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<th>Propagation of singularities for a hyperbolic system with double characteristics</th>
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PROPAGATION OF SINGULARITIES FOR A HYPERBOLIC SYSTEM WITH DOUBLE CHARACTERISTICS

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(Received April 22, 1980)

0. Introduction

Consider the Cauchy problem for a hyperbolic operator

\[ L = D_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(t, X, D_x) + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(t, X, D_x) \quad \text{on } [0, T] \times \mathbb{R}^n, \]

where \( D_t \) denotes \(-\sqrt{-1}\partial_t\), functions \( \lambda_i(t, x, \xi) \) are real valued and belong to \( B^\infty([0, T]; S^1) \) and \( b_{jk}(t, x, \xi) \) belong to \( B^\infty([0, T]; S^0) \). Throughout this paper we assume that

\[ \{\tau + \lambda_i(t, x, \xi)\} = 0 \quad \text{on } [0, T] \times \mathbb{R}^n, \]

\( (i, j, k = 1, 2) \)

where for \( f, g \in C^1(R^{(\xi+1)}_s) \) \( \{f, g\}(t, x; t, \xi) \) denotes the Poisson bracket:

\[ (\partial_x f \partial_x g - \partial_x f \partial_x g + \nabla_t f \cdot \nabla_t g - \nabla_t f \cdot \nabla_t g)(t, x; t, \xi). \]

Recently, using Fourier integral operators with multi-phase functions, Kumano-go -Taniguchi-Tozaki in [10] and Kumano-go -Taniguchi in [11] constructed the fundamental solution for a hyperbolic system with diagonal principal part (Theorem 3.1 in [11]). In these papers the propagation of singularities of solutions was investigated by using an infinite number of phase functions (Theorem 3.4 in [11] or Theorem 3.1 in the present paper).

In the present paper we prove that the propagation of singularities can be described by means of five phase functions \( \phi_1, \phi_2, \phi_1 \# \phi_2, \phi_2 \# \phi_1 \) and \( \phi_1 \# \phi_2 \# \phi_1 \), when the assumption (0.2) is satisfied (Theorem 3.2). We note that the characteristic roots satisfying (0.2) are not necessarily involutive. For examples, \( \lambda_i = -t\xi \) and \( \lambda_2 = t\xi \) for \( n = 1 \) satisfy (0.2), but

\[ \{\tau + \lambda_1, \tau + \lambda_2\} = 2\xi \neq 0 \quad (\xi \neq 0). \]

Other examples will be given in Section 2.

The propagation of singularities of solutions has been investigated by
many authors [1], [2], [3], [4], [6], [8], [12], [13], [14], [15], [16], [17], [18], [19] etc. In particular, in [2], [6], [14], [15], [16], [17], [19] operators with involutive characteristics are treated. Alinhac in [1] and Taniguchi-Tozaki in [18] give the precise descriptions for singularities of solutions for operators on \( \mathbb{R}^n \) with principal part \( \partial_t^2 - e^{2t} \partial_x^2 \) (\( l \) is a positive integer) which are not involutive.

In Section 1 we exhibit main results on the theory of Fourier integral operators in [10] and [11] needed later. In Section 2 under the assumption (0.2) we contract the multi-product \( \Phi_{j_1,\ldots,j_{v+1}}(t_0,\ldots, t_{v+1}; x, \xi) \) \((j_k = 1, 2)\) of phase functions \( \phi_{j_k}(t, s; x, \xi) \) \((j_k = 1, 2)\) (see (1.11)), which are the solutions of the eiconal equations for \( \tau + \lambda_{j_k}(t, x, \xi) \) (see (1.10)) (Theorem 2.4). In Section 3 we prove the main theorem (Theorem 3.2).

The author would like to express his sincere gratitude to Professor H. Kumano-go for his advice and encouragements.

1. Fourier integral operators

For a multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \) of non-negative integers \( \alpha_j \) and points \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \), \( y = (y_1, \cdots, y_n) \in \mathbb{R}^n \) we use the usual notation:

\[
|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j},
\]

\[
D^*_t = D^*_{t_1} \cdots D^*_{t_{v+1}}, \quad D_s = -\sqrt{-1} \partial_s, \quad \nabla_s = (\partial_{s_1}, \cdots, \partial_{s_n}),
\]

\[
\langle x \rangle = (1 + |x|^2)^{\nu/2}, \quad x \cdot y = x_1 y_1 + \cdots + x_n y_n.
\]

For \( f(x) = (f_1, \cdots, f_s) \) \((f_j(x) \in C^1(\mathbb{R}^n))\) we denote

\[
\partial_s f = \nabla_s f = (\partial_{s_1} f_1; \cdots, \frac{i}{k \to 1, \cdots, n}).
\]

Let \( \mathcal{S} \) on \( \mathbb{R}^n \) denote the Schwartz space of rapidly decreasing functions and let \( \mathcal{S}' \) denote the dual space of \( \mathcal{S} \). For \( u \in \mathcal{S}' \), the Fourier transform \( \hat{u}(\xi) = F[u](\xi) \) is defined by

\[
F[u](\xi) = \int e^{-ix \cdot \xi} u(x) dx,
\]

and then, for \( \hat{u}(\xi) \in \mathcal{S}'_\mathbb{R} \) the inverse Fourier transform \( F[\hat{u}](x) \) is defined by

\[
F[\hat{u}](x) = \int e^{ix \cdot \xi} \hat{u}(\xi) d\xi, \quad d\xi = (2\pi)^{-n} d\xi.
\]

For real \( s \) we define the Sobolev space \( H_s \), as the completion of \( \mathcal{S} \) in the norm \( \|u\|_s = \{\langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi\}^{1/2} \).

**Definition 1.1.** We say that a \( C^\infty \)-function \( p(x, \xi) \) in \( \mathbb{R}^{2s} = \mathbb{R}^t \times \mathbb{R}^\xi \) belongs to the class \( S^m \) \((-\infty < m < \infty)\), when
where 
\[
pl(x, \xi) = \partial x D_x^2 p(x, \xi).
\]

The class \( S^m \) makes a Fréchet space with semi-norms
\[
|p|^{(m)}(\vec{\xi}, \vec{x}) = \max_{|\alpha| + |\beta| \leq l} \sup_{x, \xi} \{|p_{\alpha, \beta}(x, \xi)| \langle \xi \rangle^{m-|\alpha|} \} \quad (l = 0, 1, 2, \ldots).
\]

We set \( S^{-\infty} = \bigcap_{m \in \mathbb{Z}} S^m \) and \( S^\infty = \bigcup_{m \in \mathbb{Z}} S^m \).

The pseudo-differential operator \( p(X, D_x) \in S^m \) with symbol \( p(x, \xi) \in S^m \) is defined by
\[
(1.2) \quad p(X, D_x) u = 0.0 - \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i(x-x') \cdot \xi} p(x, \xi) u(x') dx' d\xi
\]
\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i(x-x') \cdot \xi} \chi(\epsilon x', \epsilon \xi) p(x, \xi) u(x') dx' d\xi,
\]
where \( \chi(x, \xi) \in \mathcal{S}(\mathbb{R}^{2n}) \) such that \( \chi(0, 0) = 1 \) (c.f. [7]).


**Definition 1.2.** If \( 0 \leq \tau < 1 \), we denote by \( \mathcal{D}(\tau) \) the set of real valued \( C^\omega \)-functions \( \phi(x, \xi) \) in \( \mathbb{R}^{2n} \) such that \( f(x, \xi) - \xi \cdot x \) belongs to \( S^1 \) and
\[
(1.3) \quad \sum_{|\alpha| + |\beta| \leq 2} \sup_{x, \xi} \{|J_{\alpha, \beta}^{(\tau)}(x, \xi)/\langle \xi \rangle^{1-|\alpha|} \} \leq \tau.
\]

**Remark 1.1.** In [10] \( \mathcal{D}(\tau) \) denoted the class of \( C^2 \)-functions. The above definition is due to [11].

We define the Fourier integral operator \( p_\phi(X, D_x) \) with symbol \( p(x, \xi) \in S^m \) and phase function \( \phi(x, \xi) \in \mathcal{D}(\tau) \) by
\[
(1.4) \quad p_\phi(X, D_x) u(x) = \int_{\mathbb{R}^{2n}} e^{i\phi(x, \xi) \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}.
\]

**Definition 1.3.** Let \( \phi_j \in \mathcal{D}(\tau_j), \ j = 1, \ldots, \nu + 1, \ldots, \tau_\infty \equiv \sum_{j=1}^{\infty} \tau_j \leq \tau_0 \) for a sufficiently small fixed \( \tau_0 \) with \( 0 < \tau_0 \leq 1/8 \). We define the multi-product \( \Phi_{\nu+1}(x, \xi) = (\phi_1 \# \cdots \# \phi_{\nu+1})(x, \xi) \) of phase functions \( \phi_j(x, \xi) \) \((j = 1, \ldots, \nu + 1)\) by
\[
\Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \sum_{j=1}^{\nu} (\phi_j(X^j_\xi \cdot \Xi^j_\xi) - X^j_\xi \cdot \Xi^j_\xi) + \Phi_{\nu+1}(X^\nu_\xi, \xi^0 + X^\nu_\xi)
\]
where \( \{X^j_\xi, \Xi^j_\xi\} \) is defined as the solution of the equation
\[
\begin{cases}
\xi^j_i = \nabla_{\xi} \phi_j(\xi^{j-1}, \xi^j), \\
\xi^{j+1}_i = \nabla_{\xi} \phi_{j+1}(\xi^j, \xi^{j+1}) ,
\end{cases} \quad j = 1, \ldots, \nu.
\]
**Proposition 1.4** (Theorem 1.8 and Theorem 1.9 in [10]). Let $\phi_j \in \mathcal{D}(\tau_j)$, $j=1, \cdots, \nu+1, \cdots, \tau_\infty \leq \tau_0 \leq 1/8$. Then, $\Phi_{\nu+1}(x, \xi)$ of (1.5) is well defined and belongs to $\mathcal{D}(c_0 \tau_{\nu+1})$, $\tau_{\nu+1} = \tau_1 + \cdots + \tau_{\nu+1}$, with a constant $c_0 > 0$ independent of $\nu$ and $\tau_0$. We also get

\[
\begin{align*}
(1.7) & \quad \nabla_x \Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \nabla_x \phi_1(x^0, \Xi_1(x^0, \xi^{\nu+1})), \\
(1.8) & \quad \phi_1 \# \phi_2 \# \phi_3 = (\phi_1 \# \phi_2) \# \phi_3 = (\phi_1 \# \phi_2 \# \phi_3).
\end{align*}
\]

Consider a hyperbolic equation

\[
(1.9) \quad (D_t + \lambda(t, X, D_x))u = 0 \quad \text{on } [0, T]
\]

\[
(\lambda(t, x, \xi) \in B^\infty([0, T]; S^1), \text{ real valued}).
\]

Let $\phi = \phi(t, s) = \phi(t, s; x, \xi)$ be the solution of the eiconal equation

\[
(1.10) \quad \begin{cases}
\partial_t \phi + \lambda(t, x, \nabla_x \phi) = 0 & \text{on } [0, T], \\
\phi |_{t=s} = x \cdot \xi.
\end{cases}
\]

Then, we have

**Proposition 1.5** (Theorem 3.1 in [9]). For a small $T_0$ ($0 < T_0 \leq T$) we get $\phi(t, s) \in \mathcal{D}(c(t-s))$ ($0 \leq s \leq t \leq T_0$) with a constant $c > 0$.

We fix such a $T_0$ in what follows. Take $\lambda_j$ ($j=1, \cdots, \nu+1, \cdots$) as $\lambda$ of (1.9) such that $\{\lambda_j\}_{j=1}^\infty$ is bounded in $B^\infty([0, T]; S^1)$ and let $\phi_j$ be the solutions of (1.10) corresponding to $\lambda_j$. We define $\Phi = \Phi_{1, 2, \cdots, \nu+1}(t_0, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})$ ($0 \leq t_{\nu+1} \leq \cdots \leq t_0 \leq T_0 \leq T$) by

\[
(1.11) \quad \Phi(t_0, \cdots, t_{\nu+1}) = \phi_1(t_0, t_1) \# \cdots \# \phi_{\nu+1}(t_\nu, t_{\nu+1}),
\]

and define $\{X_j^i, \Xi_j^i\}_{j=1}^{\nu+1}(t_0, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})$ as the solution of

\[
(1.12) \quad \begin{cases}
x^j_i = \nabla_x \phi_j(t_{j-1}, t_j; x^{j-1}, \xi^j), \\
\xi^j_i = \nabla_x \xi_j(t_j, t_{j+1}; x^j_i, \xi^j_i),
\end{cases} \quad j = 1, \cdots, \nu,
\]

where $T_0 > 0$ is a constant independent of $\nu$ in Proposition 1.4 and Proposition 1.5. Then, we have

**Proposition 1.6** (Theorem 2.3 in [10]). $\Phi(t_0, \cdots, t_{\nu+1})$ of (1.11) satisfies

1°. $\partial_{t_j} \Phi = \lambda_j(t_j, X_j^i, \Xi_j^i) - \lambda_{j+1}(t_j, X_j^i, \Xi_j^i)$

\[
(j = 0, \cdots, \nu+1, \lambda_0 = \lambda_{\nu+2} = 0, \quad X_0^i = x^0, \Xi_0^i = \nabla_x \phi_1, \quad X_{\nu+1}^i = \nabla_{t_{\nu+1}} \Phi, \Xi_{\nu+1}^i = \xi^{\nu+1}).
\]

2°. If $t_j = t_{j+1}$ for some $j$, we have


\[ \Phi_{1,2,\ldots,v+1}(t_0, \ldots, t_j, t_{j+1}, \ldots, t_{v+1}) = \Phi_{1,2,\ldots,j+1,\ldots,v+1}(t_0, \ldots, t_j, t_{j+2}, \ldots, t_{v+1}). \]

3°. If \( \lambda_j = \lambda_{j+1} \) for some \( j \), we have

\[ \Phi_{1,2,\ldots,v+1}(t_0, \ldots, t_{v+1}) = \Phi_{1,2,\ldots,j+1,\ldots,v+1}(t_0, \ldots, t_{j+1}, t_{j+1}, \ldots, t_{v+1}). \]

Now let \((q, p)(t, s; y, \eta) = ((q_1, \ldots, q_s), (p_1, \ldots, p_s))(t, s; y, \eta) \) \((0 \leq s \leq t \leq T)\) be the bicharacteristic strip for (1.9), that is, \((q, p) (t, s)\) is the solution of

\[
\begin{cases}
\frac{dq}{dt} = -\nabla_\xi \lambda(t, q, p), \\
\frac{dp}{dt} = -\nabla_x \lambda(t, q, p), \\
(q, p) \big|_{t=s} = (y, \eta).
\end{cases}
\]  

(1.13)

Then, we can solve (1.13) in full interval \( s \leq t \leq T \) by the Gronwall inequality, since \( |\nabla_\xi \lambda(t, q, p)| \leq C_1 \) and \( |\nabla_x \lambda(t, q, p)| \leq C_1 < \infty \) \((0 \leq t \leq T)\) for a constant \( C_1 > 0 \). We state propositions on the bicharacteristic strips.

**Lemma 1.7.** Let \( \phi(x, \xi) \in \mathcal{P}(\tau) \). Then, for any \( y, \eta \in \mathbb{R}^n \) (resp. \( x, \xi \)) there exists a point \( (x, \xi) \in \mathbb{R}^n \) (resp. \( (y, \eta) \)) such that

\[
y = \nabla_\xi \phi(x, \eta), \quad \xi = \nabla_x \phi(x, \eta).
\]  

(1.14)

Proof. Set \( F(x) = F(x; y, \eta) = -\nabla_\xi \phi(x, \eta) + x + y \). We have

\[
|F(x') - F(x)| \leq \int_0^1 |\nabla_x \nabla_\xi \phi(x + \theta(x' - x), \eta) - I| \, d\theta \|x' - x\| \leq \tau \|x' - x\|,
\]

where \( I \) is a unit matrix and for a matrix \( A = (a_{ij}; \quad \overset{i \rightarrow}{j \rightarrow} 1, \ldots, n) \) the norm \( \|A\| \) is defined by \( \sum_{i,j} |a_{ij}|^2 \). Then, we can apply the fixed point theorem, and \( x = x(y, \eta) \) satisfying \( y = \nabla_\xi \phi(x, \eta) \) is determined as the fixed point. Then, \( \xi(y, \eta) \) is determined by \( \nabla_x \phi(x(y, \eta), \eta) \).

Similarly, \( (y(x, \xi), \xi(x, \xi)) \) is determined. Q.E.D.

**Lemma 1.8.** Let \((q, p)(t, s; y, \eta) \) \((0 \leq s \leq t \leq T)\) be the bicharacteristic strip defined by (1.13) and \( \phi(t, s; x, \xi) \) \((0 \leq s \leq t \leq T_0)\) be the solution of the eiconal equation (1.10). Then, it follows that

\[
y = \nabla_\xi \phi(t, s; q(t, s), \eta), \quad p(t, s) = \nabla_x \phi(t, s; q(t, s), \eta)
\]  

\((0 \leq s \leq t \leq T_0)\).

(1.15)

Proof. By Lemma 1.7 we can define \((q', p')(t, s; y, \eta) \) \((0 \leq s \leq t \leq T_0)\) by

\[
y = \nabla_\xi \phi(t, s; q'(t, s), \eta), \quad p'(t, s) = \nabla_x \phi(t, s; q'(t, s), \eta).
\]  

(1.16)
Differentiate both sides of (1.16) in \( t \), respectively. Then, using (1.10) we get

\[
\begin{align*}
\frac{dq'(t, s)}{dt} &= \nabla_t \lambda(t, q(t, s), p(t, s)), \\
\frac{dp'(t, s)}{dt} &= -\nabla_s \lambda(t, q(t, s), p(t, s)).
\end{align*}
\]

Since \( q'(s, s) = y \) and \( p'(s, s) = \eta \) from (1.16), we can see that \( q'(t, s) = q(t, s) \) and \( p'(t, s) = p(t, s) \) (0 \( \leq s \leq t \leq T_0 \)). Q.E.D.

Take \( \chi_j \) \((j = 1, \ldots, \nu + 1)\) as \( \lambda \) of (1.9) and define \( \Phi = \Phi_{1, \ldots, \nu+1} (t_0, \ldots, t_{\nu+1}; x, \xi) (0 \leq t_{\nu+1} \leq \cdots \leq t_0 \leq T_0 \leq T) \) by (1.11) corresponding to \( \{ \chi_j \}_{j=1}^{\nu+1} \). For a set \( \{ t_0, \ldots, t_{\nu+1} \} \subset [0, T_0] \) such that \( t_0 \leq t_0' \leq \cdots \leq t_{\nu+1}' \) (resp. \( t_0 \leq t_0' \leq \cdots \leq t_{\nu+1}' \)) we define a trajectory \( (Q, P)(\sigma) = (Q_{1, \ldots, \nu+1}, P_{1, \ldots, \nu+1})(\sigma; t_0', \ldots, t_{\nu+1}'; y, \eta) \) in \( t_0' \leq \sigma \leq t_{\nu+1}' \) (resp. \( t_0' \leq \sigma \leq t_{\nu+1}' \)) as follows: \((Q, P)\) (\( \sigma \)) are continuous functions on \([t_0', t_0']\) (resp. \([t_0', t_{\nu+1}']\)) such that \( (Q, P)(t_{\nu+1}') = (y, \eta) \) and for \( \sigma \in (t_0', t_{\nu+1}') \) (resp. \( \sigma \in (t_{\nu+1}', t_0') \)) \((Q, P)(\sigma)\) satisfy

\[
\frac{dQ}{d\sigma} = \nabla_t \lambda_4 (\sigma, Q, P), \quad \frac{dP}{d\sigma} = -\nabla_s \lambda_4 (\sigma, Q, P).
\]

Then, we obtain

**Proposition 1.9.** Let \( T_0 \geq t_0 \geq \cdots \geq t_{\nu+1} \geq 0 \). Using Lemma 1.7, for any point \((y, \eta)\) take a point \( x \) satisfying

\[
y = \nabla_t \Phi_{1, \ldots, \nu+1} (t_0, \ldots, t_{\nu+1}; \nu, \eta).
\]

Then, we have

\[
(1.19) \quad (Q_{1, \ldots, \nu+1}, P_{1, \ldots, \nu+1})(t_0', \ldots, t_{\nu+1}; y, \eta) = (X_0', \Xi_0')(t_0', \ldots, t_{\nu+1}; x, \nu),
\]

where \( \{ X_0', \Xi_0' \}_{j=1}^{\nu+1} \) is the solution of (1.12) corresponding to \( \Phi = \Phi_{1, \ldots, \nu+1} \) and

\[
(1.20) \quad \left\{ \begin{array}{l}
X_0' = x, \quad \Xi_0' = \nabla_s \Phi_{1, \ldots, \nu+1} (t_0, \ldots, t_{\nu+1}; \nu, \eta), \\
X_{\nu+1}' = y, \quad \Xi_{\nu+1}' = \eta.
\end{array} \right.
\]

**Proof.** Relation (1.7) in Proposition 1.4 shows that

\[
\begin{align*}
&\nabla_t \Phi (t_0, \ldots, t_{\nu+1}; x, \eta) = \nabla_t \phi_{\nu+1} (t_0, t_{\nu+1}; X_\nu', \eta), \\
&\nabla_s \Phi (t_0, \ldots, t_{\nu+1}; x, \eta) = \nabla_s \phi_1 (t_0, t_1; x, \Xi_0').
\end{align*}
\]

Together with (1.12) and (1.18) we get

\[
(1.21) \quad \left\{ \begin{array}{l}
X_0' = \nabla_t \phi_1 (t_0, t_0', \nu; X_0'^1, \Xi_0'^1), \\
\Xi_0'^1 = \nabla_s \phi_\nu (t_{\nu'-1}, t_0; X_0'^1, \Xi_0'^1), \\
k = 1, \ldots, \nu+1.
\end{array} \right.
\]
Now when \( k = v + 1 \), (1.19) is valid. From the definition of \((Q, P)(\sigma) = (Q_{1, \ldots, v+1}, P_{1, \ldots, v+1})(\sigma)\) and by Lemma 1.8 we have
\[
\begin{align*}
\{ y = \nabla_t \phi_{v+1}(t, t_{v+1}; Q(t), \eta), \\
P(t) = \nabla_x \phi_{v+1}(t, t_{v+1}; Q(t), \eta).
\end{align*}
\]

Compare the above relation with \( X'_\sigma \) and \( \Xi'_\sigma \) of (1.21). Setting \( X'^{v+1}_\sigma = y, \Xi'^{v+1}_\sigma = \eta \), we get by Lemma 1.7
\[
Q(t) = X'_\sigma, \quad P(t) = \Xi'_\sigma.
\]

In a similar way we can prove (1.19), inductively. Q.E.D.

2. Contraction of multi-phase functions

Let \( \lambda_j(t, x, \xi) \in B^\infty([0, T]; S^i) \) \((j = 1, 2)\) and be real valued functions. Throughout this section we assume that
\[
(*) \quad \{ \tau + \lambda_i, \{ \tau + \lambda_j, \tau + \lambda_k \} \}(t, x, \xi) = 0 \quad \text{on } [0, T] \times R^2_{\xi, t},
\]
where for \( f, g \in C^1(R^2_{\xi, t}) \) \( \{ f, g \}(t, x; \tau, \xi) \) denotes the Poisson bracket
\[
(2.1) \quad \{ f, g \}(t, x; \tau, \xi) = (\partial_t f \partial_x g - \partial_t g \partial_x f + \nabla_t f \cdot \nabla_x g - \nabla_t g \cdot \nabla_x f)(t, x; \tau, \xi).
\]

Let \( \phi_j(t, s; x, \xi) \) \((j = 1, 2, 0 \leq s \leq t \leq T_0)\) be the solutions of the eiconal equation (1.10) corresponding to \( \lambda_j \) and define \( \Phi = \Phi_{j_1, \ldots, j_{v+1}}(t_0, \ldots, t_{v+1}) \in \mathcal{L}(c_0(t_0 - t_{v+1})) \) \((0 \leq t_0 \leq \cdots \leq t_{v+1} \leq T_0, j = 1, 2)\) by \( \Phi = \phi_{i_1}(t_0, t_1) \# \cdots \# \phi_{i_{v+1}}(t, t_{v+1}), \) where \( c_0 > 0 \) and \( T_0 > 0 \) are constants independent of \( \nu \) (see Proposition 1.4 and Proposition 1.5). We fix such a \( T_0 \) in what follows. It is easy to see that

**Lemma 2.1.** Let \( H(t, x, \xi) \in C^1(R^{2v+1}) \) and \((q, p)(t) = (q, p)(t, s; y, \eta)\) \((0 \leq s \leq t \leq T_0)\) be the bicharacteristic strip defined by (1.13) for \( \tau + \lambda(t, x, \xi) \) of (1.9). Then, we have
\[
(2.2) \quad \frac{d}{d\sigma} H(\sigma, q(\sigma), p(\sigma)) = -\{ H, \tau + \lambda \}(\sigma, q(\sigma), p(\sigma)) \quad (t \leq \sigma \leq T_0).
\]

**Lemma 2.2.** For \( J = (j_1, \ldots, j_{v+1}) \) \((j_k = 1, 2)\) and a set \( \{ t_0, \ldots, t_{v+1} \} \) \((T \geq t_0 \geq \cdots \geq t_{v+1} \geq 0)\) let \((Q, P)(\sigma) = (Q_{j_1, \ldots, j_{v+1}}, P_{j_1, \ldots, j_{v+1}})(\sigma; t_0, \ldots, t_{v+1}; y, \eta)\) be the solution of (1.17) corresponding to \( \{ \lambda_j \}_{j = 1}^{v+1} \). Set
\[
(2.3) \quad \nu(\sigma) = (\lambda_2 - \lambda_1)(\sigma, Q(\sigma), P(\sigma)) \quad (t_{v+1} \leq \sigma \leq t_0).
\]

Then, we get
\[
(2.4) \quad \frac{d}{d\sigma} \nu(\sigma) = \{ \tau + \lambda_1, \tau + \lambda_2 \}(\sigma, Q(\sigma), P(\sigma)) \quad (t_{v+1} \leq \sigma \leq t_0).
\]
Proof. For $\sigma \in (t_k, t_{k-1})$ it follows from Lemma 2.1 that

$$
\frac{d}{d\sigma} v(\sigma) = -\{\lambda_2, \tau + \lambda_{j_k}\} + \{\lambda_1, \tau + \lambda_{j_k}\}
= -\{\tau + \lambda_2, \tau + \lambda_{j_k}\} + \{\tau + \lambda_1, \tau + \lambda_{j_k}\}.
$$

Then, we get (2.4) in both cases $j_k=1$ and 2. Q.E.D.

**Lemma 2.3.** Assume that the assumption (*) holds. Then, for $v(\sigma)$ defined by (2.3) we get

$$
v(\sigma) = a\sigma + b \quad (t_{v+1} \leq \sigma \leq t_0),
$$
where $a = \{\tau + \lambda_1, \tau + \lambda_2\}(t_{v+1}, y, \eta)$ and $b = (\lambda_2 - \lambda_1)(t_{v+1}, y, \eta) - at_{v+1}$.

Proof. We can see from Lemma 2.2 that $v(\sigma)$ belongs to $C^1([t_{v+1}, t_0])$. From (2.4) and Lemma 2.1 it follows that

$$
\frac{d^2}{d\sigma^2} v(\sigma) = -\{\tau + \lambda_1, \tau + \lambda_2\}, \tau + \lambda_{j_k}\} = 0 \quad (t_k < \sigma < t_{k-1}).
$$

Hence, we get (2.5). Q.E.D.

**Remark 2.1.** If the assumption (*) is satisfied, $v(\sigma)$ defined by (2.3) depends only on $\sigma$, $t_{v+1}$, $y$ and $\eta$, and is independent of the choice of $J = (j_1, \ldots, j_{v+1}) (v = 1, 2 \ldots)$ and $\{t_0, \ldots, t_s\}$.

**Theorem 2.4.** Assume that the assumption (*) holds. For $\{t, t_1, t_2, s\}$ ($0 \leq s < t_1 < t_1 < t \leq T_0$) we define functions $(\psi_1, \psi_2)(t, t_1, t_2, s)$ by

$$
\begin{aligned}
\psi_1(t, t_1, t_2, s) &= t - \frac{(t_1 - t_2)(t_2 - s)}{t_1 + t_2 - s}, \\
\psi_2(t, t_1, t_2, s) &= t_1 - t_2 + s - \frac{(t_1 - t_2)(t_2 - s)}{t_1 + t_2 - s}.
\end{aligned}
$$

Then, we obtain

$$
\Phi_{1,2,1}(t, \psi_1, \psi_2, s; x, \xi) = \Phi_{2,1,2}(t, t_1, t_2, s; x, \xi).
$$

Proof. We shall determine $\psi_j(t, t_1, t_2, s) (j=1, 2)$ of (2.6) as the functions satisfying (2.7). From Proposition 1.6 we get $\Phi_{2,1,2}(t, t_1, t_2, s; x, \xi)$ as the solution of

$$
\begin{aligned}
\frac{\partial}{\partial t} \Phi_{2,1,2} + \lambda_2(t, x, \nabla x, \Phi_{2,1,2}) = 0, \\
\Phi_{2,1,2}|_{t=t_1} = \Phi_{1,2}(t_1, t_2, s; x, \xi).
\end{aligned}
$$

So, we have only to determine $\psi_j (j=1, 2)$ depending only on $t, t_1, t_2$ and $s$ such that for $\Phi_{1,2,1}(t, t_1, t_2, s) = \Phi_{1,2,1}(t, t_1, t_2, s; x, \xi)$.
PROPAGATION OF SINGULARITIES FOR A HYPERBOLIC SYSTEM

\[ (2.8) \begin{cases} \partial_t(\Phi_{1,2,1}(t, \psi_1, \psi_2, s)) + \lambda_2(t, x, \nabla_x \Phi_{1,2,1}(t, \psi_1, \psi_2, s)) = 0, \\ \Phi_{1,2,1}(t, \psi_1, \psi_2, s)|_{t=t_1} = \Phi_{1,2}(t_1, t_2, s; x, \xi) \end{cases} \]

holds.

Suppose that for \( \psi_j \ (j=1, 2) \) \( (2.7) \) holds. Set \( \Delta = (t, \psi_1, \psi_2, s; x, \xi) \) and \( \psi_j' = \partial_t \psi_j \ (j=1, 2) \). Then, from \( (2.8) \) and Proposition 1.6 we have

\[ (2.9) \quad 0 = (\partial_t \Phi_{1,2,1}(\Delta) + (\partial_t \Phi_{1,2,1}(\Delta)) \psi_1' + \\
(\partial_t \Phi_{1,2,1}(\Delta)) \psi_2' + \lambda_2(t, x, \nabla_x \Phi_{1,2,1}(\Delta)) \\
- (\lambda_2 - \lambda_1)(t, x, \nabla_x \Phi_{1,2,1}(\Delta)) - \\
(\lambda_2 - \lambda_1)(\psi_1, X^1(\Delta), \Xi^1(\Delta)) \psi_1' + (\lambda_2 - \lambda_1)(\psi_2, X^2(\Delta), \Xi^2(\Delta)) \psi_2' , \]

where \( \{X^1, \Xi^1\} \) is the solution of

\[ x^k = \nabla_x \Phi_{j_k}(t_{k-1}, t_k; \xi^k+1, \xi^k), \xi^k = \nabla_x \Phi_{j_k}(t_{k-1}, t_k, \xi^k+1; x^k, \xi^k+1); \]

\[ (k = 1, 2, x^0 = x, \xi^3 = \xi, j_1 = 1, j_2 = 2, j_3 = 1) . \]

Take a point \( y \) such that

\[ y = \nabla_x \Phi_{1,2,1}(\Delta) = \nabla_x \Phi_{1,2,1}(t, \psi_1, \psi_2, s; x, \xi) . \]

Let \( (Q, P)(\sigma) = (Q_{1,2,1}(\sigma; t, \psi_1, \psi_2, s; y, \xi) \) be the solution of \( (1.17) \) and set

\[ v(\sigma) = (\lambda_2 - \lambda_1)(\sigma, Q(\sigma), P(\sigma)) . \]

Then, by Proposition 1.9 we can write \( (2.9) \) in the form

\[ (2.9)' \quad 0 = v(t) - v(\psi_1) \psi_1' + v(\psi_2) \psi_2' . \]

Take account of the assumption \((*)\). Since from Lemma 2.3 \( v(\sigma) \) has the form \( a\sigma + b \), we get

\[ (2.9)'' \quad 0 = (at+b) - (a\psi_1+b)\psi_1' + (a\psi_2+b)\psi_2' \\
= -a(\psi_1\psi_1' - \psi_2\psi_2' - t) - b(\psi_1' - \psi_2' - 1) . \]

Now we take \( \psi_j \) such that \( \psi_j \) satisfy

\[ (2.10) \quad \psi_1' - \psi_2' = 1, \quad \psi_1\psi_1' - \psi_2\psi_2' = t . \]

If \( \psi_1|_{t=t_1}=t_2 \) and \( \psi_2|_{t=t_1}=s \), the second equality of \( (2.8) \) is also satisfied by Proposition 1.6. Hence, we obtain

\[ (2.11) \quad \psi_1 - \psi_2 = t - t_1 + t_2 - s, \quad \psi_1^2 - \psi_2^2 = t^2 - t_1^2 + t_2^2 - s^2 . \]

Solving \( (2.11) \), we get the functions of \( (2.6) \) satisfying \( (2.7) \). Q.E.D.

**Remark 2.2.** For real constants \( a_j \) and \( b_j \), \( \lambda_1 = -\sum_{i=1}^\delta a_i\xi_i \) and \( \lambda_2 = -2t\sum_{i=1}^\delta b_i\xi_i \), on \( R_{1,\xi}^\delta \) satisfy the assumption \((*)\). Then, we have
\[
\begin{align*}
\Phi_{1,2,1}(t, t_1, t_2, s; x, \xi) &= \sum_{i=1}^{n} \{a_i(t-t_i-t_2-s)+b_i(t_i^2-t_2^2)\} \xi_i + x \cdot \xi, \\
\Phi_{2,1,2}(t, t_1, t_2, s; x, \xi) &= \sum_{i=1}^{n} \{a_i(t_1-t_2)+b_i(t_i^2-t_1^2-t_2^2-s)\} \xi_i + x \cdot \xi.
\end{align*}
\]

From these multi-phase functions we see that \(\psi_j\) \((j=1, 2)\) of (2.6) are uniquely determined functions which satisfy (2.7) for any \(a_j\) and \(b_j\).

**Remark 2.3.** Set \(\Delta_2 = \{(t_1, t_2) ; 0 \leq t_1 < t_2 < t \leq T_0]\). Consider the mapping \(M : \Delta_2 \rightarrow (\psi_1, \psi_2)\) with \((t, s)\) as a parameter. It is clear that the image of the mapping \(M\) is included in \(\Delta_2\). Since from (2.11)

\[
t_1-t_2 = t_2-\psi_1+\psi_2-s, \quad t_1^2-t_2^2 = t_2^2-\psi_1^2+\psi_2^2-s^2,
\]

\(M^2 = I\) (identity map) holds. This implies that the mapping \(M : \Delta_2 \rightarrow \Delta_2\) is one to one and onto. Make the change of variables with \((t, s)\) as a parameter

\[
t'_1 = \psi_1(t, t_1, t_2, s), \quad t'_2 = \psi_2(t, t_1, t_2, s).
\]

Then, we get

\[
\begin{align*}
\int_{t_1}^{t_2} \int_{t_2}^{t_1} \exp \{i \Phi_{2,1,2}(t, t_1, t_2, s; x, \xi)\} dt_2 dt_1 \\
= \int_{t_1}^{t_2} \int_{t_2}^{t_1} \exp \{i \Phi_{1,2,1}(t', t_1, t_2, s; x, \xi)\} \frac{t_1-t_2}{t-t_1+t_2-s} dt' dt_1.
\end{align*}
\]

We note that the functions \(\psi_1, \psi_2\) and \((t_1-t_2)/(t-t_1+t_2-s)\) have singular points \((t_1-t, t_2-s)\). So it seems that it is not easy to construct the fundamental solution by using Fourier integral operators with a finite number of phase functions, if we only follow the method in [10], [11], [15] and [17].

Let \((Q_{j_1, ..., j_{v+1}}, P_{j_1, ..., j_{v+1}})(\sigma; t_0, ..., t_{v+1}; y, \eta)\) be the solution of (1.17) corresponding to \(\{\lambda_j\}_{j=1}^{2n+1}\) and a set \(\{t_0, ..., t_{v+1}\} \subset [0, T_0]\).

**Corollary 2.5.** Assume that \((*)\) holds. Then, for any \(v \geq 2\), \{(j_1, ..., j_{v+1})

\((j_1=1, 2, j_k+j_{k+1})\) and \(\{t_0, ..., t_{v+1}\} (T_0 \geq t_0 \geq ... \geq t_{v+1} \geq 0)\) we get

\[
(2.12) \quad \Phi_{j_1, ..., j_{v+1}}(t_0, ..., t_{v+1}; x, \xi) = \Phi_{1,2,1}(t_0, t'_1, t'_2, t_{v+1}; x, \xi),
\]

for some \(t'_j (j=1, 2, t_0 \geq t'_1 \geq t'_2 \geq t_{v+1})\) independent of \(x\) and \(\xi\). By using the same \(t'_j (j=1, 2)\) we also get

\[
(2.13) \quad (Q_{j_1, ..., j_{v+1}}, P_{j_1, ..., j_{v+1}})(t_0, t_0, ..., t_{v+1}; y, \eta) = (Q_{1,2,1}, P_{1,2,1})(t_0, t_1, t'_2, t_{v+1}; y, \eta)
\]

for any point \((y, \eta) \in \mathbb{R}^{2n}\).
Proof. We can get (2.12) by Proposition 1.6 and Theorem 2.4, inductively. Then, we obtain (2.13) by using (2.12) and Proposition 1.9. Q.E.D.

Remark 2.4. For \( \lambda_j(t, x, \xi) \) \((j=1, 2)\) in Remark 2.2 we have

\[
\begin{align*}
\Phi_1(t, s) &= \sum_{i=1}^{n} a_i(t-s)\xi_i + x \cdot \xi, \\
\Phi_2(t, s) &= \sum_{i=1}^{n} b_i(t^2-s^2)\xi_i + x \cdot \xi, \\
\Phi_{1,2}(t, t_1, s) &= \sum_{i=1}^{n} \{a_i(t-t_1)+b_i(t^2-s^2)\} \xi_i + x \cdot \xi, \\
\Phi_{2,1}(t, t_1, s) &= \sum_{i=1}^{n} \{a_i(t-s)+b_i(t^2-t_1^2)\} \xi_i + x \cdot \xi.
\end{align*}
\]

Comparing (2.14) with \( \Phi_{1,2,1} \) and \( \Phi_{2,1,2} \) in Remark 2.2, we can see that we can generally contract \( \Phi_{1,2,1}(t, t_1, t_2, s) \) and \( \Phi_{2,1,2}(t, t_1, t_2, s) \) \((t > t_1 > t_2 > s)\) no more. Furthermore, from Proposition 1.9 we can also see that we can generally contract \( (Q_{1,2,1}, P_{1,2,1})(t, t_1, t_2, s) \) and \( (Q_{2,1,2}, P_{2,1,2})(t, t_1, t_2, s)(t > t_1 > t_2 > s) \) no more.

Examples. We give examples of \( \lambda_k(t, x, \xi) \) \((k=1, 2)\) satisfying (\( (*) \)) on \([0, T] \times R^d_t \times \mathbb{R}^n_\xi \) except \( \lambda_k \) in Remark 2.2 below. They are not involutive, since \{\( \tau+\lambda_1, \tau+\lambda_2 \)\} \((t, x, \xi)\) does not identically vanish on a set \( \{(t, x, \xi); \lambda_1(t, x, \xi)=\lambda_2(t, x, \xi)\} \).

1. \( \lambda_1(t, x, \xi)=\xi_1, \lambda_2(t, x, \xi)=x_1\xi_2+\xi_3. \)
2. \( \lambda_1(t, x, \xi)=x_2\xi_1, \lambda_2(t, x, \xi)=t\xi_2. \)
3. \( \lambda_1(t, x, \xi)=x_2\xi_1+\xi_3, \lambda_2(t, x, \xi)=-x_2\xi_1+\xi_2. \)

3. Propagation of singularities

Consider a hyperbolic system with diagonal principal part

\[
L = D_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(t, X, D_x) + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(t, X, D_x)
\]

on \([0, T] \times \mathbb{R}^d_t \times \mathbb{R}^n_\xi \) \( (\lambda_j(t, x, \xi) \in B^\infty([0, T]; S^1)), \)

real valued, \( b_{ij}(t, x, \xi) \in B^\infty([0, T]; S^0)) \).

We assume that for a constant \( M > 0 \) we have

\[
\lambda_j(t, x, \delta \xi) = \delta \lambda_j(t, x, \xi) \quad (|\xi| \geq M, \delta \geq 1). \]

We also assume that (\( * \)) of Section 2 holds.

We study the Cauchy problem

\[
\begin{cases}
LU(t, x) = 0 & \text{on } [0, T], \\
U|_{t=0} = G(x),
\end{cases}
\]
where \( U(t, x) = \langle u(t, x), u_0(t, x) \rangle \) and \( G(x) = \langle g_1(x), g_2(x) \rangle g_3(x) \in H_{-\infty} = \bigcup_\delta H_\delta \).

Let \( \phi_j(t, s; x, \xi) (0 \leq s \leq t \leq T_0 \leq T) \) be the solutions of the eiconal equations (1.10) corresponding to \( \lambda_j \) and define \( \Phi^\lambda = \Phi_{\lambda_1, \ldots, \lambda_{v+1}}(0, \ldots, t_{v+1}) \) \( (j = 1, 2) \) by \( \Phi = \Phi_j(t_0, t_1) \# \cdots \# \Phi_{j_{v+1}}(t_v, t_{v+1}) \) (see (1.11)).

If we apply Theorem 3.1 in Kumano-go-Taniguchi [11] to \( L \) of (3.1), then, for a small \( T_0 \) \( (0 < T_0 \leq T) \) we can get the fundamental solution \( E(t, s) \) \( (0 \leq s \leq t \leq T_0) \) of \( L \) (i.e. \( L E(t, s) = 0 \) on \([0, T_0]\) and \( E(s, s) = I \) (unit matrix)), which is represented by means of Fourier integral operators with multi-phase functions \( \Phi_{\lambda, \ldots, \lambda_{v+1}}(\nu = 0, 1, \ldots) \). We fix such a \( T_0 \) in what follows. We will apply the theory in [11] for the propagation of singularities of solutions (Theorem 3.4 in [11]) to the Cauchy problem (3.3).

For \( \lambda_j, \ldots, \lambda_{v+1}, (y, \eta) \) and a fixed \( 0 \leq \varepsilon < 1 \) we define an \( \varepsilon \)-station chain \( \{t_1, \ldots, t_v\} \) as the point \( t > t_1 > \cdots > t_v > 0 \) such that for \( k = 1, \ldots, v \)

\[
|\lambda_j(t_k, x^k, \xi^k) - \lambda_{j_{k+1}}(t_k, x^k, \xi^k) | \leq \varepsilon \langle \xi^k \rangle \\
\text{at } (x^k, \xi^k) = (Q_{j_{k-1}, \ldots, j_{v+1}}, P_{j_{k-1}, \ldots, j_{v+1}})(t_k; t, t_1, \ldots, t_v, 0; y, \eta),
\]

where \((Q_{j_{k-1}, \ldots, j_{v+1}}, P_{j_{k-1}, \ldots, j_{v+1}})(\sigma; t_0, \ldots, t_v, 0; y, \eta) \) is the solution of (1.17) corresponding to \( \{\lambda_j\}_{k+1} \) and \( \{t_0, \ldots, t_{v+1}\} \) \( (t_0 = t, t_{v+1} = 0) \). Define the \( \varepsilon \)-station set \( \Lambda_{\varepsilon, j_{k-1}, \ldots, j_{v+1}}(t; y, \eta) \) by the set of all \( \varepsilon \)-station chains \( \{t_1, \ldots, t_v\} \).

We set \( WF(G) = \bigcup_{j=1}^{2} WF(g_j) \) for the wave front set \( WF(g_j) \) of \( g_j \). For \( J = (j_1, \ldots, j_{v+1}) \) we set

\[
\Lambda_{\varepsilon}^J(t; y, \eta) = \{(Q_{j_{k-1}, \ldots, j_{v+1}}, P_{j_{k-1}, \ldots, j_{v+1}})(t; t, t_1, \ldots, t_v, 0; y, \eta); \\
\{t_1, \ldots, t_v\} \in \Lambda_{\varepsilon, j_{k-1}, \ldots, j_{v+1}}(t; y, \eta) \},
\]

and set

\[
\Gamma_{t, \varepsilon} = \{ \delta \Lambda_{\varepsilon}^J(t; y, \eta); (y, \eta) \in WF_\delta(G), J = (j_1, \ldots, j_{v+1}) \},
\]

\[
\{ \delta \Lambda_{\varepsilon}^J(t; y, \eta); (y, \eta) \in WF_\delta(G); J = (j_1, \ldots, j_{v+1}) \}
\]

\[
\{ \delta \Lambda_{\varepsilon}^J(t; y, \eta); (y, \eta) \in WF_\delta(G); J = (j_1, \ldots, j_{v+1}) \}
\]

\[
\text{for a large constant } M_0 > 0 \text{ depending on } M \text{ of (3.2)}. \] Then, Theorem 3.4 in [11] says without the assumption (*)

**Theorem 3.1.** \( \bigcap_{0 < \varepsilon < 1} \Gamma_{t, \varepsilon} \) is closed and for the solution \( U(t, x) \) of the Cauchy problem (3.3) we have

\[
WF(U(t)) \subset \bigcap_{0 < \varepsilon < 1} \Gamma_{t, \varepsilon} \quad (0 \leq t \leq T_0).
\]

If we add the assumption (*), then, setting

\[
\Gamma_{t, 0} = \{ \delta \Lambda_{\varepsilon}^J(t; y, \eta); (y, \eta) \in WF(G), \delta > 0, \}
\]

\[
\{ \delta \Lambda_{\varepsilon}^J(t; y, \eta); (y, \eta) \in WF(G), \delta > 0, \}
\]

\[
|\eta| \geq M_0, J = (1), (2), (1, 2), (2, 1), (1, 2, 1) \}
\]

\[
|\eta| \geq M_0, J = (1), (2), (1, 2), (2, 1), (1, 2, 1) \}
\]
we get the main theorem.

**Theorem 3.2.** Assume that the assumption (*) holds. Then, for the solution $U(t, x)$ of the Cauchy problem (3.3) we get

$$WF(U(t)) \subset \Gamma_{t, 0} \quad (0 \leq t \leq T_0).$$

**Proof.** By Theorem 3.1 we have only to prove that

$$\bigcap_{0 < t < T} \bigcap_{t} \Gamma_{t, r} = \Gamma_{t, 0}.$$  

It is easy to see that $\bigcap_{0 < t < T} \bigcap_{t} \Gamma_{t, r} \supset \Gamma_{t, 0}$. So, we prove that

$$\bigcap_{0 < t < T} \bigcap_{t} \Gamma_{t, r} \subset \Gamma_{t, 0}.$$  

We fix $0 < t \leq T_0$ and take a point $(x^0, \xi^0) \in \bigcap_{0 < t < T} \bigcap_{t} \Gamma_{t, r}$ and fix it. If we take $|\xi^0|$ sufficiently large, then, setting $\xi^k = \xi_{j_{x^0+1}, \ldots, j_{x^0}}(t_k; 0, t, \ldots, t_0; x^0, \xi^0)$ ($k = 1, \ldots, \nu + 1, t_{x^0+1} = 0$), we have

$$C^{-1} \leq |\xi^k| \leq C \quad (k = 0, \ldots, \nu + 1).$$

Here, the positive constant $C$ is independent of the choice of $\mathcal{J} = (j_1, \ldots, j_{\nu+1})$ and a set $\{t_0, \ldots, t_\nu\} \subset [0, t]$. Since $(x^0, \xi^0)$ belongs to $\bigcap_{0 < t < T} \bigcap_{t} \Gamma_{t, r}$, for any $\varepsilon_m = 2^{-m}$ there exist $J_m^m = (j_m^m, \ldots, j_{m+1}^m)$ ($j_m^m = 1, 2, j_{m+1}^m = j_{m+1}^m$), $(y_m, \eta_m) \in WF_{e_m}(G)$ and $\{t_1^m, \ldots, t_{\nu+1}^m\} \in \Lambda_{t^m, \mathcal{J}^m}$ such that

$$\left(\begin{array}{l}
\xi^k = (y_1^m, \ldots, y_{\nu+1}^m, P_{j_{x^0+1}^m, \ldots, j_{x^0}^m}(t_k; t, t, \ldots, t_0; y_0, \eta_0)) \\
\end{array}\right).$$

We consider $(x^0, \xi^0)$ dividing into two cases as follows.

I) The case where we can take a subsequence $l = \{m_\nu\}_{\nu=1}^\infty$ and a point $\sigma_1 (0 \leq \sigma_1 \leq t)$ such that $t_{1, \nu} \rightarrow \sigma_1$ and $t_{\nu} \rightarrow \sigma_1$ as $l \rightarrow \infty$.

II) The other case.

I). We show that $(x^0, \xi^0)$ belongs to $\Gamma_{t, 0}$, when $0 < \sigma_1 < t$. In the other case $\sigma_1 = 0$ or $t$ we can also prove this by the similar way. By the assumption I) we can also take a subsequence $\gamma = \{l_\nu\}_{\nu=1}^\infty$ of $l = \{m_\nu\}_{\nu=1}^\infty$ such that

$$(j_1^\gamma, j_{\nu+1}^\gamma) = (1, 1), (1, 2), (2, 1) \text{ or } (2, 2).$$

We may assume that $j_1^\gamma = 1$ and $j_{\nu+1}^\gamma = 2$, since we can prove similarly in the other cases. Now, take a point $(y^0, \eta^0)$ ($|\eta^0| \geq C^{-1}$, see (3.11)) such that

$$(y^0, \eta^0) = (Q_{1, 2}, P_{2, 1})(t; 0, \sigma_1, t; x^0, \xi^0).$$

We note that

$$(3.13)' \quad (x^0, \xi^0) = (Q_{1, 2}, P_{1, 2})(t; t, \sigma_1, 0; y^0, \eta^0).$$
Then, it is easy to see that

\[ y^0 = x^0 + \int_{t_1}^{t_1} \nabla_t \lambda_1(\sigma, Q_{t_1}(\sigma; 0, \sigma_1, t; x^0, \xi_0), P_{t_1}(\sigma; 0, \sigma_1, t; x^0, \xi_0)) \, d\sigma \]

\[ + \int_{t_1}^{0} \nabla_t \lambda_2(\sigma, Q_{t_2}(\sigma; 0, \sigma_1, t; x^0, \xi_0), P_{t_2}(\sigma; 0, \sigma_1, t; x^0, \xi_0)) \, d\sigma. \]

Using the assumption of this case, for any small \( \delta > 0 \) there exists \( N \) such that for any \( \gamma \geq N \) we have

\[ \{ t_1^\gamma, \ldots, t_N^\gamma \} \subset [\sigma_1 - \delta, \sigma_1 + \delta]. \]

Since for any \( y^\gamma \) we have the similar equality to (3.14), we get

\[ |y^\gamma - y^\gamma| \leq C_1 \delta \quad (\gamma \geq N) \]

for a constant \( C_1 > 0 \) independent of \( \delta \) and \( \gamma \). By the similar way we get

\[ |\eta^\gamma - \eta^\gamma| \leq C_1 \delta \quad (\gamma \geq N). \]

Consequently, we can see that \( (y^\gamma, \eta^\gamma) \to (y^0, \eta^0) \) as \( \gamma \to \infty \) and

\[ (y^0, \eta^0) \in W^p(G). \]

Next, since \( \{ t_1^\gamma, \ldots, t_N^\gamma \} \subset [t_1, \ldots, t_N^\gamma] \), it follows from (3.11) and (3.12) that

\[ |(\gamma_2 - \gamma_1)(t_1^\gamma, Q_1(t_1^\gamma; t_1^\gamma, t; x^\gamma, \xi^\gamma), P_1(t_1^\gamma; t_1^\gamma, t; x^\gamma, \xi^\gamma))| \leq C \varepsilon \gamma \]

for a constant \( C \) of (3.11). Here, noting that \( j_1^\gamma = 1 \) and \( j_2^\gamma = 2 \), we used

\[ (Q_1(t_1^\gamma; t_1^\gamma, t; x^\gamma, \xi^\gamma), P_1(t_1^\gamma; t_1^\gamma, t; x^\gamma, \xi^\gamma)) = (Q_1, P_1)(t_1^\gamma; t_1^\gamma, t; x^\gamma, \xi^\gamma). \]

When \( \gamma \to \infty \), we get from (3.13)

\[ 0 = (\gamma_2 - \gamma_1)(\sigma_1, Q_1(\sigma_1; t_1^\gamma, t; x^\gamma, \xi^\gamma), P_1(\sigma_1; t_1^\gamma, t; x^\gamma, \xi^\gamma)) \]

\[ = (\gamma_2 - \gamma_1)(\sigma_1, Q_1, P_1(\sigma_1; t, \sigma_1, t; x^\gamma, \xi^\gamma), P_1(\sigma_1; t, \sigma_1, t; x^\gamma, \xi^\gamma)). \]

Together with (3.13)' and (3.16) this implies that

\[ (x^\gamma, \xi^\gamma) \in \{ \Lambda_0^{(1,2)}(t; y, \eta); (y, \eta) \in W^p(G) \} \]

\[ \subset \Gamma_{t_0}. \]

II). We can take a subsequence \( l = \{ m_n \}_{n=1}^\infty \) and points \( \sigma_1, \sigma_2 (0 \leq \sigma_2 < \sigma_1 \leq t) \) such that \( t_1^\gamma \to \sigma_1 \) and \( t_2^\gamma \to \sigma_2 \) as \( l \to \infty \). We set

\[ \nu(\sigma; l) = (\gamma_2 - \gamma_1)(\sigma; Q_1(t_1^\gamma; t_1^\gamma, t_1^\gamma, \ldots, t_N^\gamma; 0; y^\gamma, \eta^\gamma), P_1(t_1^\gamma; t_1^\gamma, t_1^\gamma, \ldots, t_N^\gamma; 0; y^\gamma, \eta^\gamma), (0 \leq \sigma \leq t). \]
For large $l$ we have

$$t'_1 - t'_i \geq \frac{1}{2} (\sigma_1 - \sigma_2) > 0,$$

and then, noting that $\{t'_1, \ldots, t'_i\} \in \Lambda_{t_1, t'_1, \ldots, t'_i}(y', \eta')$, we have by (3.11)

$$|v(t'_1; l)|, |v(t'_i; l)| \leq C \varepsilon_i.$$

Consequently, since $v(\sigma; l)$ of (3.17) has the form

$$v(\sigma; l) = a\sigma + b \quad (0 \leq \sigma \leq t)$$

from Lemma 2.3 in Section 2, it follows that

$$|v(\sigma; l)| \leq 2C \varepsilon_i T_0 (t'_1 - t'_i) \leq 4C \varepsilon_i T_0 (\sigma_1 - \sigma_2) \quad (0 \leq \sigma \leq t).$$

Now, by Corollary 2.5 there exist some $\bar{t}'_1, \bar{t}'_2 (t' > \bar{t}'_1 > \bar{t}'_2 > 0)$ such that

$$(3.20) \quad (\xi', \eta') = (Q_{1, 2, 1}, P_{1, 2, 1})(t; t, \bar{t}'_1, \bar{t}'_2, 0; y', \eta').$$

Then, we note that

$$(3.20)' \quad (y', \eta') = (Q_{1, 2, 1}, P_{1, 2, 1})(0; 0, \bar{t}'_2, \bar{t}'_1, 0; \xi', \eta').$$

We set

$$v_i(\sigma; l) = (\lambda_2 - \lambda_1)(\sigma; Q_{1, 2, 1}(\sigma; t, \bar{t}'_1, \bar{t}'_2, 0; y', \eta'),$$

$$P_{1, 2, 1}(\sigma; t, \bar{t}'_1, \bar{t}'_2, 0; y', \eta')).$$

Since $v_i(\sigma; l) = v(\sigma; l)$ by Lemma 2.3 and Remark 2.1, from (3.19) we obtain

$$|v_i(\sigma; l)| \leq \frac{4C}{\sigma_1 - \sigma_2} \varepsilon_i T_0.$$

Next, let $\sigma_i (i=1, 2, \sigma_1 \leq \sigma_2)$ be the accumulating points of sets $\{\bar{t}'_i\}_{i=1}$, respectively and take some subsequence $\{\gamma = \lambda_i\}_{i=1}$ such that $\bar{t}'_1 \rightarrow \sigma_1$ and $\bar{t}'_2 \rightarrow \sigma_2$ as $\gamma \rightarrow \infty$. Then, it follows from (3.20)' that there exists $(\xi^0, \eta^0)$ such that

$$(y'', \eta'') \rightarrow (\xi^0, \eta^0) = (Q_{1, 2, 1}, P_{1, 2, 1})(0; 0, \sigma_2, \sigma_1, t; \xi^0, \xi^0)$$

as $\gamma \rightarrow \infty$, and

$$(\xi^0, \eta^0) \in WF(G).$$

We note that

$$(3.23) \quad (\xi^0, \eta^0) \in WF(G).$$

$$(\xi^0, \eta^0) = (Q_{1, 2, 1}, P_{1, 2, 1})(t; t, \sigma, \sigma_2, 0; \xi^0, \eta^0).$$
By using (3.22) we obtain

\[(\lambda_1 - \lambda_2)(\sigma, Q_{1,2,1}(\sigma; t, \sigma_1, \sigma_2, 0; \tilde{\eta}^0, \eta^0), P_{1,2,1}(\sigma; t, \sigma_1, \sigma_2, 0; \tilde{\eta}^0, \eta^0)) \]

\[= \lim_{\tau \to \sigma} v_1(\sigma; \gamma) \quad (0 \leq \sigma \leq t).\]

This implies with (3.23) and (3.24) that

\[(x^0, \xi^0) \in \tilde{\Gamma}_{t,0}\]

which means (3.9) together with the result of 1). Q.E.D.

References


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