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Osaka University
0. Introduction

Consider the Cauchy problem for a hyperbolic operator

\begin{equation}
L = D_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(t, X, D_x) + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(t, X, D_x) \quad \text{on } [0, T] \times \mathbb{R}^n,
\end{equation}

where \( D_t \) denotes \(-\sqrt{-1} \partial_t \), functions \( \lambda_i(t, x, \xi) \) are real valued and belong to \( B^\infty([0, T]; \mathcal{S}^1) \) and \( b_{jk}(t, x, \xi) \) belong to \( B^\infty([0, T]; \mathcal{S}^0) \). Throughout this paper we assume that

\begin{equation}
\{\tau + \lambda_i, \{\tau + \lambda_j, \tau + \lambda_k\}\}(t, x, \xi) = 0 \quad \text{on } [0, T] \times \mathbb{R}^n_{x,t},
\end{equation}

\((i, j, k = 1, 2)\)

where for \( f, g \in C^1(\mathbb{R}^n_{x,t}) \) \( \{f, g\}(t, x; \tau, \xi) \) denotes the Poisson bracket:

\[
(\partial_x f \partial_{\xi} g - \partial_t f \partial_x g + \nabla_\delta f \cdot \nabla_\delta g - \nabla_x f \cdot \nabla_\xi g)(t, x; \tau, \xi).
\]

Recently, using Fourier integral operators with multi-phase functions, Kumano-go -Taniguchi-Tozaki in [10] and Kumano-go -Taniguchi in [11] constructed the fundamental solution for a hyperbolic system with diagonal principal part (Theorem 3.1 in [11]). In these papers the propagation of singularities of solutions was investigated by using an infinite number of phase functions (Theorem 3.4 in [11] or Theorem 3.1 in the present paper).

In the present paper we prove that the propagation of singularities can be described by means of five phase functions \( \phi_1, \phi_2, \phi_1 \# \phi_2, \phi_2 \# \phi_1 \) and \( \phi_1 \# \phi_2 \# \phi_1 \), when the assumption (0.2) is satisfied (Theorem 3.2). We note that the characteristic roots satisfying (0.2) are not necessarily involutive. For examples, \( \lambda_1 = -t \xi \) and \( \lambda_2 = t \xi \) for \( n=1 \) satisfy (0.2), but

\[
\{\tau + \lambda_1, \tau + \lambda_2\}(=2\xi) \neq 0 \quad (\xi \neq 0).
\]

Other examples will be given in Section 2.

The propagation of singularities of solutions has been investigated by
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many authors [1], [2], [3], [4], [6], [8], [12], [13], [14], [15], [16], [17], [18], [19] etc. In particular, in [2], [6], [14], [15], [16], [17], [19] operators with involutive characteristics are treated. Alinhac in [1] and Taniguchi-Tozaki in [18] give the precise descriptions for singularities of solutions for operators on $\mathbb{R}^n$ with principal part $\partial_t^l - \partial^2\partial_t^l (l \text{ is a positive integer})$ which are not involutive.

In Section 1 we exhibit main results on the theory of Fourier integral operators in [10] and [11] needed later. In Section 2 under the assumption (0.2) we contract the multi-product $\Phi_{t_1, \ldots, t_{v+1}}(t_0, \ldots, t_{v+1}; x, \xi) (j_k = 1, 2)$ of phase functions $\phi_{j_k}(t, s; x, \xi) (j_k = 1, 2)$ (see (1.11)), which are the solutions of the eiconal equations for $\tau + \lambda_{j_k}(t, x, \xi)$ (see (1.10)) (Theorem 2.4). In Section 3 we prove the main theorem (Theorem 3.2).

The author would like to express his sincere gratitude to Professor H. Kumano-go for his advice and encouragements.

1. Fourier integral operators

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers $\alpha_j$ and points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we use the usual notation:

$$
|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j},
$$

$$
D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D_{x_j} = -\sqrt{-1} \partial_{x_j}, \quad \nabla_x = (\partial_{x_1}, \ldots, \partial_{x_n}),
$$

$$
\langle x \rangle = (1 + |x|^2)^{1/2}, \quad \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.
$$

For $f(x) = (f_1, \ldots, f_s) (f_j(x) \in C^1(\mathbb{R}^n))$ we denote

$$
\partial_s f = \nabla_s f = (\partial_{x_j} f_j; \ j = 1, \ldots, n).
$$

Let $\mathcal{S}$ on $\mathbb{R}^n$ denote the Schwartz space of rapidly decreasing functions and let $\mathcal{S}'$ denote the dual space of $\mathcal{S}$. For $u \in \mathcal{S}'$, the Fourier transform $\hat{u}(\xi) = F[u](\xi)$ is defined by

$$
F[u](\xi) = \int e^{-ix\cdot\xi} u(x) dx,
$$

and then, for $\hat{u}(\xi) \in \mathcal{S}_\xi$ the inverse Fourier transform $F[\hat{u}](x)$ is defined by

$$
F[\hat{u}](x) = \int e^{ix\cdot\xi} \hat{u}(\xi) d\xi, \quad d\xi = (2\pi)^{-n} d\xi.
$$

For real $s$ we define the Sobolev space $H_s$ as the completion of $\mathcal{S}$ in the norm $||u||_s = \left\{ \int \langle x \rangle^{2s} |u(\xi)|^2 d\xi \right\}^{1/2}$.

**Definition 1.1.** We say that a $C^\infty$-function $p(x, \xi)$ in $\mathbb{R}^{2n} = \mathbb{R}_x^n \times \mathbb{R}_\xi^n$ belongs to the class $S^m$ ($-\infty < m < \infty$), when
where \( p^m_0(x, \xi) = \partial_t D_x^k p(x, \xi) \).

The class \( S^m \) makes a Fréchet space with semi-norms

\[
| p |^m = \max_{|a + b| \leq l} \sup_{x, \xi} \{| p^m_0(x, \xi) | \langle \xi \rangle^{|a|} \} \quad (l = 0, 1, 2, \ldots).
\]

We set \( S^{-\infty} = \bigcap_{m < \infty} S^m \) and \( S^\infty = \bigcup_{m = -\infty} S^m \).

The pseudo-differential operator \( p(X, D_x) \in S^m \) with symbol \( p(x, \xi) \in S^m \) is defined by

\[
(1.2) \quad p(X, D_x) u = 0 - \int_{\mathbb{R}^{2n}} e^{i(x-x') \cdot \xi} p(x, \xi) u(x') \, dx' \, d\xi
\]

\[
= \lim_{t \to 0} \int_{\mathbb{R}^{2n}} e^{i(x-x') \cdot \xi} \chi(x', \xi) p(x, \xi) u(x') \, dx' \, d\xi,
\]

where \( \chi(x, \xi) \in \mathcal{S}(\mathbb{R}^{2n}) \) such that \( \chi(0, 0) = 1 \) (c.f. [7]).


**Definition 1.2.** If \( 0 \leq \tau < 1 \), we denote by \( \mathcal{P}(\tau) \) the set of real valued \( C^\infty \)-functions \( \phi(x, \xi) \) in \( \mathbb{R}^{2n} \) such that \( f(x, \xi) = \phi(x, \xi) - x \cdot \xi \) belongs to \( S^1 \) and

\[
\sum_{|a + b| \leq 2} \sup_{x, \xi} \{ | J^m_0(x, \xi) \langle \xi \rangle^{-|a|} \} \leq \tau.
\]

**Remark 1.1.** In [10] \( \mathcal{P}(\tau) \) denoted the class of \( C^2 \)-functions. The above definition is due to [11].

We define the Fourier integral operator \( p_\phi(X, D_x) \) with symbol \( p(x, \xi) \in S^m \) and phase function \( \phi(x, \xi) \in \mathcal{P}(\tau) \) by

\[
(1.4) \quad p_\phi(X, D_x) u(x) = \int_{\mathbb{R}^{2n}} e^{i\phi(x, \xi)} p(x, \xi) \hat{u}(\xi) \, d\xi,
\]

where \( \hat{u} \in \mathcal{S} \).

**Definition 1.3.** Let \( \phi_j \in \mathcal{P}(\tau_j), j = 1, \ldots, v+1, \ldots, \tau_v \equiv \sum_{j=1}^v \tau_j \leq \tau_0 \) for a sufficiently small fixed \( \tau_0 \) with \( 0 < \tau_0 \leq 1/8 \). We define the *multi-product* \( \Phi_{v+1}(x, \xi) = (\phi_1 \# \cdots \# \phi_{v+1})(x, \xi) \) of phase functions \( \phi_j(x, \xi) \) \( (j = 1, \ldots, v+1) \) by

\[
\Phi_{v+1}(x^0, \xi^{v+1}) = \sum_{j=1}^v (\phi_j(X_j^{-1}, \Xi_j) - X_j \cdot \Xi_j) + \phi_{v+1}(X_v, \xi^{v+1})
\]

\[
(X_v = x^0),
\]

where \( \{ X_j, \Xi_j \}_{j=1}^{v+1}(x^0, \xi^{v+1}) \) is defined as the solution of the equation

\[
\begin{cases}
\xi^j = \nabla_x \phi_j(x^{j-1}, \xi^j), \\
\xi^j = \nabla_x \phi_{j+1}(x^j, \xi^{j+1}), \quad j = 1, \ldots, v.
\end{cases}
\]
Proposition 1.4 (Theorem 1.8 and Theorem 1.9 in [10]). Let \( \phi_j \in \mathcal{D}(\tau_j) \), \( j=1, \ldots, v+1 \), with \( \tau_0 \leq \tau_0 \leq 1/8 \). Then, \( \Phi_{v+1}(x, \xi) \) of (1.5) is well defined and belongs to \( \mathcal{D}(c_0 \tau_{v+1}) \) with a constant \( c_0 > 0 \) independent of \( v \) and \( \tau_0 \). We also get

\[
\begin{align*}
\nabla_x \Phi_{v+1}(x^0, \xi^{v+1}) &= \nabla_x \phi_1(x^0, \Xi_1(x^0, \xi^{v+1})) \\
\nabla_\xi \Phi_{v+1}(x^0, \xi^{v+1}) &= \nabla_\xi \phi_1(x^0, \xi^{v+1}, \xi^{v+1}),
\end{align*}
\]

(1.7)

\[
\phi_1 \# \phi_2 \# \phi_3 = (\phi_1 \# \phi_2) \# \phi_3 = (\phi_1 \# \phi_2 \# \phi_3).
\]

Consider a hyperbolic equation

\[
(D_t + \lambda(t, X, D_x))u = 0 \quad \text{on } [0, T]
\]

\[
(\lambda(t, x, \xi) \in B^{-\infty}([0, T]; S^1), \text{ real valued}).
\]

Let \( \phi = \phi(t, s) = \phi(t, s; x, \xi) \) be the solution of the eiconal equation

\[
\begin{align*}
\partial_t \phi + \lambda(t, x, \nabla_x \phi) &= 0 \quad \text{on } [0, T], \\
\phi|_{t=s} &= x \cdot \xi.
\end{align*}
\]

(1.10)

Then, we have

Proposition 1.5 (Theorem 3.1 in [9]). For a small \( T_0 \) \((0 < T_0 \leq T)\) we get \( \phi(t, s) \in \mathcal{D}(c(t-s)) \) \((0 \leq s \leq t \leq T_0)\) with a constant \( c > 0 \).

We fix such a \( T_0 \) in what follows. Take \( \lambda_j \) \((j=1, \ldots, v+1, \ldots)\) as \( \lambda \) of (1.9) such that \( \{\lambda_j\}_{j=1}^v \) is bounded in \( B^{-\infty}([0, T]; S^1) \) and let \( \phi_j \) be the solutions of (1.10) corresponding to \( \lambda_j \). We define \( \Phi = \Phi_{1, 2, \ldots, v+1}(t_0, \ldots, t_{v+1}; x^0, \xi^{v+1}) \) \((0 \leq t_{v+1} \leq \cdots \leq t_0 \leq T_0 \leq T)\) by

\[
\Phi(t_0, \ldots, t_{v+1}) = \phi_1(t_0, t_1) \# \cdots \# \phi_{v+1}(t_v, t_{v+1}),
\]

(1.11)

and define \( \{X_j, \Xi_j\}_{j=1}^{v+1}(t_0, \ldots, t_{v+1}; x^0, \xi^{v+1}) \) as the solution of

\[
\begin{align*}
x^j &= \nabla_x \phi_j(t_{j-1}, t_j; x^{j-1}, \xi^j), \\
\xi^j &= \nabla_\xi \phi_{j+1}(t_j, t_{j+1}; x^j, \xi^{j+1}), \quad j = 1, \ldots, v,
\end{align*}
\]

(1.12)

where \( T_0 > 0 \) is a constant independent of \( v \) in Proposition 1.4 and Proposition 1.5. Then, we have

Proposition 1.6 (Theorem 2.3 in [10]). \( \Phi(t_0, \ldots, t_{v+1}) \) of (1.11) satisfies

1°. \( \partial_j \Phi = \lambda_j(t_j, X_j^j, \Xi_j^j) - \lambda_{j+1}(t_{j+1}, X_j, \Xi_j) \)

\((j = 0, \ldots, v+1, \lambda_0 = \lambda_{v+2} = 0, X_0^j = x^0, \Xi_0^j = \nabla_x \phi_1, X_{v+1}^j = \nabla_{\xi_{v+1}} \Phi, \Xi_{v+1}^j = \xi^{v+1})\).

2°. If \( t_j = t_{j+1} \) for some \( j \), we have
If \( \lambda_j = \lambda_{j+1} \) for some \( j \), we have
\[
\Phi_{1,2,\ldots,v+1}(t_0, \ldots, t_j, t_{j+1}, \ldots, t_{v+1}) = \Phi_{1,2,\ldots,j+j+2,\ldots,v+1}(t_0, \ldots, t_j, t_{j+2}, \ldots, t_{v+1}).
\]

Let \( (q, p)(t, s; y, \eta) \) be the bicharacteristic strip for (1.9), that is, \( (q, p)(t, s) \) is the solution of
\[
\begin{align*}
\frac{dq}{dt} &= \nabla_x \lambda(t, q, p), \\
\frac{dp}{dt} &= -\nabla_x \lambda(t, q, p), \\
(q, p)|_{t=t_0} &= (y, \eta).
\end{align*}
\]
(1.13)

Then, we can solve (1.13) in full interval \( s \leq t \leq T \) by the Gronwall inequality, since \( |\nabla_x \lambda(t, q, p)| \leq C_1 \) and \( |\nabla_x \lambda(t, q, p)| \leq C_1 < p \) \( (0 \leq t \leq T) \) for a constant \( C_1 > 0 \). We state propositions on the bicharacteristic strips.

**Lemma 1.7.** Let \( \phi(x, \xi) \in \mathcal{O}(\tau) \). Then, for any \( y, \eta \in \mathbb{R}^n \) (resp. \( x, \xi \)) there exists a point \( (x, \xi) \in \mathbb{R}^n \) (resp. \( (y, \eta) \)) such that
\[
y = \nabla_x \phi(x, \eta), \quad \xi = \nabla_x \phi(x, \eta).
\]
(1.14)

Proof. Set \( F(x) = F(x; y, \eta) = -\nabla_x \phi(x, \eta) + x + y \). We have
\[
|F(x') - F(x)| \leq \int_0^1 \|\nabla_x \nabla_x \phi(x + \theta(x' - x), \eta)\|d\theta \|x' - x\| \leq \tau \|x' - x\|,
\]
where \( I \) is a unit matrix and for a matrix \( A = (a_{ij}; \quad i, j \rightarrow 1, \ldots, n) \) the norm \( \|A\| \) is defined by \( \sum_{i,j} |a_{ij}|^2 \). Then, we can apply the fixed point theorem, and \( x = x(y, \eta) \) satisfying \( y = \nabla_x \phi(x, \eta) \) is determined as the fixed point. Then, \( \xi(y, \eta) \) is determined by \( \nabla_x \phi(x(y, \eta), \eta) \).

Similarly, \( (y(x, \xi), \xi(x, \xi)) \) is determined. Q.E.D.

**Lemma 1.8.** Let \( (q, p)(t, s; y, \eta)(0 \leq s \leq t \leq T) \) be the bicharacteristic strip defined by (1.13) and \( \phi(t, s; x, \xi)(0 \leq s \leq t \leq T_0) \) be the solution of the eiconal equation (1.10). Then, it follows that
\[
y = \nabla_x \phi(t, s; q(t, s), \eta), \quad p(t, s) = \nabla_x \phi(t, s; q(t, s), \eta)
\]
(1.15)

(0 \leq s \leq t \leq T_0).

Proof. By Lemma 1.7 we can define \( (q', p')(t, s; y, \eta)(0 \leq s \leq t \leq T_0) \) by
\[
y = \nabla_x \phi(t, s; q'(t, s), \eta), \quad p'(t, s) = \nabla_x \phi(t, s; q'(t, s), \eta).
\]
(1.16)
Differentiate both sides of (1.16) in $t$, respectively. Then, using (1.10) we get

$$
\frac{dq'(t, s)}{dt} = \nabla_t \lambda(t, q'(t, s), p'(t, s)),
$$

$$
\frac{dp'(t, s)}{dt} = -\nabla_s \lambda(t, q'(t, s), p'(t, s)).
$$

Since $q'(s, s) = y$ and $p'(s, s) = \eta$ from (1.16), we can see that $q'(t, s) = q(t, s)$ and $p'(t, s) = p(t, s) \ (0 \leq s \leq t \leq T_0)$. Q.E.D.

Take $\lambda_j \ (j = 1, \ldots, \nu + 1)$ as $\lambda$ of (1.9) and define $\Phi = \Phi_{1, \ldots, \nu + 1}(t_0, \ldots, t_{\nu + 1}; \ x, \ \xi) \ (0 \leq t_{\nu + 1} \leq \cdots \leq t_0 \leq T \leq T_0)$ by (1.11) corresponding to $\{\lambda_{j}\}_{j=1}^{\nu+1}$. For a set $\{t_0, \ldots, t_{\nu + 1}\} \subset [0, T_0]$ such that $t_0 \geq t_1 \geq \cdots \geq t_{\nu + 1}$ (resp. $t_0 \leq t_1 \leq \cdots \leq t_{\nu + 1}$) we define a trajectory $(Q, P)(\sigma) = (Q, P, P_{1, \ldots, \nu + 1})(\sigma; t_0, \ldots, t_{\nu + 1}; y, \eta)$ in $t_0 \geq \sigma \geq t_{\nu + 1}$ (resp. $t_0 \leq \sigma \leq t_{\nu + 1}$) as follows: $(Q, P)(\sigma)$ is continuous functions on $[t_{\nu + 1}, t_0]$ (resp. $[t_0, t_{\nu + 1}]$) such that $(Q, P)(t_{\nu + 1}) = (y, \eta)$ and for $\sigma \in (t_0, t_{\nu + 1})$ (resp. $\sigma \in (t_{\nu + 1}, t_0)$) $(Q, P)(\sigma)$ satisfy

$$
\frac{dQ}{d\sigma} = \nabla_t \lambda_4(\sigma, Q, P), \quad \frac{dP}{d\sigma} = -\nabla_s \lambda_4(\sigma, Q, P).
$$

Then, we obtain

**Proposition 1.9.** Let $T \geq T_0 \geq t_0 \geq \cdots \geq t_{\nu + 1} \geq 0$. Using Lemma 1.7, for any point $(y, \eta)$ take a point $x$ satisfying

$$
y = \nabla_t \Phi_{1, \ldots, \nu + 1}(t_0, \ldots, t_{\nu + 1}; x, \eta).
$$

Then, we have

$$
(Q_{1, \ldots, \nu + 1}, P_{1, \ldots, \nu + 1})(t_0, \ldots, t_{\nu + 1}; y, \eta) = (X_k, \Xi_k)(t_0, \ldots, t_{\nu + 1}; x, \eta) \quad (k = 0, \ldots, \nu + 1),
$$

where $\{X_k, \Xi_k\}_{k=1}^{\nu+1}$ is the solution of (1.12) corresponding to $\Phi = \Phi_{1, \ldots, \nu + 1}$ and

$$
\begin{cases}
X_0^v = x, & \Xi_0^v = \nabla_s \Phi_{1, \ldots, \nu + 1}(t_0, \ldots, t_{\nu + 1}; x, \eta), \\
X_{\nu + 1}^v = y, & \Xi_{\nu + 1}^v = \eta.
\end{cases}
$$

**Proof.** Relation (1.7) in Proposition 1.4 shows that

$$
\begin{aligned}
\nabla_t \Phi(t_0, \ldots, t_{\nu + 1}; x, \eta) &= \nabla_t \phi_{\nu + 1}(t_0, t_{\nu + 1}; X_s^v, \eta), \\
\nabla_s \Phi(t_0, \ldots, t_{\nu + 1}; x, \eta) &= \nabla_s \phi_1(t_0, t_1; x, \Xi_k^v).
\end{aligned}
$$

Together with (1.12) and (1.18) we get

$$
\begin{aligned}
X_k^v &= \nabla_t \phi_k(t_{k-1}, t_k; X_k^{k-1}, \Xi_k^v), \\
\Xi_k^{k-1} &= \nabla_s \phi_k(t_{k-1}, t_k; X_k^{k-1}, \Xi_k^v), \quad k = 1, \ldots, \nu + 1.
\end{aligned}
$$
Now when \( k = v + 1 \), (1.19) is valid. From the definition of \((Q, P)(\sigma) = (Q_{i_1, \ldots, i_{v+1}}, P_{i_1, \ldots, i_{v+1}})(\sigma)\) and by Lemma 1.8 we have

\[
\begin{align*}
\begin{cases}
y = \nabla t \phi_{v+1}(t_\nu, t_{v+1}; Q(t_\nu), \eta), \\
P(t_\nu) = \nabla t \phi_{v+1}(t_\nu, t_{v+1}; Q(t_\nu), \eta).
\end{cases}
\end{align*}
\]

Compare the above relation with \( X^*_\nu \) and \( \Xi^*_\nu \) of (1.21). Setting \( X^{v+1}_\nu = y, \Xi^{v+1}_\nu = \eta \), we get by Lemma 1.7

\[ Q(t_\nu) = X^*_\nu, \quad P(t_\nu) = \Xi^*_\nu. \]

In a similar way we can prove (1.19), inductively. Q.E.D.

2. Contraction of multi-phase functions

Let \( \lambda_j(t, x, \xi) \in B^\infty([0, T]; S^i) \) (\( j = 1, 2 \)) and be real valued functions. Throughout this section we assume that

\((*)\) \[ \{\tau + \lambda_i, \{\tau + \lambda_j, \tau + \lambda_k\}\}(t, x, \xi) = 0 \quad \text{on} \ [0, T] \times \mathbb{R}^{2n}_{x, \xi} \]

\( (i, j, k = 1, 2) \),

where for \( f, g \in C^i(R^{2n}_{x, \xi}) \) \( \{f, g\}(t, x; \tau, \xi) \) denotes the Poisson bracket

\[ \{f, g\}(t, x; \tau, \xi) = (\partial_\tau f \partial_t g - \partial_t f \partial_\tau g + \nabla_\tau f \cdot \nabla_\xi g - \nabla_t f \cdot \nabla_\xi g)(t, x; \tau, \xi). \]

Let \( \phi_j(t, s; x, \xi) \) (\( j = 1, 2 \), \( 0 \leq s \leq t \leq T_0 \)) be the solutions of the eiconal equation (1.10) corresponding to \( \lambda_j \) and define \( \Phi = \Phi_{j_1, \ldots, j_{v+1}}(t_0, \ldots, t_{v+1}) \in \mathcal{P}(c_0(t_0 - t_{v+1})) \) \( (0 \leq t_{v+1} \leq \cdots \leq t_0 \leq T_0, j_k = 1, 2) \) by \( \Phi = \phi_{j_1}(t_0, t_1)\# \cdots \# \phi_{j_{v+1}}(t_v, t_{v+1}) \), where \( c_0 > 0 \) and \( T_0 > 0 \) are constants independent of \( v \) (see Proposition 1.4 and Proposition 1.5). We fix such a \( T_0 \) in what follows. It is easy to see that

**Lemma 2.1.** Let \( H(t, x, \xi) \in C^i(R^{2n+1}) \) and \( (q, p)(t) = (q, p)(t, s; y, \eta) \) \( (0 \leq s \leq t \leq T_0) \) be the bicharacteristic strip defined by (1.13) for \( \tau + \lambda(t, x, \xi) \) of (1.9). Then, we have

\[ \frac{d}{d\sigma} H(\sigma, q(\sigma), p(\sigma)) = -\{H, \tau + \lambda\}(\sigma, q(\sigma), p(\sigma)) \quad (s \leq \sigma \leq T_0). \]

**Lemma 2.2.** For \( J = (j_i, \ldots, j_{v+1}) \) \( (j_k = 1, 2) \) and a set \( \{t_0, \ldots, t_{v+1}\} \) \( (T \geq t_0 \geq \cdots \geq t_{v+1} \geq 0) \) let \( (Q, P)(\sigma) = (Q_{j_1, \ldots, j_{v+1}}, P_{j_1, \ldots, j_{v+1}})(\sigma; t_0, \ldots, t_{v+1}; y, \eta) \) be the solution of (1.17) corresponding to \( \{\lambda_{j_1, \ldots, j_{v+1}}\}^1_{k=1} \). Set

\[ v(\sigma) = (\lambda_2 - \lambda_1)(\sigma, Q(\sigma), P(\sigma)) \quad (t_{v+1} \leq \sigma \leq t_0). \]

Then, we get

\[ \frac{d}{d\sigma} v(\sigma) = \{\tau + \lambda_1, \tau + \lambda_2\}(\sigma, Q(\sigma), P(\sigma)) \quad (t_{v+1} \leq \sigma \leq t_0). \]
Proof. For \( \sigma \in (t_h, t_{h-1}) \) it follows from Lemma 2.1 that
\[
\frac{d}{d\sigma} v(\sigma) = -\{\lambda_2, \tau + \lambda_j\} + \{\lambda_1, \tau + \lambda_{j_k}\}
\]
\[
= -\{\tau + \lambda_2, \tau + \lambda_{j_k}\} + \{\tau + \lambda_1, \tau + \lambda_{j_k}\}.
\]

Then, we get (2.4) in both cases \( j_h = 1 \) and 2. Q.E.D.

Lemma 2.3. Assume that the assumption \((*)\) holds. Then, for \( v(\sigma) \) defined by (2.3) we get
\[
v(\sigma) = a(\sigma + b) \quad (t_{v+1} \leq \sigma \leq t_0),
\]
where \( a = \{\tau + \lambda_1, \tau + \lambda_2\}(t_{v+1}, y, \eta) \) and \( b = (\lambda_2 - \lambda_1)(t_{v+1}, y, \eta) - at_{v+1} \).

Proof. We can see from Lemma 2.2 that \( v(\sigma) \) belongs to \( C^1([t_{v+1}, t_0]) \). From (2.4) and Lemma 2.1 it follows that
\[
\frac{d^2}{d\sigma^2} v(\sigma) = -\{\tau + \lambda_1, \tau + \lambda_2\}, \tau + \lambda_{j_k}\} = 0 \quad (t_h < \sigma < t_{h-1}).
\]
Hence, we get (2.5). Q.E.D.

Remark 2.1. If the assumption \((*)\) is satisfied, \( v(\sigma) \) defined by (2.3) depends only on \( \sigma, t_{v+1}, y \) and \( \eta \), and is independent of the choice of \( J = (j_1, \cdots, j_{v+1}) \) \((v = 1, 2 \cdots)\) and \( \{t_0, \cdots, t_s\} \).

Theorem 2.4. Assume that the assumption \((*)\) holds. For \( \{t, t_1, t_2, s\} \)
\((0 \leq s < t_3 < t_1 < t \leq T_0)\) we define functions \( (\psi_1, \psi_2)(t, t_1, t_2, s) \) by
\[
\begin{align*}
\psi_1(t, t_1, t_2, s) &= \frac{(t_1 - t_2)(t_2 - s)}{t - t_1 + t_2 - s}, \\
\psi_2(t, t_1, t_2, s) &= t_1 - t_2 + s - \frac{(t_1 - t_2)(t_2 - s)}{t - t_1 + t_2 - s}.
\end{align*}
\]
Then, we obtain
\[
\Phi_{1,2,1}(t, \psi_1, \psi_2, s; x, \xi) = \Phi_{2,1,2}(t, t_1, t_2, s; x, \xi).
\]

Proof. We shall determine \( \psi_j(t, t_1, t_2, s) \) \((j = 1, 2)\) of (2.6) as the functions satisfying (2.7). From Proposition 1.6 we get \( \Phi_{2,1,2}(t, t_1, t_2, s; x, \xi) \) as the solution of
\[
\begin{align*}
\partial_t \Phi_{2,1,2} + \lambda_2(t, x, \nabla_x \Phi_{2,1,2}) &= 0, \\
\Phi_{2,1,2}|_{t=t_1} &= \Phi_{1,2}(t_1, t_2, s; x, \xi).
\end{align*}
\]
So, we have only to determine \( \psi_j \) \((j = 1, 2)\) depending only on \( t, t_1, t_2 \) and \( s \) such that for \( \Phi_{1,2,1}(t, t_1, t_2, s) = \Phi_{1,2,1}(t, t_1, t_2, s; x, \xi) \).
\begin{equation}
\begin{aligned}
\frac{\partial (\Phi_{1,2,1}(t, \psi_1, \psi_2, s))}{\partial t} + \lambda_2(t, x, \nabla_s \Phi_{1,2,1}(t, \psi_1, \psi_2, s)) &= 0, \\
\Phi_{1,2,1}(t, \psi_1, \psi_2, s) |_{t-t_1} = \Phi_{1,2}(t_1, t_2, s; x, \xi)
\end{aligned}
\end{equation}

holds.

Suppose that for \( \psi_j \) \((j=1, 2)\) (2.7) holds. Set \( \Delta=(t, \psi_1, \psi_2, s; x, \xi) \) and \( \psi_j'=\partial_t \psi_j \) \((j=1, 2)\). Then, from (2.8) and Proposition 1.6 we have

\begin{equation}
0 = (\partial_t \Phi_{1,2,1})(\Delta)+(\partial_t \Phi_{1,2,1})(\Delta)\psi_1' + \lambda_2(t, x, \nabla_s \Phi_{1,2,1}(\Delta))
\end{equation}

\begin{equation}
= (\lambda_2-\lambda_1)(t, x, \nabla_s \Phi_{1,2,1}(\Delta))-
\end{equation}

\begin{equation}
(\lambda_2-\lambda_1)(\psi_1, X^1(\Delta), \Xi^1(\Delta))\psi_1' + (\lambda_2-\lambda_1)(\psi_2, X^2(\Delta), \Xi^2(\Delta))\psi_2',
\end{equation}

where \( \{X^1, \Xi^1\}_{t-1} \) is the solution of

\begin{equation}
x^k = \nabla_\xi \Phi_{j_k}(t_k, \xi^k), \quad x^k = \nabla_\xi \Phi_{j_k+1}(t_k, \xi^k+1)
\end{equation}

\begin{equation}
(k = 1, 2, x^0 = x, \xi^0 = \xi, j_1 = 1, j_2 = 2, j_3 = 1).
\end{equation}

Take a point \( y \) such that

\begin{equation}
y = \nabla_\xi \Phi_{1,2,1}(\Delta) = \nabla_\xi \Phi_{1,2,1}(t, \psi_1, \psi_2, s; x, \xi).
\end{equation}

Let \( (Q, P)(\sigma)=(Q_{1,2,1}, Q_{1,2,1})(\sigma; t, \psi_1, \psi_2, s; y, \xi) \) be the solution of (1.17) and set

\begin{equation}
v(\sigma) = (\lambda_2-\lambda_1)(\sigma, Q(\sigma), P(\sigma)).
\end{equation}

Then, by Proposition 1.9 we can write (2.9) in the form

\begin{equation}
0 = \nu(t)-\nu(\psi_1)\psi_1'+\nu(\psi_2)\psi_2'.
\end{equation}

Take account of the assumption (*). Since from Lemma 2.3 \( v(\sigma) \) has the form \( a\sigma + b \), we get

\begin{equation}
0 = (at+b)-(a\psi_1+b)\psi_1'+(a\psi_2+b)\psi_2'
\end{equation}

\begin{equation}
= -a(\psi_1'\psi_2' - t) - b(\psi_1' - \psi_2' - 1).
\end{equation}

Now we take \( \psi_j \) such that \( \psi_j \) satisfy

\begin{equation}
\psi_1' - \psi_2' = 1, \quad \psi_1\psi_2' - \psi_2\psi_1' = t.
\end{equation}

If \( \psi_1 |_{t-t_1} = t_2 \) and \( \psi_2 |_{t-t_1} = s \), the second equality of (2.8) is also satisfied by Proposition 1.6. Hence, we obtain

\begin{equation}
\psi_1 - \psi_2 = t - t_1 + t_2 - s, \quad \psi_1^2 - \psi_2^2 = t^2 - t_1^2 + t_2^2 - s^2.
\end{equation}

Solving (2.11), we get the functions of (2.6) satisfying (2.7). Q.E.D.

**Remark 2.2.** For real constants \( a_j \) and \( b_j \), \( \lambda_1 = -\sum_{i=1}^s a_i \xi_i \) and \( \lambda_2 = -2t \sum_{i=1}^s b_i \xi_i \) on \( R^2_{i,\xi} \) satisfy the assumption (*). Then, we have
From these multi-phase functions we see that \( \psi_j \) \((j=1, 2)\) of (2.6) are uniquely determined functions which satisfy (2.7) for any \( a_j \) and \( b_j \).

**REMARK 2.3.** Set \( \Delta_2 = \{(t_1, t_2); 0 \leq s < t_2 < t_1 < \xi \leq T_0 \} \). Consider the mapping \( M: \Delta_2 \to (\psi_1, \psi_2) \) with \((t, s)\) as a parameter. It is clear that the image of the mapping \( M \) is included in \( \Delta_2 \). Since from (2.11)

\[
 t_1 - t_2 = t - \psi_1 + \psi_2 - s, \quad t_1' - t_2' = t' - \psi_1' + \psi_2' - s',
\]

\( M^2 = I \) (identity map) holds. This implies that the mapping \( M: \Delta_2 \to \Delta_2 \) is one to one and onto. Make the change of variables with \((t, s)\) as a parameter

\[
 t_1 = \psi_1(t, s), \quad t_2 = \psi_2(t, s).
\]

Then, we get

\[
 \int_{t_0}^{t_1} \int_{s_0}^{s_1} \exp \{i\Phi_{1,2,1}(t, s; x, \xi)\} dt_1 ds_1 = \int_{t_0'}^{t_1'} \int_{s_0'}^{s_1'} \exp \{i\Phi_{1,2,1}(t, s; x, \xi)\} \frac{t_1' - t_2'}{t_1' + t_2' - s'} dt_2' ds_2'.
\]

We note that the functions \( \psi_1, \psi_2 \) and \((t_1-t_0)/(t_1-t_0+s)\) have singular points \((t_1-t, t_2-s)\). So it seems that it is not easy to construct the fundamental solution by using Fourier integral operators with a finite number of phase functions, if we only follow the method in [10], [11], [15] and [17].

Let \( (Q_{j_1, \ldots, j_{v+1}}, P_{j_1, \ldots, j_{v+1}})(\sigma; t_0, \ldots, t_{v+1}; y, \eta) \) be the solution of (1.17) corresponding to \( \{\lambda_{j_k}\}_{k=1}^v \) and a set \( \{t_0, \ldots, t_{v+1}\} \subset [0, T_0] \).

**Corollary 2.5.** Assume that (\( \star \)) holds. Then, for any \( v \geq 2 \), \( \{j_1, \ldots, j_{v+1}\} \) \((j_k=1, 2, j_k+j_{k+1})\) and \( \{t_0, \ldots, t_{v+1}\} \) \((T_0 \geq t_0 > \cdots > t_{v+1} \geq 0)\) we get

\[
 \Phi_{j_1, \ldots, j_{v+1}}(t_0, \ldots, t_{v+1}; x, \xi) = \Phi_{1,2,1}(t_0, t_1', t_2', x, \xi).
\]

for some \( t_1' \) \((j=1, 2)\) independent of \( x \) and \( \xi \). By using the same \( t_1' \) \((j=1, 2)\) we also get

\[
 (Q_{j_1, \ldots, j_{v+1}}, P_{j_1, \ldots, j_{v+1}})(t_0, \ldots, t_{v+1}; y, \eta) = (Q_{1,2,1}, P_{1,2,1})(t_0, t_1', t_2', t_{v+1}; y, \eta)
\]

for any point \((y, \eta) \in \mathbb{R}^2\).
Proof. We can get (2.12) by Proposition 1.6 and Theorem 2.4, inductively. Then, we obtain (2.13) by using (2.12) and Proposition 1.9. Q.E.D.

Remark 2.4. For $\lambda_j(t, x, \xi) (j = 1, 2)$ in Remark 2.2 we have
\[
\begin{align*}
\Phi_1(t, s, \xi) &= \sum_{i=1}^{n} a_i(t-s)\xi_i + x \cdot \xi, \\
\Phi_2(t, s, \xi) &= \sum_{i=1}^{n} b_i(t^2 - s^2)\xi_i + x \cdot \xi, \\
\Phi_1,2(t, t_1, s) &= \sum_{i=1}^{n} \{a_i(t-t_1) + b_i(t^2 - s^2)\} \xi_i + x \cdot \xi, \\
\Phi_2,1(t, t_1, s) &= \sum_{i=1}^{n} \{a_i(t_1-s) + b_i(t^2 - t_1^2)\} \xi_i + x \cdot \xi.
\end{align*}
\]
Comparing (2.14) with $\Phi_{1,2,1}$ and $\Phi_{2,1,2}$ in Remark 2.2, we can see that we can generally contract $\Phi_{1,2,1}(t, t_1, t_2, s)$ and $\Phi_{2,1,2}(t, t_1, t_2, s) (t > t_1 > t_2 > s)$ no more. Furthermore, from Proposition 1.9 we can also see that we can generally contract $(Q_{1,2,1}, P_{1,2,1})(t, t_1, t_2, s)$ and $(Q_{2,1,2}, P_{2,1,2})(t, t_1, t_2, s) (t > t_1 > t_2 > s)$ no more.

Examples. We give examples of $\lambda_k(t, x, \xi) (k = 1, 2)$ satisfying (*) on $[0, T] \times \mathbb{R}_x^3$ except $\lambda_k$ in Remark 2.2 below. They are not involutive, since $\{\tau + \lambda_1, \tau + \lambda_2\}(t, x, \xi)$ does not identically vanish on a set $\{(t, x, \xi); \lambda_1(t, x, \xi) = \lambda_2(t, x, \xi)\}$.

1. $\lambda_1(t, x, \xi) = \xi_1, \lambda_2(t, x, \xi) = x_2\xi_1 + \xi_3$.
2. $\lambda_1(t, x, \xi) = x_1\xi_1, \lambda_2(t, x, \xi) = t\xi_2$.
3. $\lambda_1(t, x, \xi) = x_2\xi_1 + \xi_3, \lambda_2(t, x, \xi) = -x_1\xi_1 + \xi_2$.

3. Propagation of singularities

Consider a hyperbolic system with diagonal principal part
\[
L = D_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(t, X, D_x) + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(t, X, D_x)
\]
on $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ with \((\lambda_j(t, x, \xi) \in \mathcal{B}^\infty([0, T]; \mathcal{S})\), real valued, $b_{jk}(t, x, \xi) \in \mathcal{B}^\infty([0, T]; \mathcal{S})$).

We assume that for a constant $M > 0$ we have
\[
\lambda_j(t, x, \xi) = \delta \lambda_j(t, x, \xi) \quad (|\xi| \geq M, \delta \geq 1).
\]
We also assume that (*) of Section 2 holds.

We study the Cauchy problem
\[
\begin{align*}
LU(t, x) &= 0 \quad \text{on } [0, T], \\
U|_{t=0} &= G(x),
\end{align*}
\]
where \( U(t, x) = (u_1(t, x), u_2(t, x)) \) and \( G(x) = (g_1(x), g_2(x))(g_3(x) \in H_\infty = \bigcup H_s) \).

Let \( \phi_j(t, s; x, \xi) \) \( 0 \leq s \leq t \leq T \) be the solutions of the eiconal equations (1.10) corresponding to \( \lambda_j \) and define \( \Phi = \Phi_{\lambda_1, \ldots, \lambda_{v+1}}(t_0, \ldots, t_{v+1}) \) \( j = 1, 2 \) by \( \Phi = \Phi_{\lambda_1}(t_0, t_1) \# \cdots \# \Phi_{\lambda_{v+1}}(t_v, t_{v+1}) \) (see (1.11)).

If we apply Theorem 3.1 in Kumano-go-Taniguchi [11] to \( L \) of (3.1), then, for a small \( T_0 \) \( 0 < T_0 \leq T \) we can get the fundamental solution \( E(t, s) \) \( 0 \leq s \leq t \leq T_0 \) of \( L \) (i.e. \( LE(t, s) = 0 \) on \( [0, T_0] \)) and \( E(s, s) = I \) (unit matrix), which is represented by means of Fourier integral operators with multi-phase functions \( \Phi_{\lambda_1, \ldots, \lambda_{v+1}} \) \( (v = 0, 1, \ldots) \). We fix such a \( T_0 \) in what follows. We will apply the theory in [11] for the propagation of singularities of solutions (Theorem 3.4 in [11]) to the Cauchy problem (3.3).

For \( \lambda_{j_1}, \ldots, \lambda_{j_{v+1}}, (y, \eta) \) and a fixed \( 0 < \epsilon < 1 \) we define an \( \epsilon \)-station chain \( \{t_i, \ldots, t_v\} \) as the point \( t > t_i > \cdots > t_v > 0 \) such that for \( k = 1, \ldots, v \)

\[
|\lambda_{j_k}(t_k, x^k, \xi^k) - \lambda_{j_{k+1}}(t_k, x^k, \xi^k)| \leq \epsilon \xi^k
\]

\[\text{at} (x^k, \xi^k) = (Q_{j_1, \ldots, j_{v+1}, \nu_j}, \lambda_{j_1, \ldots, j_{v+1}}(t_k; t, t_1, \ldots, t_v, 0; y, \eta)),\]

where \( (Q_{j_1, \ldots, j_{v+1}}, \lambda_{j_1, \ldots, j_{v+1}})(t_0, \ldots, t_v, 0; y, \eta) \) is the solution of (1.17) corresponding to \( \{\lambda_{j_1}, \ldots, \lambda_{j_{v+1}}\} \) \( \{t_0, \ldots, t_{v+1}\} \) \( (t_0 = t, t_{v+1} = 0) \). Define the \( \epsilon \)-station set \( \Lambda_{\epsilon, \lambda_1, \ldots, \lambda_{v+1}}(t; y, \eta) \) by the set of all \( \epsilon \)-station chains \( \{t_1, \ldots, t_v\} \).

We set \( WF(G) = \bigcup_{j=1}^g WF(g_j) \) for the wave front set \( WF(g_j) \) of \( g_j \). For \( J = (j_1, \ldots, j_{v+1}) \) we set

\[
\Lambda^\epsilon_J(t; y, \eta) = \{(Q_{j_1, \ldots, j_{v+1}}, \lambda_{j_1, \ldots, j_{v+1}})(t; t, t_1, \ldots, t_v, 0; y, \eta); \}
\{t_1, \ldots, t_v\} \in \Lambda_{\epsilon, \lambda_1, \ldots, \lambda_{v+1}}(t; y, \eta)\}
\]

and set

\[
\Gamma_{t, \epsilon} = \{\delta \Lambda^\epsilon_J(t; y, \eta); (y, \eta) \in WF_\delta(G), J = (j_1, \ldots, j_{v+1}) \}
\]

\[
(\delta J = 1, 2, \delta > 1, \delta > 0, |\eta| \geq M_0)
\]

\[
(\delta J = 1, 2, \delta > 1, \delta > 0, |\eta| \geq M_0, J = (1, 2), (1, 2), (2, 1), (1, 2, 1))
\]

for a large constant \( M_0 > 0 \) depending on \( M \) of (3.2). Then, Theorem 3.4 in [11] says without the assumption \( (*) \)

**Theorem 3.1.**  \( \bigcap_{\epsilon < 1} \Gamma_{t, \epsilon} \) is closed and for the solution \( U(t, x) \) of the Cauchy problem (3.3) we have

\[
WF(U(t)) \subset \bigcap_{\epsilon < 1} \Gamma_{t, \epsilon} \quad (0 \leq t \leq T_0).
\]

If we add the assumption \( (*) \), then, setting

\[
\Gamma_{t, 0} = \{\delta \Lambda^\epsilon_J(t; y, \eta); (y, \eta) \in WF(G), \delta > 0, |\eta| \geq M_0, J = (1, 2), (1, 2), (2, 1), (1, 2, 1)\}
\]
we get the main theorem.

**Theorem 3.2.** Assume that the assumption (*) holds. Then, for the solution \( U(t, x) \) of the Cauchy problem (3.3) we get

\[(3.9) \quad \text{WF}(U(t)) \subseteq \Gamma_{t,0} (0 \leq t \leq T_0).\]

Proof. By Theorem 3.1 we have only to prove that

\[(3.10) \quad \bigcap_{0 < t < T_0} \Gamma_{t,\epsilon} = \Gamma_{t,0}.\]

It is easy to see that \( \bigcap_{0 < t < T_0} \Gamma_{t,\epsilon} \supseteq \Gamma_{t,0} \). So, we prove that

\[(3.11) \quad \bigcap_{0 < t < T_0} \Gamma_{t,\epsilon} \subseteq \Gamma_{t,0}.\]

We fix \( 0 < t \leq T_0 \) and take a point \((x^0, \xi^0) \in \bigcap_{0 < t < T_0} \Gamma_{t,\epsilon} \) and fix it. If we take \(|\xi^0| \) sufficiently large, then, setting \( \xi^k = P_{j_{k+1}, \ldots, j_{k+1}}(t_k, 0, t_k, \ldots, 0, x^0, \xi^0) \) \((k = 1, \ldots, \nu + 1, t_{\nu + 1} = 0)\), we have

\[(3.12) \quad C^{-1} \leq |\xi^k| \leq C \quad (k = 0, \ldots, \nu + 1).\]

Here, the positive constant \( C \) is independent of the choice of \( J = (j_1, \ldots, j_{\nu + 1}) \) and a set \( \{t_0, \ldots, t_{\nu + 1}\} \subset [0, t] \). Since \((x^0, \xi^0)\) belongs to \( \bigcap_{0 < t < T_0} \Gamma_{t,\epsilon} \), for any \( \epsilon_n = 2^{-n} \) there exist \( J_m^m = (j_m^m, \ldots, j_m^m) \) \((j_m^m = 1, 2, j_m^m = j_m^{m+1})\), \((y_m^m, \eta_m^m) \in \text{WF}(G) \) and \( \{\eta_m^m, \ldots, \eta_m^m\} \subseteq L_{t_m^m, j_m^m, \ldots, j_m^{m+1}}(y_m^m, \eta_m^m) \) such that

\[(3.13) \quad (x^0, \xi^0) = (Q_{1,1}, \ldots, Q_{m,1}, P_{1,1}, \ldots, P_{m,1})(t; t, t, \ldots, t, x^0, \xi^0).\]

We consider \((x^0, \xi^0)\) dividing into two cases as follows.

I) The case where we can take a subsequence \( l = \{m_n\}_{n=1}^\infty \) and a point \( \sigma_1 (0 \leq \sigma_1 \leq t) \) such that \( t^l_1 \rightarrow \sigma_1 \) and \( t^l_1 \rightarrow \sigma_1 \) as \( l \rightarrow \infty \).

II) The other case.

I). We show that \((x^0, \xi^0)\) belongs to \( \Gamma_{t,0} \), when \( 0 < \sigma_1 < t \). In the other case \( \sigma_1 = 0 \) or \( t \) we can also prove this by the similar way. By the assumption I) we can also take a subsequence \( \gamma = \{l_n\}_{n=1}^\infty \) of \( l = \{m_n\}_{n=1}^\infty \) such that

\[(j_1^l, j_2^{l_1+1}) = (1, 1), (1, 2), (2, 1) \text{ or } (2, 2).\]

We may assume that \( j_1^l = 1 \) and \( j_2^{l_1+1} = 2 \), since we can prove similarly in the other cases. Now, take a point \((\bar{x}^0, \bar{\eta}^0) \) \(||\bar{\eta}^0|| \geq C^{-1}, \text{ see (3.11)}\) such that

\[(3.14) \quad (\bar{x}^0, \bar{\eta}^0) = (Q_{2,1}, P_{1,1})(0; t, \sigma_1, t, x^0, \xi^0).\]

We note that

\[(3.14)' \quad (x^0, \xi^0) = (Q_{1,2}, P_{1,2})(t; t, \sigma_1, 0; \bar{x}^0, \bar{\eta}^0).\]
Then, it is easy to see that
\begin{equation}
\tilde{y}^0 = x^0 + \int_{t_1}^{t_1} \nabla_t \lambda_1(\sigma, Q_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0), P_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0)) d\sigma + \int_{t_1}^0 \nabla_t \lambda_2(\sigma, Q_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0), P_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0)) d\sigma.
\end{equation}

Using the assumption of this case, for any small $\delta > 0$ there exists $N$ such that for any $\gamma \geq N$ we have
\begin{equation}
\{t_1^\gamma, \ldots, t_N^\gamma\} \subset [\sigma_1 - \delta, \sigma_1 + \delta].
\end{equation}

Since for any $y^\gamma$ we have the similar equality to (3.14), we get
\begin{equation}
|y^\gamma - y^\gamma| \leq C_1 \delta \quad (\gamma \geq N)
\end{equation}
for a constant $C_1 > 0$ independent of $\delta$ and $\gamma$. By the similar way we get
\begin{equation}
|\eta^\gamma - \eta^\gamma| \leq C_1 \delta \quad (\gamma \geq N).
\end{equation}

Consequently, we can see that $(y^\gamma, \eta^\gamma) \to (y^0, \eta^0)$ as $\gamma \to \infty$ and
\begin{equation}
(y^0, \eta^0) \in WF(G).
\end{equation}

Next, since $\{t_1^\gamma, \ldots, t_N^\gamma\} \in \Lambda_{t_1^\gamma, \ldots, t_N^\gamma, t_{\gamma+1}^\gamma}(y^\gamma, \eta^\gamma)$, it follows from (3.11) and (3.12) that
\begin{equation}
|(\lambda_2 - \lambda_1)(t_1^\gamma, Q_1(t_1^\gamma; \eta^\gamma); t; x^\gamma, \xi^0), P_1(t_1^\gamma; \eta^\gamma); t; x^\gamma, \xi^0))| \leq C \varepsilon_\gamma
\end{equation}
for a constant $C$ of (3.11). Here, noting that $j_1^\gamma = 1$ and $j_{\gamma+1}^\gamma = 2$, we used
\begin{equation}
(Q_1, P_1)(t_1^\gamma; t_1^\gamma, t; x^\gamma, \xi^0)
\end{equation}

When $\gamma \to \infty$, we get from (3.13)
\begin{equation}
0 = (\lambda_2 - \lambda_1)(\sigma_1, Q_1(\sigma_1; \sigma_1; t; x^0, \xi^0), P_1(\sigma_1; \sigma_1; t; x^0, \xi^0))
= (\lambda_2 - \lambda_1)(\sigma_1, Q_1(\sigma_1; \sigma_1; t, \sigma_1, 0; y^0, \eta^0), P_1(\sigma_1; \sigma_1, 0; y^0, \eta^0)).
\end{equation}

Together with (3.13)' and (3.16) this implies that
\begin{equation}
(x^0, \xi^0) \in \{\Lambda_0^{(1,2)}(t; y, \eta); (y, \eta) \in WF(G)\}
\end{equation}
\[\subset \Gamma_{t_0}.
\]

II). We can take a subsequence $l=\{m_\mu\}_{\mu=1}^\infty$ and points $\sigma_1, \sigma_2 (0 \leq \sigma_2 < \sigma_1 \leq t)$ such that $t_1^l \to \sigma_1$ and $t_{\gamma}^l \to \sigma_2$ as $l \to \infty$. We set
\begin{equation}
v(\sigma; l) = (\lambda_2 - \lambda_1)(\sigma; Q_1(\sigma; \sigma_1; t; t_1^l, \sigma, 0; y^l, \eta^l), P_1(\sigma; \sigma_1; t; t_1^l, \sigma, 0; y^l, \eta^l)) \quad (0 \leq \sigma \leq t).
\end{equation}
For large $l$ we have
\[ t_1' - t_i' \geq \frac{1}{2} (\sigma_1 - \sigma_2) > 0, \]
and then, noting that \( t_i', \ldots, t_{i+1}' \in \Lambda_{t_i, t_{i+1}}(y', \eta') \), we have by (3.11)
\[ \|v(t_i' ; l)\|, \|v(t_i' ; l)\| \leq C \epsilon_1. \]
Consequently, since \( v(\sigma ; l) \) of (3.17) has the form
\[ v(\sigma ; l) = a\sigma + b \quad (0 \leq \sigma \leq t) \]
from Lemma 2.3 in Section 2, it follows that
\[ |v(\sigma ; l)| \leq 2C \epsilon_1 T_0 (t_1' - t_i'), \]
\[ \leq 4C \epsilon_1 T_0 (\sigma_1 - \sigma_2) \quad (0 \leq \sigma \leq t). \]

Now, by Corollary 2.5 there exist some \( t_{i_1}, t_{i_2} \) \((t > t_{i_2} > t_{i_1} > 0)\) such that
\[ (x^0, \xi^0) = (Q_{1,2,1}, P_{1,2,1})(t; t, t_{i_1}, t_{i_2}, 0; y', \eta'). \]
Then, by Lemma 2.3 and Remark 2.1, from (3.19) we obtain
\[ v_i(\sigma; l) = (\lambda_2 - \lambda_1)(\sigma; Q_{1,2,1}(\sigma; t, t_{i_1}, t_{i_2}, 0; y', \eta'), P_{1,2,1}(\sigma; t, t_{i_1}, t_{i_2}, 0; y', \eta')). \]
Since \( v_i(\sigma; l) = v(\sigma; l) \) by Lemma 2.3 and Remark 2.1, from (3.19) we obtain
\[ |v_i(\sigma; l)| \leq \frac{4C}{\sigma_1 - \sigma_2} \epsilon_1 T_0. \]

Next, let \( \sigma_i (i = 1, 2, \sigma_1 \geq \sigma_2) \) be the accumulating points of sets \( \{i_i\}_{i \in \mathbb{N}} \), respectively and take some subsequence \( \{\gamma = l_\nu\}_{\nu \in \mathbb{N}} \) such that \( t_l \to \sigma_1 \) and \( t_{i_2} \to \sigma_2 \) as \( \gamma \to \infty \). Then, it follows from (3.20)' that there exists \( (\tilde{y}^0, \tilde{\eta}^0) \) such that
\[ (y^\gamma, \eta^\gamma) \to (\tilde{y}^0, \tilde{\eta}^0) = (Q_{1,2,1}, P_{1,2,1})(0; 0, \sigma_2, \sigma_1, t; x^0, \xi^0) \]
as \( \gamma \to \infty \), and
\[ (\tilde{y}^0, \tilde{\eta}^0) \in \text{WF}(G). \]
We note that
\[ (x^0, \xi^0) = (Q_{1,2,1}, P_{1,2,1})(t; t, \sigma_1, \sigma_2, 0; \tilde{y}^0, \tilde{\eta}^0). \]
By using (3.22) we obtain

\[(\lambda_1 - \lambda_2)(\sigma, Q_{1,2,1}(\sigma; t, \sigma_1, \sigma_2, 0; \bar{\xi}, \bar{\eta}), P_{1,2,1}(\sigma; t, \sigma_1, \sigma_2, 0; \bar{\xi}, \bar{\eta})) = \lim_{\gamma \to \infty} \gamma_v(\sigma; \gamma)
= 0 \quad (0 \leq \sigma \leq t).\]

This implies with (3.23) and (3.24) that

\[(x^0, \xi) \in \Gamma_{t,0}\]

which means (3.9) together with the result of 1). Q.E.D.

References


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