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<td>Author(s)</td>
<td>Nakaoka, Minoru</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 7(2) P.443-P.449</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1970</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/8083">https://doi.org/10.18910/8083</a></td>
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<td>DOI</td>
<td>10.18910/8083</td>
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**NOTE ON A THEOREM DUE TO MILNOR**

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(Received April 17, 1970)

1. Introduction

J. Milnor [1] has proved the following theorem: Let $N$ be a closed topological manifold which is a mod 2 homology $n$-sphere, and $T$ be a fixed point free involution on $N$. Then, for every continuous map $f:N \rightarrow N$ such that $f_*:H_n(N; \mathbb{Z}_2) \rightarrow H_n(N; \mathbb{Z}_2)$ is not trivial, there exists a point $y \in N$ such that $fT(y) = Tf(y)$.

In the present paper, we shall show that this result can be generalized as follows:

**Theorem 1.** Let $N$ and $M$ be topological $n$-manifolds on each of which there is given a fixed point free involution $T$. Assume that $N$ has the mod 2 homology of the $n$-sphere. Then, for every continuous map $f:N \rightarrow M$ such that $f_*:H_n(N; \mathbb{Z}_2) \rightarrow H_n(M; \mathbb{Z}_2)$ is not trivial, there exists a point $y \in N$ such that $fT(y) = Tf(y)$.

Our method is different from Milnor [1], and we shall apply the method we used in [2] to prove a generalization of Borsuk-Ulam theorem.

**Remark.** The theorem is regarded in some sense as a converse of Corollary 1 of the main theorem in [2].

Throughout this paper, all chain complexes and hence all homology and cohomology groups will be considered on $\mathbb{Z}_2$.

2. The chain map

Let $Y$ be a Hausdorff space on which there is given a fixed point free involution $T$. Denote by $\pi$ the cyclic group of order 2 generated by $T$. We shall denote by $Y_\pi$ the orbit space of $Y$, and by $p: Y \rightarrow Y_\pi$ the projection. Consider the induced homomorphisms $T_*: S(Y) \rightarrow S(Y)$ and $p_*: S(Y) \rightarrow S(Y_\pi)$ of singular complexes. Then a chain map

$$\phi: S(Y_\pi) \rightarrow S(Y)$$

can be defined by

$$\phi(\partial) = \partial + T_*(\partial), \quad p_*(\partial) = \partial,$$
where \( c \in S(Y_\pi), \tilde{c} \in S(Y) \). Obviously \( \phi \) is functorial with respect to equivariant continuous maps. Therefore \( \phi \) induces homomorphisms

\[
\phi_* : H_\pi(Y_\pi) \to H_\pi(Y), \quad \phi^* : H^*(Y) \to H^*(Y_\pi)
\]

of homology and cohomology, which are functorial with respect to equivariant continuous maps.

As for the homomorphism \( p^* : H^*(Y_\pi) \to H^*(Y) \) and the cap product, we have

**Lemma 1.** \( \phi_*(\alpha \smile a) = p^*(\alpha) \smile \phi_*(a) \) for \( \alpha \in H^*(Y_\pi), \ a \in H_\pi(Y_\pi) \).

**Proof.** Let \( u \) be a singular cochain of \( Y_\pi \), and \( c \) a singular chain of \( Y_\pi \). Take a singular chain \( \tilde{c} \) of \( Y \) such that \( p_\pi(\tilde{c}) = c \). Since

\[
u \smile c = u \smile p_\pi(\tilde{c}) = p_\pi(p^u \smile c),
\]

it follows that

\[
\phi(u \smile c) = p^u \smile \tilde{c} + T_\pi(p^u \smile \tilde{c})
\]

\[
= p^u \smile \tilde{c} + T_\pi p^u \smile \tilde{c}
\]

\[
= p^u \smile \tilde{c} + p^u \smile T_\pi \tilde{c}
\]

\[
= p^u \smile (\tilde{c} + T_\pi \tilde{c})
\]

\[
= p^u \phi(c).
\]

This proves the desired lemma.

We have also

**Lemma 2.** If \( Y \) is a closed topological \( n \)-manifold, then \( \phi_* : H_\pi(Y_\pi) \to H_n(Y) \) sends the (mod 2) fundamental class of \( Y_\pi \) to that of \( Y \).

**Proof.** Let \( y \) be any point of \( Y \). Then \( \phi \) induces a homomorphism \( \phi_* : H_\pi(Y_\pi, Y_\pi - p(y)) \to H_\pi(Y, Y - \{y, T(y)\}) \), and the following commutative diagram holds:

\[
\begin{array}{ccc}
H_\pi(Y_\pi) & \xrightarrow{j_{1*}} & H_\pi(Y_\pi, Y_\pi - p(y)) \\
\downarrow \phi_* & & \downarrow \phi_* \\
H_\pi(Y) & \xrightarrow{j_{2*}} & H_\pi(Y, Y - \{y, T(y)\}) \\
\downarrow j_{3*} & & \downarrow j_{4*} \\
& & H_\pi(Y, Y - y)
\end{array}
\]

where \( j_{i*} \) (\( i = 1, 2, 3, 4 \)) are induced by the inclusions. If \( w \in H_\pi(Y_\pi) \) is the fundamental class, then \( j_{1*}(w) \) is the generator of \( H_\pi(Y_\pi, Y_\pi - p(y)) \). It is easily seen that \( j_{3*} \circ \phi_* \) sends the generator of \( H_\pi(Y_\pi, Y_\pi - p(y)) \) to that of \( H_\pi(Y, Y - y) \). Therefore \( j_{3*} \phi_* (w) \) is the generator of \( H_\pi(Y, Y - y) \). Consequently \( \phi_* (w) \) is the
fundamental class of $H_n(Y)$. This completes the proof of Lemma 2.

Remark. $\phi$ is a kind of transfer map.

3. The element $\theta'$

Let $N$ and $M$ be connected closed topological manifolds, on each of which there is given a fixed point free involution $T$. Consider the product manifolds $N \times M$ and $N \times M^2 = N \times M \times M$ on which $\pi$ acts without fixed point by

$$T(y, x) = (T(y), T(x)), \quad T(y, x, x') = (T(y), x', x)$$

($y \in N$, $x \in M$). Let $N \times M$, $N \times M^2$ denote the orbit spaces; these are connected closed topological manifolds.

Define a continuous map $d'_o: N \times M \to N \times M^2$ by

$$d'_o(y, x) = (y, x, T(x))$$

($y \in N$, $x \in M$). Then $d'_o$ induces a continuous map $d'_o: N \times M \to N \times M^2$, and hence a homomorphism $d'_{o*}: H_*(N \times M) \to H_*(N \times M^2)$. Let $\tau \in H_{m+n}(N \times M)$ denote the fundamental class of the manifold $N \times M$ and define

$$\theta'_o \in H^m(N \times M^2)$$

to be the element which is the Poincaré dual of $d'_{o*}(\tau)$, where $n = \dim N$, $m = \dim M$.

Assume now that $n \geq m$ and $N$ has the mod 2 homology of the sphere ($n \geq 1$). Consider the space $N^\infty$ constructed in §5 of [2]. Then it follows from Theorem 6 of [2] that there exists a unique element $\theta' \in H^m(N^\infty \times M^2)$ such that

$$i^*(\theta') = \theta'_o$$

for the homomorphism $i^*: H^m(N^\infty \times M^2) \to H^m(N \times M^2)$ induced by the inclusion.

With the notation in [2], we have

**Theorem 2.**

$$\theta' = P(1, \overline{\mu}) + \delta',$$

where $\overline{\mu} \in H^m(M)$ is the generator, and $\delta'$ is a linear combination of elements of the type $P(\alpha, \beta)$ with $\deg \alpha > 0$, $\deg \beta > 0$. (Compare Theorem 7 in [2].)

Proof. Consider the orbit space $N^\infty \times M$ of $N^\infty \times M$ on which $\pi$ acts by

$$T(y, x) = (T(y), T(x)), \quad (y \in N^\infty, x \in M).$$

Then the projection $N^\infty \times M \to M$ defines a fibration $q: N^\infty \times M \to M$ with fibre $N^\infty$. Since $\tilde{H}_*(N^\infty) = 0$, it follows that

$$q_*: H_*(N^\infty \times M) \cong H_*(M)$$

and in particular $H_{m+n}(N^\infty \times M) = 0$.

For the continuous map $d': N^\infty \times M \to N^\infty \times M^2$ defined similarly to $d'_o$, the
following commutative diagram holds:

\[
\begin{array}{ccc}
H_{n+m}(N \times M) & \xrightarrow{d'_{0*}} & H_{n+m}(N \times M^2) \\
\downarrow i_* & & \downarrow i_* \\
H_{n+m}(N \times M) & \xrightarrow{d'_{0*}} & H_{n+m}(N \times M^2),
\end{array}
\]

where \(i\) are the inclusions. Hence we have

\[i_\ast d'_{0*}(\tau) = d'_{0*}i_\ast(\tau) = 0.\]

For the generator \(\lambda\) of \(H_{n+2m}(N \times M^2)\), we have

\[d'_{0*}(\tau) = \theta' \circ \lambda.\]

Therefore it follows that

\[0 = i_\ast(\theta' \circ \lambda) = i_\ast(i_\ast(\theta') \circ \lambda) = \theta' \circ i_\ast(\lambda).\]

Let \(\mu \in H_m(M)\) be the generator, and \(\{\alpha_1, \alpha_2, \ldots, \alpha_l\}\) be a basis of the module \(H^*(M)\) such that \(\alpha_1 = 1\) and \(\alpha_l = \mu\). Then, with the notations in [2], we have

\[i_\ast(\lambda) = P_n(\mu)\]

(see §6 of [2]), and

\[\theta' = \sum_{i,j} g_{ij} P_i(\alpha_j) + \sum_{j,k} h_{jk} P(\alpha_j, \alpha_k)\]

where \(g_{ij}, h_{jk} \in \mathbb{Z}\) (see Theorem 4 of [2]). Thus it follows from Theorem 4 of [2] that

\[0 = \theta' \circ P_n(\mu) = \sum_{i,j} g_{ij} P_{n-1}(\alpha_j \circ \mu)\]

and hence \(g_{ij} = 0\).

It remains now to prove that \(h_{11} = 1\). To do this, we consider the following diagram:

\[
\begin{array}{ccc}
H_{n+m}(N \times M^3) & \xrightarrow{d'_{0*}} & H_{n+m}(N \times M^4) \\
\phi_* & & \phi_* \\
\end{array}
\]

\[
\begin{array}{ccc}
H_{n+m}(N \times M^3) & \xrightarrow{d'_{0*}} & H_{n+m}(N \times M^2) \\
\phi_* & & \phi_* \\
\end{array}
\]

where \(p: N \times M^2 \to N \times M^2\) is the projection. It follows from Lemma 1 that the diagram is commutative, and from Lemma 2 that

\[\phi_\ast(\tau) = \nu \times \mu, \quad \phi_\ast(\lambda) = \nu \times \mu \times \mu\]

where \(\nu \in H_n(N)\) is the generator. Therefore we have
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\[ d'_{\theta \phi} (v \times \mu) = d'_{\theta \phi} \phi (\tau) = p^*(\theta') \cap \phi (\lambda) \]
\[ = p^* i^* (\theta') \cap (v \times \mu \times \mu). \]

Consider next the commutative diagram

\[
\begin{array}{ccc}
H^m(N^m \times M^3) & \xrightarrow{i^*} & H^m(N \times M^3) \\
p^* \downarrow & & \downarrow p^* \\
H^m(N^m \times M^3) & \xrightarrow{i^*} & H^m(N \times M^3),
\end{array}
\]

where \( p \) are the projections and \( i \) are the inclusions. Since it is obvious that

\[ p^* (P(\alpha_j, \alpha_k)) = 1 \times \alpha_j \times \alpha_k + 1 \times \alpha_k \times \alpha_j, \]

we have

\[ d'_{\theta \phi} (v \times \mu) = i^* p^* (\theta') \cap (v \times \mu \times \mu) \]
\[ = i^* p^* \left( \sum_{j \leq k} h_{jk} P(\alpha_j, \alpha_k) \right) \cap (v \times \mu \times \mu) \]
\[ = v \times \left( \sum_{j \leq k} h_{jk} (a_j \times a_k + a_k \times a_j) \right), \]

where \( a_i = \alpha_i \cap \mu. \)

Let \( \Delta_*: H_*(M) \rightarrow H_*(M \times M) \) denote the homomorphism induced by the diagonal map. Then we have

\[ d'_{\theta \phi} (v \times \mu) = v \times (1 \times T)_* \Delta_*(\mu). \]

Therefore it holds that

\[ (1 \times T)_* \Delta_*(\mu) = \sum_{j \leq k} h_{jk} (a_j \times a_k + a_k \times a_j). \]

Thus it follows that

\[ 1 = \langle \bar{\mu}, \mu \rangle = \langle 1 \times \bar{\mu}, \Delta_*(\mu) \rangle \]
\[ = \langle 1 \times T^*(\bar{\mu}), \sum_{j \leq k} h_{jk} (a_j \times a_k + a_k \times a_j) \rangle \]
\[ = \sum_{j \leq k} h_{jk} \langle 1 \times \bar{\mu}, a_j \times a_k \rangle + \langle 1 \times \bar{\mu}, a_k \times a_j \rangle \]
\[ = h_{11}. \]

This completes the proof of Theorem 2.

4. Proof of Theorem 1

In what follows we shall prove Theorem 1.

We note first that \( M \) may be assumed to be a connected closed topological manifold.

Consider continuous maps \( s: N \rightarrow N \times M^2 \) and \( k: N \rightarrow N^m \times N^2 \) defined by
\[ s(y) = (y, f(y), fT(y)), \]
\[ k(y) = (y, y, T(y)), (y \in N). \]

Then, as in the proof of Lemma 4 in [2], we have by Theorem 2
\[ s^*(\theta') = s^* s^i* (\theta') = k^* (1 \times f^*)^* (\theta') \]
\[ = k^* (1 \times f^*)^* (P(1, \bar{\nu}) + \delta') \]
in \( H^*(N_\nu). \) From this and the hypothesis, it follows that
\[ s^*(\theta') = k^* P(1, \bar{\nu}), \]
where \( \bar{\nu} \in H^*(N) \) is the generator.

We have a commutative diagram

\[ \begin{array}{ccc}
H^*(N^3) & \rightarrow & H^*(N_\nu) \\
\downarrow {\phi^*} & & \downarrow {\phi^*} \\
H^*(N^* \times N^2) & \rightarrow & H^*(N) \\
\downarrow {k^*} & & \downarrow {\phi^*} \\
H^*(N^* \times N^2) & \rightarrow & H^*(N_\nu),
\end{array} \]

where \( N^3 = N \times N \times N \) and \( \Delta: N \rightarrow N^3 \) is the diagonal map. It is easily seen that
\[ P(1, \bar{\nu}) = \phi^* (1 \times 1 \times \bar{\nu}). \]
Therefore we have
\[ k^* P(1, \bar{\nu}) = k^* \phi^* (1 \times 1 \times \bar{\nu}) \]
\[ = \phi^* \Delta^* (1 \times 1 \times T^* (\bar{\nu})) \]
\[ = \phi^* \Delta^* (1 \times 1 \times \bar{\nu}) \]
\[ = \phi^* (\bar{\nu}), \]
which proves
\[ s^*(\theta') \neq 0. \]

Put
\[ A'(f) = \{ y \in N \mid f T(y) = T f(y) \} \]
and
\[ B'(f) = \text{Image of } A'(f) \text{ under the projection } N \rightarrow N_\nu. \]

Then we have the following commutative diagram which is similar to the diagram in the proof of Lemma 5 in [2]:
\[ H_{2n}(d_0^\sigma(N \times M)) \quad j_* \rightarrow H_{2n}(N \times M^2) \]
\[ \downarrow \gamma_2 \quad \downarrow \gamma_1 \]
\[ H^*(N \times M^2, N \times M^2 - d_0^\sigma(N \times M)) \quad j_* \rightarrow H^*(N \times M^2) \]
\[ \downarrow s^* \quad \downarrow s^* \]
\[ H^*(N \times N - B'(f)) \quad j_* \rightarrow H^*(N_\alpha). \]

Therefore \( s_\sigma(\theta'_0) \neq 0 \) implies \( H^*(N_\alpha, N_\alpha - B'(f)) \neq 0 \), which shows \( B'(f) \neq \phi \). Thus \( A'(f) \neq \phi \) and the proof completes.

**References**

