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QUANTITIES AND REAL NUMBERS

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Introduction

The theory of real numbers, as a basis of mathematical analysis, had been already completed in the nineteenth century in several ways (cf. [1], [2], [3]), and now we seem to have nothing to do newly with it. These mathematical theories have been established as the completion of the system of rational numbers, while the intimate relation between the quantity and the number has been rather neglected.

Here we shall start from the characterization of the system of positive quantities and derive the system of positive real numbers as the set of automorphisms of the system of positive quantities. Then, the extension of the system of positive real numbers to the whole system of real numbers can be easily carried out.

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1. System of positive quantities

A. Axioms of the system of positive quantities. Let Q be a system of quantities of the same kind with the following properties with respect to the addition:

(I_1) If a and $b \in Q$, then $a+b \in Q$ ($a+b$ is uniquely determined).
(I_2) $a+b=b+a$ if a and $b \in Q$.
(I_3) $(a+b)+c=a+(b+c)$ $(a, b, c \in Q)$.
(I_4) If $a+c=b+c$ $(a, b, c \in Q)$, then $a=b$.

The system of quantities Q is said to be *positive,* if the following conditions are fulfilled:

(II_1) If a and $b \in Q$, then $a+b \neq a$.

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(II_2) If a and $b \in Q$ and $a \neq b$, then there exists $c \in Q$ such that either $a+c=b$ or $a=b+c$.

DEFINITION. Let Q be a positive system of quantities and let $a, b \in Q$. b is said to be *larger* than a: denoted by $b>a$ (a is said to be *smaller* than b: $a<b$), if and only if there exists $a' \in Q$ such that $b=a+a'$.

Proposition 1.0. *If $a<b$ $(a, b \in Q)$, then there exists $a' \in Q$ uniquely such that $a'+a=b$. In this case we write $a'=b-a$.*

Proposition 1.1. *For any given pair of elements $a, b \in Q$, just one of the following three cases happens*

1) $a=b$, 2) $a<b$, 3) $b<a$.

Proposition 1.2. i) *Let $a<b$ and $b<c$ $(a, b, c \in Q)$, then $a<c$.* ii) *$a+c<b+c$ holds, if and only if $a<b$.*

B. Axiom of the continuity of Q. Let Q be a positive system of quantities. Q is said to be *continuous* if Q satisfies the following axioms:

(III_1) For any $a \in Q$, there exists $a' \in Q$ such that $a'<a$.

(III_2) (A pair of non-empty subsets Q^-, Q^+, of Q is called *Dedekind's pair,* if and only if $Q^- \cup Q^+ = Q$, $Q^- \cap Q^+ = \emptyset$ (empty set) and $a_1 \in Q^-$, $a_2 \in Q^+$ always implies $a_1<a_2$.) For any Dedekind's pair Q^-, Q^+ of Q, there exists an element $c \in Q$ such that $a_1 \in Q^-$ and $a_2 \in Q^+$ implies $a_1 \leqq c \leqq a_2$.

From (III_1) we get

Proposition 1.3. *Let $a, b \in Q$ and $a<b$. Then there eixsts an element $c \in Q$ such that $a<c<b$.*

From (III_1) and (III_2) we get

Proposition 1.4. *Let Q^-, Q^+ be Dedekind's pair of Q. Then the element $c \in Q$ in* (III_2) *is uniquely determined. And c is either the largest element of Q^- or the smallest element of Q^+.*

We call c the *cut element* of the Dedekind's pair Q^-, Q^+ and we write $c=(Q^- | Q^+)$.

For any $a \in Q$ and any natural number n $(n \in \mathbf{N})$, we define $na \in Q$ by induction: $1a=a$ and $(n+1)a=na+a$.

Proposition 1.5 (Archimedes). *Let a and $b \in Q$. Then there exists $n \in \mathbf{N}$ such that $na>b$.*

Proof. Define Q^- and Q^+ by $Q^- = \{q \in Q;\ ^{\exists}n \in \boldsymbol{N},\ na > q\}$, $Q^+ = \{q' \in Q;\ ^{\forall}n \in \boldsymbol{N}, na \leqq q'\}$, respectively. If Proposition 1.5 was false, then Q^-, Q^+ would be Dedekind's pair ($Q^{\pm} \neq \emptyset$), and $c = (Q^- | Q^+) \in Q$ would lead us to a contradiction.

2. Linear mapping and automorphism

A. Let Q and Q' be positive systems of quantities satisfying the axiom of continuity. Let Φ be a mapping of Q into Q', i.e., by Φ to every $q \in Q$ there corresponds uniquely an element $q' \in Q'$: $q' = \Phi(q)$ (a function of the variable element $q \in Q$).

A mapping Φ of Q into Q' is said to be *linear* (homomorphism), if and only if

$$\Phi(a_1 + a_2) = \Phi(a_1) + \Phi(a_2) \text{ for any pair } a_1, a_2 \in Q\,.$$

A mapping Φ of Q into Q' is said to be 1–1 (one to one) onto Q', if and only if to *every* $a' \in Q'$, there exists *uniquely* an $a \in Q$ such that $\Phi(a) = a'$. In this case the inverse mapping Φ^{-1} of Q' into Q is 1–1 onto Q.

Proposition 2.1 (Theorem of inversion). *Let Φ be a linear mapping of Q into Q'. Then Φ is* 1–1 *onto Q', and the inverse mapping Φ^{-1} of Q' onto Q is also linear.*

To prove Proposition 2.1 we use the following lemmas.

Lemma 2.1.1. *Let Φ be a mapping in Proposition* 2.1. *Then $\Phi(a_1) < \Phi(a_2)$ (a_1, $a_2 \in Q$), if and only if $a_1 < a_2$.*

Lemma 2.1.2. *For any $a \in Q$ and any $n \in \boldsymbol{N}$, there exists $a_n \in Q$ such that $na_n < a$.*

(Use mathematical induction with respect to n).

Lemma 2.1.3. *For any $b \in Q'$, there exist a_1 and $a_2 \in Q$ such that $\Phi(a_1) < b < \Phi(a_2)$.*

Proof of Proposition 2.1. For any fixed $a' (\in Q')$ we define, subsets of Q, Q^- and Q^+ by

$$Q^- = \{q \in Q;\ \Phi(q) < a'\} \text{ and } Q^+ = \{q \in Q;\ \Phi(q) \geqq a'\}\,.$$

Then, Q^- and Q^+ form *Dedekind's* pair of Q.

We put $a = (Q^- | Q^+)$ ($a \in Q$), and we shall show $\Phi(a) = a'$. If it was $\Phi(a) < a'$, then a should be the largest element of Q^-. But there would exist $a_1 \in Q$ such that $a < a_1$ and $\Phi(a_1) < a'$, contradicting to that a is the largest element of Q^-. Because, by Lemma 2.1.3 there would exist $c \in Q$ such that $\Phi(c) < a' - \Phi(a)$ ($\in Q'$), hence $\Phi(a+c) < a'$ $(a_1 = a+c)$.

If it was $\Phi(a) > a'$, then a should be the smallest element of Q^+. But, by Lemma 2.1.3 there exists $c \in Q$ such that $\Phi(c) < \Phi(a) - a'$, hence $c < a$ and $a' < \Phi(a-c)$ $(a_1 = a-c)$, contradiction. Thus, we have shown that Φ maps Q *onto* Q'. By Lemma 2.1.1 we see that Φ is a 1–1 mapping of Q onto Q'. We can easily see that Φ^{-1} is linear. Q.E.D.

B. Rational automorphism. Let Q be a positive system of quantities satisfying the axiom of continuity. A *linear mapping* of Q onto Q itself is called an *automorphism* of Q.

Let a mapping Φ of Q onto Q be defined by

$$\Phi(q) = mq \quad (q \in Q)$$

with a given $m \in \boldsymbol{N}$. Then Φ^{-1} is an automorphism of Q, and by Proposition 2.1 Φ^{-1} is also an automorphism of Q. Thus we write

$$\Phi^{-1}(q) = m^{-1}q\,.$$

For any automorphism Φ of Q (or any linear mapping of Q into Q) we have

$$\Phi(nq) = n\Phi(q) \quad (n \in \boldsymbol{N})\,.$$

Hence

$$n^{-1}\Phi(q) = \Phi(n^{-1}q) \quad ({}^{\forall}q \in Q)\,.$$

Thus, for any m and $n \in \boldsymbol{N}$,

$$n^{-1}(mq) = m(n^{-1}q) \quad ({}^{\forall}q \in Q)\,.$$

The mapping Φ of Q onto Q defined by

$$\Phi(q) = n^{-1}(mq) = m(n^{-1}q) \quad ({}^{\forall}q \in Q) \quad (m,\, n \in \boldsymbol{N})$$

is also an automorphism of Q. For this automorphism of Q we write

$$\frac{m}{n}\,q = m(n^{-1}q) = n^{-1}(mq) \quad ({}^{\forall}q \in Q)\,.$$

An automorphism Φ of Q given by $\Phi(q) = \frac{m}{n}\,q$ $({}^{\forall}q \in Q)$ $(m,\, n \in \boldsymbol{N})$ is called a rational automorphism of Q.

Proposition 2.2. *Let Φ be a linear mapping of Q into Q'.*

Then, for any rational automorphism $\frac{m}{n}$ (of Q and Q'),

$$\Phi\left(\frac{m}{n}q\right)=\frac{m}{n}\Phi(q) \quad (^{\forall}q\in Q).$$

Proposition 2.3. *Let a, $b\in Q$ such that $a<b$. Then for any $c\in Q$ there exists a rational automorphism of Q, $\frac{m}{n}$, such that $a<\frac{m}{n}c<b$.*

Proof. Let $d=b-a$ $(a<b)$. Then there exists $n\in \boldsymbol{N}$ such that $nd>c$ (by Proposition 1.5). Hence $d>n^{-1}c$. Putting $c_i=\frac{i}{n}c$ $(i\in \boldsymbol{N})$ we have $c_i=ic_1$ and $c_i<c_{i+1}$.

There exists $j\in \boldsymbol{N}$ (by Proposition 1.5) such that $c_j=jc_1>a$. Let $j=m$ be the smallest natural number with this property.
Then, we get easily

$$a<c_m<b \text{ with } c_m=\frac{m}{n}c.$$

Proposition 2.4. *Let Φ_1 and Φ_2 be automorphisms of Q such that $\Phi_1(a)=\Phi_2(a)$ for some $a\in Q$. Then $\Phi_1(q)=\Phi_2(q)$ (identically) for all $q\in Q$.*

Proof. Was $b\in Q$ such that $\Phi_1(b)<\Phi_2(b)$. Then, putting $\Phi_1(a)=\Phi_2(a)=c$, by Proposition 2.3 there would exist a rational automorphism $\frac{m}{n}$ such that

$$\Phi_1(b)<\frac{m}{n}c<\Phi_2(b).$$

Thus, as $\Phi_1(b)<\frac{m}{n}c=\Phi_1\left(\frac{m}{n}a\right)$ and $\Phi_2\left(\frac{m}{n}a\right)=\frac{m}{n}c<\Phi_2(b)$, we get, by Lemma 2.1.1, $b<\frac{m}{n}a<b$, an absurd conclusion. Similarly the assumption $\Phi_2(b)<\Phi_1(b)$ would lead us to a contradiction. Consequently, we have $\Phi_1(b)=\Phi_2(b)$ for every $b\in Q$. Q.E.D.

Proposition 2.5. *Let Φ_1 and Φ_2 be automorphisms of Q such that $\Phi_1(a)<\Phi_2(a)$ for some $a\in Q$. Then $\Phi_1(q)<\Phi_2(q)$ for all $q\in Q$. (In this case we write $\Phi_1<\Phi_2$ simply).*

Proof. If the statement was not true, we could assume the existence of some $b\in Q$ such that $\Phi_1(b)>\Phi_2(b)$, (If $\Phi_1(b)=\Phi_2(b)$ then $\Phi_1(q)=\Phi_2(q)$ for all $q\in Q$).

Put $c=\Phi_1(b)-\Phi_2(b)$ $(\in Q)$, then by Proposition 1.5 there exists $n\in \boldsymbol{N}$ such that $nc>\Phi_1(a)$ and $nb>a$. Hence $n^{-1}a<b$.
Then, there exists $m\in \boldsymbol{N}$ such that $\frac{m}{n}a<b\leqq\frac{m+1}{n}a$.
Thus,

$$\Phi_1(b)\leqq\Phi_1\left(\frac{m+1}{n}a\right)=\frac{m+1}{n}\Phi_1(a)<\frac{m}{n}\Phi_2(a)+n^{-1}\Phi_1(a)$$

$$<\Phi_2(b)+n^{-1}\Phi_1(a)<\Phi_2(b)+c\,.$$

This contradicts with the equality $\Phi_1(b)=\Phi_2(b)+c$. Q.E.D.

Let $\boldsymbol{P}$ be the set of all rational automorphism of Q. By Proposition 2.5 in $\boldsymbol{P}$ is defined order of elements r_1 and $r_2 \in \boldsymbol{P}$ such that $r_1 q < r_2 q$ for any $q \in Q \Rightarrow r_1 < r_2$. In $\boldsymbol{P}$ addition and multiplication are defined by

$$r_1+r_2 \in \boldsymbol{P} \Leftrightarrow (r_1+r_2)(q) = r_1(q)+r_2(q) \quad \text{for all } q \in Q$$
$$r_1 \cdot r_2 \in \boldsymbol{P} \Leftrightarrow r_2(r_1 q) = r_2(r_1 q) \quad \text{for all } q \in Q\,.$$

(cf. Propositions 2.2 and 2.4)

In $\boldsymbol{P}$ the addition and multiplication satisfy the wellknown rules of addition and multiplication of rational numbers.

A pair of subsets $\boldsymbol{P}^-$ and $\boldsymbol{P}^+$ of rational automorphism $\boldsymbol{P}$ are called Dedekind's pair of $\boldsymbol{P}$, if and only if $\boldsymbol{P}^-$ and $\boldsymbol{P}^+$ are not empty, $\boldsymbol{P}^- \cup \boldsymbol{P}^+ = \boldsymbol{P}$, $\boldsymbol{P}^- \cap \boldsymbol{P}^+ = \emptyset$ (empty) and $r_1 \in \boldsymbol{P}^-$, $r_2 \in \boldsymbol{P}^+ \Rightarrow r_1 < r_2$.

Proposition 2.6. *Let Q be a positive system of quantities satisfying the axiom of continuity and let $a \in Q$. If $\boldsymbol{P}^-$ and $\boldsymbol{P}^+$ are Dedekind's pair of $\boldsymbol{P}$, then there exists just one $c \in Q$ such that $r_1 a \leqq c \leqq r_2 a$ for every $r_1 \in \boldsymbol{P}^-$ and for every $r_2 \in \boldsymbol{P}^+$.*

Proof. Let Q^- and Q^+ be subsets of Q defined by

$$Q^- = \{q \in Q;\ q < ra \text{ for all } r \in \boldsymbol{P}^+\},$$
$$Q^+ = \{q \in Q;\ {}^{\exists}r \in \boldsymbol{P}^+,\ ra \leqq q\}\,.$$

Then Q^- and Q^+ are Dedekind's pair of Q.

Put $$c = (Q^- \mid Q^+)\,.$$

First assume c is the largest element of Q^-. Then, $c < ra$ for all $r \in \boldsymbol{P}^+$, and $c \geqq ra$ for all $r \in \boldsymbol{P}^-$. (If $c < r_1 a$ for some $r_1 \in \boldsymbol{P}^-$, then as $r_1 a \in Q^-$, this contradicts to that c is the largest element of Q^-.)

Second assume c is the smallest element of Q^+. Then, $c = r_0 a$ for some $r_0 \in \boldsymbol{P}^+$, and $ra < r_0 a = c$ for all $r \in \boldsymbol{P}^-$. Further, as $\{ra;\ r \in \boldsymbol{P}^+\} \subset Q^+$, we have $c \leqq ra$ for all $r \in \boldsymbol{P}^+$.

The uniqueness of c in Proposition 2.6 follows from Proposition 2.3.

Q.E.D.

C. Theorem of isomorphism. Let Q and Q' be positive systems of quantities satisfying the axiom of continuity.

Proposition 2.7 (Theorem of isomorphism). *Let $a \in Q$ and $a' \in Q'$ be* *given. Then, there exists just one linear mapping (isomorphism) Φ of Q onto Q' such that $\Phi(a)=a'$.*

Proof. To define the desired linear mapping Φ, take an arbitrary $q\in Q$ and let $\boldsymbol{P}^-_{(q)}$ and $\boldsymbol{P}^-_{(q)}$ be defined by

$$\boldsymbol{P}^-_{(q)}=\{r\in\boldsymbol{P};\ ra<q\} \text{ and } \boldsymbol{P}^+_{(q)}=\{r\in\boldsymbol{P};\ ra\geqq q\} \text{ respectively}.$$

Then $\boldsymbol{P}^-_{(q)}$ and $\boldsymbol{P}^+_{(b)}$ form Dedekind's pair of rational automorphisms.

Thus, by Proposition 2.6 there exists just one $q'\in Q'$ (depending on q) such that $r_1a'\leqq q'\leqq r_2a'$ for every $r_1\in\boldsymbol{P}^-_{(q)}$ and every $r_2\in\boldsymbol{P}^+_{(q)}$.
Hence we define the mapping Φ in such a way that $\Phi(q)=q'$. Clearly we have $\Phi(a)=a'$.

Now let $\Phi(q_i)=q'_i$ $(i=1, 2)$. Then we have to show that Φ is linear:

$$\Phi(q_1+q_2)=q'_1+q'_2\,.$$

Let $q_3=q_1+q_2$ and $\boldsymbol{P}^-_i=\{r\in\boldsymbol{P};\ ra<q_i\}$ $(i=1, 2, 3)$. Then we get $\boldsymbol{P}^-_3=\{r_1+r_2;\ r_1\in\boldsymbol{P}^-_1,\ r_2\in\boldsymbol{P}^-_2\}$. Because, first we easily see $\{r_1+r_2;\ r_1\in\boldsymbol{P}^-_1,\ r_2\in\boldsymbol{P}^-_2\}\subset\boldsymbol{P}^-_3$. To show $\boldsymbol{P}^-_3\subset\{r_1+r_2;\ r_1\in\boldsymbol{P}^-_1,\ r_2\in\boldsymbol{P}^-_2\}$, let $r\in\boldsymbol{P}^-_3$. Then, as $ra<q_1+q_2$, putting $d=\mathrm{Min}\{q_1+q_2-ra,\ q_1,\ q_2\}\in Q$, by Proposition 2.3 there exist $r_i'\in\boldsymbol{P}(i=1, 2)$ such that $q_i-\dfrac{1}{2}d<r_i'a<q_i$ $(i=1, 2)$. Then, $r_i'\in\boldsymbol{P}^-_i$ and $ra\leqq q_1+q_2-d<(r_1'+r_2')a<q_1+q_2$. Thus, putting $r_i=\dfrac{r_i'}{r_1'+r_2'}r(i=1, 2)$, we have $r_i\in\boldsymbol{P}^-_i$ and $r=r_1+r_2$. For $r_ia=\dfrac{r_i'}{r_1'+r_2'}ra\leqq\dfrac{r_i'}{r_1'+r_2'}(r_1'+r_2')a=r_i'a<q_i$.

Now let $\boldsymbol{P}^+_3=\{r\in\boldsymbol{P};\ ra\geqq q_3=q_1+q_2\}$. Then $\boldsymbol{P}^-_3$ and $\boldsymbol{P}^+_3$ are Dedekind's pair of rational numbers. And by the definition of Φ, putting $\Phi(q_3)=q_3'$, we get

$$r_1a'<q_3'\leqq r_2a' \quad \text{for every } r_1\in\boldsymbol{P}^-_3 \text{ and every } r_2\in\boldsymbol{P}^+_3.$$

Thus, we have to prove $q_3'=q_1'+q_2'$.

If it was $q_3'<q_1'+q_2'$, there would exist $r'\in\boldsymbol{P}$ such that

$$q_3'<r'a'<q_1'+q_2'\,.$$

By the similar method as the case $ra<q_1+q_2$ before, there exist $r_i'\in\boldsymbol{P}^-_i$ $(i=1, 2)$ such that $r_1'+r_2'=r'$. Thus $r'\in\boldsymbol{P}^-_3=\{r\in\boldsymbol{P};\ ra<q_3\}=\{r\in\boldsymbol{P};\ ra'<q_3'\}$ contradicting to $q_3'<r'a'$.

If it was $q_3'>q_1'+q_2'$ there would exist $r'\in\boldsymbol{P}$ such that

$$q_3'>r'a'>q_1'+q_2'\,.$$

And similarly as above there exist $r_i'\in\boldsymbol{P}^+_i$ $(i=1, 2)$ such that $r_1'+r_2'=r'$. Hence $r'\in\boldsymbol{P}^+_3=\{r\in\boldsymbol{P};\ ra\geqq q_1+q_2\}$ contradicting to $r'a'<q_3'$.

The uniquness of Φ is clear, by Proposition 2.4. Q.E.D.

3. Ring of automorphisms and field of real numbers

A. Semi-ring of automorphisms of Q. Let Q be a positive system of quantities satisfying the axiom of continuity, and let $\boldsymbol{\Phi}$ be the set of all automorphisms of Q (the set of all linear mappings of Q onto Q).

In $\boldsymbol{\Phi}$ are defined sum and product as follows:

$$\Phi_1+\Phi_2 \text{ by } (\Phi_1+\Phi_2)(q) = \Phi_1(q)+\Phi_2(q) \quad \text{for all } q\in Q\,.$$
$$\Phi_2\circ\Phi_1 \text{ by } (\Phi_2\circ\Phi_1)(q) = \Phi_2(\Phi_1(q)) \quad \text{for all } q\in Q\,.$$

Proposition 3.1. *For addition (summation) in $\boldsymbol{\Phi}$ hold the commutative and associative laws:*

$$\Phi_1+\Phi_2 = \Phi_2+\Phi_1, \quad (\Phi_1+\Phi_2)+\Phi_3 = \Phi_1+(\Phi_2+\Phi_3)\,.$$

Proposition 3.2. *For summation and product in $\boldsymbol{\Phi}$ hold the distributive and associative laws*

$$\Phi_1\circ(\Phi_2+\Phi_3) = \Phi_1\circ\Phi_2+\Phi_1\circ\Phi_3,$$
$$(\Phi_1+\Phi_2)\circ\Phi_3 = \Phi_1\circ\Phi_3+\Phi_2\circ\Phi_3,$$
$$(\Phi_1\circ\Phi_2)\circ\Phi_3 = \Phi_1\circ(\Phi_2\circ\Phi_3)\,.$$

Proposition 3.3. *For the product in $\boldsymbol{\Phi}$ holds the commutative law* $\Phi_1\circ\Phi_2=\Phi_2\circ\Phi_1$.

Proof. Let $a\in Q$, then we have to show $\Phi_1\circ\Phi_2(a)=\Phi_2\circ\Phi_1(a)$.

If it was not so we might assume $\Phi_1\circ\Phi_2\,(a)<\Phi_2\circ\Phi_1(a)$. Thus, by Proposition 2.3 there would exist a rational automorphism $r=\dfrac{m}{n}$ such that

$$\Phi_1\circ\Phi_2(a)<r\Phi_1(a)<\Phi_2\circ\Phi_1(a)\,.$$

As by Proposition 2.2 $r\Phi_1(a)=\Phi_1(ra)>\Phi_1\circ\Phi_2(a)$, we get by Lemma 2.1.1 $ra>\Phi_2(a)$ and hence $r>\Phi_2$. Further, as $rb<\Phi_2(b)$ with $b=\Phi_1(a)$, we get $r<\Phi_2$, contradicting to the above consequence $r>\Phi_2$.

Similar assumption $\Phi_1\circ\Phi_2(a)>\Phi_2\circ\Phi_1(a)$ would give us a contradiction.

Q.E.D.

B. Field of real numbers. Now let us introduce 0 (zero) and negative elements $-\phi$ ($\phi\in\boldsymbol{\Phi}$) as follows:

We define 0 as an ideal element such that

$$\phi+0 = 0+\phi = \phi \quad \text{for all } \phi\in\boldsymbol{\Phi}\,.$$

Further for every given $\phi\in\boldsymbol{\Phi}$ we define $-\phi$ by

$$\phi+(-\phi) = (-\phi)+\phi = 0\,.$$

Proposition 3.4. *The set* $\boldsymbol{R}$ *of all elements* ϕ, 0 *and* $-\phi$ *with* $\phi \in \boldsymbol{\Phi}$ *forms a commutative group with respect to the addition.*

In $\boldsymbol{R}$, an extension of $\boldsymbol{\Phi}$, we define the product by

1) If $\phi_1, \phi_2 \in \boldsymbol{\Phi}$ the product $\phi_1 \circ \phi_2$ remains the same as in $\boldsymbol{\Phi}$
2) $(-\phi_1) \circ \phi_2 = \phi_2 \circ (-\phi_1) = -(\phi_1 \circ \phi_2)$ with any $\phi_1, \phi_2 \in \boldsymbol{\Phi}$
3) $(-\phi_1) \circ (-\phi_2) = \phi_1 \circ \phi_2$ with any $\phi_1, \phi_2 \in \boldsymbol{\Phi}$
4) $\phi \circ 0 = 0 \circ \phi = 0$ with any $\phi \in \boldsymbol{\Phi}$

Proposition 3.5. *The sum and product defined above make* $\boldsymbol{R}$ *a commutative field.* $\boldsymbol{R}$ *is essentially independent of* Q ($\boldsymbol{R}_Q$ *is isomorphic to* $\boldsymbol{R}_{Q'}$, *only if* Q *and* Q' *are positive systems of quantities with axiom of continuity*)

Supplement to Proposition 3.4. Let $\phi_i \in \boldsymbol{\Phi}$ $(i=1, 2)$. We define the addition in $\boldsymbol{R}$ as follows.

(1) $\phi_1 + \phi_2$ remains the same as in $\boldsymbol{\Phi}$

(2) $$\phi_1 + (-\phi_2) = (-\phi_2) + \phi_1 = \begin{cases} \phi_1 - \phi_2 & \text{if } \phi_2 < \phi_1 \\ 0 & \text{if } \phi_1 = \phi_2 \\ -(\phi_2 - \phi_1) & \text{if } \phi_1 < \phi_2 \end{cases}$$

(3) $(-\phi_1) + (-\phi_2) = (-\phi_2) + (-\phi_1) = -(\phi_1 + \phi_2)$

(4) $(-\phi) + 0 = 0 + (-\phi) = -\phi$ and $0 + 0 = 0$.

Supplement to Proposition 3.5. Proof. Let $a \in Q$ be fixed then the 1–1 correspondence $\phi \leftrightarrow \phi(a) = q$ $(\phi \in \boldsymbol{\Phi}_Q, q \in Q)$ gives an isomorphism (1–1 linear mapping) of $\boldsymbol{\Phi}_Q$ with Q with respect to the addition: $\boldsymbol{\Phi}_Q \simeq Q$. As $Q \simeq Q'$ by Proposition 2.7, we see $\boldsymbol{\Phi}_Q \simeq Q \simeq Q' \simeq \boldsymbol{\Phi}_{Q'}$ with respect to the addition (and the order).

Further regarding $\boldsymbol{\Phi}_Q$ and $\boldsymbol{\Phi}_{Q'}$ to make them isomorphic also with respect to the product, we can conclude that to $1_Q \in \boldsymbol{\Phi}_Q$ must correspond $1_{Q'} \in \boldsymbol{\Phi}_{Q'}$, since $1_Q^2 = 1_Q$ and $1'^2_{Q'} = 1'_{Q'}$. Hence to $n_Q \in \boldsymbol{\Phi}_Q (n \in \boldsymbol{N})$ must correspond $n_Q' \in \boldsymbol{\Phi}_{Q'}$ with the same $n \in \boldsymbol{N}$. Thus to $r_Q \in \boldsymbol{\Phi}_Q$ with a rational $r \in R$ must correspond $r_{Q'} \in \boldsymbol{\Phi}_{Q'}$ with the same r.

Now let $\phi \in \boldsymbol{\Phi}_Q$ and assume $\phi \notin \boldsymbol{P}$ (rational automorphism). Let $\boldsymbol{P}^-$ and $\boldsymbol{P}^+$ be defined by $\boldsymbol{P}^- = \{r \in \boldsymbol{P};\ r < \phi\}$ and $\boldsymbol{P}^+ = \{r \in \boldsymbol{P};\ r > \phi\}$. Then $\boldsymbol{P}^-$ and $\boldsymbol{P}^+$ form Dedekind's pair and there exists just one $\phi' \in \boldsymbol{\Phi}_{Q'}$ such that

$$r_{-(Q')} \leqq \phi' \leqq r_{+(Q')} \quad \text{for every } r_- \in \boldsymbol{P}^- \text{ and every } r_+ \in \boldsymbol{P}^+ .$$

By the isomorphism of $\boldsymbol{\Phi}_Q$ with $\boldsymbol{\Phi}_{Q'}$ with respect to the order, (because of additoin) we see that to ϕ must correspond ϕ'. Consequently we can regard $\boldsymbol{\Phi}_Q$ and $\boldsymbol{\Phi}_{Q'}$ coincide as systems of addition and product (semi-rings) in abstract sense.

As $\boldsymbol{R}$ is uniquely derived from $\boldsymbol{\Phi}$ we see that $\boldsymbol{R}_Q$ and $\boldsymbol{R}_{Q'}$ coincide as fields in abstract sense. We call thus the abstract system $\boldsymbol{R}$ the system of real numbers and $\boldsymbol{\Phi}$ the system of positive real numbers.

C. Logarithmic function. Let Φ be the positive system of real numbers.

Proposition 3.6. *Let $\boldsymbol{\Psi}$ be defined by $\boldsymbol{\Psi}=\{\phi\in\boldsymbol{\Phi};\ \phi>1\}$. Then the set $\boldsymbol{\Psi}$ satisfies the postulates $\mathrm{I}_1\sim\mathrm{I}_4$, $\mathrm{II}_{1,2}$, $\mathrm{III}_{1,2}$, if we replace the multiplication-symbol in $\boldsymbol{\Psi}$ by the addition-symbol. Thus we can regard $\boldsymbol{\Psi}$ as a positive system of qunatities with the axiom of continuity replacing the symbol $\circ$ in $\boldsymbol{\Psi}$ by the symbol $+$.*

Proposition 3.7. *Let $\boldsymbol{\Psi}$ be the same one given in Proposition 3.6. Let $b\in\boldsymbol{\Psi}$ $(b>1)$, then there exists uniquely a 1–1 correspondence f of $\boldsymbol{\Psi}$ with $\boldsymbol{\Phi}[f(\psi)=\phi,\ \psi\in\boldsymbol{\Psi},\ \phi\in\boldsymbol{\Phi}]$ such that $f(b)=1$ and $f(\psi_1\circ\psi_2)=f(\psi_1)+f(\psi_2)$.*

The mapping f: $\boldsymbol{\Psi}\to\boldsymbol{\Phi}$ is called the *logarithmic function* with the basis b, and we write

$$f(\psi)=\log_b\psi \quad ({}^{\forall}\psi\in\boldsymbol{\Psi}).$$

We extend the logarithmic function onto $\boldsymbol{\Phi}$ (the positive system of real numbers) as follows

(1) If $\phi>1$ then $\log_b\phi$ remains the same as above.

(2) If $\phi=1$ then we define $\log_b\phi=0$.

(3) If $\phi<1(\phi\in\boldsymbol{\Phi})$ we define $\log_b\phi=-\log_b(\phi^{-1})$.

Proposition 3.8. *The logarithmic function on $\boldsymbol{\Phi}$ satisfies* $(b>1)$

$$\log_b(\phi_1\circ\phi_2)=\log_b(\phi_1)+\log_b(\phi_2),$$
$$\log_b(b)=1,\quad \log_b 1=0.$$

References

[1] R. Dedekind: Was sind und was sollen die Zahlen ?, F. Vieweg, Braunschweig, 1887.

[2] E.G.H. Landau: Grundlagen der Analysis, Akademische Verlag, Leipzig, 1930 (Chelsea, 1960).

[3] T. Takagi: The concept of numbers, Iwanami Shoten, Tokyo, 1949 (in Japanease).