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ON ONE-SIDED QF-2 RINGS I

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We first consider a right artinian ring. Then every projective module P is a direct sum of indecomposable submodules; $P=\sum_I \oplus P_\alpha$. Furthermore for any simple module A in $P/\mathrm{J}(P)$ there exists a direct summand P_0 of P such that $(P_0+\mathrm{J}(P))/\mathrm{J}(P)=A$, where $\mathrm{J}(P)$ is the Jacobson radical of P. It is clear that $P\neq\sum_J \oplus P_\beta$ for any proper subset J of I.

In this paper we shall study those properties on injectives E with the condition (**) in [3] and [4], e.g. QF-2 algebra [11] (see §1). If $E=\sum_I E'_\alpha$ and $E\neq\sum_J E'_\beta$ for $J\subsetneqq I$, we say $\sum_I E'_\alpha$ be an irredundant sum. We shall give structure theorems of artinian rings over which every irredundant sum of injective in E is injective and every simple module in $E/J(E)$ is lifted to an indecomposable submodule of E. We have studied perfect rings satisfying (*) (see §1) in [4]. We shall show that they satisfy the above properties and they are right artinian from these facts.

We shall extend those ideas to more general modules in [5] and study the dual properties on projectives in [6].

1. Preliminaries

Throughout we consider a ring R with identity and every module is a unitary right R-module. Let M be an R-module. We shall denote the *Jacobson radical* and an *injective envelope* of M by $J(M)$ and $E(M)$, respectively. If M is a small sumbodule in $\mathrm{E}(M)$, M is called a *small module* [7] and [9] and otherwise we call M a *non-small module* [3]. If M contains a non-zero injective submodule, M is clearly non-small. We consider the converse case, namely

(*) *Every non-small module contains a non-zero injective submodule* [4].

In [4] we have studied perfect rings with (*). We shall show that such rings are right artinian in §4. Furthermore, we shall give some weaker conditions than (*) and show that rings satisfying new conditions give us new classes of rings.

Let M be a module and $\{M_\alpha\}_I$ a set of submodules of M. If $\sum_I M_\alpha \supsetneqq \sum_{I'} M_\alpha$ for any proper subset I' of I, then we say the sum $\sum_I M_\alpha$ be *irredundant*. It is clear that every direct sum is irredundant.

From now on, we assume R is a left and right perfect ring [1]. Let $\{g_i\}_1^t$ be a complete set of mutually orthogonal primitive idempotents with $1=\sum g_i$. If g_iR is a small (resp. non-small) module, we call g_i a *small* (resp. *non-small*) *idempotent*. Then we obtain a partition $\{g_i\}_1^t=\{e_i\}_1^n \cup \{f_j\}_1^m$, where the e_j is non-small and the f_j is small [3]. We have the following lemma from [4], Theorem 2.3.

Lemma 1. *If R is a left and right perfect ring with* (*), *then every indecomposable injective module is of a form $e_iR/e_i\mathrm{l}(J^k)$, where $J=\mathrm{J}(R)$, e_i is non-small and* $\mathrm{l}(J^k)=\{x\in R \mid xJ^k=0\}$.

In this paper we always consider injective modules which are related to the above form. We note the injective $e_iR/e_i\mathrm{l}(J^k)$ contains a unique maximal submodule $e_iJ/e_i\mathrm{l}(J^k)$ and every epimorphism of $e_iR/e_i\mathrm{l}(J^k)$ onto itself is isomorphic, since $\mathrm{End}_R(e_iR/e_i\mathrm{l}(J^k))$ is a homomorphic image of the local ring e_iRe_i and $\mathrm{l}(J^k)$ is a two-sided ideal. Hence, we quote here a condition in [3] and [4];

(**) *Every indecomposable and injective module contians a unique maximal submodule,* i.e. a cyclic hollow module.

Furthermore, we consider a new condition;

(E-I) *Every epimorphism of an R-module onto itself is isomorphic* (cf. [5]).

If R is a finite dimensional algebra over a field K, then we can consider the duality. The above condition (**) is dual to

(**)* *Every indecomposable and projective module contains a unique minimal submodule.*

If R further satisfies (**)* for every left projective module, we call R a *QF-2 ring* following Thrall [11]. Hence, in general, we shall call a ring satisfying (**) a *right QF-2** ring in this note. We shall study a right QF-2 ring (satisfying (**)*) in [6].

From now on, we always assume (**). Then if E_α is indecomposable and injective, $\mathrm{J}(E_\alpha)=E_\alpha J$ is a unique maximal submodule and $E_\alpha/E_\alpha J$ is simple, since R is perfect. Let x be any element in $E_\alpha - E_\alpha J$, then $xR=E_\alpha$ and so $E_\alpha \approx eR/eA$, where e is a non-small idempotent and A is a right ideal of R. We denote E_α by $\mathrm{E}(S_\alpha)$, where S_α is a simple submodule. Let M be an R-

module. We denote M/MJ and the natural epimorphism of M onto M/MJ by $\bar{M}$ and φ_M, respectively. If M_1 is a submodule of M, $\varphi_M|M_1=\rho\varphi_{M1}$; $\rho: M_1/M_1J\to M/MJ$. If there are no confusions, we denote $\varphi_M(M_1)$ by $\bar{M}_1$ (actually $\bar{M}_1=M_1/M_1J$). If M_1 is a direct smmand, $\varphi_M|M_1=\varphi_{M_1}$.

Lemma 2. *We assume an R-module M is equal to a sum of injective submodules $\{\mathrm{E}(S_\alpha)\}_I$. Then $\sum_I \mathrm{E}(S_\alpha)$ is irredundant if and only if $\varphi_M(M)=\sum_I\oplus\varphi_M(\mathrm{E}(S_\alpha))$.*

Proof. It is clear that $\bar{M}=\sum_I\overline{\mathrm{E}(S_\alpha)}$ and the lemma is trivial since MJ is a small submodule in M [1].

Lemma 3. *Let R be a right* QF-2* *and artinian ring. Then for every non-small primitive idempotent e there exists a right ideal A such that eR/eA is injective.*

Proof. Let $\mathrm{E}(eR)=\sum_{i=1}^k\oplus e_i'R/e_i'A_i$, where e_i' is non-small and the $e_i'A_i$ is a right ideal. Since eR is non-small and $e_i'R/e_i'A_i$ is hollow, $\pi_i(eR)=e_i'R/e_i'A_i$ for some i, where $\pi_i: \mathrm{E}(eR)\to e_i'R/e_i'A_i$ is the projection. Hence, $eR\approx e_i'R$.

2. Right artinian rings with (*)

We shall show in §4 that perfect rings with (*) are right artinian. Hence we shall first add here a characterization of such rings (cf. [4], Theorem 2.3).

Let R be a right artinian. Then we have a standard decomposition of R:

$$R=\sum_{i=1}^{p}\sum_{j=1}^{\rho(i)}\oplus g_{ij}R\,,$$

where $\{g_{ij}\}$ a set of mutually orthogonal primitive idempotents such that $g_{ij}R\approx g_{ij'}R$ and $g_{ij}R\not\approx g_{kj'}R$ if $i\neq k$. As in §1 we denote non-small idempotent by e_{ij} and put $E_i=\sum_{j=1}^{\rho(i)}e_{ij}$.

Now it is clear that R *satisfies* (*) *if and only if every module M is a direct sum of an injective submodule and a small submodule.* We can restate [4], Theorems 2.3 and 2.4 as follows:

Theorem 1. *Let R be right artinian. Then the following conditions are equivalent.*

1) *R satisfies* (*).

2) *There exists $n_i>0$ for each non-small idempotent $e_i=e_{i1}$ such that $e_iR/e_iJ^{k_i}$ is injective for all $n_i\leqslant k_i$ and $e_iR/e_iJ^{n_i-1}$ is small.*

3) *$R/r(J)J^k$ is a direct sum of an injective module and a small projective*

module for all $k>0$ as R-modules.

4) *R/A is a direct sum of an injective module and a small module for every right ideal A contained in $r(J)$.*

In this case $A=\sum\oplus A_i$ and the A_i is a right ideal in $E_ir(J)$ where $J=\mathrm{J}(R)$ and $r(J)=\{x\in R\,|\,Jx=0\}$.

Proof. Let $R=\sum_{i=1}^{n}\oplus e_iR\oplus\sum_{j=1}^{m}f_iR$ as in §1 and $D=r(J)$. Since the f_jR is small, $f_jD=0$ by [9], Proposition 4.8. Hence, $D=\sum\oplus e_iD$ and $DJ^a=\sum\oplus e_iDJ^a$.

1)$\leftrightarrow$2). We assume that $e_i=e$ and eR is injective and $eJ^{q-1}\neq 0$, $eJ^q=0$. Then eJ^{q-1} is a unique minimal submodule in eR. Hence, $eJ^{q-1}=e\mathrm{l}(J)$. Similarly, we obtain $eJ^{q-t}=e\mathrm{l}(J^t)$ if eR/eJ^{q-t+1} is injective. If eR/eJ^s is small and eR/eJ^{s+1} is injective, $eD\subset eJ^s$ and hence, $eD=eJ^s$, since eJ^s/eJ^{s+1} is unique minimal. Hence, we have proved 1)$\leftrightarrow$2) by [4], Theorem 2.3.

1)$\rightarrow$3). $D=\sum\oplus e_iD=\sum\oplus e_iJ^{n_i}$ for some n_i by the above. Hence, $R/DJ^k=\sum\oplus f_jR\oplus\sum_{i=1}^{n}\oplus e_iR/e_iJ^{n_i+k}$ and the $e_iR/e_iJ^{n_i+k}$ is injective for $k>0$ by [4], Theorem 2.3.

3)$\rightarrow$1). We always have $DJ^k=\sum\oplus e_iDJ^k$. Hence, $R/DJ^k=\sum_{j=1}^{m}\oplus f_jR\oplus\sum_{i=1}^{n}\oplus e_iR/e_iDJ^k$. Therefore, the e_iR/e_iDJ^k is injective for any $k>0$ by Krull-Remak-Schmidt theorem, since e_iR is non-small. If $e_iDJ^t=0$ and $e_iDJ^{t-1}\neq 0$, e_iR is injective and e_iDJ^{t-1} is a unique minimal submodule in e_iR and $e_iDJ^{t-1}=e_i\mathrm{l}(J)$. Repeating those arguments as in the proof of [4], Theorem 2.3, there exist an integer n_i and a unique series of submodules $e_i\mathrm{l}(J^t)$ of e_iR such that $e_iR/e_i\mathrm{l}(J^t)$ is injective for $t<n_i$ and $e_iD=e_i\mathrm{l}(J^{n_i})$. Therefore, R satisfies (*) by [4], Theorem 2.3.

1)$\rightarrow$4). It is clear from the fact mentioned before the theorem.

4)$\rightarrow$1). Since $DJ^k\subsetneq D$ for $k>0$, e_iR/e_iDJ^k is not small by [9], Proposition 4.8. Hence, we can use the same method in 3)$\rightarrow$1).

Finally we assume (*) and $e_i=e_{i1}$. Let $e_iD\supset e_iA_i\supsetneq e_iB_i$ be right ideals. Then $e_iB_i=e_iJ^{t_i}$ for some t_i and e_iR/e_iB_i is injective by [4], Theorem 2.3. If $e_iA_i/e_iB_i\overset{\psi}{\approx}e_jA_j/e_jB_j$, we can extend ψ to an isomorphism of e_iR/e_iB_i to e_jR/e_jB_j, since e_pR/e_pB_p is indecomposable and injective for $p=i, j$. Hence $i=j$. Since $e_{ij}D\approx e_{ij'}D$, the set of simple factor modules of composition series of E_iD coincides with one of $e_{i1}D$. Therefore, $A=\sum\oplus(E_iD\cap A)$.

Example 1. There exists a commutative ring R with R/J artinian, which satisfies the condition in [4], Theorem 2.3. We quote the example in [7], p. 378. Let $R=Z_{(p)}\oplus Z_{p^\infty}$, where $Z_{(p)}=\mathrm{End}_Z(Z_{p^\infty})$. Then R has a ring structure as usual. $\mathrm{J}(R)=pZ_{(p)}\oplus Z_{p^\infty}$ and $r(J)=\{(0, 1/p)\}$. Furthermore, $0\rightarrow\{(0, 1/p)\}\rightarrow$

$\to R \to pZ_{(p)} \oplus Z_{p^\infty} (\approx \mathrm{J}(R)) \to 0$ is exact. Hence, R is self-injective and $R/r(J)$ is a small module. However R does not satisfy (*) by [4], Lemma 2.1, since $Q_{(p)}$ does not contain any cyclic injective modules.

3. Lifting property on injectives

In this section we assume R is a right artinian ring. Let M be an R-module. If for any simple submodule A_α of M/MJ there exists a direct summand M_α of M such that $\varphi(M_\alpha)=A_\alpha$, then we say M have the *lifting property of simple module.*

Now we shall study injective modules over right QF-2* ring. We define two weaker conditions than (*).

(*1) *Every non-small module which is a homomorphic image of an injective module contains a non-zero injective module.*

And

(*2) *For every non-small module M there exists an indecomposable direct summand E_1 of* $\mathrm{E}(M)$ *such that* $\overline{\mathrm{E}(M)} \supset \bar{M} \supset \bar{E}_1$.

We know from (**) that every indecomposable and injective module is of a form eR/eA, where e is a non-small primitive idempotent and eA is a right ideal of R.

Theorem 2. *Let R be a right* QF-2* *and artinian ring. Then the following conditions are equivalent.*

1) *R satisfies* (*1).

2) *Every irredundant sum of direct summand E_α in an injective module E such that $\varphi_E(\sum E_\alpha)=\sum\oplus\varphi_E(E_\alpha)$ is injective.*

3) *For each non-small idempotent e, there exist right ideals $eA_1 \subset eA_2 \subset \cdots \subset eA_{t-1} \subset eA_t$ such that*

i) *eA_t/eA_1 is a uni-serial module such that eA_i/eA_{i-1} is the socle of eR/eA_{i-1}.*

ii) *eR/eB is small for all right ideals $eB \supsetneqq eA_t$.*

iii) *The set of those eR/eA_i is the representative set of indecomposable injectives.*

(It is possible that eR contains more than one such increasing series).

Proof. 1)$\to$2). Let E be injective and $\sum E_\alpha$ an irredundant sum in E with E_α injective and indecomposable as in 2). Since R is artinian (hence noetherian) we may show that $\sum_K E_\alpha$ is injective for every finite subset K of I. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ be a finite subset of I and $E(p)=\sum_{i=1}^{p} E_{\alpha_i}$. We shall show the above fact by inducton on p. We assume $E(p-1)$ is injective and $E=E(p-1)\oplus K'$. Let $\pi\colon E\to K'$ be the projection. Then $\pi(E_{\alpha_p})$ is not a small submodule in K', since $\varphi_E(\sum E_\alpha)=\sum\oplus\varphi_E(E_\alpha)$. Hence, $\pi(E_{\alpha_p})$ is injective by (*1) for E_{α_p} is hollow. Therefore, $E(p)=E(p-1)\oplus\pi(E_{\alpha_p})$ is injective.

2)→3). We shall show that if eR/eA is injective and eR/eB is non-small for $eB\supset eA$, then eR/eB is injective for $e=e_i$. Put $E=\mathrm{E}(eR/eB)$ and $F=eR/eA\oplus E$. Let $f\colon eR/eA\to eR/eB$ be the natural epimorphism and $G=\{x+f(x)\,|\,x\in eR/eA\}\subset F$. Then $G\approx eR/eA$ and $\varphi_F(eR/eA+G)=\varphi_F(eR/eA)\oplus\varphi_F(G)$ since eR/eB is non-small and hence $\overline{f(x)}\neq 0$ for some x. Hence, $eR/eA+G=eR/eA\oplus f(eR/eA)$ is injective. Therefore, $eR/eB=f(eR/eA)$ is injective. From Lemma 3 we have injective eR/eC_i for each e. We may assume that eA_1 is a minimal one among eC_i. Let eA_2/eA_1 be the socle of eR/eA_1. If eR/eA_2 is non-small, eR/eA_2 is injective by the above. Repeating those arguments, we get a series of right ideals $eA_1\subset eA_2\subset\cdots\subset eA_t$ such that eR/eA_j is injective and eR/eB is small for all $eB\supsetneqq eA_t$ by [4], Lemma 1.1. Hence, we have proved 2)→3) by (**).

3)→1). Let M be a non-small module which is a homomorphic image of injective E. Let $E=\sum\oplus\mathrm{E}(S_\alpha)$. Then there exists $\mathrm{E}(S_\alpha)\,(\approx eR/eA_i)$ whose image is a non-small submodule M_0 of M. Hence, $M_0\approx eR/eC$, $eC\supset eA_i$. On the other hand, either $eC=eA_j\supset eA_i$ for some j or $eC\supsetneqq eA_t$ from 3) i). Hence, $eC=eA_j$ by 3) i).

Corollary. *Let R be right artinian. Then R satisfies* (*) *if and only if R is a right* QF-2* *and* QF-3 *ring* [11] *satisfying* (*1), (see Example 2).

Proof. It is clear from Theorems 1 and 2 and [4], Theorem 1.3.

We have considered special irredundant sums in an injective module in Theorem 2. If we drop the assumption in 2) of Theorem 2, we have the well known theorem:

Let R be a right noetherian ring. Then the following conditions are equivalent.

1) *R is right hereditary.*

2) *Every irredundant sum of indecomposable injective submodules in an injective module is injective.*

We shall give a proof for the sake of completeness.

1)→2). It is clear.

2)→1). Let E be injective and let E_1, E_2 be injective submodules. We shall show E_1+E_2 is injective. We have decompositions $E_i=\sum_{I_i}\oplus E_{i\alpha}\ (i=1,2)$ where the $E_{i\alpha}$ is indecomposable. Since R is right noetherian, we may show $\sum_{K_1}E_{1\alpha}+\sum_{K_2}E_{2\beta}$ is injective for finite subsets K_i. Hence, since we may assume that its sum is irredundant, E_1+E_2 is injective. Let A be a submodule of E. Then we can show by the same argument as in the proof 2)→3) of Theorem 2 that E/A is injective. Therefore, R is right hereditary.

Using the above and Theorem 2 we have

Theorem 2′. *Let R be a right artinian* QF-2* *ring. Then the following conditions are equivalent.*

1) *Every homomorphic image of an injective module is injective* (*R is hereditary*).

2) *Every irredundant sum of indecomposable injective submodules in an injective modules is injective.*

3) *For each non-small idempotent e, there exists a right ideal eA such that eR/eA is a uni-serial module and eR/eB is injective for all $eR\supseteqq eB\supseteqq eA$. The set $\{eR/eB\}_{e,B}$ is the representative set of indecomposable injective modules* (cf. Example 3).

Proof. 2)→3). It is clear from Theorem 2 and the above.

3)→2). Since every homomorphic image of eR/eB is injective, we can prove the implication by the same argument as in Theorem 2.

We assume that a module M has the lifting property of simple module. Then for any decomposition $\bar{M}=\sum_I \oplus A_\alpha$ with A_α simple, there exists a set of direct summands M_α of M such that $\bar{M}_\alpha=A_\alpha$ and $M=\sum_I M_\alpha$ is an irredundant sum (cf. Lemma 2).

Theorem 3. *Let R be a right* (QF-2* *and*) *artinian ring. Then the following conditions are equivalent.*

1) *R satisfies* (*2).

2) *Every injective module has the lifting property of simple module.*

3) i) *For each non-small idempotent e there exists a chain of right ideals A_i such that eR/eA_i is injective and $eA_1\subset eA_2\subset\cdots\subset eA_t$ and each element in* $\mathrm{End}_R(eR/eJ)$ *is induced from some element in* $\mathrm{Hom}_R(eR/eA_i, eR/eA_j)$ *for any $i\geqslant j$.*

ii) *The set of $\{e_iR/e_iA_{ij}\}_{i=1}^{n}{}_{j=1}^{t_i}$ is the representative set of indecomposable injectives.*

Proof. 1)→2). Let E be injective and $E/EJ=\sum\oplus A_\alpha$; the A_α is simple. We take an element x in E such that $\overline{Rx}=A_\alpha$. Then xR is non-small by the definition. Hence, there exists an indecomposable direct summand E_α of E such that $A_\alpha=\varphi_E(xR)\supset\varphi_E(E_\alpha)$ by (*2). Therefore, $\bar{E}_\alpha=A_\alpha$.

2)→3). We know from (**) and Lemma 3 that the representative set of indecomposable injectives is $\{e_iR/e_iB_{ij}\}_{i=i}^{n}{}_{j=1}^{\kappa(i)}$. We put $E(i)=\sum_{j=1}^{\kappa(i)}\oplus e_iR/e_iB_{ij}$, $e_i=e$ and $B_{ij}=B_j$. Then there exists an epimorphism either $f_{lm}\colon eR/eB_m\to eR/eB_l$ or $f_{ml}\colon eR/eB_l\to eR/eB_m$ for any pair l and m by 2) and [5], Corollary to

Theorem 2. We denote this situation by $eR/eB_m \succcurlyeq$(resp. $\preccurlyeq$) eR/eB_l. Then the relation $\succcurlyeq$ is linear. We take the maximal one among eR/eB_j, say $eB_1(=eA_1)$ with respect to the relation $\succcurlyeq$. Let eR/eB_2 be the second one. Since there exists an epimorphism f_{21}, there exists a right ideal A_2 such that $eA_2 \supset eA_1$ and $eR/eA_2 \approx eR/eB_2$. Repeating those arguments, we have a chain of right ideals A_i satisfying i) by [5], Theorem 2.

3)$\rightarrow$2). It is clear from 3) and [5], Theorem 2.

2)$\rightarrow$1). It is clear from the definition.

4. Perfect rings with (*)

We shall show, in this section, that a left and right perfect ring satisfying (*) is right artinian.

Theorem 4. *Let R be a perfect and right* QF-2* *ring. We assume every indecomposable injective module satisfies (E-I). Further if every injective module E has the lifting property of simple module, then R is right artinian.*

Proof. Let E be an indecomposable injective module, say $E=\mathrm{E}(S)$; S is simple. We shall show that E is $\sum$-injective [2]. Put $T=\sum\oplus E_\alpha$; $E_\alpha=E$ and $Q=\mathrm{E}(T)$. It is clear that $\sum_I\oplus\bar{E}_\alpha$ is a direct summand in $\bar{Q}=Q/QJ$. Now we can express $\bar{Q}=\sum_I\oplus\bar{E}_\alpha\oplus\sum_J\oplus A_\beta$; the A_β is simple. Then $Q=\sum_I\oplus E_\alpha+\sum_L \mathrm{E}(S_\beta)$ by the assumption and Lemma 2 and $\overline{\mathrm{E}(S_\beta)}=A_\beta$. We shall show $L=\phi$. Otherwise, there would exist $A_\beta\neq 0$. $\sum_I\oplus S$ is the socle of T and hence of Q. Hence, $S_\beta\approx S$ and $\mathrm{E}(S_\beta)\approx E$. Let $S_\beta\subset\sum_K\oplus S$ for some finite subset K of I and $E_K=\sum_K\oplus E_\gamma$. Then $Q=E_K\oplus F$. We denote the projection of Q onto F by π. $\ker\pi\,|\,\mathrm{E}(S_\beta)\supset S_\beta$ and $\pi(\mathrm{E}(S_\beta))\not\subset FJ$ since $Q\neq\sum_I\oplus E_\alpha+\sum_{\gamma\neq\beta}\mathrm{E}(S_\gamma)$. It is clear $\pi(\mathrm{E}(S_\beta)J)\subset FJ$. Hence, $F/FJ=C/CJ\oplus B_2\oplus\cdots$, where $C=\pi(\mathrm{E}(S_\beta))$ and the B_i is simple. Then we have an irredundant sum $\sum E_\delta'$ of indecomposable injective submodules of F such that $\varphi_F(E_{\delta_1}')=C/CJ$. It is clear as above that $E_{\delta_1}'\approx E$. Put $F=E_{\delta_1}'\oplus L$ and π_1 the projection of F onto E_{δ_1}'. Since $C+FJ=E_{\delta_1}'+FJ$, $\pi_1\pi\colon \mathrm{E}(S_\beta)\rightarrow E_{\delta_1}'$ is an epimorphism. Hence, $\pi_1\pi\,|\,\mathrm{E}(S_\beta)$ is isomorphic by the assumption on E. However, $\ker\pi_1\pi\,|\,\mathrm{E}(S_\beta)\supset S_\beta$, which is a contradiction. Thus, E is $\sum$-injective. Let V be any injective module and $\sum_1^t\sum_{I_i}\oplus S_{i\alpha_i}$ the socle of V, where $S_{i\alpha}\approx S_{i\beta}$ and $S_{i\alpha}\not\approx S_{j\beta}$ if $i\neq j$. Then $V=\mathrm{E}(\sum_{i=1}^t\sum_{I_i}\oplus S_{i\alpha_i})=\sum_1^t\oplus\mathrm{E}(\sum_{I_i}\oplus S_{i\alpha_i})=\sum_1^t\oplus\sum_{I_i}\mathrm{E}(S_{i\alpha_i})$ by the above. Hence, R is right artinian by [10], Theorem 4.5 in p. 85.

Theorem 5. *Let R be a left and right perfect ring. If R satisfies (*), R is right artinian.*

Proof. It is clear from Lemma 1 and Theorems 3 and 4.

Proposition 1. *Let R and E be as above. We assume $E=\sum_I \mathrm{E}(S_\alpha)$ is irredundant. Then there exists a set of epimorphisms $\psi_\alpha\colon \mathrm{E}(S_\alpha)\to\mathrm{E}(S_\alpha')$ for all $\alpha\in I$ such that $E=\sum_I\oplus\mathrm{E}(S_\alpha')$.*

Proof. We denote the injective $e_iR/e_i\mathrm{l}(J^t)$ by $E(e_i, t)$. We assume $E=\sum_{J(i,0)} F(e_1, 0)_\alpha+\sum_{J(1,1)} F(e_1, 1)_\alpha+\cdots+\sum_{J(1,s_1)} F(e_1, s_1)_\alpha+\cdots+\sum_{J(n,s_n)} F(e_n, s_n)_\alpha$ is an irredundant sum of E, where $F(e_i, j)_\alpha\approx E(e_i, j)$. Let F_0 be an indecomposable direct summand of $F_{j,t}=\sum_{J(i,t)} F(e_j, t)_\alpha$. $F_{j,t}$ is injective by Theorem 2 and Lemma 2 and $F_{j,t}=F_0\oplus E'$. Let π be the projection of $F_{j,t}$ onto F_0. Then there exists some $F(e_j, t)$ such that $\pi(F(e_j, t)_\beta)=F_0$ and so $F_0\approx E(e_j, p)$ for $p\geqslant t$. Let $F_{1\,s_1}+F_{1\,s_1-1}=F_{1\,s_1}\oplus L$ and $F_{1,s_1}=\sum_{I(1,s_1)}\oplus\mathrm{E}(S_\gamma)$. Then $\mathrm{E}(S_\gamma)\approx E(e_1, s_1)$ by the above and $|I(1, s_1)|=|J(1, s_1)|$ by Lemma 2, where $|P|$ means the cardinal of P. Hence, there is a set of epimorphisms $\psi_\alpha'\colon F(e_1, s_1)_\alpha\to\mathrm{E}(S_\alpha)$. Let $\pi'\colon F_{1,s_1}+F_{1,s_1-1}\to L$ be the projection. Since F_{1,s_1-1} is an irredundant sum, $\pi'(F(e_1, s_1-1)_\alpha)$ is injective by Theorem 1. Let $L=\sum_{I(1,s_1-1)}\oplus\mathrm{E}(S_\delta')$. Then $\mathrm{E}(S_\delta')$ is a homomorphic image of $\pi'(F(e_1, s_1-1))$ and hence of $F(e_1, s_1-1)_\alpha$. Further $|I(1, s_1-1)|=|J(1, s_1-1)|$. Hence, we obtain a set of epimorphisms $\psi_\alpha''\colon F(e_1, s_1-1)_\alpha\to\mathrm{E}(S_\alpha')$. Repeating those arguments, we obtain the proposition.

Proposition 2. *Let R and E be as above. We assume E is a direct sum of $\mathrm{E}(S_\alpha)$ whose proper homomorphic images are not injective. Then every irredundant sum of indecomposable injectives of E is a direct sum.*

Proof. Let $E=\sum E_\alpha$ be an irredundant sum of indecomposable injectives. We put $E(n)=\sum_{i=1}^n E_{\alpha_i}$ for a finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. We show $E(n)=\sum_{i=1}^n\oplus E_{\alpha_i}$ by the induction on n. We assume $E(n-1)=\sum_{i=1}^{n-1}\oplus E_{\alpha_i}$ and $E(n)=E(n-1)\oplus K$. Let π be the projection of $E(n)$ onto K. Then $\pi|E_{\alpha_n}$ is epimorphic. Since K is injective by Theorem 2, $\pi|E_{\alpha_n}$ is isomorphic by the assumption. Hence, $0=\ker\pi\cap E_{\alpha_n}=E(n-1)\cap E_{\alpha_n}$. Therefore, $E(n)=\sum_{i=1}^n\oplus E_{\alpha_i}$.

Proposition 3. *Let R be as above. Then R is a* QF *ring if and only if every irredundant sum $\sum_I \mathrm{E}(S_\alpha)$ of injective module is a direct sum.*

Proof. If R is a QF ring, each e_iR has no proper homomorphic injective images, since every injective module is projective. Hence, we obtain "only if" by Proposition 2. If R is not a QF ring, then there exists a non-small idempotent e such that eR and eR/eA are injective by Theorem 1, where eA is a proper ideal of eR. Then $E=eR\oplus eR/eA$ has an irredundant sum $(eR, 0)+\{(er, \bar{e}r\,|\,r\in R\}$, where $\bar{e}r$ denotes the image of er by the natural epimorphism: $eR\to eR/eA$. $\{(er, \bar{e}r)\,|\,r\in R\}\approx eR$ and $(eR, 0)\cap\{(er, \bar{e}r)\,|\,r\in R\}=(eA, 0)\neq 0$.

We shall give some examples which show that (*1) and (*2) are independent. It is clear that (*) implies (*1) and (*2).

Example 2. Let K be a field and $C=K\oplus M$; $M=K$, the trivial extension and

$$R=\begin{pmatrix} C & C \\ 0 & C \end{pmatrix}.$$

Put $e_i=e_{ii}$ (matrix units). Then $e_1R=\mathrm{Hom}_C(Re_2, C)$. Since C is self-injective, $\{e_1R, e_1R/(0, C)\}$ is the complete set of indecomposable injectives. Hence, R is right QF-2* and QF-3. Furthermore, R satisfies (*2) by Theorem 3 but not (*1) by 3, ii) in Theorem 2.

Example 3. Put

$$R=\begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}.$$

Then $e_1R/(0, K, K)$, $e_1R/(0, K, 0)$ and $e_1R/(0, 0, K)$ is the complete set of indecomposable injectives. Hence, R is right QF-2* and satisfies (*1) by Theorem 2. Since $(0, K, 0)\cup(0, 0, K)=(0, K, K)$, R does not satisfy (*2) by Theorem 3.

Example 4. Put

$$R=\begin{pmatrix} K & K & K & K \\ & K & 0 & K \\ & & K & K \\ 0 & & & K \end{pmatrix}.$$

Then e_1R, e_1R/e_1J, $e_1R/(0, 0, K, K)$ and $e_1R/(0, K, 0, K)$ is the complete set of indecomposable injectives. Hence R is right QF-2* and furthermore, every indecomposable projective is uniform and so R is QF-2. However, R satisfies neither (*1) nor (*2).

Example 5. Put

$$R=\begin{pmatrix} K & K & 0 & K \\ & K & K & K \\ & & K & 0 \\ 0 & & & K \end{pmatrix}\quad(e_{12}\cdot e_{23}=0).$$

Then e_1R, e_1R/e_1J, e_1R/e_1J^2 and $e_2R/(0, 0, 0, K)$ is the complete set of indecomposable injectives. Hence, R is right QF-2* and satisfies (*1) and (*2). However, (*) is not satisfied. We note R is not QF-3 (cf. Corollary to Theorem 2).

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