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NOTE ON A THEOREM DUE TO MILNOR

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1. Introduction

J. Milnor [1] has proved the following theorem: Let N be a closed topological manifold which is a mod 2 homology n -sphere, and T be a fixed point free involution on N . Then, for every continuous map $f: N \rightarrow N$ such that $f_*: H_n(N; \mathbf{Z}_2) \rightarrow H_n(N; \mathbf{Z}_2)$ is not trivial, there exists a point $y \in N$ such that $fT(y) = Tf(y)$.

In the present paper, we shall show that this result can be generalized as follows:

Theorem 1. *Let N and M be topological n -manifolds on each of which there is given a fixed point free involution T ($n \geq 1$). Assume that N has the mod 2 homology of the n -sphere. Then, for every continuous map $f: N \rightarrow M$ such that $f_*: H_n(N; \mathbf{Z}_2) \rightarrow H_n(M; \mathbf{Z}_2)$ is not trivial, there exists a point $y \in N$ such that $fT(y) = Tf(y)$.*

Our method is different from Milnor [1], and we shall apply the method we used in [2] to prove a generalization of Borsuk-Ulam theorem.

REMARK. The theorem is regarded in some sense as a converse of Corollary 1 of the main theorem in [2].

Throughout this paper, all chain complexes and hence all homology and cohomology groups will be considered on \mathbf{Z}_2 .

2. The chain map

Let Y be a Hausdorff space on which there is given a fixed point free involution T . Denote by π the cyclic group of order 2 generated by T . We shall denote by Y_π the orbit space of Y , and by $p: Y \rightarrow Y_\pi$ the projection. Consider the induced homomorphisms $T_*: S(Y) \rightarrow S(Y)$ and $p_*: S(Y) \rightarrow S(Y_\pi)$ of singular complexes. Then a chain map

$$\phi: S(Y_\pi) \rightarrow S(Y)$$

can be defined by

$$\phi(c) = \tilde{c} + T_*(\tilde{c}), \quad p_*(\tilde{c}) = c,$$

where $c \in S(Y_\pi)$, $\tilde{c} \in S(Y)$. Obviously ϕ is functorial with respect to equivariant continuous maps. Therefore ϕ induces homomorphisms

$$\phi_*: H_*(Y_\pi) \rightarrow H_*(Y), \quad \phi^*: H^*(Y) \rightarrow H^*(Y_\pi)$$

of homology and cohomology, which are functorial with respect to equivariant continuous maps.

As for the homomorphism $p^*: H^*(Y_\pi) \rightarrow H^*(Y)$ and the cap product, we have

Lemma 1. $\phi_*(\alpha \frown a) = p^*(\alpha) \frown \phi_*(a)$ for $\alpha \in H^*(Y_\pi)$, $a \in H_*(Y_\pi)$.

Proof. Let u be a singular cochain of Y_π , and c a singular chain of Y_π . Take a singular chain \tilde{c} of Y such that $p_*(\tilde{c}) = c$. Since

$$u \frown c = u \frown p_*(\tilde{c}) = p_*(p^*u \frown \tilde{c}),$$

it follows that

$$\begin{aligned} \phi(u \frown c) &= p^*u \frown \tilde{c} + T_*(p^*u \frown \tilde{c}) \\ &= p^*u \frown \tilde{c} + T^*p^*u \frown T_*\tilde{c} \\ &= p^*u \frown \tilde{c} + p^*u \frown T_*\tilde{c} \\ &= p^*u \frown (\tilde{c} + T_*\tilde{c}) \\ &= p^*u \frown \phi(c). \end{aligned}$$

This proves the desired lemma.

We have also

Lemma 2. If Y is a closed topological n -manifold, then $\phi_*: H_n(Y_\pi) \rightarrow H_n(Y)$ sends the (mod 2) fundamental class of Y_π to that of Y .

Proof. Let y be any point of Y . Then ϕ induces a homomorphism $\phi_*: H_*(Y_\pi, Y_\pi - p(y)) \rightarrow H_*(Y, Y - \{y, T(y)\})$, and the following commutative diagram holds:

$$\begin{array}{ccc} H_n(Y_\pi) & \xrightarrow{j_1^*} & H_n(Y_\pi, Y_\pi - p(\mathfrak{y})) \\ \downarrow \phi_* & & \downarrow \phi_* \\ H_n(Y) & \xrightarrow{j_2^*} & H_n(Y, Y - \{\mathfrak{y}, T(\mathfrak{y})\}) \\ & \searrow j_3^* & \downarrow j_4^* \\ & & H_n(Y, Y - y) \end{array}$$

where j_i^* ($i=1, 2, 3, 4$) are induced by the inclusions. If $w \in H_n(Y_\pi)$ is the fundamental class, then $j_{1*}(w)$ is the generator of $H_n(Y_\pi, Y_\pi - p(y))$. It is easily seen that $j_{4*} \circ \phi_*$ sends the generator of $H_n(Y_\pi, Y_\pi - p(y))$ to that of $H_n(Y, Y - y)$. Therefore $j_{3*} \phi_*(w)$ is the generator of $H_n(Y, Y - y)$. Consequently $\phi_*(w)$ is the

fundamental class of $H_n(Y)$. This completes the proof of Lemma 2.

REMARK. ϕ is a kind of transfer map.

3. The element θ'

Let N and M be connected closed topological manifolds, on each of which there is given a fixed point free involution T . Consider the product manifolds $N \times M$ and $N \times M^2 = N \times M \times M$ on which π acts without fixed point by

$$T(y, x) = (T(y), T(x)), \quad T(y, x, x') = (T(y), x', x)$$

($y \in N, x, x' \in M$). Let $N \times_{\pi} M, N \times_{\pi} M^2$ denote the orbit spaces; these are connected closed topological manifolds.

Define a continuous map $d'_0: N \times M \rightarrow N \times M^2$ by

$$d'_0(y, x) = (y, x, T(x))$$

($y \in N, x \in M$). Then d'_0 induces a continuous map $d'_0: N \times_{\pi} M \rightarrow N \times_{\pi} M^2$, and hence a homomorphism $d'_{0*}: H_*(N \times_{\pi} M) \rightarrow H_*(N \times_{\pi} M^2)$. Let $\tau \in H_{m+n}(N \times_{\pi} M)$ denote the fundamental class of the manifold $N \times_{\pi} M$ and define

$$\theta'_0 \in H^m(N \times_{\pi} M^2)$$

to be the element which is the Poincaré dual of $d'_{0*}(\tau)$, where $n = \dim N, m = \dim M$.

Assume now that $n \geq m$ and N has the mod 2 homology of the sphere ($n \geq 1$). Consider the space N^∞ constructed in § 5 of [2]. Then it follows from Theorem 6 of [2] that there exists a unique element $\theta' \in H^m(N^\infty \times_{\pi} M^2)$ such that

$$i^*(\theta') = \theta'_0$$

for the homomorphism $i^*: H^m(N^\infty \times_{\pi} M^2) \rightarrow H^m(N \times_{\pi} M^2)$ induced by the inclusion.

With the notation in [2], we have

Theorem 2. $\theta' = P(1, \bar{\mu}) + \delta'$,

where $\bar{\mu} \in H^m(M)$ is the generator, and δ' is a linear combination of elements of the type $P(\alpha, \beta)$ with $\deg \alpha > 0, \deg \beta > 0$. (Compare Theorem 7 in [2].)

Proof. Consider the orbit space $N^\infty \times_{\pi} M$ of $N^\infty \times M$ on which π acts by $T(y, x) = (T(y), T(x)), (y \in N^\infty, x \in M)$. Then the projection $N^\infty \times M \rightarrow M$ defines a fibration $q: N^\infty \times_{\pi} M \rightarrow M_{\pi}$ with fibre N^∞ . Since $\tilde{H}_*(N^\infty) = 0$, it follows that $q_*: H_*(N^\infty \times_{\pi} M) \cong H_*(M_{\pi})$ and in particular $H_{n+m}(N^\infty \times_{\pi} M) = 0$.

For the continuous map $d': N^\infty \times_{\pi} M \rightarrow N^\infty \times_{\pi} M^2$ defined similarly to d'_0 , the

following commutative diagram holds:

$$\begin{CD} H_{n+m}(N \times_{\pi} M) @>d'_{0*}>> H_{n+m}(N \times_{\pi} M^2) \\ @V i_* VV @VV i_* V \\ H_{n+m}(N^{\infty} \times_{\pi} M) @>d'_{*}>> H_{n+m}(N^{\infty} \times_{\pi} M^2), \end{CD}$$

where i are the inclusions. Hence we have

$$i_* d'_{0*}(\tau) = d'_{*} i_*(\tau) = 0.$$

For the generator λ of $H_{n+2m}(N \times_{\pi} M^2)$, we have

$$d'_{0*}(\tau) = \theta'_0 \frown \lambda.$$

Therefore it follows that

$$0 = i_*(\theta'_0 \frown \lambda) = i_*(i^*(\theta') \frown \lambda) = \theta' \frown i_*(\lambda).$$

Let $\mu \in H_m(M)$ be the generator, and $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be a basis of the module $H^*(M)$ such that $\alpha_1 = 1$ and $\alpha_l = \bar{\mu}$. Then, with the notations in [2], we have

$$i_*(\lambda) = P_n(\mu)$$

(see §6 of [2]), and

$$\theta' = \sum_{i,j} g_{ij} P_i(\alpha_j) + \sum_{j < k} h_{jk} P(\alpha_j, \alpha_k)$$

where $g_{ij}, h_{jk} \in Z_2$ (see Theorem 4 of [2]). Thus it follows from Theorem 4 of [2] that

$$0 = \theta' \frown P_n(\mu) = \sum_{i,j} g_{ij} P_{n-i}(\alpha_j \frown \mu)$$

and hence $g_{ij} = 0$.

It remains now to prove that $h_{ij} = 1$. To do this, we consider the following diagram:

$$\begin{CD} H_{n+m}(N \times_{\pi} M) @>d'_{0*}>> H_{n+m}(N \times_{\pi} M^2) @<<\frown \lambda<< H^m(N \times_{\pi} M^2) \\ @V \phi_* VV @VV \phi_* V @VV p^* V \\ H_{n+m}(N \times M) @>d'_{0*}>> H_{n+m}(N \times M^2) @<<\frown \phi_*(\lambda)<< H^m(N \times M^2), \end{CD}$$

where $p: N \times M^2 \rightarrow N \times_{\pi} M^2$ is the projection. It follows from Lemma 1 that the diagram is commutative, and from Lemma 2 that

$$\phi_*(\tau) = \nu \times \mu, \quad \phi_*(\lambda) = \nu \times \mu \times \mu$$

where $\nu \in H_n(N)$ is the generator. Therefore we have

$$\begin{aligned} d'_{0*}(v \times \mu) &= d'_{0*}\phi_*(\tau) = p^*(\theta'_0) \frown \phi_*(\lambda) \\ &= p^*i^*(\theta') \frown (v \times \mu \times \mu). \end{aligned}$$

Consider next the commutative diagram

$$\begin{array}{ccc} H^m(N^\infty \times M^2) & \xrightarrow{i^*} & H^m(N \times M^2) \\ \downarrow p^* & & \downarrow p^* \\ H^m(N^\infty \times M^2) & \xrightarrow{i^*} & H^m(N \times M^2), \end{array}$$

where p are the projections and i are the inclusions. Since it is obvious that

$$p^*(P(\alpha_j, \alpha_k)) = 1 \times \alpha_j \times \alpha_k + 1 \times \alpha_k \times \alpha_j,$$

we have

$$\begin{aligned} d'_{0*}(v \times \mu) &= i^*p^*(\theta') \frown (v \times \mu \times \mu) \\ &= i^*p^*(\sum_{j < k} h_{jk}P(\alpha_j, \alpha_k)) \frown (v \times \mu \times \mu) \\ &= v \times (\sum_{j < k} h_{jk}(a_j \times a_k + a_k \times a_j)), \end{aligned}$$

where $a_i = \alpha_i \frown \mu$.

Let $\Delta_*: H_*(M) \rightarrow H_*(M \times M)$ denote the homomorphism induced by the diagonal map. Then we have

$$d'_{0*}(v \times \mu) = v \times (1 \times T)_* \Delta_*(\mu).$$

Therefore it holds that

$$(1 \times T)_* \Delta_*(\mu) = \sum_{j < k} h_{jk}(a_j \times a_k + a_k \times a_j).$$

Thus it follows that

$$\begin{aligned} 1 &= \langle \bar{\mu}, \mu \rangle = \langle 1 \times \bar{\mu}, \Delta_*(\mu) \rangle \\ &= \langle 1 \times T^*(\bar{\mu}), \sum_{j < k} h_{jk}(a_j \times a_k + a_k \times a_j) \rangle \\ &= \sum_{j < k} h_{jk}(\langle 1 \times \bar{\mu}, a_j \times a_k \rangle + \langle 1 \times \bar{\mu}, a_k \times a_j \rangle) \\ &= h_{\lambda'}. \end{aligned}$$

This completes the proof of Theorem 2.

4. Proof of Theorem 1

In what follows we shall prove Theorem 1.

We note first that M may be assumed to be a connected closed topological manifold.

Consider continuous maps $s: N \rightarrow N \times M^2$ and $k: N \rightarrow N^\infty \times N^2$ defined by

$$s(y) = (y, f(y), fT(y)),$$

$$k(y) = (y, y, T(y)), (y \in N).$$

Then, as in the proof of Lemma 4 in [2], we have by Theorem 2

$$s^*(\theta'_0) = s^*i^*(\theta') = k^*(1 \times f^2)^*(\theta')$$

$$= k^*(1 \times f^2)^*(P(1, \bar{\mu}) + \delta')$$

in $H^n(N_\pi)$. From this and the hypothesis, it follows that

$$s^*(\theta'_0) = k^*P(1, \bar{\nu}),$$

where $\bar{\nu} \in H^n(N)$ is the generator.

We have a commutative diagram

$$\begin{array}{ccc}
 & & H^n(N^3) \\
 & \nearrow (1 \times 1 \times T)^* & \searrow \Delta_* \\
 H^n(N^\infty \times N^2) & \xrightarrow{k^*} & H^n(N) \\
 \downarrow \phi^* & \xrightarrow{k^*} & \downarrow \phi^* \\
 H^n(N^\infty_\pi \times N^2) & \xrightarrow{k^*} & H^n(N_\pi)
 \end{array}$$

where $N^3 = N \times N \times N$ and $\Delta: N \rightarrow N^3$ is the diagonal map. It is easily seen that

$$P(1, \bar{\nu}) = \phi^*(1 \times 1 \times \bar{\nu}).$$

Therefore we have

$$\begin{aligned}
 k^*P(1, \bar{\nu}) &= k^*\phi^*(1 \times 1 \times \bar{\nu}) \\
 &= \phi^*\Delta^*(1 \times 1 \times T^*(\bar{\nu})) \\
 &= \phi^*\Delta^*(1 \times 1 \times \bar{\nu}) \\
 &= \phi^*(\bar{\nu}),
 \end{aligned}$$

which proves

$$s^*(\theta'_0) \neq 0.$$

Put

$$A'(f) = \{y \in N \mid fT(y) = Tf(y)\}$$

and

$$B'(f) = \text{Image of } A'(f) \text{ under the projection } N \rightarrow N_\pi.$$

Then we have the following commutative diagram which is similar to the diagram in the proof of Lemma 5 in [2]:

$$\begin{array}{ccc}
 H_{2m}(d'_0(N \times_\pi M)) & \xrightarrow{j^*} & H_{2m}(N \times_\pi M^2) \\
 \downarrow \backslash \gamma'_2 & & \downarrow \backslash \gamma_1 \\
 H^n(N \times_\pi M^2, N \times_\pi M^2 - d'_0(N \times_\pi M)) & \xrightarrow{j^*} & H^n(N \times_\pi M^2) \\
 \downarrow s^* & & \downarrow s^* \\
 H^n(N_\pi, N_\pi - B'(f)) & \xrightarrow{j^*} & H^n(N_\pi).
 \end{array}$$

Therefore $s^*(\theta'_0) \neq 0$ implies $H^n(N_\pi, N_\pi - B'(f)) \neq 0$, which shows $B'(f) \neq \phi$. Thus $A'(f) \neq \phi$ and the proof completes.

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References

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