) > \delta/2$. There are $\theta_1, \dots, \theta_p \in K$ such that the intervals $\left(\theta_i - \frac{\pi}{n}, \theta_i + \frac{\pi}{n}\right)$ cover K and no more than two of them intersect at any point. Since the union of these intervals is contained in $(-2\pi, 2\pi)$, it follows that $p \frac{2\pi}{n} \leq 8\pi$, and hence $p \leq 4n$. Therefore

$$\nu(K) \leq \sum_{i=1}^p \nu\left(\theta_i - \frac{\pi}{n}, \theta_i + \frac{\pi}{n}\right) \leq p \left[\frac{C_2}{C_1}\right]^2 \frac{1}{n} \varepsilon \leq 4 \left[\frac{C_2}{C_1}\right]^2 \varepsilon,$$

and $\delta < 8(C_2/C_1)^2 \varepsilon$, proving the Lemma and hence also the Theorem.

Corollary 1. *If ν is a positive measure on $[-\pi, \pi)$ and (n_j) is an increasing sequence of positive integers, then*

$$\limsup_{j \rightarrow \infty} \frac{1}{n_j} \int_{-\pi}^{\pi} \left| \frac{1 - e^{in_j(\lambda - \theta)}}{1 - e^{i(\lambda - \theta)}} \right|^2 d\nu(\lambda) > 0 \quad \text{for } \nu\text{-almost all } \theta.$$

Proof. For each $\varepsilon > 0$, $\{\theta: \limsup A_{n_j}(\theta) = 0\} \subset \{\theta: A_{n_j}(\theta) < \varepsilon \text{ for all large enough } j\}$, a set of measure less than ε/C by the Lemma.

Corollary 2. *Let H be a Hilbert space, $T: H \rightarrow H$ a unitary operator, and $0 \neq f \in H$. Then there exists a frequency θ at which the "mean power" of f , defined by*

$$\bar{P}(\theta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{-ik\theta} T^k f \right\|^2,$$

is positive.

Corollary 3. *Let H be a Hilbert space and $T: H \rightarrow H$ a unitary operator. For each $\theta \in [-\pi, \pi)$ let*

$$\mathcal{B}_\theta = \{e^{i\theta}g - Tg: g \in H\}$$

be the space of " θ -twisted coboundaries" for T . Then $\bigcap_{\theta \in [-\pi, \pi)} \mathcal{B}_\theta = \{0\}$.

Proof. If $f \in \mathcal{B}_\theta$, then $\{\|S_n^\theta f\|: n=1, 2, \dots\}$ is bounded.

REMARK. As in [2], by considering fixed points of the operator $V_j^\theta g = e^{-i\theta}(f + Tg)$, one can show that in fact $f \in \mathcal{B}_\theta$ if and only if $\{\|S_n^\theta f\|: n=1, 2, \dots\}$ is bounded. For further developments in this direction, see [1].

Corollary 4. *As in [3], define the "spectral notch" subshift $\Sigma(r, \theta)$ corresponding to $r > 0$ and $\theta \in [-\pi, \pi)$ to be the set of all those $x \in \{-1, 1\}^{\mathbb{Z}}$ for which*

$$\left| \sum_{k=m}^{m+n} x_k e^{-ik\theta} \right| < r \quad \text{for all } m \in \mathbb{Z} \text{ and all } n \geq 0.$$

Then $\bigcap_{\theta \in [-\pi, \pi)} \bigcup_{r > 0} \Sigma(r, \theta) = \phi$.

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