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Osaka University
1. Introduction

T. Kamae has asked (personal communication) whether it is possible to find a sequence \((a_k)\) of \(\pm 1\)'s such that the sums

\[
\sum_{k=m}^{m+n} a_k e^{-ik\theta}
\]

stay bounded (for all integers \(m\) and \(n\) with \(n \geq 0\)) for all \(\theta \in [-\pi, \pi)\) (with the bound possibly depending on \(\theta\)). We show that there is no such sequence. In fact, the only such real-valued sequences must be "essentially zero" in a sense explained below.

This conclusion is reached by adopting a dynamical viewpoint, applying the Spectral Theorem, and showing that every nonzero element of \(L^2\) must have nonzero mean power at some frequency. This latter observation is equivalent to the triviality of the intersection of all the spaces of "twisted coboundaries" for a unitary operator.

2. Results

Suppose that \(a=(a_k) \in \mathbb{R}^\mathbb{Z}\) is a doubly infinite sequence with the property that

\[
|\sum_{k=m}^{m+n} a_k e^{-ik\theta}| \leq c(\theta) < \infty \quad \text{for all } m \in \mathbb{Z}, \text{ all } n \geq 0, \text{ and all } \theta \in [-\pi, \pi).
\]

Taking \(n=\theta=0\), we see that \(a\) is bounded and so takes values in a compact interval \(I\). Let \(X\) denote the closure of the orbit of \(a\) under the shift transformation \(\sigma\) in the compact metric space \(I^\mathbb{Z}\). Let \(\mu\) be a shift-invariant Borel probability measure on \(X\).

Given \(x \in X\) and a block \(B=b_0 \cdots b_n\) which appears in \(x\), we can find a block \(D=d_0 \cdots d_n\) in \(a\) such that \(|b_i - d_i| < 1/(n+1)\) for \(i=0, \cdots, n\). Consequently

\[
|\sum_{k=0}^{n} b_k e^{-ik\theta}| \leq c(\theta) + 1 \quad \text{for all } \theta.
\]

If \(Tg=g \circ \sigma\) for \(g \in L^2(X, \mu)\) and \(f(x) = \pi_0 x = x_0\) for \(x \in X\), we have then that
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\[ \| \sum_{k=0}^{n} T^k f e^{-i\theta k} \|_2 \leq c(\theta) + 1 \quad \text{for all } \theta \text{ and all } n \geq 0. \]

We will see that this is impossible unless $f=0$ in $L^2$. Since this cannot happen for a sequence $a$ which assumes only finitely many values, all nonzero, the original question will be settled. For general sequences, the conclusion is that boundedness against all $\theta$ is possible only if projection onto the central coordinate is 0 a.e. with respect to every invariant measure on the orbit closure $X$ of the sequence; that is, the only invariant probability measure on $X$ is concentrated on the fixed point $0^\circ$. In this case we say that the sequence $(a_k)$ is essentially zero.

**Theorem.** Let $H$ be a Hilbert space and $T: H \to H$ a unitary operator. For each $n=1, 2, \cdots, \in \mathbb{Z}$, and $f \in H$ let

\[ S^\theta f = \sum_{k=0}^{n} e^{-i\theta k} T^k f. \]

If $\sup_{\theta} \| S^\theta f \| < \infty$ for all $\theta$, then $f=0$.

**Proof.** Applying the Spectral Theorem with common notations and conventions, we may write

\[ T f = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda) f, \]

\[ S^\theta f = \int_{-\pi}^{\pi} e^{i\lambda} \sum_{k=0}^{n} e^{-i\theta k} dE(\lambda) f = \int_{-\pi}^{\pi} \frac{1 - e^{i(n-\theta)}}{1 - e^{i(\lambda-\theta)}} dE(\lambda) f, \]

and

\[ \| S^\theta f \|^2 = \int_{-\pi}^{\pi} \left| \frac{1 - e^{i(n-\theta)}}{1 - e^{i(\lambda-\theta)}} \right|^2 d\|Ef\|^2(\lambda). \]

The following Lemma will show that such expressions cannot stay bounded for any positive measure (such as $\nu = \|E(\cdot)\|^2$ if $f \neq 0$ a.e.), thereby completing the proof.

**Lemma.** There is a constant $C > 0$ such that if $\nu$ is a positive measure on $[-\pi, \pi)$, $n$ is a positive integer, $\varepsilon > 0$, and

\[ A_n(\theta) = \frac{1}{n} \sum_{k=0}^{n-1} \left| \frac{1 - e^{i(n-\theta)}}{1 - e^{i(\lambda-\theta)}} \right|^2 d\nu(\lambda), \]

then $\nu \{ \theta \in [-\pi, \pi) : A_n(\theta) < \varepsilon \} < \frac{\varepsilon}{C}$.

**Proof.** Let $C_1$ and $C_2$ be positive constants such that $|\alpha| < \pi$ implies that $C_1 |\alpha| \leq |1 - e^{i\alpha}| \leq C_2 |\alpha|$. Then

\[ A_n(\theta) \geq \frac{1}{n} \int_{\theta-(\pi/n)}^{\theta+(\pi/n)} \left| \frac{1 - e^{i(n-\theta)}}{1 - e^{i(\lambda-\theta)}} \right|^2 d\nu(\lambda) \geq \left( \frac{C_1}{C_2} \right)^2 n\nu(\theta - \frac{\pi}{n}, \theta + \frac{\pi}{n}). \]
Let $\varepsilon > 0$ and $n > 0$, and let $\delta = \delta(\varepsilon) = \nu \{ \theta : A_n(\theta) < \varepsilon \}$. Suppose that $\delta > 0$, since otherwise we are finished. Choose a compact set $K \subset \{ \theta : A_n(\theta) < \varepsilon \}$ with $\nu(K) > \delta/2$. There are $\theta_1, \ldots, \theta_p \in K$ such that the intervals $(\theta_i - \frac{\pi}{n}, \theta_i + \frac{\pi}{n})$ cover $K$ and no more than two of them intersect at any point. Since the union of these intervals is contained in $(-2\pi, 2\pi)$, it follows that $p \frac{2\pi}{n} \leq 8\pi$, and hence $p \leq 4n$. Therefore

$$\nu(K) \leq \sum_{i=1}^{p} \nu \left( \theta_i - \frac{\pi}{n}, \theta_i + \frac{\pi}{n} \right) \leq \nu \left[ \frac{C_2}{C_1} \right] \frac{1}{n} \varepsilon \leq 4 \left( \frac{C_2}{C_1} \right) \varepsilon,$$

and $\delta < 8(C_2/C_1)^2 \varepsilon$, proving the Lemma and hence also the Theorem.

**Corollary 1.** If $\nu$ is a positive measure on $[-\pi, \pi)$ and $(n_j)$ is an increasing sequence of positive integers, then

$$\limsup_{j \to \infty} \frac{1}{n_j} \int_{-\pi}^{\pi} \frac{1 - e^{i n_j (\lambda - \theta)}}{1 - e^{i (\lambda - \theta)}} d\nu(\lambda) > 0 \quad \text{for } \nu\text{-almost all } \theta.$$

Proof. For each $\varepsilon > 0$, $\{ \theta : \lim sup A_{n_j}(\theta) = 0 \} \subset \{ \theta : A_{n_j}(\theta) < \varepsilon \text{ for all large enough } j \}$, a set of measure less than $\varepsilon/C$ by the Lemma.

**Corollary 2.** Let $H$ be a Hilbert space, $T: H \to H$ a unitary operator, and $0 \neq f \in H$. Then there exists a frequency $\theta$ at which the "mean power" of $f$, defined by

$$P(\theta) = \limsup_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{-i\theta} T^k f \right\|^2,$$

is positive.

**Corollary 3.** Let $H$ be a Hilbert space and $T: H \to H$ a unitary operator. For each $\theta \in [-\pi, \pi)$ let

$$\mathcal{B}_\theta = \{ e^{i\theta} g - T g : g \in H \}$$

be the space of "$\theta$-twisted coboundaries" for $T$. Then $\bigcap_{\theta \in [\pi, \pi)} \mathcal{B}_\theta = \{0\}$.

Proof. If $f \in \mathcal{B}_\theta$, then $\{ \| S_{n}^\theta f \| : n=1, 2, \ldots \}$ is bounded.

**Remark.** As in [2], by considering fixed points of the operator $V^\theta f = e^{-i\theta} (f + T g)$, one can show that in fact $f \in \mathcal{B}_\theta$ if and only if $\{ \| S_{n}^\theta f \| : n=1, 2, \ldots \}$ is bounded. For further developments in this direction, see [1].

**Corollary 4.** As in [3], define the "spectral notch" subshift $\{ \lambda^\theta \}$ corresponding to $r > 0$ and $\theta \in [-\pi, \pi)$ to be the set of all those $x \in \{-1, 1\}^\mathbb{Z}$ for which
\[ |\sum_{k=0}^{m} x_k e^{-ik\theta}| < r \quad \text{for all } m \in \mathbb{Z} \text{ and all } n \geq 0. \]

Then \( \bigcap_{\theta \in [0, n]} \bigcup_{r > 0} \sum(r, \theta) = \emptyset. \)

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