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INFINITE OUTER GALOIS THEORY OF NON COMMUTATIVE RINGS

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In [4], T. Nagahara presented infinite Galois theory of commutative rings with no non-trivial idempotent. On the other hand, Y. Miyashita studied in [3] finite outer Galois theory of non commutative rings.

We shall introduce the notion of infinte outer Galois extension of non commutative rings and obtain a generalization of the fundamental theorem of Galois theory.

In the first place, we recall the definition of finite Galois extension of non commutative rings. Let Γ be a ring with identity 1, Λ a subring with the same identity 1 and G a finite group of automorphisms of Γ. Then Γ is called a (finite) Galois extension of Λ relative to a group G if the following conditions hold:

(1) There exists an element z of Γ such that $t_G(z)=1$ where $t_G(x)=\sum_{\sigma\in G}\sigma(x)$ for any element x of Γ.

(2) $\Lambda=\Gamma^G$ where Γ^G is the fixed ring of Γ by G, i.e. Γ^G is the set of all elements of Γ left invariant by G.

(3) There are elements $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ of Γ such that for all σ in G

$$\sum_{i=1}^{n} x_i\sigma(y_i) = \begin{cases} 1 & (\sigma=1) \\ 0 & (\sigma\neq 1) \end{cases}$$

If Γ is a finite Galois extension of Λ relative to group G and $V_\Gamma(\Lambda)$ is the center C of Γ where $V_\Gamma(\Lambda)$ is the commutor ring of Λ in Γ, then Γ is called a finite outer Galois extension of Λ relative to a group G [cf. 3]. This notion will be extended to the following case.

Let Γ be a ring with identity 1, Ω a subring of Γ and σ, τ are two mappings of Ω to Γ. If there exists $\omega\in\Omega$ such that $\sigma(\omega)e\neq\tau(\omega)e$ for any central idempotent e of Γ, we say that the mappings σ and τ are strongly distinct. Moreover let G be a group of automorphisms of Γ (not necessarily finite). Then by G-strong subring we mean a subring Ω of Γ to which the restrictions of any two elements of G are either equal or strongly distinct as mappings of Ω to Γ. Fixing a representative system $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ of the right cosets of H in G for any finite index subgroup H of G, $t_{G/H}$ means $t_{G/H}(x)=\sum_{i=1}^{n}\sigma_i(x)$ for $x\in\Gamma$.

DEFINITION. Let Γ and G be as above and Λ a subring of Γ with the same identity 1. Then it is said that Γ is *an outer Galois extension of* Λ *relative to a group* G if the following conditions (from now on, we shall call them the outer Galois conditions) are satisfied:

(1) $t_{G/N^*}(\Gamma^N)=\Lambda$ for any finite index subgroup N of G where $N^*=\{\sigma\mid\sigma\in G,\ \sigma(x)=x \text{ for all } x\in\Gamma^N\}$.

(2) For any finite subset F of Γ, there exists a subring Ω of Γ containing Λ such that a) $F\subset\Omega$, b) Ω is a separable extension[1] of Λ, c) Ω is G-strong, and d) H is a finite index subgroup of G where $H=\{\sigma\mid\sigma\in G,\ \sigma(x)=x \text{ for any } x\in\Omega\}$ and there exists an element ω_K of Ω such that $t_{K/H}(\omega_K)=1$ for any subgroup K of G containing H.

(3) $V_\Gamma(\Lambda)=C$ where C is the center of Γ.

Throughout this paper, we assume that Γ *is an outer Galois extension of* Λ *relative to a group* G and Λ-module means right Λ-module.

First we shall present a characterization of outer Galois extensions.

From the Definition we obtain clearly next Lemma.

Lemma 1. $\#\{\sigma(\gamma)\mid\sigma\in G\}$ *is finite for any* $\gamma\in\Gamma$.

Corollary. *If* Ω *is a subring of* Γ *finitely generated as* Λ*-module, then* $\#(G\mid\Omega)$ *is finite.*

Lemma 2. *Let* Ω *be a subring of* Γ *such that* Ω *is a separable extension of* Λ *and is* G*-strong. If* $\#(G\mid\Omega)$ *is finite, then the following statements hold*: 1) $\Omega=\Gamma^H$ *where* $H=\{\sigma\mid\sigma\in G,\ \sigma(x)=x \text{ for all } x\in\Omega\}$. 2) Ω *is a finitely generated projective* Λ*-module.*

Proof. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be elements of Ω satisfying the separability conditions. If we write e the image of $\sum_{i=1}^{n}x_i\otimes y_i$ by the natural mapping of $\Omega\otimes_\Lambda\Omega$ to $\Omega\otimes_\Lambda\Gamma$ and set $e_\sigma=(1\otimes\sigma)(e)$ for $\sigma\in G$, it is clear that $xe=ex$ and $xe_\sigma=e_\sigma\sigma(x)$ for any $x\in\Omega$. Let φ be a mapping of $\Omega\otimes_\Lambda\Gamma$ onto Γ by $\varphi(x\otimes y)=xy$ for any $x\otimes y\in\Omega\otimes\Gamma$. Then $\varphi(e_\sigma)$ belongs to the center C of Γ

1) Let Γ be a ring with identity 1, Λ a subring of Γ. Γ is called a separable extension of Λ if there exist $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ of Γ such that $\sum_{i=1}^{n}x_iy_i=1$ and $\sum_{i=1}^{n}zx_i\otimes y_i=\sum_{i=1}^{n}x_i\otimes y_iz$ for any $z\in\Gamma$ where $\sum_{i=1}^{n}x_i\otimes y_i\in\Gamma\otimes_\Lambda\Gamma$. In this case, we shall say that $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ satisfy the separablity conditions.

since $x\varphi(e_\sigma)=\varphi(e_\sigma)\sigma(x)$ for any $x\in\Omega$. We have that $\varphi(e_\sigma)=(\sum_{i=1}^n x_i y_i)\varphi(e_\sigma)=(\sum_{i=1}^n x_i\sigma(y_i))\varphi(e_\sigma)=\varphi(e_\sigma)^2$. Therefore for all $\sigma\in G$

$$\sum_{i=1}^n x_i\sigma(y_i) = \begin{cases} 1 & (\sigma\in H) \\ 0 & (\sigma\notin H) \end{cases}$$

since Ω is G-strong. Since the index of H in G is finite, we have $\omega=\sum_{i=1}^n x_i t_{G/H}(y_i\omega)$ for any $\omega\in\Omega$. Thus Ω is a finitely generated projective Λ-module. The remaining part is trivial from the fact that $\gamma=\sum_{i=1}^n x_i t_{G/H}(y_i\gamma)$ for any $\gamma\in\Gamma^H$.

Lemma 3. *Let F be any finite subset of Γ. Then there exists a normal subgroup N of G such that the index of N in G is finite, $F\subset\Gamma^N$ and Γ^N is a (finite) outer Galois extension of Λ relative to G/N.*

Proof. If $F^*=\{\sigma(x)\mid\sigma\in G,\ x\in F\}$, F^* is finite. Let Ω be a subring of Γ satisfying the outer Galois conditions (2) for a finite subset F^* of Γ. Then $\Omega=\Gamma^H$ where $H=\{\sigma\mid\sigma\in G,\ \sigma(x)=x \text{ for all } x\in\Omega\}$. If N is the normal subgroup of G generated by H, we have $F\subset\Gamma^N$. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be elements of Ω satisfying the separability conditions. Then we have already known

$$\sum_{i=1}^n x_i\sigma(y_i) = \begin{cases} 1 & (\sigma\in H) \\ 0 & (\sigma\notin H) \end{cases}$$

for all $\sigma\in G$. Since there exists $\omega_N\in\Omega$ such that $t_{N/H}(\omega_N)=1$, we obtain

$$\sum_{i=1}^n t_{N/H}(\omega_N x_i)\sigma(t_{N/H}(y_i)) = \begin{cases} 1 & (\sigma\in N) \\ 0 & (\sigma\notin N) \end{cases}$$

for all $\sigma\in G$. Hence $N=\{\sigma\mid\sigma\in G,\ \sigma(x)=x \text{ for all } x\in\Gamma^N\}$, so that there exists $\gamma_N\in\Gamma^N$ such that $t_{G/N}(\gamma_N)=1$. Since $V_{\Gamma^N}(\Lambda)$ is clearly the center of Γ^N, Γ^N is a (finite) outer Galois extension of Λ relative to G/N.

Lemma 4. (cf. [3]). *Let Γ be a finite outer Galois extension of Λ relative to G. Then if H is any subgroup of G, Γ^H is a separable extension of Λ finitely generated as Λ-module and G-strong. Moreover if H is a normal subgroup of G, Γ^H is a (finite) outer Galois extension of Λ relative to G/H.*

Proposition 2. *If H is a subgroup of G such that the index of H in G is finite, then we obtain that Γ^H is a separable extension of Λ finitely generated as Λ-module and G-strong.*

Proof. Let γ_1 be one of generators of Γ^H as Λ-module. Then there exists a normal subgroup N_1 of G such that $\gamma_1\in\Gamma^{N_1}$ and Γ^{N_1} is a Galois extension of Λ relative to G/N_1. Assume that N_{k-1} exists. If we can take out γ_k being one of generators of Γ^H as Λ-module not included in $\Gamma^{N_{k-1}}$, N_k is a normal subgroup of G such that $\gamma_k \in \Gamma^{N_k}$, $\Gamma^{N_{k-1}} \subset \Gamma^{N_k}$ and Γ^{N_k} is a finite Galois extension of Λ relative to G/N_k. Then we have a chain

$$G \supset HN_1 \supset HN_2 \supset \cdots \supset HN_k \supseteqq H .$$

Hence

$$[G:H] > [HN_1:H] > [HN_2:H] > \cdots > [HN_k:H] \geqq 1 .$$

Since $[G:H]$ is finite, there is a rational integer k_0 such that $\Gamma^H \subset \Gamma^{N_{k_0}}$. Γ^H is the fixed ring of $\Gamma^{N_{k_0}}$ by HN_{k_0}/N_{k_0}, so that Γ^H is a separable extension of Λ finitely generated as Λ-module and G-strong.

Corollary. *If N is a normal subgroup of finite index in G, Γ^N is a (finite) outer Galois extension of Λ relative to a factor group of G.*

Proof. (cf. [3]).
Now we summarize a characterization of outer Galois extensions.

Proposition 3. *Let $\tilde{\Gamma}$ be a ring with identity* 1, *$\tilde{\Lambda}$ a subring of $\tilde{\Gamma}$ with same identity* 1 *and G a group of automorphisms of $\tilde{\Gamma}$.*

Then $\tilde{\Gamma}$ is an outer Galois extension of $\tilde{\Lambda}$ relative to G if and only if the following conditions hold:

(1) $\tilde{\Gamma}^G = \tilde{\Lambda}$.

(2) *For any finite subset F of $\tilde{\Gamma}$, there exists a normal subgroup $\tilde{N}$ of $\tilde{G}$ such that $F \subset \tilde{\Gamma}^N$, the index of $\tilde{N}$ in $\tilde{G}$ is finite and $\tilde{\Gamma}^N$ is a finite outer Galois extension of $\tilde{\Lambda}$ relative to $\tilde{G}/\tilde{N}$.*

Proof. Necessity. It is obvious from Lemma 3 and 4.

Sufficiency. It follows from the proof of Proposition 2 that Γ^H is finitely generated as Λ-module for any finite index subgroup H of G. Then there exists a normal subgroup N of G such that $\Gamma^H \subset \Gamma^N$ and Γ^N is a finite Galois extension of Λ relative to G/N. Hence we have $t_{G/\bar{H}}(\Gamma^H) = \Lambda$ where $\bar{H} = \{\sigma \mid \sigma \in G,\ \sigma(x) = x \text{ for all } x \in \Gamma^H\}$. The remainder of the proof is obvious.

Lemma 5. *Let Ω be a subring of Γ which is a separable extension of Λ finitely generated as Λ-module and G-strong. If M is a left free Γ-module $\sum_{i=1}^{n} \oplus \Gamma\sigma'_i$ where $G \mid \Omega = \{\sigma'_1, \sigma'_2, \cdots, \sigma'_m\}$, we may regard M as a right Ω-module by $x \cdot \sigma'_i \cdot y = x\sigma'_i(y)\sigma'_i$ for $x \in \Gamma$, $y \in \Omega$.*

Then if ψ is a mapping of M to $\mathrm{Hom}(\Omega_\Lambda, \Gamma_\Lambda)$ *by $\psi(\sum_{i=1}^{m} \gamma_i \cdot \sigma'_i)(y) = \sum_{i=1}^{m} \gamma_i \cdot \sigma'_i(y)$ for $\sum_{i=1}^{m} \gamma_i \sigma'_i \in M$, $y \in \Omega$, ψ* is *Γ-Ω-isomorphism.*

Proof. (cf. [4]).

REMARK. In the above Lemma, if $\Lambda=\Gamma^G$ and $\#\{\sigma(\gamma)\mid\sigma\in G\}$ is finite for any $\gamma\in\Gamma$, we may omit the assumption that Γ is an outer Galois extension of Λ relative to G.

Proposition 4. *Let G^* be the closure of G (with respect to the finite topology). Then Γ is an outer Galois extension of Λ relative to G^*.*

Proof. For any finite subset F of Γ, there exists a normal subgroup N of G such that Γ^N is a finite outer Galois extension of Λ relative to G/N. Then we have $G\mid\Gamma^N=G^*\mid\Gamma^N$. Hence Γ^{N^*} is a finite Galois extension of Λ relative to G^*/N^* where $N^*=\{\sigma\mid\sigma\in G^*,\ \sigma(x)=x$ for all $x\in\Gamma^N\}$.

DEFINITION. Let Ω be a subring of Γ containing Λ. Then we shall call Ω is *a locally separable G-strong extension of* Λ if, for any finite subset F of Ω, there exists a subring Ω' of Ω containing F which is a separable extension of Λ finitely generated as Λ-module and G-strong.

Proposition 5. *If H is a closed subgroup of G (with respect to the finite topology), then Γ^H is a locally separable G-strong extension of* Λ and $H=H^*$ *where $H^*=\{\sigma\mid\sigma\in G,\ \sigma(x)=x$ for all $x\in\Gamma^H\}$.*

Proof. Let F be a finite subset of Γ^H. Then there exists a normal subgroup N of G such that the index of N in G is finite, $F\subset\Gamma^N$ and Γ^N is an outer Galois extension of Λ relative to G/N. Since $\Gamma^H\cap\Gamma^N=\Gamma^{HN}=(\Gamma^N)^{HN/N}$, $\Gamma^H\cap\Gamma^N$ is a separable extension of Λ finitely generated as Λ-module and G-strong. Hence Γ^H is a locally separable G-strong extension of Λ. We shall show the remaining part. Let F be any finite subset of Γ. Then there exists a subring Ω of Γ which is a finite outer Galois extension of Λ relative to a factor group of G and contain F. Then $H\mid\Omega=H^*\mid\Omega$ by finite Galois theory (cf. [2]), so that $H\mid F=H^*\mid F$. Thus we have $H=H^*$ since H is dense in H^*.

Corollary 1. *Let H_1, H_2 be two closed subgroup of G. If $\Gamma^{H_1}\supset\Gamma^{H_2}$, we have $H_1\subset H_2$.*

Corollary 2. *Let H_1, H_2 be as above. If $H_1\neq H_2$, then $\Gamma^{H_1}\neq\Gamma^{H_2}$.*

Now we may exhibit the fundamental theorem of outer Galois theory.

Theorem. 1) *Let Γ be an outer Galois extension of Λ relative to a group G and assume that G is compact (with respect to the finite topology). Then there is one-to-one lattice-inverting correspondence between closed subgroups of G and subrings of Γ which are locally separable G-strong extension of Λ. If Ω is a locally separable G-strong extension of Λ which is a subring of Γ, then the corresponding subgroup is $H_\Omega=\{\sigma\mid\sigma\in G,\ \sigma(x)=x$ for all $x\in\Omega\}$.*

2) *A closed subgroup N of G is normal in G if and only if Γ^N is mapped onto itself by every elements of G, in which case Γ^N is an outer Galois extension of Λ relative to G/N.*

Proof. 2) is obvious. We need only show that if Ω is a subring of Γ which is a locally separable G-strong extension of Λ, then $\Omega=\Gamma^H$ where $H=\{\sigma\mid\sigma\in G,\ \sigma(x)=x$ for all $x\in\Omega\}$. Suppose that there exists $\gamma\in\Gamma^H$ such that $\gamma\notin\Omega$. If $C=\{\sigma\mid\sigma\in G,\ \sigma(\gamma)\neq\gamma\}$, C is closed in G. Let X be the set of subring Ω_α of Ω which is separable extension of Λ finitely generated as Λ-module and G-strong (Let I be the set of suffixes of Ω_α's). If $H_\alpha=\{\sigma\mid\sigma\in G,\ \sigma(x)=x$ for all $x\in\Omega_\alpha\}$ for each $\Omega_\alpha\in X$, H_α is closed subgroup of G. Let $\{\Omega_1, \Omega_2, \cdots, \Omega_n\}$ be any finite subset of X. Then there exists a subring Ω' of Ω such that $\Omega_i\subset\Omega'$ $(i=1, 2, \cdots, n)$, and Ω' is a separable extension of Λ finitely generated as Λ-module and G-strong. If $K=\{\sigma\mid\sigma\in G,\ \sigma(x)=x$ for all $x\in\Omega'\}$, $\Omega'=\Gamma^K$ and so $K\subseteq\bigcap_{i=1}^n H$. Since $C\cap K\neq\phi$, $\bigcap_{i=1}^n(C\cap H_i)=C\cap(\bigcap_{i=1}^n H_i)\neq\phi$. Furthermore we obtain $\bigcap_{\alpha\in I}(C\cap H_\alpha)\neq\phi$ since G is compact. Hence $C\cap H\neq\phi$. This is contradiction. Therefore $\Omega=\Gamma^H$, completing the proof.

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