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NOTE ON A THEOREM DUE TO MILNOR

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(Received April 17, 1970)

1. Introduction

J. Milnor [1] has proved the following theorem: Let \( N \) be a closed topological manifold which is a mod 2 homology \( n \)-sphere, and \( T \) be a fixed point free involution on \( N \). Then, for every continuous map \( f:N \to N \) such that \( f_*: H_n(N; \mathbb{Z}_2) \to H_n(N; \mathbb{Z}_2) \) is not trivial, there exists a point \( y \in N \) such that \( fT(y) = Tf(y) \).

In the present paper, we shall show that this result can be generalized as follows:

**Theorem 1.** Let \( N \) and \( M \) be topological \( n \)-manifolds on each of which there is given a fixed point free involution \( T \) (\( n \geq 1 \)). Assume that \( N \) has the mod 2 homology of the \( n \)-sphere. Then, for every continuous map \( f:N \to M \) such that \( f_*: H_n(N; \mathbb{Z}_2) \to H_n(M; \mathbb{Z}_2) \) is not trivial, there exists a point \( y \in N \) such that \( fT(y) = Tf(y) \).

Our method is different from Milnor [1], and we shall apply the method we used in [2] to prove a generalization of Borsuk-Ulam theorem.

**REMARK.** The theorem is regarded in some sense as a converse of Corollary 1 of the main theorem in [2].

Throughout this paper, all chain complexes and hence all homology and cohomology groups will be considered on \( \mathbb{Z}_2 \).

2. The chain map

Let \( Y \) be a Hausdorff space on which there is given a fixed point free involution \( T \). Denote by \( \pi \) the cyclic group of order 2 generated by \( T \). We shall denote by \( Y_\pi \) the orbit space of \( Y \), and by \( p: Y \to Y_\pi \) the projection. Consider the induced homomorphisms \( T_\#: S(Y) \to S(Y) \) and \( p_#: S(Y) \to S(Y_\pi) \) of singular complexes. Then a chain map

\[ \phi: S(Y_\pi) \to S(Y) \]

can be defined by

\[ \phi(c) = \bar{c} + T_\#(\bar{c}), \quad p_#(\bar{c}) = c, \]
where $c \in S(Y_\ast)$, $\partial \in S(Y)$. Obviously $\phi$ is functorial with respect to equivariant continuous maps. Therefore $\phi$ induces homomorphisms

$$
\phi_* : H_\ast(Y_\ast) \to H_\ast(Y), \quad \phi^* : H^\ast(Y) \to H^\ast(Y_\ast)
$$

of homology and cohomology, which are functorial with respect to equivariant continuous maps.

As for the homomorphism $p^* : H^\ast(Y_\ast) \to H^\ast(Y)$ and the cap product, we have

**Lemma 1.** $\phi^* (\alpha \cap a) = p^* (\alpha) \cap \phi_* (a)$ for $\alpha \in H^\ast(Y_\ast)$, $a \in H_\ast(Y_\ast)$.

**Proof.** Let $u$ be a singular cochain of $Y_\ast$, and $c$ a singular chain of $Y_\ast$. Take a singular chain $\bar{c}$ of $Y$ such that $p_\ast (\bar{c}) = c$. Since

$$
u \cap c = u \cap p_\ast (\bar{c}) = p_\ast (p^\ast u \cap c),$$

it follows that

$$
\phi (u \cap c) = p^\ast u \cap \bar{c} + T_4 (p^\ast u \cap \bar{c})
$$

$$= p^\ast u \cap \bar{c} + T^4 p^\ast u \cap T^4 \bar{c}
$$

$$= p^\ast u \cap (\bar{c} + T^4 \bar{c})
$$

$$= p^\ast u \cap \phi (c).
$$

This proves the desired lemma.

We have also

**Lemma 2.** If $Y$ is a closed topological $n$-manifold, then $\phi_* : H_n(Y_\ast) \to H_n(Y)$ sends the (mod 2) fundamental class of $Y_\ast$ to that of $Y$.

**Proof.** Let $y$ be any point of $Y$. Then $\phi$ induces a homomorphism $\phi_* : H_\ast(Y_\ast, Y_\ast - p(y)) \to H_\ast(Y, Y - \{y, T(y)\})$, and the following commutative diagram holds:

$$
\begin{array}{ccc}
H_\ast(Y_\ast) & \xrightarrow{j_{1\ast}} & H_\ast(Y_\ast, Y_\ast - p(y)) \\
\downarrow \phi_* & & \downarrow \phi_* \\
H_\ast(Y) & \xrightarrow{j_{2\ast}} & H_\ast(Y, Y - \{y, T(y)\}) \\
\downarrow j_{i\ast} & & \downarrow j_{i\ast} \\
& & H_\ast(Y, Y - y)
\end{array}
$$

where $j_{i\ast}$ ($i = 1, 2, 3, 4$) are induced by the inclusions. If $w \in H_n(Y_\ast)$ is the fundamental class, then $j_{1\ast} (w)$ is the generator of $H_n(Y_\ast, Y_\ast - p(y))$. It is easily seen that $j_{3\ast} \circ \phi_*$ sends the generator of $H_n(Y_\ast, Y_\ast - p(y))$ to that of $H_n(Y, Y - y)$. Therefore $j_{3\ast} \circ \phi_* (w)$ is the generator of $H_n(Y, Y - y)$. Consequently $\phi_* (w)$ is the
fundamental class of \( H_n(Y) \). This completes the proof of Lemma 2.

**Remark.** \( \phi \) is a kind of transfer map.

### 3. The element \( \theta' \)

Let \( N \) and \( M \) be connected closed topological manifolds, on each of which there is given a fixed point free involution \( T \). Consider the product manifolds \( N \times M \) and \( N \times M^2 = N \times M \times M \) on which \( \pi \) acts without fixed point by

\[
T(y, x) = (T(y), T(x)), \quad T(y, x, x') = (T(y), x', x)
\]

\((y \in N, x, x' \in M)\). Let \( N \times M, N \times M^2 \) denote the orbit spaces; these are connected closed topological manifolds.

Define a continuous map \( d_0' : N \times M \to N \times M^2 \) by

\[
d_0'(y, x) = (y, x, T(x))
\]

\((y \in N, x \in M)\). Then \( d_0' \) induces a continuous map \( d_0'' : N \times M \to N \times M^2 \), and hence a homomorphism \( d_0''_* : H_*(N \times M) \to H_*(N \times M^2) \). Let \( \tau \in H_{m+n}(N \times M) \) denote the fundamental class of the manifold \( N \times M \) and define

\[
\theta_0' \in H^m(N \times M^2)
\]
to be the element which is the Poincaré dual of \( d_0''(\tau) \), where \( n = \dim N, \ m = \dim M \).

Assume now that \( n \geq m \) and \( N \) has the mod 2 homology of the sphere \( (n \geq 1) \). Consider the space \( N^\infty \) constructed in §5 of [2]. Then it follows from Theorem 6 of [2] that there exists a unique element \( \theta' \in H^m(N^\infty \times M^2) \) such that

\[
i^*(\theta') = \theta_0'
\]

for the homomorphism \( i^* : H^m(N^\infty \times M^2) \to H^m(N \times M^2) \) induced by the inclusion.

With the notation in [2], we have

**Theorem 2.** \( \theta' = P(1, \overline{\mu}) + \delta' \), where \( \overline{\mu} \in H^m(M) \) is the generator, and \( \delta' \) is a linear combination of elements of the type \( P(\alpha, \beta) \) with \( \deg \alpha > 0, \deg \beta > 0 \). (Compare Theorem 7 in [2].)

**Proof.** Consider the orbit space \( N^\infty \times M \) of \( N^\infty \times M \) on which \( \pi \) acts by \( T(y, x) = (T(y), T(x)), (y \in N^\infty, x \in M) \). Then the projection \( N^\infty \times M \to M \) defines a fibration \( q : N^\infty \times M \to M_\pi \) with fibre \( N^\infty \). Since \( \overline{H_*(N^\infty)} = 0 \), it follows that \( q_* : H_*(N^\infty \times M) \cong H_*(M_\pi) \) and in particular \( H_{n+m}(N^\infty \times M) = 0 \).

For the continuous map \( d' : N^\infty \times M \to N^\infty \times M^2 \) defined similarly to \( d_0' \), the
following commutative diagram holds:

\[
\begin{array}{ccc}
H_{n+m}(N \times M) & \xrightarrow{d'_0} & H_{n+m}(N \times M^2) \\
\downarrow i_* & & \downarrow i_* \\
H_{n+m}(N_\pi \times M) & \xrightarrow{d'_b} & H_{n+m}(N_\pi \times M^2),
\end{array}
\]

where \(i\) are the inclusions. Hence we have

\[i_*d'_0(\tau) = d'_b i_*(\tau) = 0.\]

For the generator \(\lambda\) of \(H_{n+2m}(N \times M^2)\), we have

\[d'_0(\tau) = \theta'_0 \circ \lambda.\]

Therefore it follows that

\[0 = i_*(\theta'_0 \circ \lambda) = i_*(i'^*(\theta') \circ \lambda) = \theta' \circ i_*(\lambda).\]

Let \(\mu \in H_m(M)\) be the generator, and \(\{\alpha_1, \alpha_2, \ldots, \alpha_l\}\) be a basis of the module \(H^*(M)\) such that \(\alpha_1 = 1\) and \(\alpha_i = \overline{\mu}\). Then, with the notations in [2], we have

\[i_*(\lambda) = P_n(\mu)\]

(see §6 of [2]), and

\[\theta' = \sum_{i,j} g_{i,j} P_\lambda(\alpha_i) + \sum_{j,k} h_{j,k} P(\alpha_j, \alpha_k)\]

where \(g_{i,j}, h_{j,k} \in \mathbb{Z}\) (see Theorem 4 of [2]). Thus it follows from Theorem 4 of [2] that

\[0 = \theta' \circ P_n(\mu) = \sum_{i,j} g_{i,j} P_{n-i}(\alpha_j \circ \mu)\]

and hence \(g_{i,j} = 0.\)

It remains now to prove that \(h_{l,l} = 1.\) To do this, we consider the following diagram:

\[
\begin{array}{ccc}
H_{n+m}(N \times M) & \xrightarrow{d'_0} & H_{n+m}(N \times M^2) & \xrightarrow{\lambda} & H^m(N \times M^2) \\
\downarrow \phi_* & & \downarrow \phi_* & & \downarrow p_* \\
H_{n+m}(N \times M) & \xrightarrow{d'_b} & H_{n+m}(N \times M^2) & \xrightarrow{\phi_*(\lambda)} & H^m(N \times M^2),
\end{array}
\]

where \(p: N \times M^2 \to N \times M^2\) is the projection. It follows from Lemma 1 that the diagram is commutative, and from Lemma 2 that

\[\phi_*(\tau) = \nu \times \mu, \quad \phi_*(\lambda) = \nu \times \mu \times \mu\]

where \(\nu \in H_n(N)\) is the generator. Therefore we have
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\[ d'_0(v \times \mu) = d'_0 \phi_0(\tau) = p^*(\theta') \wedge \phi_0(\lambda) \]
\[ = p^* i^*(\theta') \wedge (v \times \mu \times \mu). \]

Consider next the commutative diagram

\[
\begin{array}{cccc}
H^m(N^w \times M^2) & \xrightarrow{i^*} & H^m(N \times M^2) \\
p^* & \downarrow & p^* \\
H^m(N^w \times M^2) & \xrightarrow{i^*} & H^m(N \times M^2),
\end{array}
\]

where \( p \) are the projections and \( i \) are the inclusions. Since it is obvious that
\[ p^*(P(\alpha_j, \alpha_k)) = 1 \times \alpha_j \times \alpha_k + 1 \times \alpha_k \times \alpha_j, \]
we have
\[ d'_0(v \times \mu) = i^* p^*(\theta') \wedge (v \times \mu \times \mu) \]
\[ = i^* p^*(\sum_{j \leq k} h_{j,k} P(\alpha_j, \alpha_k)) \wedge (v \times \mu \times \mu) \]
\[ = v \times (\sum_{j \leq k} h_{j,k}(a_j \times a_k + a_k \times a_j)), \]
where \( a_i = \alpha_i \wedge \mu. \)

Let \( \Delta_*: H_*(M) \to H_*(M \times M) \) denote the homomorphism induced by the diagonal map. Then we have
\[ d'_0(v \times \mu) = v \times (1 \times T)_* \Delta_*(\mu). \]
Therefore it holds that
\[ (1 \times T)_* \Delta_*(\mu) = \sum_{j \leq k} h_{j,k}(a_j \times a_k + a_k \times a_j). \]

Thus it follows that
\[ 1 = h\mu, \mu\rangle = \langle 1 \times \mu, \Delta_*(\mu) \rangle \]
\[ = \langle 1 \times T^*(\mu), \sum_{j \leq k} h_{j,k}(a_j \times a_k + a_k \times a_j) \rangle \]
\[ = \sum_{j \leq k} h_{j,k}(\langle 1 \times \mu, a_j \times a_k \rangle + \langle 1 \times \mu, a_k \times a_j \rangle) \]
\[ = h_{1,1}. \]

This completes the proof of Theorem 2.

4. Proof of Theorem 1

In what follows we shall prove Theorem 1.

We note first that \( M \) may be assumed to be a connected closed topological manifold.

Consider continuous maps \( s: N \to N \times M^2 \) and \( k: N \to N^w \times N^2 \) defined by
Then, as in the proof of Lemma 4 in [2], we have by Theorem 2
\[
s^*(\theta_0) = s^*i^*(\theta') = k^*(1 \times f^2)^*(\theta') = k^*(1 \times f^2)^*(P(1, \overline{v}) + \delta')
\]
in \(H^*(N_\ast)\). From this and the hypothesis, it follows that
\[
s^*(\theta_0) = k^*P(1, \overline{v}),
\]
where \(\overline{v} \in H^*(N)\) is the generator.

We have a commutative diagram

\[
\begin{align*}
H^*(N^3) & \xrightarrow{(1 \times 1 \times T)^*} H^*(N^3) \\
\phi^* & \downarrow \quad k^* \quad \phi^* \downarrow \\
H^*(N \times N^2) & \quad H^*(N) \quad H^*(N_\ast)
\end{align*}
\]

where \(N^3 = N \times N \times N\) and \(\Delta: N \to N^3\) is the diagonal map. It is easily seen that
\[
P(1, \overline{v}) = \phi^*(1 \times 1 \times \overline{v}).
\]
Therefore we have
\[
k^*P(1, \overline{v}) = k^*\phi^*(1 \times 1 \times \overline{v}) = \phi^*\Delta^*(1 \times 1 \times T^*(\overline{v})) = \phi^*\Delta^*(1 \times 1 \times \overline{v}) = \phi^*(\overline{v}),
\]
which proves
\[
s^*(\theta_0) \neq 0.
\]

Put
\[
A'(f) = \{y \in N \mid fT(y) = Tf(y)\}
\]
and
\[
B'(f) = \text{Image of } A'(f) \text{ under the projection } N \to N_\ast.
\]

Then we have the following commutative diagram which is similar to the diagram in the proof of Lemma 5 in [2]:
Therefore $s^{*}(\theta_0) \neq 0$ implies $H^*(N_\pi^*, N_\pi^* - B'(f)) \neq 0$, which shows $B'(f) \neq \phi$. Thus $A'(f) \neq \phi$ and the proof completes.

References

