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PROPAGATION OF SINGULARITIES FOR A HYPERBOLIC SYSTEM WITH DOUBLE CHARACTERISTICS

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0. Introduction

Consider the Cauchy problem for a hyperbolic operator

$$(0.1)\qquad L=D_t+\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}(t, X, D_x)+\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}(t, X, D_x)\qquad \text{on } [0, T]\times R^n,$$

where D_t denotes $-\sqrt{-1}\partial_t$, functions $\lambda_i(t, x, \xi)$ are real valued and belong to $B^\infty([0, T]; S^1)$ and $b_{jk}(t, x, \xi)$ belong to $B^\infty([0, T]; S^0)$. Throughout this paper we assume that

$$(0.2)\qquad \{\tau+\lambda_i, \{\tau+\lambda_j, \tau+\lambda_k\}\}(t, x, \xi)=0 \qquad \text{on } [0, T]\times R^{2n}_{x,\xi}, \quad (i, j, k=1, 2)$$

where for $f, g\in C^1(R^{2(n+1)}_{t,x,\tau,\xi})$ $\{f, g\}(t, x; \tau, \xi)$ denotes the Poisson bracket: $(\partial_\tau f\partial_t g-\partial_t f\partial_\tau g+\nabla_\xi f\cdot\nabla_x g-\nabla_x f\cdot\nabla_\xi g)(t, x; \tau, \xi)$.

Recently, using Fourier integral operators with multi-phase functions, Kumano-go -Taniguchi-Tozaki in [10] and Kumano-go -Taniguchi in [11] constructed the fundamental solution for a hyperbolic system with diagonal principal part (Theorem 3.1 in [11]). In these papers the propagation of singularities of solutions was investigated by using an infinite number of phase functions (Theorem 3.4 in [11] or Theorem 3.1 in the present paper).

In the present paper we prove that the propagation of singularities can be described by means of five phase functions ϕ_1, ϕ_2, $\phi_1\#\phi_2$, $\phi_2\#\phi_1$ and $\phi_1\#\phi_2\#\phi_1$, when the assumption (0.2) is satisfied (Theorem 3.2). We note that the characteristic roots satisfying (0.2) are not necessarily involutive. For examples, $\lambda_1=-t\xi$ and $\lambda_2=t\xi$ for $n=1$ satisfy (0.2), but

$$\{\tau+\lambda_1, \tau+\lambda_2\}(=2\xi)\neq 0 \qquad (\xi\neq 0).$$

Other examples will be given in Section 2.

The propagation of singularities of solutions has been investigated by many authors [1], [2], [3], [4], [6], [8], [12], [13], [14], [15], [16], [17], [18], [19] etc.. In particular, in [2], [6], [14], [15], [16], [17], [19] operators with involutive characteristics are treated. Alinhac in [1] and Taniguchi-Tozaki in [18] give the precise descriptions for singularities of solutions for operators on R^1_x with principal part $\partial_t^2 - t^{2l}\partial_x^2$ (l is a positive integer) which are not involutive.

In Section 1 we exhibit main results on the theory of Fourier integral operators in [10] and [11] needed later. In Section 2 under the assumption (0.2) we contract the multi-product $\Phi_{j_1,\cdots,j_{\nu+1}}(t_0, \cdots, t_{\nu+1}; x, \xi)$ ($j_k = 1, 2$) of phase functions $\phi_{j_k}(t, s; x, \xi)$ ($j_k=1, 2$) (see (1.11)), which are the solutions of the eiconal equations for $\tau + \lambda_{j_k}(t, x, \xi)$ (see (1.10)) (Theorem 2.4). In Section 3 we prove the main theorem (Theorem 3.2).

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1. Fourier integral operators

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ of non-negative integers α_j and points $x = (x_1, \cdots, x_n) \in R^n$, $y = (y_1, \cdots, y_n) \in R^n$ we use the usual notation:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,\ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n},\ \partial_{x_j} = \frac{\partial}{\partial x_j},$$
$$D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n},\ D_{x_j} = -\sqrt{-1}\,\partial_{x_j},\ \nabla_x = (\partial_{x_1}, \cdots, \partial_{x_n}),$$
$$\langle x \rangle = (1 + |x|^2)^{1/2},\ x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

For $f(x) = (f_1, \cdots, f_n)$ ($f_j(x) \in C^1(R^n)$) we denote

$$\partial_x f = \nabla_x f = (\partial_{x_k} f_j;\ \overset{j\downarrow}{\underset{k\rightarrow}{}} 1, \cdots, n).$$

Let $\mathcal{S}$ on R^n denote the Schwartz space of rapidly decreasing functions and let $\mathcal{S}'$ denote the dual space of $\mathcal{S}$. For $u \in \mathcal{S}_x$ the Fourier transform $\hat{u}(\xi) = F[u](\xi)$ is defined by

$$F[u](\xi) = \int e^{-ix\cdot\xi} u(x) dx,$$

and then, for $\hat{u}(\xi) \in \mathcal{S}_\xi$ the inverse Fourier transform $\bar{F}[\hat{u}](x)$ is defined by

$$\bar{F}[\hat{u}](x) = \int e^{ix\cdot\xi} \hat{u}(\xi) \bar{d}\xi,\ \bar{d}\xi = (2\pi)^{-n} d\xi.$$

For real s we define the Sobolev space H_s as the completion of $\mathcal{S}$ in the norm $\|u\|_s = \{\int \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 \bar{d}\xi\}^{1/2}$.

DEFINITION 1.1. We say that a C^∞-function $p(x, \xi)$ in $R^{2n} = R^n_x \times R^n_\xi$ belongs to *the class* S^m ($-\infty < m < \infty$), when

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}\,, \tag{1.1}$$

where $p_{(\beta)}^{(\alpha)}(x,\xi)=\partial_\xi^\alpha D_x^\beta p(x,\xi)$.

The class S^m makes a Fréchet space with semi-norms

$$|p|_l^{(m)}=\max_{|\alpha+\beta|\leqq l}\sup_{x,\xi}\{|p_{(\beta)}^{(\alpha)}(x,\xi)|/\langle\xi\rangle^{m-|\alpha|}\}\qquad (l=0,1,2,\cdots).$$

We set $S^{-\infty}=\bigcap_{-\infty<m<\infty}S^m$ and $S^\infty=\bigcup_{-\infty<m<\infty}S^m$.

The pseudo-differential operator $p(X,D_x)\in S^m$ with symbol $p(x,\xi)\in S^m$ is defined by

$$\begin{aligned}p(X,D_x)u&=0_s\text{-}\iint_{R^{2n}}e^{i(x-x')\cdot\xi}p(x,\xi)u(x')dx'd\xi\\&=\lim_{\varepsilon\to 0}\iint_{R^{2n}}e^{i(x-x')\cdot\xi}\chi(\varepsilon x',\varepsilon\xi)p(x,\xi)u(x')dx'd\xi\,,\end{aligned}\tag{1.2}$$

where $\chi(x,\xi)\in\mathcal{S}(R^{2n})$ such that $\chi(0,0)=1$ (c.f. [7]).

Now we state definitions and theorems in Kumano-go-Taniguchi-Tozaki [10] and Kumano-go-Taniguchi [11] without proofs (see also [5]).

DEFINITION 1.2. If $0\leqq\tau<1$, we denote by $\mathcal{P}(\tau)$ the set of real valued C^∞-functions $\phi(x,\xi)$ in R^{2n} such that $J(x,\xi)=\phi(x,\xi)-x\cdot\xi$ belongs to S^1 and

$$\sum_{|\alpha+\beta|\leqq 2}\sup_{x,\xi}\{|J_{(\beta)}^{(\alpha)}(x,\xi)/\langle\xi\rangle^{1-|\alpha|}|\}\leqq\tau\,. \tag{1.3}$$

REMARK 1.1. In [10] $\mathcal{P}(\tau)$ denoted the class of C^2-functions. The above definition is due to [11].

We define the Fourier integral operator $p_\phi(X,D_x)$ with symbol $p(x,\xi)\in S^m$ and phase function $\phi(x,\xi)\in\mathcal{P}(\tau)$ by

$$p_\phi(X,D_x)u(x)=\int_{R^n}e^{i\phi(x,\xi)}p(x,\xi)\hat{u}(\xi)d\xi\,,\qquad u\in\mathcal{S}\,. \tag{1.4}$$

DEFINITION 1.3. Let $\phi_j\in\mathcal{P}(\tau_j)$, $j=1,\cdots,\nu+1,\cdots$, $\bar{\tau}_\infty\equiv\sum_{j=1}^\infty\tau_j\leqq\tau_0$ for a sufficiently small fixed τ_0 with $0<\tau_0\leqq 1/8$. We define *the multi-product* $\Phi_{\nu+1}(x,\xi)=(\phi_1\#\cdots\#\phi_{\nu+1})(x,\xi)$ of phase functions $\phi_j(x,\xi)$ $(j=1,\cdots,\nu+1)$ by

$$\Phi_{\nu+1}(x^0,\xi^{\nu+1})=\sum_{j=1}^{\nu}(\phi_j(X_\nu^{j-1},\Xi_\nu^j)-X_\nu^j\cdot\Xi_\nu^j)+\phi_{\nu+1}(X_\nu^\nu,\xi^{\nu+1})\qquad (X_\nu^0=x^0)\,, \tag{1.5}$$

where $\{X_\nu^j,\Xi_\nu^j\}_{j=1}^{\nu}(x^0,\xi^{\nu+1})$ is defined as the solution of the equation

$$\begin{cases}x^j=\nabla_\xi\phi_j(x^{j-1},\xi^j)\,,\\ \xi^j=\nabla_x\phi_{j+1}(x^j,\xi^{j+1})\,,\qquad j=1,\cdots,\nu\,.\end{cases}\tag{1.6}$$

Proposition 1.4 (Theorem 1.8 and Theorem 1.9 in [10]). *Let* $\phi_j \in \mathcal{P}(\tau_j)$, $j=1, \cdots, \nu+1, \cdots$, $\tau_\infty \leqq \tau_0 \leqq 1/8$. *Then,* $\Phi_{\nu+1}(x, \xi)$ *of* (1.5) *is well defined and belongs to* $\mathcal{P}(c_0 \bar{\tau}_{\nu+1})$, $\bar{\tau}_{\nu+1} = \tau_1 + \cdots + \tau_{\nu+1}$, *with a constant* $c_0 > 0$ *independent of* ν *and* τ_0. *We also get*

$$(1.7) \qquad \begin{cases} \nabla_x \Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \nabla_x \phi_1(x^0, \Xi_\nu^1(x^0, \xi^{\nu+1})), \\ \nabla_\xi \Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \nabla_\xi \phi_{\nu+1}(X_\nu^\nu(x^0, \xi^{\nu+1}), \xi^{\nu+1}), \end{cases}$$

$$(1.8) \qquad \phi_1 \# \phi_2 \# \phi_3 = (\phi_1 \# \phi_2) \# \phi_3 = (\phi_1 \# \phi_2 \# \phi_3).$$

Consider a hyperbolic equation

$$(1.9) \qquad (D_t + \lambda(t, X, D_x))u = 0 \qquad \text{on } [0, T]$$
$$(\lambda(t, x, \xi) \in B^\infty([0, T]; S^1), \text{ real valued}).$$

Let $\phi = \phi(t, s) = \phi(t, s; x, \xi)$ be the solution of the eiconal equation

$$(1.10) \qquad \begin{cases} \partial_t \phi + \lambda(t, x, \nabla_x \phi) = 0 \qquad \text{on } [0, T], \\ \phi|_{t=s} = x \cdot \xi. \end{cases}$$

Then, we have

Proposition 1.5 (Theorem 3.1 in [9]). *For a small* T_0 $(0 < T_0 \leqq T)$ *we get* $\phi(t, s) \in \mathcal{P}(c(t-s))$ $(0 \leqq s \leqq t \leqq T_0)$ *with a constant* $c > 0$.

We fix such a T_0 in what follows. Take λ_j $(j=1, \cdots, \nu+1, \cdots)$ as λ of (1.9) such that $\{\lambda_j\}_{j=1}^\infty$ is bounded in $B^\infty([0, T]; S^1)$ and let ϕ_j be the solutions of (1.10) corresponding to λ_j. We define $\Phi = \Phi_{1,2,\cdots,\nu+1}(t_0, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})$ $(0 \leqq t_{\nu+1} \leqq \cdots \leqq t_0 \leqq T_0 \leqq T)$ by

$$(1.11) \qquad \Phi(t_0, \cdots, t_{\nu+1}) = \phi_1(t_0, t_1) \# \cdots \# \phi_{\nu+1}(t_\nu, t_{\nu+1}),$$

and define $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(t_0, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})$ as the solution of

$$(1.12) \qquad \begin{cases} x^j = \nabla_\xi \phi_j(t_{j-1}, t_j; x^{j-1}, \xi^j), \\ \xi^j = \nabla_x \phi_{j+1}(t_j, t_{j+1}; x^j, \xi^{j+1}), \qquad j = 1, \cdots, \nu, \end{cases}$$

where $T_0 > 0$ is a constant independent of ν in Proposition 1.4 and Proposition 1.5. Then, we have

Proposition 1.6 (Theorem 2.3 in [10]). $\Phi(t_0, \cdots, t_{\nu+1})$ *of* (1.11) *satisfies*

1°.
$$\partial_{t_j} \Phi = \lambda_j(t_j, X_\nu^j, \Xi_\nu^j) - \lambda_{j+1}(t_j, X_\nu^j, \Xi_\nu^j)$$
$$(j = 0, \cdots, \nu+1, \ \lambda_0 = \lambda_{\nu+2} = 0, \ X_\nu^0 = x^0, \ \Xi_\nu^0 = \nabla_{x^0} \Phi,$$
$$X_\nu^{\nu+1} = \nabla_{\xi^{\nu+1}} \Phi, \ \Xi_\nu^{\nu+1} = \xi^{\nu+1}).$$

2°. *If* $t_j = t_{j+1}$ *for some* j, *we have*

$$\Phi_{1,2,\cdots,\nu+1}(t_0, \cdots, t_j, t_{j+1}, \cdots, t_{\nu+1})$$
$$= \Phi_{1,2,\cdots,j,j+2,\cdots,\nu+1}(t_0, \cdots, t_j, t_{j+2}, \cdots, t_{\nu+1}).$$

3^0. *If* $\lambda_j=\lambda_{j+1}$ *for some* j, *we have*

$$\Phi_{1,2,\cdots,\nu+1}(t_0, \cdots, t_{\nu+1})$$
$$= \Phi_{1,2,\cdots,j-1,j+1,\cdots,\nu+1}(t_0, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{\nu+1}).$$

Now let $(q, p)(t, s; y, \eta)=((q_1, \cdots, q_n), (p_1, \cdots, p_n))(t, s; y, \eta)$ $(0\leqq s\leqq t\leqq T)$ be the bicharacteristic strip for (1.9), that is, (q, p) (t, s) is the solution of

$$\left\{\begin{aligned} \frac{dq}{dt} &= \nabla_\xi \lambda(t, q, p), \\ \frac{dp}{dt} &= -\nabla_x \lambda(t, q, p), \quad (q, p)|_{t=s} = (y, \eta). \end{aligned}\right. \tag{1.13}$$

Then, we can solve (1.13) in full interval $s\leqq t\leqq T$ by the Gronwall inequality, since $|\nabla_\xi\lambda(t, q, p)|\leqq C_1$ and $|\nabla_x\lambda(t, q, p)|\leqq C_1\langle p\rangle$ $(0\leqq t\leqq T)$ for a constant $C_1>0$. We state propositions on the bicharacteristic strips.

Lemma 1.7. *Let* $\phi(x, \xi)\in\mathscr{P}(\tau)$. *Then, for any* $y, \eta\in R^{2n}$ (*resp.* (x, ξ)) *there exists a point* $(x, \xi)\in R^{2n}$ (*resp.* (y, η)) *such that*

$$y = \nabla_\xi\phi(x, \eta), \ \xi = \nabla_x\phi(x, \eta). \tag{1.14}$$

Proof. Set $F(x)=F(x; y, \eta)=-\nabla_\xi\phi(x, \eta)+x+y$. We have

$$|F(x')-F(x)|\leqq\int_0^1||\nabla_x\nabla_\xi\phi(x+\theta(x'-x), \eta)-I||d\theta\,|x'-x|\leqq\tau|x'-x|,$$

where I is a unit matrix and for a matrix $A=(a_{ij};\ \overset{i\downarrow}{\underset{j\rightarrow}{}}1, \cdots, n)$ the norm $||A||$ is defined by $\{\sum_{i,j}|a_{ij}|^2\}^{1/2}$. Then, we can apply the fixed point theorem, and $x=x(y, \eta)$ satisfying $y=\nabla_\xi\phi(x, \eta)$ is determined as the fixed point. Then, $\xi(y, \eta)$ is determined by $\nabla_x\phi(x(y, \eta), \eta)$.

Similarly, $(y(x, \xi), \eta(x, \xi))$ is determined. Q.E.D.

Lemma 1.8. *Let* (q, p) $(t, s; y, \eta)$ $(0\leqq s\leqq t\leqq T)$ *be the bicharacteristic strip defined by* (1.13) *and* $\phi(t, s; x, \xi)$ $(0\leqq s\leqq t\leqq T_0)$ *be the solution of the eiconal equation* (1.10). *Then, it follows that*

$$y = \nabla_\xi\phi(t, s; q(t, s), \eta), \quad p(t, s) = \nabla_x\phi(t, s; q(t, s), \eta) \quad (0\leqq s\leqq t\leqq T_0). \tag{1.15}$$

Proof. By Lemma 1.7 we can define (q', p') $(t, s; y, \eta)$ $(0\leqq s\leqq t\leqq T_0)$ by

$$y = \nabla_\xi\phi(t, s; q'(t, s), \eta), p'(t, s) = \nabla_x\phi(t, s; q'(t, s), \eta). \tag{1.16}$$

Differentiate both sides of (1.16) in t, respectively. Then, using (1.10) we get

$$\begin{cases} \dfrac{dq'}{dt}(t, s) = \nabla_\xi \lambda(t, q'(t, s), p'(t, s)), \\ \dfrac{dp'}{dt}(t, s) = -\nabla_x \lambda(t, q'(t, s), p'(t, s)). \end{cases}$$

Since $q'(s, s)=y$ and $p'(s, s)=\eta$ from (1.16), we can see that $q'(t, s)=q(t, s)$ and $p'(t, s)=p(t, s)$ $(0\leqq s\leqq t\leqq T_0)$. Q.E.D.

Take λ_j $(j=1, \cdots, \nu+1)$ as λ of (1.9) and define $\Phi=\Phi_{1,\cdots,\nu+1}(t_0, \cdots, t_{\nu+1}; x, \xi)$ $(0\leqq t_{\nu+1}\leqq\cdots\leqq t_0\leqq T_0\leqq T)$ by (1.11) corresponding to $\{\lambda_j\}_{j=1}^{\nu+1}$. For a set $\{t_0', \cdots, t_{\nu+1}'\}\subset[0, T_0]$ such that $t_0'\geqq t_1'\geqq\cdots\geqq t_{\nu+1}'$ (resp. $t_0'\leqq t_1'\leqq\cdots\leqq t_{\nu+1}'$) we define a trajectory $(Q, P)(\sigma)=(Q_{1,\cdots,\nu+1}, P_{1,\cdots,\nu+1})(\sigma; t_0', \cdots, t_{\nu+1}'; y, \eta)$ in $t_0'\geqq\sigma\geqq t_{\nu+1}'$ (resp. $t_0'\leqq\sigma\leqq t_{\nu+1}'$) as follows: $(Q, P)(\sigma)$ are continuous functions on $[t_{\nu+1}', t_0']$ (resp. $[t_0', t_{\nu+1}']$) such that $(Q, P)(t_{\nu+1}')=(y, \eta)$ and for $\sigma\in(t_k', t_{k-1}')$ (resp. $\sigma\in(t_{k-1}', t_k')$) $(Q, P)(\sigma)$ satisfy

$$\frac{dQ}{d\sigma} = \nabla_\xi \lambda_k(\sigma, Q, P), \quad \frac{dP}{d\sigma} = -\nabla_x \lambda_k(\sigma, Q, P). \tag{1.17}$$

Then, we obtain

Proposition 1.9. *Let $T\geqq T_0\geqq t_0\geqq\cdots\geqq t_{\nu+1}\geqq 0$. Using Lemma 1.7, for any point (y, η) take a point x satisfying*

$$y = \nabla_\xi \Phi_{1,\cdots,\nu+1}(t_0, \cdots, t_{\nu+1}; x, \eta). \tag{1.18}$$

Then, we have

$$\begin{aligned} &(Q_{1,\cdots,\nu+1}, P_{1,\cdots,\nu+1})(t_k; t_0, \cdots, t_{\nu+1}; y, \eta) \\ &= (X_\nu^k, \Xi_\nu^k)(t_0, \cdots, t_{\nu+1}; x, \eta) \qquad (k = 0, \cdots, \nu+1), \end{aligned} \tag{1.19}$$

where $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^{\nu}$ is the solution of (1.12) corresponding to $\Phi=\Phi_{1,\cdots,\nu+1}$ and

$$\begin{cases} X_\nu^0 = x, \ \Xi_\nu^0 = \nabla_x \Phi_{1,\cdots,\nu+1}(t_0, \cdots, t_{\nu+1}; x, \eta), \\ X_\nu^{\nu+1} = y, \ \Xi_\nu^{\nu+1} = \eta. \end{cases} \tag{1.20}$$

Proof. Relation (1.7) in Proposition 1.4 shows that

$$\begin{cases} \nabla_\xi \Phi(t_0, \cdots, t_{\nu+1}; x, \eta) = \nabla_\xi \phi_{\nu+1}(t_\nu, t_{\nu+1}; X_\nu^\nu, \eta), \\ \nabla_x \Phi(t_0, \cdots, t_{\nu+1}; x, \eta) = \nabla_x \phi_1(t_0, t_1; x, \Xi_\nu^1). \end{cases}$$

Together with (1.12) and (1.18) we get

$$\begin{cases} X_\nu^k = \nabla_\xi \phi_k(t_{k-1}, t_k; X_\nu^{k-1}, \Xi_\nu^k), \\ \Xi_\nu^{k-1} = \nabla_x \phi_k(t_{k-1}, t_k; X_\nu^{k-1}, \Xi_\nu^k), \qquad k = 1, \cdots, \nu+1. \end{cases} \tag{1.21}$$

Now when $k=\nu+1$, (1.19) is valid. From the definition of $(Q, P)(\sigma)=(Q_{1,\cdots,\nu+1}, P_{1,\cdots,\nu+1})(\sigma)$ and by Lemma 1.8 we have

$$\begin{cases} y = \nabla_\xi \phi_{\nu+1}(t_\nu, t_{\nu+1}; Q(t_\nu), \eta), \\ P(t_\nu) = \nabla_x \phi_{\nu+1}(t_\nu, t_{\nu+1}; Q(t_\nu), \eta). \end{cases}$$

Compare the above relation with X^ν_ν and Ξ^ν_ν of (1.21). Setting $X^{\nu+1}_\nu=y$, $\Xi^{\nu+1}_\nu=\eta$, we get by Lemma 1.7

$$Q(t_\nu) = X^\nu_\nu, \quad P(t_\nu) = \Xi^\nu_\nu.$$

In a similar way we can prove (1.19), inductively. Q.E.D.

2. Contraction of multi-phase functions

Let $\lambda_j(t, x, \xi)\in B^\infty([0, T]; S^1)$ $(j=1, 2)$ and be real valued functions. Throughout this section we assume that

$$(*) \qquad \{\tau+\lambda_i, \{\tau+\lambda_j, \tau+\lambda_k\}\}(t, x, \xi) = 0 \quad \text{on } [0, T]\times R^{2n}_{x,\xi} \quad (i, j, k = 1, 2),$$

where for $f, g\in C^1(R^{2(n+1)}_{t,x,\tau,\xi})$ $\{f, g\}(t, x; \tau, \xi)$ denotes the Poisson bracket

$$(2.1) \quad \{f, g\}(t, x; \tau, \xi) = (\partial_\tau f\partial_t g-\partial_t f\partial_\tau g+\nabla_\xi f\cdot\nabla_x g-\nabla_x f\cdot\nabla_\xi g)(t, x; \tau, \xi).$$

Let $\phi_j(t, s; x, \xi)$ $(j=1, 2, 0\leqq s\leqq t\leqq T_0)$ be the solutions of the eiconal equation (1.10) corresponding to λ_j and define $\Phi=\Phi_{j_1,\cdots,j_{\nu+1}}(t_0, \cdots, t_{\nu+1})\in \mathcal{P}(c_0(t_0-t_{\nu+1}))$ $(0\leqq t_{\nu+1}\leqq\cdots\leqq t_0\leqq T_0, j_k=1, 2)$ by $\Phi=\phi_{j_1}(t_0, t_1)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, t_{\nu+1})$, where $c_0>0$ and $T_0>0$ are constants independent of ν (see Proposition 1.4 and Proposition 1.5). We fix such a T_0 in what follows. It is easy to see that

Lemma 2.1. *Let $H(t, x, \xi)\in C^1(R^{2n+1})$ and $(q, p)(t)=(q, p)(t, s; y, \eta)$ $(0\leqq s\leqq t\leqq T_0)$ be the bicharacteristic strip defined by* (1.13) *for $\tau+\lambda(t, x, \xi)$ of* (1.9). *Then, we have*

$$(2.2) \quad \frac{d}{d\sigma}H(\sigma, q(\sigma), p(\sigma)) = -\{H, \tau+\lambda\}(\sigma, q(\sigma), p(\sigma)) \qquad (s\leqq\sigma\leqq T_0).$$

Lemma 2.2. *For $J=(j_1, \cdots, j_{\nu+1})$ $(j_k=1, 2)$ and a set $\{t_0, \cdots, t_{\nu+1}\}$ $(T\geqq t_0\geqq\cdots\geqq t_{\nu+1}\geqq 0)$ let $(Q, P)(\sigma)=(Q_{j_1,\cdots,j_{\nu+1}}, P_{j_1,\cdots,j_{\nu+1}})(\sigma; t_0, \cdots, t_{\nu+1}; y, \eta)$ be the solution of* (1.17) *corresponding to $\{\lambda_{j_k}\}_{k=1}^{\nu+1}$. Set*

$$(2.3) \quad v(\sigma) = (\lambda_2-\lambda_1)(\sigma, Q(\sigma), P(\sigma)) \qquad (t_{\nu+1}\leqq\sigma\leqq t_0).$$

Then, we get

$$(2.4) \quad \frac{d}{d\sigma}v(\sigma) = \{\tau+\lambda_1, \tau+\lambda_2\}(\sigma, Q(\sigma), P(\sigma)) \qquad (t_{\nu+1}\leqq\sigma\leqq t_0).$$

Proof. For $\sigma\in(t_k, t_{k-1})$ it follows from Lemma 2.1 that

$$\begin{aligned}\frac{d}{d\sigma}v(\sigma) &= -\{\lambda_2, \tau+\lambda_{j_k}\}+\{\lambda_1, \tau+\lambda_{j_k}\}\\ &= -\{\tau+\lambda_2, \tau+\lambda_{j_k}\}+\{\tau+\lambda_1, \tau+\lambda_{j_k}\}\,.\end{aligned}$$

Then, we get (2.4) in both cases $j_k=1$ and 2. Q.E.D.

Lemma 2.3. *Assume that the assumption* (*) *holds. Then, for $v(\sigma)$ defined by* (2.3) *we get*

$$v(\sigma) = a\sigma+b \qquad (t_{\nu+1}\leqq\sigma\leqq t_0)\,, \tag{2.5}$$

where $a=\{\tau+\lambda_1, \tau+\lambda_2\}(t_{\nu+1}, y, \eta)$ *and* $b=(\lambda_2-\lambda_1)(t_{\nu+1}, y, \eta)-at_{\nu+1}$.

Proof. We can see from Lemma 2.2 that $v(\sigma)$ belongs to $C^1([t_{\nu+1}, t_0])$. From (2.4) and Lemma 2.1 it follows that

$$\frac{d^2}{d\sigma^2}v(\sigma) = -\{\{\tau+\lambda_1, \tau+\lambda_2\}, \tau+\lambda_{j_k}\} = 0 \qquad (t_k<\sigma<t_{k-1})\,.$$

Hence, we get (2.5). Q.E.D.

Remark 2.1. If the assumption (*) is satisfied, $v(\sigma)$ defined by (2.3) depends only on σ, $t_{\nu+1}$, y and η, and is independent of the choice of $\boldsymbol{J}=(j_1, \cdots, j_{\nu+1})$ $(\nu=1, 2 \cdots)$ and $\{t_0, \cdots, t_\nu\}$.

Theorem 2.4. *Assume that the assumption* (*) *holds. For* $\{t, t_1, t_2, s\}$ $(0\leqq s<t_2<t_1<t\leqq T_0)$ *we define functions* (ψ_1, ψ_2) (t, t_1, t_2, s) *by*

$$\begin{cases}\psi_1(t, t_1, t_2, s) = t-\dfrac{(t_1-t_2)(t_2-s)}{t-t_1+t_2-s}\,,\\[2ex] \psi_2(t, t_1, t_2, s) = t_1-t_2+s-\dfrac{(t_1-t_2)(t_2-s)}{t-t_1+t_2-s}\,.\end{cases} \tag{2.6}$$

Then, we obtain

$$\Phi_{1,2,1}(t, \psi_1, \psi_2, s; x, \xi) = \Phi_{2,1,2}(t, t_1, t_2, s; x, \xi)\,. \tag{2.7}$$

Proof. We shall determine $\psi_j(t, t_1, t_2, s)$ $(j=1, 2)$ of (2.6) as the functions satisfying (2.7). From Proposition 1.6 we get $\Phi_{2,1,2}(t, t_1, t_2, s; x, \xi)$ as the solution of

$$\begin{cases}\partial_t\Phi_{2,1,2}+\lambda_2(t, x, \nabla_x\Phi_{2,1,2}) = 0\,,\\ \Phi_{2,1,2}|_{t=t_1} = \Phi_{1,2}(t_1, t_2, s; x, \xi)\,.\end{cases}$$

So, we have only to determine ψ_j $(j=1, 2)$ depending only on t, t_1, t_2 and s such that for $\Phi_{1,2,1}(t, t_1, t_2, s)=\Phi_{1,2,1}(t, t_1, t_2, s; x, \xi)$

$$
(2.8) \qquad \begin{cases} \partial_t(\Phi_{1,2,1}(t, \psi_1, \psi_2, s)) + \lambda_2(t, x, \nabla_x \Phi_{1,2,1}(t, \psi_1, \psi_2, s)) = 0, \\ \Phi_{1,2,1}(t, \psi_1, \psi_2, s)|_{t=t_1} = \Phi_{1,2}(t_1, t_2, s; x, \xi) \end{cases}
$$

holds.

Suppose that for ψ_j $(j=1, 2)$ (2.7) holds. Set $\Delta=(t, \psi_1, \psi_2, s; x, \xi)$ and $\psi_j'=\partial_t\psi_j$ $(j=1, 2)$. Then, from (2.8) and Proposition 1.6 we have

$$
\begin{aligned}
(2.9) \quad 0 &= (\partial_t\Phi_{1,2,1})(\Delta) + (\partial_{t_1}\Phi_{1,2,1})(\Delta)\psi_1' + \\
&\qquad (\partial_{t_2}\Phi_{1,2,1})(\Delta)\psi_2' + \lambda_2(t, x, \nabla_x\Phi_{1,2,1}(\Delta)) \\
&= (\lambda_2-\lambda_1)(t, x, \nabla_x\Phi_{1,2,1}(\Delta)) - \\
&\qquad (\lambda_2-\lambda_1)(\psi_1, X_2^1(\Delta), \Xi_2^1(\Delta))\psi_1' + (\lambda_2-\lambda_1)(\psi_2, X_2^2(\Delta), \Xi_2^2(\Delta))\psi_2',
\end{aligned}
$$

where $\{X_2^i, \Xi_2^i\}_{i=1}^2(t_0, t_1, t_2, t_3; x, \xi)$ is the solution of

$$
\begin{gathered}
x^k = \nabla_\xi \phi_{j_k}(t_{k-1}, t_k; x^{k-1}, \xi^k), \ \xi^k = \nabla_x \phi_{j_{k+1}}(t_k, t_{k+1}; x^k, \xi^{k+1}) \\
(k = 1, 2, \ x^0 = x, \ \xi^3 = \xi, \ j_1 = 1, \ j_2 = 2, \ j_3 = 1).
\end{gathered}
$$

Take a point y such that

$$
y = \nabla_\xi\Phi_{1,2,1}(\Delta) = \nabla_\xi\Phi_{1,2,1}(t, \psi_1, \psi_2, s; x, \xi).
$$

Let $(Q, P)(\sigma)=(Q_{1,2,1}, P_{1,2,1})(\sigma; t, \psi_1, \psi_2, s; y, \xi)$ be the solution of (1.17) and set

$$
v(\sigma) = (\lambda_2-\lambda_1)(\sigma, Q(\sigma), P(\sigma)).
$$

Then, by Proposition 1.9 we can write (2.9) in the form

$$
(2.9)' \qquad 0 = v(t) - v(\psi_1)\psi_1' + v(\psi_2)\psi_2'.
$$

Take account of the assumption $(*)$. Since from Lemma 2.3 $v(\sigma)$ has the form $a\sigma+b$, we get

$$
\begin{aligned}
(2.9)'' \qquad 0 &= (at+b) - (a\psi_1+b)\psi_1' + (a\psi_2+b)\psi_2' \\
&= -a(\psi_1\psi_1' - \psi_2\psi_2' - t) - b(\psi_1' - \psi_2' - 1).
\end{aligned}
$$

Now we take ψ_j such that ψ_j satisfy

$$
(2.10) \qquad \psi_1' - \psi_2' = 1, \quad \psi_1\psi_1' - \psi_2\psi_2' = t.
$$

If $\psi_1|_{t=t_1}=t_2$ and $\psi_2|_{t=t_1}=s$, the second equality of (2.8) is also satisfied by Proposition 1.6. Hence, we obtain

$$
(2.11) \qquad \psi_1 - \psi_2 = t - t_1 + t_2 - s, \quad \psi_1^2 - \psi_2^2 = t^2 - t_1^2 + t_2^2 - s^2.
$$

Solving (2.11), we get the functions of (2.6) satisfying (2.7). Q.E.D.

REMARK 2.2. For real constants a_j and b_j $\lambda_1=-\sum_{i=1}^n a_i\xi_i$ and $\lambda_2=-2t\sum_{i=1}^n b_i\xi_i$ on $R_{x,\xi}^{2n}$ satisfy the assumption $(*)$. Then, we have

$$\begin{cases} \Phi_{1,2,1}(t, t_1, t_2, s; x, \xi) = \sum_{i=1}^{n} \{a_i(t-t_1+t_2-s)+b_i(t_1^2-t_2^2)\}\xi_i + x\cdot\xi , \\ \Phi_{2,1,2}(t, t_1, t_2, s; x, \xi) = \sum_{i=1}^{n} \{a_i(t_1-t_2)+b_i(t^2-t_1^2+t_2^2-s^2)\}\xi_i + x\cdot\xi . \end{cases}$$

From these multi-phase functions we see that ψ_j $(j=1, 2)$ of (2.6) are uniquely determined functions which satisfy (2.7) for any a_j and b_j.

REMARK 2.3. Set $\Delta_2=\{(t_1, t_2); 0\leqq s<t_2<t_1<t\leqq T_0\}$. Consider the mapping M: $\Delta_2\ni(t_1, t_2)\to(\psi_1, \psi_2)$ with (t, s) as a parameter. It is clear that the image of the mapping M is included in Δ_2. Since from (2.11)

$$t_1-t_2 = t-\psi_1+\psi_2-s,\ t_1^2-t_2^2 = t^2-\psi_1^2+\psi_2^2-s^2 ,$$

$M^2=I$ (identity map) holds. This implies that the mapping M: $\Delta_2\to\Delta_2$ is one to one and onto. Make the change of variables with (t, s) as a parameter

$$t_1' = \psi_1(t, t_1, t_2, s), \quad t_2' = \psi_2(t, t_1, t_2, s) .$$

Then, we get

$$\begin{aligned} &\int_s^t\int_s^{t_1} \exp\{i\Phi_{2,1,2}(t, t_1, t_2, s; x, \xi)\}\, dt_2 dt_1 \\ &= \int_s^t\int_s^{t_1'} \exp\{i\Phi_{1,2,1}(t, t_1', t_2', s; x, \xi)\} \frac{t_1'-t_2'}{t-t_1'+t_2'-s}\, dt_2' dt_1' . \end{aligned}$$

We note that the functions ψ_1, ψ_2 and $(t_1-t_2)/(t-t_1+t_2-s)$ have singular points $(t_1=t, t_2=s)$. So it seems that it is not easy to construct the fundamental solution by using Fourier integral operators with a finite number of phase functions, if we only follow the method in **[10]**, **[11]**, **[15]** and **[17]**.

Let $(Q_{j_1,\cdots,j_{\nu+1}}, P_{j_1,\cdots,j_{\nu+1}})(\sigma; t_0, \cdots, t_{\nu+1}; y, \eta)$ be the solution of (1.17) corresponding to $\{\lambda_{j_k}\}_{k=1}^{\nu+1}$ and a set $\{t_0, \cdots, t_{\nu+1}\}\subset[0, T_0]$.

Corollary 2.5. *Assume that* $(*)$ *holds. Then, for any* ν $(\geqq 2)$, $\{j_1, \cdots, j_{\nu+1}\}$ $(j_k=1, 2, j_k\neq j_{k+1})$ *and* $\{t_0, \cdots, t_{\nu+1}\}$ $(T_0\geqq t_0>\cdots>t_{\nu+1}\geqq 0)$ *we get*

$$\begin{aligned} &\Phi_{j_1,\cdots,j_{\nu+1}}(t_0, \cdots, t_{\nu+1}; x, \xi) \\ &= \Phi_{1,2,1}(t_0, t_1', t_2', t_{\nu+1}; x, \xi) , \end{aligned} \tag{2.12}$$

for some t_j' $(j=1, 2, t_0>t_1'>t_2'>t_{\nu+1})$ *independent of* x *and* ξ. *By using the same* t_j' $(j=1, 2)$ *we also get*

$$\begin{aligned} &(Q_{j_1,\cdots,j_{\nu+1}}, P_{j_1,\cdots,j_{\nu+1}})(t_0; t_0, \cdots, t_{\nu+1}; y, \eta) \\ &= (Q_{1,2,1}, P_{1,2,1})(t_0; t_0, t_1', t_2', t_{\nu+1}; y, \eta) \end{aligned} \tag{2.13}$$

for any point $(y, \eta)\in R^{2n}$.

Proof. We can get (2.12) by Proposition 1.6 and Theorem 2.4, inductively. Then, we obtain (2.13) by using (2.12) and Proposition 1.9. Q.E.D.

REMARK 2.4. For $\lambda_j(t, x, \xi)$ $(j=1, 2)$ in Remark 2.2 we have

$$(2.14) \qquad \begin{cases} \phi_1(t, s) = \sum_{i=1}^{n} a_i(t-s)\xi_i + x\cdot\xi\,, \\ \phi_2(t, s) = \sum_{i=1}^{n} b_i(t^2-s^2)\xi_i + x\cdot\xi\,, \\ \Phi_{1,2}(t, t_1, s) = \sum_{i=1}^{n} \{a_i(t-t_1)+b_i(t_1^2-s^2)\}\xi_i + x\cdot\xi\,, \\ \Phi_{2,1}(t, t_1, s) = \sum_{i=1}^{n} \{a_i(t_1-s)+b_i(t^2-t_1^2)\}\xi_i + x\cdot\xi\,. \end{cases}$$

Comparing (2.14) with $\Phi_{1,2,1}$ and $\Phi_{2,1,2}$ in Remark 2.2, we can see that we can gererally contract $\Phi_{1,2,1}(t, t_1, t_2, s)$ and $\Phi_{2,1,2}(t, t_1, t_2, s)$ $(t>t_1>t_2>s)$ no more. Furthermore, from Proposition 1.9 we can also see that we can generally contract $(Q_{1,2,1}, P_{1,2,1})(t, t_1, t_2, s)$ and $(Q_{2,1,2}, P_{2,1,2})(t, t_1, t_2, s)(t>t_1>t_2>s)$ no more.

EXAMPLES. We give examples of $\lambda_k(t, x, \xi)$ $(k=1, 2)$ satisfying $(*)$ on $[0, T]\times R^6_{x,\xi}$ except λ_k in Remark 2.2 below. They are not involutive, since $\{\tau+\lambda_1, \tau+\lambda_2\}(t, x, \xi)$ doe snot identically vanish on a set $\{(t, x, \xi);\ \lambda_1(t, x, \xi) = \lambda_2(t, x, \xi)\}$.

1. $\lambda_1(t, x, \xi)=\xi_1$, $\lambda_2(t, x, \xi) = x_1\xi_2+\xi_3$.
2. $\lambda_1(t, x, \xi)=x_1\xi_1$, $\lambda_2(t, x, \xi) = t\xi_2$.
3. $\lambda_1(t, x, \xi)=x_2\xi_1+\xi_3$, $\lambda_2(t, x, \xi)=-x_3\xi_1+\xi_2$.

3. Propagation of singularities

Consider a hyperbolic system with diagonal principal part

$$(3.1) \qquad L = D_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(t, X, D_x) + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(t, X, D_x)$$

$$\text{on } [0, T]\times R^n \quad (\lambda_j(t, x, \xi)\in B^\infty([0, T];\ S^1)\,,\ \text{real valued},\ b_{jk}(t, x, \xi)\in B^\infty([0, T];\ S^0))\,.$$

We assume that for a constant $M>0$ we have

$$(3.2) \qquad \lambda_j(t, x, \delta\xi) = \delta\lambda_j(t, x, \xi) \qquad (|\xi|\geqq M,\ \delta\geqq 1)\,.$$

We also assume that $(*)$ of Section 2 holds.

We study the Cauchy problem

$$(3.3) \qquad \begin{cases} LU(t, x) = 0 & \text{on } [0, T]\,, \\ U|_{t=0} = G(x)\,, \end{cases}$$

where $U(t, x)={}^t(u_1(t, x), u_2(t, x))$ and $G(x)={}^t(g_1(x), g_2(x))(g_k(x)\in H_{-\infty}=\bigcup_\sigma H_\sigma)$. Let $\phi_j(t, s; x, \xi)$ $(0\leqq s\leqq t\leqq T_0\leqq T)$ be the solutions of the eiconal equations (1.10) corresponding to λ_j and define $\Phi=\Phi_{j_1,\cdots,j_{\nu+1}}(t_0, \cdots, t_{\nu+1})$ $(j_k=1, 2)$ by $\Phi=\phi_{j_1}(t_0, t_1)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, t_{\nu+1})$ (see (1.11)).

If we apply Theorem 3.1 in Kumano-go-Taniguchi [11] to L of (3.1), then, for a small T_0 $(0<T_0\leqq T)$ we can get the fundamental solution $E(t, s)$ $(0\leqq s\leqq t\leqq T_0)$ of L (i.e. $LE(t, s)=0$ on $[0, T_0]$ and $E(s, s)=I$ (unit matrix)), which is represented by means of Fourier integral operators with multi-phase functions $\Phi_{j_1,\cdots,j_{\nu+1}}$ $(\nu=0, 1, \cdots)$. We fix such a T_0 in what follows. We will apply the theory in [11] for the propagation of singularities of solutions (Theorem 3.4 in [11]) to the Cauchy problem (3.3).

For $\lambda_{j_1}, \cdots, \lambda_{j_{\nu+1}}$, (y, η) and a fixed $0\leqq\varepsilon<1$ we define an ε-station chain $\{t_1, \cdots, t_\nu\}$ as the point $t>t_1>\cdots>t_\nu>0$ such that for $k=1, \cdots, \nu$

$$\begin{aligned}(3.4)\qquad &|\lambda_{j_k}(t_k, x^k, \xi^k)-\lambda_{j_{k+1}}(t_k, x^k, \xi^k)|\leqq\varepsilon\langle\xi^k\rangle\\ &\text{at } (x^k, \xi^k)=(Q_{j_1,\cdots,j_{\nu+1}}, P_{j_1,\cdots,j_{\nu+1}})(t_k; t, t_1,\cdots, t_\nu, 0; y, \eta)\,,\end{aligned}$$

where $(Q_{j_1,\cdots,j_{\nu+1}}, P_{j_1,\cdots,j_{\nu+1}})(\sigma; t_0, \cdots, t_\nu, 0; y, \eta)$ is the solution of (1.17) corresponding to $\{\lambda_{j_k}\}_{k=1}^{\nu+1}$ and $\{t_0, \cdots, t_{\nu+1}\}$ $(t_0=t, t_{\nu+1}=0)$. Define the ε-station set $\Lambda_{\varepsilon,j_1,\cdots,j_{\nu+1}}(t; y, \eta)$ by the set of all ε-station chains $\{t_1, \cdots, t_\nu\}$.

We set $WF(G)=\bigcup_{j=1}^{2} WF(g_j)$ for the wave front set $WF(g_j)$ of g_j. For $\boldsymbol{J}=(j_1, \cdots, j_{\nu+1})$ we set

$$\begin{aligned}(3.5)\qquad \Lambda_\varepsilon^{\boldsymbol{J}}(t; y, \eta)= \{&(Q_{j_1,\cdots,j_{\nu+1}}, P_{j_1,\cdots,j_{\nu+1}})(t; t, t_1, \cdots, t_\nu, 0; y, \eta);\\ &\{t_1, \cdots, t_\nu\}\in\Lambda_{\varepsilon,j_1,\cdots,j_{\nu+1}}(t; y, \eta)\}\,,\end{aligned}$$

and set

$$\begin{aligned}\Gamma_{t,\varepsilon}= \{&\delta\Lambda_\varepsilon^{\boldsymbol{J}}(t; y, \eta); (y, \eta)\in WF_\varepsilon(G), \boldsymbol{J}=(j_1, \cdots, j_{\nu+1})\,,\\ (3.6)\qquad &j_k=1, 2, \nu=0, 1, \cdots, \delta>0, |\eta|\geqq M_0\}\\ &(WF_\varepsilon(G)=\{(y, \eta); \operatorname{dis}\{(y, |\eta|^{-1}\eta), WF(G)\}\leqq\varepsilon\})\,,\end{aligned}$$

for a large constant $M_0>0$ depending on M of (3.2). Then, Theorem 3.4 in [11] says without the assumption (*)

Theorem 3.1. $\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}$ *is closed and for the solution $U(t, x)$ of the Cauchy problem* (3.3) *we have*

$$(3.7)\qquad WF(U(t))\subset\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}\qquad (0\leqq t\leqq T_0)\,.$$

If we add the assumption (*), then, setting

$$\begin{aligned}(3.8)\qquad \tilde{\Gamma}_{t,0}= \{&\delta\Lambda_0^{\boldsymbol{J}}(t; y, \eta); (y, \eta)\in WF(G), \delta>0\,,\\ &|\eta|\geqq M_0, \boldsymbol{J}=(1),(2), (1, 2), (2, 1), (1, 2, 1)\}\,,\end{aligned}$$

we get the main theorem.

Theorem 3.2. *Assume that the assumption* (*) *holds. Then, for the solution* $U(t, x)$ *of the Cauchy problem* (3.3) *we get*

$$WF(U(t))\subset\tilde{\Gamma}_{t,0} \qquad (0\leqq t\leqq T_0)\,. \tag{3.9}$$

Proof. By Theorem 3.1 we have only to prove that

$$\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}=\tilde{\Gamma}_{t,0}\,. \tag{3.10}$$

It is easy to see that $\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}\supset\tilde{\Gamma}_{t,0}$. So, we prove that

$$\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}\subset\tilde{\Gamma}_{t,0}\,.$$

We fix $0<t\leqq T_0$ and take a point $(x^0, \xi^0)\in\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}$ and fix it. If we take $|\xi^0|$ sufficiently large, then, setting $\xi^k=P_{j_{\nu+1},\cdots,j_1}(t_k;\, 0, t_\nu, \cdots, t_0;\, x^0, \xi^0)$ $(k=1, \cdots, \nu+1,\ t_{\nu+1}=0)$, we have

$$C^{-1}\leqq|\xi^k|\leqq C \qquad (k=0, \cdots, \nu+1)\,. \tag{3.11}$$

Here, the positive constant C is independent of the choice of $\boldsymbol{J}=(j_1, \cdots, j_{\nu+1})$ and a set $\{t_0, \cdots, t_\nu\}\subset[0, t]$. Since (x^0, ξ^0) belongs to $\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}$, for any $\varepsilon_m=2^{-m}$ there exist $\boldsymbol{J}^m_{\nu_m}=(j^m_1, \cdots, j^m_{\nu_m+1})$ $(j^m_k=1, 2,\ j^m_k\neq j^m_{k+1})$, $(y^m, \eta^m)\in WF_{\varepsilon_m}(G)$ and $\{t^m_1, \cdots, t^m_{\nu_m}\}\in\Lambda_{\varepsilon_m, j^m_1,\cdots,j^m_{\nu_m+1}}(y^m, \eta^m)$ such that

$$(x^0, \xi^0)=(Q_{j^m_1,\cdots,j^m_{\nu_m+1}}, P_{j^m_1,\cdots,j^m_{\nu_m+1}})(t;\, t, t^m_1, \cdots, t^m_{\nu_m}, 0;\, y^m, \eta^m)\,. \tag{3.12}$$

We consider (x^0, ξ^0) deviding into two cases as follows.

I) The case where we can take a subsequence $l=\{m_\mu\}_{\mu=1}^\infty$ and a point σ_1 $(0\leqq\sigma_1\leqq t)$ such that $t^l_1\to\sigma_1$ and $t^l_{\nu_l}\to\sigma_1$ as $l\to\infty$.

II) The other case.

I). We show that (x^0, ξ^0) belongs to $\tilde{\Gamma}_{t,0}$, when $0<\sigma_1<t$. In the other case $\sigma_1=0$ or t we can also prove this by the similar way. By the assumption I) we can also take a subsequence $\gamma=\{l_\mu\}_{\mu=1}^\infty$ of $l=\{m_\mu\}_{\mu=1}^\infty$ such that

$$(j^\gamma_1, j^\gamma_{\nu_\gamma+1})=(1, 1), (1, 2), (2, 1) \text{ or } (2, 2)\,.$$

We may assume that $j^\gamma_1=1$ and $j^\gamma_{\nu_\gamma+1}=2$, since we can prove similarly in the other cases. Now, take a point $(\bar{y}^0, \bar{\eta}^0)$ $(|\bar{\eta}^0|\geqq C^{-1}$, see (3.11)) such that

$$(\bar{y}^0, \bar{\eta}^0)=(Q_{2,1}, P_{2,1})(0;\, 0, \sigma_1, t;\, x^0, \xi^0)\,. \tag{3.13}$$

We note that

$$(x^0, \xi^0)=(Q_{1,2}, P_{1,2})(t;\, t, \sigma_1, 0;\, \bar{y}^0, \bar{\eta}^0)\,. \tag{3.13$'$}$$

Then, it is easy to see that

$$\bar{y}^0 = x^0 + \int_t^{\sigma_1} \nabla_\xi \lambda_1(\sigma, Q_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0), P_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0)) d\sigma$$
$$+ \int_{\sigma_1}^{0} \nabla_\xi \lambda_2(\sigma, Q_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0), P_{2,1}(\sigma; 0, \sigma_1, t; x^0, \xi^0)) d\sigma. \tag{3.14}$$

Using the assumption of this case, for any small $\delta > 0$ there exists N such that for any $\gamma \geqq N$ we have

$$\{t_1^\gamma, \cdots, t_{\nu_\gamma}^\gamma\} \subset [\sigma_1 - \delta, \sigma_1 + \delta]. \tag{3.15}$$

Since for any y^γ we have the similar equality to (3.14), we get

$$|\bar{y}^0 - y^\gamma| \leqq C_1 \delta \qquad (\gamma \geqq N)$$

for a constant $C_1 > 0$ independent of δ and γ. By the similar way we get

$$|\bar{\eta}^0 - \eta^\gamma| \leqq C_1 \delta \qquad (\gamma \geqq N).$$

Consequently, we can see that $(y^\gamma, \eta^\gamma) \to (\bar{y}^0, \bar{\eta}^0)$ as $\gamma \to \infty$ and

$$(\bar{y}^0, \bar{\eta}^0) \in WF(G). \tag{3.16}$$

Next, since $\{t_1^\gamma, \cdots, t_{\nu_\gamma}^\gamma\} \in \Lambda_{\varepsilon_\gamma, j_1^\gamma, \cdots, j_{\nu_\gamma+1}^\gamma}(y^\gamma, \eta^\gamma)$, it follows from (3.11) and (3.12) that

$$|(\lambda_2 - \lambda_1)(t_1^\gamma, Q_1(t_1^\gamma; t_1^\gamma, t; x^0, \xi^0), P_1(t_1^\gamma; t_1^\gamma, t; x^0, \xi^0))| \leqq C \varepsilon_\gamma$$

for a constant C of (3.11). Here, noting that $j_1^\gamma = 1$ and $j_{\nu_\gamma+1}^\gamma = 2$, we used

$$\begin{aligned}&(Q_{j_1^\gamma, \cdots, j_{\nu_\gamma+1}^\gamma}, P_{j_1^\gamma, \cdots, j_{\nu_\gamma+1}^\gamma})(t_1^\gamma; t, t_1^\gamma, \cdots, t_{\nu_\gamma}^\gamma, 0; y^\gamma, \eta^\gamma)\\ &= (Q_1, P_1)(t_1^\gamma; t_1^\gamma, t; x^0, \xi^0).\end{aligned}$$

When $\gamma \to \infty$, we get from (3.13)

$$\begin{aligned}0 &= (\lambda_2 - \lambda_1)(\sigma_1, Q_1(\sigma_1; \sigma_1, t; x^0, \xi^0), P_1(\sigma_1; \sigma_1, t; x^0, \xi^0))\\ &= (\lambda_2 - \lambda_1)(\sigma_1, Q_{1,2}(\sigma_1; t, \sigma_1, 0; \bar{y}^0, \bar{\eta}^0), P_{1,2}(\sigma_1; t, \sigma_1, 0; \bar{y}^0, \bar{\eta}^0)).\end{aligned}$$

Together with (3.13)′ and (3.16) this implies that

$$\begin{aligned}(x^0, \xi^0) &\in \{\Lambda_0^{(1,2)}(t; y, \eta); (y, \eta) \in WF(G)\}\\ &\subset \tilde{\Gamma}_{t,0}.\end{aligned}$$

II). We can take a subsequence $l = \{m_\mu\}_{\mu=1}^\infty$ and points σ_1, σ_2 $(0 \leqq \sigma_2 < \sigma_1 \leqq t)$ such that $t_1^l \to \sigma_1$ and $t_{\nu_l}^l \to \sigma_2$ as $l \to \infty$. We set

$$\begin{aligned}v(\sigma; l) = (\lambda_2 - \lambda_1)(\sigma; &Q_{j_1^l, \cdots, j_{\nu_l+1}^l}(\sigma; t, t_1^l, \cdots, 0; y^l, \eta^l),\\ &P_{j_1^l, \cdots, j_{\nu_l+1}^l}(\sigma; t, t_1^l, \cdots, 0; y^l, \eta^l) \qquad (0 \leqq \sigma \leqq t).\end{aligned} \tag{3.17}$$

For large l we have

$$t_1^l - t_{\nu_l}^l \geqq \frac{1}{2}(\sigma_1 - \sigma_2) > 0\,,$$

and then, noting that $\{t_1^l, \cdots, t_{\nu_l}^l\} \in \Lambda_{\varepsilon_l, j_1^l, \cdots, j_{\nu_l+1}^l}(y^l, \eta^l)$, we have by (3.11)

$$|v(t_1^l; l)|,\ |v(t_{\nu_l}^l; l)| \leqq C\varepsilon_l\,.$$

Consequently, since $v(\sigma; l)$ of (3.17) has the form

$$v(\sigma; l) = a\sigma + b \qquad (0 \leqq \sigma \leqq t) \tag{3.18}$$

from Lemma 2.3 in Section 2, it follows that

$$\begin{aligned} |v(\sigma; l)| &\leqq 2C\varepsilon_l T_0(t_1^l - t_{\nu_l}^l) \\ &\leqq 4C\varepsilon_l T_0(\sigma_1 - \sigma_2) \qquad (0 \leqq \sigma \leqq t)\,. \end{aligned} \tag{3.19}$$

Now, by Corollary 2.5 there exist some $\bar{t}_1^l$, $\bar{t}_2^l$ $(t > \bar{t}_2^l > \bar{t}_2^l > 0)$ such that

$$(x^0, \xi^0) = (Q_{1,2,1}, P_{1,2,1})(t; t, \bar{t}_1^l, \bar{t}_2^l, 0; y^l, \eta^l)\,. \tag{3.20}$$

Then, we note that

$$(y^l, \eta^l) = (Q_{1,2,1}, P_{1,2,1})(0; 0, \bar{t}_2^l, \bar{t}_1^l, t; x^0, \xi^0)\,. \tag{3.20$'$}$$

We set

$$\begin{aligned} v_1(\sigma; l) = (\lambda_2 - \lambda_1)(\sigma; Q_{1,2,1}(\sigma; t, \bar{t}_1^l, \bar{t}_2^l, 0; y^l, \eta^l), \\ P_{1,2,1}(\sigma; t, \bar{t}_1^l, \bar{t}_2^l, 0; y^l, \eta^l))\,. \end{aligned} \tag{3.21}$$

Since $v_1(\sigma; l) = v(\sigma; l)$ by Lemma 2.3 and Remark 2.1, from (3.19) we obtain

$$|v_1(\sigma; l)| \leqq \frac{4C}{\sigma_1 - \sigma_2}\varepsilon_l T_0\,. \tag{3.22}$$

Next, let $\bar{\sigma}_i$ $(i=1, 2, \bar{\sigma}_1 \geqq \bar{\sigma}_2)$ be the accumulating points of sets $\{\bar{t}_i^l\}_{l=1}^\infty$, respectively and take some subsequence $\{\gamma = l_\mu\}_{\mu=1}^\infty$ such that $\bar{t}_1^\gamma \to \bar{\sigma}_1$ and $\bar{t}_2^\gamma \to \bar{\sigma}_2$ as $\gamma \to \infty$. Then, it follows from (3.20)$'$ that there exists $(\bar{y}^0, \bar{\eta}^0)$ such that

$$(y^\gamma, \eta^\gamma) \to (\bar{y}^0, \bar{\eta}^0) = (Q_{1,2,1}, P_{1,2,1})(0; 0, \bar{\sigma}_2, \bar{\sigma}_1, t; x^0, \xi^0)$$

as $\gamma \to \infty$, and

$$(\bar{y}^0, \bar{\eta}^0) \in WF(G)\,. \tag{3.23}$$

We note that

$$(x^0, \xi^0) = (Q_{1,2,1}, P_{1,2,1})(t; t, \bar{\sigma}_1, \bar{\sigma}_2, 0; \bar{y}^0, \bar{\eta}^0)\,. \tag{3.24}$$

By using (3.22) we obtain

$$\begin{aligned}&(\lambda_1-\lambda_2)(\sigma,\, Q_{1,2,1}(\sigma;\, t,\, \bar\sigma_1,\, \bar\sigma_2,\, 0;\, \bar y^0,\, \bar\eta^0),\, P_{1,2,1}(\sigma;\, t,\, \bar\sigma_1,\, \bar\sigma_2,\, 0;\, \bar y^0,\, \bar\eta^0))\\ &=\lim_{\gamma\to\infty} v_1(\sigma;\, \gamma)\\ &=0 \qquad\qquad (0\leqq\sigma\leqq t)\,.\end{aligned}$$

This implies with (3.23) and (3.24) that

$$(x^0,\, \xi^0)\in\tilde\Gamma_{t,0}$$

which means (3.9) together with the result of I). Q.E.D.

References

[1] S. Alinhac: *Branching of singularities for a class of hyperbolic operators,* Indiana Univ. Math. J. **27** (1978), 1027–1037.

[2] S. Alinhac: *A class of hyperbolic operators with double involutive characteristics of Fuchsian type,* Comm. Partial Differential Equations **3** (1978), 877–905.

[3] J. Chazarain: *Opérateurs hyperboliques à caractéristiques de multiplicité constante,* Ann. Inst. Fourier **24** (1) (1974), 173–202.

[4] J. Chazarain: *Propagation des singularités pour une classe d'opérateurs à caractéristiques multiples et résobulité locale,* Ann. Inst. Fourier **24** (1) (1974), 203–223.

[5] A.K. Gautesen and D. Ludwig: *Tangential characteristics and coupling of waves,* J. Math. Anal. Appl. **38** (1972), 430–466.

[6] M. Hata: *On the Cauchy problem for hyperbolic operators with characteristic roots of variable multiplicity,* Master thesis, Osaka Univ., (1977).

[7] H. Kumano-go: Pseudo-differential operators, Iwanami Shoten, Tokyo, 1974 (in Japanese).

[8] H. Kumano-go: *Factorizations and fundamental solutions for differential operators of elliptic-hyperbolic type,* Proc. Japan Acad. **52** (1976), 480–483.

[9] H. Kumano-go: *A calculus of Fourier integral operators on R^n and the fundamental solution for an operator of hyperbolic type,* Comm. Partial Differential Equations **1** (1976), 1–44.

[10] H. Kumano-go, K. Taniguchi and Y. Tozaki: *Multi-products of phase functions for Fourier integral operators with an application,* Comm. Partial Differential Equations **3** (1978), 349–380.

[11] H. Kumano-go and K. Taniguchi: *Fourier integral operators of multi-phase and the fundamental solution for a hyperbolic system,* Funkcial. Ekvac. **22** (1979), 161–196.

[12] P.D. Lax: *Asymptotic solutions of oscillatory initial value problem,* Duke Math. J. **24** (1957), 627–646.

[13] D. Ludwig: *Exact and asymptotic solutions of the Cauchy problem,* Comm. Pure Appl. Math. **13** (1960), 473–508.

[14] D. Ludwig and B. Granoff: *Propagation of singularities along characteristics with nonuniform multiplicity,* J. Math. Anal. Appl. **21** (1968), 556–574.

[15] Y. Morimoto: *Fundamental solution for a hyperbolic equation with involutive characteristics of variable multiplicity,* Comm. Partial Differential Equations **4** (1979), 609–643.

[16] J.C. Nosmas: *Parametrix du problèm de Cauchy pour une classe de systèmes hyperboliques symétrisables à caractérisitques involutives de multiplicité variable,* C.R. Acad. Sci. Paris Sér. A **288** (1979), 129–132.

[17] K. Taniguchi: *Multi-products of Fourier integral operators and the fundamental solution for a hyperbolic system with involutive characteristics,* to appear.

[18] K. Taniguchi and Y. Tozaki: *A hyperbolic equation with double characteristics which has a solution with branching singularities,* Math. Japonica **25** (1980), 279–300.

[19] G.A. Uhlmann: *Pseudo-differential operators with involutive characteristics,* Comm. Partial Differential Equations **2** (1977), 713–779.

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