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THE SYMBOL CALCULUS FOR THE FUNDAMENTAL SOLUTION OF A DEGENERATE PARABOLIC SYSTEM WITH APPLICATIONS

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(Received December 24, 1975)

Introduction

In the paper [2] S. D. Eidelman has constructed the fundamental solution of a system of partial differential operators which is parabolic in the sense of Petrovski with sufficiently smooth coefficients. A few years later, the assumptions on the smoothness of the coefficients have been weakened to uniform Hölder continuity. The bibliography and bibliographical remarks concerning this topics are found in A. Friedman’s book [4]. The applications of the fundamental solution to the study of the Cauchy problem and other related problems are found in the above book and S. D. Eidelman’s book [3]. On the other hand, if the coefficients are sufficiently smooth, the recent results of the theory of pseudo-differential operators, especially that of H. Kumano-go [8] and [9], have enabled us to construct a symbol of the fundamental solution of a parabolic operator which may be of degenerate type through only the symbol calculus. (See the paper C. Tsutsumi [18].)

In the present paper we shall, using a method similar to that of [18], construct the fundamental solution of a degenerate parabolic system \( L = \partial_t + p(t; X, D_x) \) which has the property (F) (See the Definition 2.2). A system of partial differential operators which is parabolic in the sense of Petrowski with \( C^\infty \)-coefficients has this property, and so do the operators treated in T. Matsuzawa [11], B. Helffer [6], C. Tsutsumi [18] and M. Miyake [12]. In the papers [11], [6] and [12], a family of parametrices \( K_0 + K_1 + \cdots + K_j \) of the operator \( L \) is constructed so that they satisfy the equation \( L_{t,s}(\sum_{k=0}^j K_k(x, y, t, t')) = \delta(x-y, t-t') + F_j(x, y, t, t'), \) and \( K_1, \cdots, K_j \) and \( F_j \) are very regular. In [18] and the present paper, however, the fundamental solution is constructed in the class of pseudo-differential operators.

In section 1 we shall give some lemmas on the symbol calculus. In section 2 the matrix \( \epsilon(t, s; x, \xi) \) of symbols of fundamental solution will be constructed and its asymptotic expansion will be given in a very natural form (See the formula (2.23)). In section 3 the general result of section 2 is applied to a
degenerate parabolic operator of higher order of the form

\[ L = \partial_t^M + \sum_{j=0}^l t^j a_j(t; X, D_x) \partial_x^{M-j} + \cdots + \sum_{k=0}^M t^k a_k(t; X, D_x) , \]

where \( l \) is a positive integer and \( a_j, k \in \mathcal{D}'(\mathbb{R}^{(j+k)(M+j)}), j=1, 2, \ldots, M, k=1, 2, \ldots, j, l \), to obtain a result including that of M. Miyake [12]. In section 4 using the symbol of the fundamental solution and following the idea of Y. Kannai [7], we shall give sufficient conditions for the operator \( L \) to be hypoelliptic. This gives an example of a hypoelliptic operator with multiple characteristics (See Theorem II.1.1 in F. Treves [17]). It will also be shown that the operator treated by B. Helffer [6] satisfies our hypothesis under some additional restriction.

The results of the present paper have been announced partly in [15] and [16].

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1. Definitions and lemmas

Let \( x \in \mathbb{R}^n \), \( \xi \in \mathbb{R}^n \) and let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a multi-index of non-negative integers. We use the following notation:

\[ |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} , \quad \langle x \rangle = (1 + |x|^2)^{1/2} , \]

\[ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n , \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! , \]

\[ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} , \quad x^\xi = x_1^{\xi_1} + x_2^{\xi_2} + \cdots + x_n^{\xi_n} , \]

\[ \partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \cdots \partial_{\xi_n}^{\alpha_n} \quad \text{where} \quad \partial_{\xi_j} = \partial/\partial_{\xi_j} \]

\[ D_\xi^\alpha = D_{\xi_1}^{\alpha_1} D_{\xi_2}^{\alpha_2} \cdots D_{\xi_n}^{\alpha_n} \quad \text{where} \quad D_{\xi_j} = -i \partial/\partial_{\xi_j} . \]

According to L. Schwartz [14] we use the notation \( \mathcal{D}(\Omega), \mathcal{D} = \mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathcal{D}), \mathcal{D}' = \mathcal{D}'(\mathbb{R}^n), S, S' \) and \( \mathcal{B} = \mathcal{B}(\mathbb{R}^n) \) to denote the spaces of \( M \)-dimensional vector valued functions and distributions. For an interval \( J \) of \( \mathbb{R}^l \) we denote by \( \mathcal{E}_k(J; \mathcal{B}) \) the space of \( k \) times continuously differentiable functions of \( t \in J \) with values in \( \mathcal{B} \). We set \( \mathcal{E}_k(J; \mathcal{B}) = \bigcap_m \mathcal{E}_m^k(J; \mathcal{B}) \).

**Definition 1.1** ([10]). We say that an \( M \times M \) matrix \( p(x, \xi) \) with components \( p_{j,k}(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) belongs to \( S_{m,B}, -\infty < m < \infty, 0 < \delta < \rho < 1, \) when for any \( \alpha, \beta \) there exists a constant \( C_{m,\beta} \) such that

\[ |p(x, \xi)| \leq C_{m,\beta} \langle \xi \rangle^{m - p|\alpha| - \delta|\beta|} , \]

where \( p(x, \xi) = \partial_\xi^\alpha D_\xi^\beta p(x, \xi) \) and \( |p| \) denotes the norm of the matrix \( p \) defined by

\[ |p| = \sup \{|p y|/|y| ; y \in C^N, y \neq 0\} . \]
We define the corresponding operator \( p(X, D_x) \) by

\[
(1.2) \quad p(X, D_x)u(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi, \quad u \in S,
\]

where \( \hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx \) and \( d\xi = (2\pi)^n d\xi \). We call a linear map \( P \) of \( S \) into \( S \) a pseudodifferential operator with symbol \( p(x, \xi) \in S^{0,0}_p \) if \( P = p(X, D_x) \) and we also write \( P \in S^{0,0}_p \). We set \( S^{m,0}_p = \bigcup_m S^{m,0}_p \) and \( S^{-\infty} = \bigcap_m S^{m,0}_p \).

N. B. Throughout this paper we assume that \( \rho \) and \( \delta \) satisfy the condition \( 0 < \delta < \rho < 1 \).

**Definition 1.2 ([10]).** For a \( p(x, \xi) \in S^{0,0}_p \) we define semi-norms \(|p|^{(l)}\), \( l=0, 1, 2, \ldots \), by

\[
(1.3) \quad |p|^{(m)} = \max_{|\alpha+\beta| \leq l} \sup_{(x, \xi) \in \Omega} |p^{(m)}(X, D_x)(x, \xi)| \}
\]

Then \( S^{m,0}_p \) makes a Fréchet space with these semi-norms. For a symbol \( p(t, x, \xi) \) with a parameter \( t \) we write

\[
p(t; x, \xi) \in B^t(S^{m,0}_p) \quad \text{in} \quad (s, T),
\]

when \( p(t; x, \xi) \) is a \( k \) times continuously differentiable \( S^{m,0}_p \)-valued function in \( s \leq t \leq T \). A subset \( B \) of \( S^{m,0}_p \) is said to be a bounded set in \( S^{m,0}_p \) when

\[
\sup_{l=0, 1, 2, \ldots} \{ |p|^{(m)} \} = \infty \text{ for } l=0, 1, 2, \ldots.
\]

We write

\[
w-lim_{t \to s} p(t; x, \xi) = p(s; x, \xi) \quad \text{in} \quad S^{m,0}_p,
\]

when there exists a constant \( c \) such that \( \{ p(t; x, \xi); t \in (s, s+c) \} \) is a bounded set in \( S^{m,0}_p \) and for any and \( \alpha, \beta \) for any compact set \( K \subset R^m_t \), \( p\in(B)(t; x, \xi) \) converges to \( p\in(B)(s; x, \xi) \) uniformly in \( R^m_t \times K \) as \( t \) tends to \( s \).

In order to treat a product of pseudodifferential operators, we introduce the oscillatory integral and multiple symbols.

**Definition 1.3 ([10]).** We say that an \( M \times M \) matrix \( a(\eta, y) \) with components \( a_{ij}(\eta, y) \in C^{\alpha}(R^m_t \times R^n_\eta) \) belong to a class \( \mathcal{A}^{m,\tau}_{\delta,\tau} \), \(-\infty < m < \infty, 0 \leq \delta < 1, 0 \leq \tau \) when for any multi-index \( \alpha, \beta \), we have

\[
(1.4) \quad |\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \leq C_{\alpha,\beta} \delta^{m+|\beta|} |\eta|^\tau
\]

for a constant \( C_{\alpha,\beta} \). For an \( a(\eta, y) \in \mathcal{A}^{m,\tau}_{\delta,\tau} \) we define semi-norms \(|a|^{(l)}\), \( l=0, 1, 2, \ldots \), by

\[
(1.5) \quad |a|^{(l)} = \max_{|\alpha+\beta| \leq l} \sup_{(\eta, y) \in \Omega} |\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \delta^{m-|\beta|} |\eta|^{-\tau}.
\]

Then \( \mathcal{A}^{m,\tau}_{\delta,\tau} \) makes a Fréchet space. We set \( \mathcal{A} = \bigcup_{0 \leq \delta < 1} \bigcup_{-\infty < m < \infty} \bigcup_{0 \leq \tau} \mathcal{A}^{m,\tau}_{\delta,\tau} \). We say that a subset \( B \) of \( \mathcal{A} \) is bounded when \( B \subset \mathcal{A}^{m,\tau}_{\delta,\tau} \), for some \( \delta, m \) and \( \tau \), and
For an \( a(\eta, y) \in \mathbb{A}_{\delta, \tau}^m \) we define the oscillatory integral \( Os[e^{-iy \cdot \eta} a] \) by

\[
Os[e^{-iy \cdot \eta} a] = Os\left( \sum_{\varepsilon > 0} e^{-iy \cdot \eta} a(\eta, y)dy d\eta \right)
\]

\[
= \lim_{\varepsilon \to 0} \sum_{\varepsilon > 0} e^{-iy \cdot \eta} \chi(\varepsilon \eta, \varepsilon y) a(\eta, y)dy d\eta ,
\]

where \( \chi(\eta, y) \in \mathcal{S}(R_{\Delta, \tau}^m) \) such that \( \chi(0, 0) = 1 \).

The following lemma proves the well-definedness of (1.6).

**Lemma 1.4** ([8]). For an \( a(\eta, y) \in \mathbb{A}_{\delta, \tau}^m \) let \( l \) and \( l' \) be positive integers such that

\[
-2l(1-\delta)+m < -\nu \eta, -2l'+\tau < \nu \eta .
\]

Then we have

\[
Os[e^{-iy \cdot \eta} a] = \int e^{-iy \cdot \eta} \langle y \rangle^{-2l'} \langle D_y \rangle^{2l'} \{ \langle \eta \rangle^{-2l} \langle D_\eta \rangle^{2l} a(\eta, y) \} dy d\eta
\]

and

\[
|Os[e^{-iy \cdot \eta} a]| \leq C |a|_{l+l'}
\]

for a constant \( C \) which is independent of \( a(\eta, y) \).

**Lemma 1.5** ([9]). Let \( \{a(t; \eta, y)\}_{0 \leq t \leq 1} \) be a bounded set of \( \mathbb{A} \). Suppose that there exists an \( a(0; \eta, y) \in \mathbb{A} \) such that \( a(t; \eta, y) \rightarrow a(0; \eta, y) \) as \( t \rightarrow 0 \) uniformly on any compact set of \( R_{\Delta, \tau}^m \). Then we have

\[
\lim_{t \to 0} Os[e^{-iy \cdot \eta} a(t)] = Os[e^{-iy \cdot \eta} a(0)].
\]

**Definition 1.6** ([9]). i) We say that an \( M \times M \) matrix \( p(x_0, \xi, x, x_1, \cdots, x_v, \xi_v) \) whose components are \( C^\infty \)-functions defined in \( R^{(2v+1)m} \) is a multiple symbol of class \( S(m_1, m_2, \cdots, m_v) \), when for any \( \alpha_1, \cdots, \alpha_v, \beta_0, \beta_1, \cdots, \beta_v \), there exists a constant \( C = C(\alpha_1, \cdots, \alpha_v, \beta_0, \beta_1, \cdots, \beta_v) \) such that

\[
|p(x_0, \xi, x, x_1, \cdots, x_v, \xi_v)| \leq C \langle \xi \rangle^{m_0} \prod_{j=1}^v \langle \xi_j \rangle^{m_j} + q(\langle \xi \rangle + \langle \xi_j \rangle) \langle \beta \rangle^{m_j},
\]

where \( \xi^{v+1} = 0 \). For a \( p(x_0, \xi, x, x_1, \cdots, x_v, \xi_v) \in S^{m_1, m_2, \cdots, m_v} \) we define semi-norms

\[
|p|_{l, m_1, m_2, \cdots, m_v} = \max_{|\alpha| \leq l, |\beta| \leq l'} \inf \{ C \}
\]

Then \( S^{m_1, m_2, \cdots, m_v} \) makes a Fréchet space.

ii) The associated pseudodifferential operator \( P = p(x_0, D_x, D_x^1, \cdots, D_x^v, X_v) \)
with multiple symbol \( \rho(x^0, \xi^1, x^1, \cdots, \xi^v, x^v) \) is defined by

(1.13) \[
Pu(x) = \text{Os} - \int e^{-i(x^1y_1 + \cdots + \eta^v)} \rho(x^1, \eta^1, x+y^1, \eta^2, \cdots, \eta^v, x+y^1+\cdots+y^v) dy^1d\eta^1 \cdots dy^vd\eta^v
\]

for \( u \in \mathcal{B} \).

Remark 1.7. i) The pseudodifferential operator \( P \) defined by (1.2) is extended to a continuous operator \( P: \mathcal{B} \to \mathcal{B} \) by setting \( v = 1 \) in (1.13).

ii) When \( \rho_j(x, \xi) \in \mathcal{S}_{p, \delta}, j = 1, 2, \cdots, \nu \), we have that \( \rho_1(x^0, \xi^1)\rho_2(x^1, \xi^2) \cdots \rho_v(x^{v-1}, \xi^v) \in \mathcal{S}_{p, \delta}(m_1, \cdots, m_v) \).

Lemma 1.8 ([10]). Let \( p(t; x, \xi) \in \mathcal{B}(S_{p, \delta}) \) in \( (s, T) \) Then we have

(1.14) if \( w-\lim_{t \to t^*} p(t; x, \xi) = p(s; x, \xi) \) in \( S_{p, \delta} \), then \( \lim_{t \to t^*} P(t)u = P(s)u \)

for any \( u \in \mathcal{B} \).

Lemma 1.9 ([9]). For a \( \rho(x^0, \xi^1, x^1, \cdots, \xi^v) \in \mathcal{S}_{p, \delta}(m_1, \cdots, m_v) \), set

(1.15) \[
q(x, \xi) = \text{Os} - \int e^{-i(x^1y_1 + \cdots + \eta^v)} \times \rho(x, \xi+y^1, x+y^1, \cdots, \xi+y^v, x+y^1+\cdots+y^v, \xi) d\eta^1 \cdots d\eta^v
\]

Then we have, for \( m = m_1 + m_2 + \cdots + m_v \),

(1.16) \[
q(x, \xi) \in \mathcal{S}^{m} \quad \text{and} \quad q(X, D_x) = \rho(X^0, D_x, \cdots, X^v, D_x).
\]

Furthermore, for any \( l \) there exists a constant \( C \) such that

(1.17) \[
|q|^{(m)} \leq C^l |\rho|^{(m_1, \cdots, m_v)}_{l_0, l_0'},
\]

where

(1.18) \[
l_0 = l+2[n/2+1], \quad l_0' = l+2[(n+\sum_{j=1}^v m_j) + pl + \delta l]/(2(1-\delta)) + 1].
\]

Lemma 1.10 ([10]). For a \( \rho(x, \xi) \in \mathcal{S}_{p, \delta} \), set

(1.19) \[
\rho^*(x, \xi) = \text{Os} - \int e^{-iy^v*} \rho(x+y, \xi+y)^* dy d\eta,
\]

where \( \rho^* \) is the conjugate transpose of the matrix \( \rho \). Then \( P^* = \rho^*(X, D_x) \) is the formal adjoint of \( P = \rho(X, D_x) \) in the sense that \( \langle Pu, v \rangle = \langle u, P^*v \rangle \) for any \( u, v \in \mathcal{S} \). Let \( L = \partial_t + \rho(X, D_x) \). Then the formal adjoint \( 'L \) of \( L \) in \( \mathbb{R}_1^l \times \mathbb{R}_x^v \) is given by

(1.20) \[
'L = -\partial_t^l + \rho^*(X, D_x).
\]
For \( p_j(x, \xi) \in S_{\rho_j}^m, j=1, 2, \ldots, \nu \), we denote by \( p_1 \circ p_2 \circ \cdots \circ p_\nu(x, \xi) \) the symbol of the product \( P_1 P_2 \cdots P_\nu \) of pseudodifferential operators \( P_j = p_j(X, D_x) \). By Lemma 1.9 we have

\[
(1.21) \quad p_1 \circ p_2 \circ \cdots \circ p_\nu(x, \xi) = \mathcal{O} - \iint e^{-i(x^1 \xi^1 + \cdots + x^\nu \xi^{\nu-1})} p_1(x, \xi + \eta^1) p_2(x+y^1, \xi + \eta^2) \cdots p(x+y^{\nu-1}, \xi) dy^1 d\eta^1 \cdots dy^{\nu-1} d\eta^{\nu-1}.
\]

For \( k=1, 2, \ldots \), we set

\[
(1.22) \quad [p_1 \circ p_2 \circ \cdots \circ p_\nu](x, \xi) = \sum_{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_\nu| = k} \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_\nu!} p_1^{(\alpha_1)} p_2^{(\alpha_2)} \cdots p_\nu^{(\alpha_\nu)}(x, \xi)
\]

and for \( k=0 \) we set

\[
(1.23) \quad [p_1 \circ p_2 \circ \cdots \circ p_\nu](x, \xi) = p_1(x, \xi) p_2(x, \xi) \cdots p_\nu(x, \xi).
\]

**Lemma 1.11 ([13]).** Let \( p_j(x, \xi) \in S_{\rho_j}^{m_j}, j=0, 1, \ldots, \nu \). Then we have:

i) **For any positive integer** \( N \),

\[
(1.24) \quad p_1 \circ p_2 \circ \cdots \circ p_\nu(x, \xi) = \sum_{k=0}^N [p_1 \circ p_2 \circ \cdots \circ p_\nu](x, \xi) \in S_{\rho_0}^{m_0 - (\rho - 1)N},
\]

where \( m = m_1 + m_2 + \cdots + m_\nu \).

ii) **For** \( k=0, 1, 2, \ldots \),

\[
(1.25) \quad [p_0 \circ p_1 \circ p_2 \circ \cdots \circ p_\nu](x, \xi) = \sum_{k=0}^N \sum_{|\alpha| = k} \frac{1}{\alpha!} p_0^{(\alpha)}(x, \xi)
\]

Note. A proof of (1.24) by using Taylor's expansion has been given in [9].

**Lemma 1.12.** Let \( p \) be an \( M \times M \) matrix and let \( \lambda_1, \lambda_2, \ldots, \lambda_M \) be the eigenvalues of the matrix \( p \), and

\[
\lambda = \min_j \text{Re} \lambda_j,
\]

where \( \text{Re} \lambda_j \) means the real part of \( \lambda_j \). Then the inequality

\[
(1.26) \quad |\exp[-tp]| \leq \sum_{l=0}^{M-1} |(2t|p|)l| \exp[-t\lambda]
\]

holds for \( t \geq 0 \).


**Lemma 1.13.** Let \( a > 1 \). Then for any \( \varepsilon > 0 \) we have the following inequality

\[
(1.27) \quad v-u \leq \varepsilon(v^a-u^a) + \varepsilon^{-1/(a-1)}
\]

for any \( u \) and \( v \) such that \( 0 \leq u < v \).
Proof. Let \( f(t)=t^r, t \geq 0 \). Since \( df(t)/dt \) is an increasing function of \( t \), we have
\[
f(v-u) - f(0) \leq f(v) - f(u).
\]
Thus we have \((v-u)^s \leq v^s - u^s\), i.e., \( v-u \leq (v-u)^{1-s}(v^s - u^s) \). If \((v-u)^{1-s} \leq \varepsilon\), we have \( v-u \leq \varepsilon(v^s - u^s) \). If \((v-u)^{1-s} \geq \varepsilon\), we have \( v-u \leq \varepsilon^{-1/(a-1)}\).

Lemma 1.14. Let \( f(t; x, \xi) \) be a non-negative continuous function if there exist constants \( C>0 \) and \( c \) such that \( 1/(1+1) \leq c < 1 \) and
\[
(1.28) \quad tf(t; x, \xi) \leq C(t^{1+1}\langle \xi \rangle^n)^{c} \quad \text{for} \quad t \geq 0,
\]
then for any \( \varepsilon > 0 \) there exists a constant \( C' \) such that
\[
(1.29) \quad \int_t^s f(\sigma; x, \xi)d\sigma \leq \varepsilon \int_t^s \sigma^{1+1}\langle \xi \rangle^n d\sigma + C' \quad \text{for} \quad 0 \leq s \leq t.
\]
Proof. Since \( f(\sigma; x, \xi) \leq C\sigma^{\varepsilon(t+1)-1}\langle \xi \rangle^n \), we have
\[
\int_t^s f(\sigma; x, \xi)d\sigma \leq C\varepsilon^{-1}(t+1)^{-1}\{\varepsilon^{\varepsilon(t+1)}-s^{\varepsilon(t+1)}\langle \xi \rangle^n\}.
\]
If we set \( v=\varepsilon^{\varepsilon(t+1)}\langle \xi \rangle^n \), \( u=\varepsilon^{\varepsilon(t+1)}\langle \xi \rangle^n \) and \( a=1/c \), then we have by Lemma 1.13 that
\[
\int_t^s f(\sigma; x, \xi)d\sigma \leq C\varepsilon^{-1}(t+1)^{-1}\{\varepsilon^{\varepsilon(t+1)}-s^{\varepsilon(t+1)}\langle \xi \rangle^n\} + C'' \).
\]
This proves the Lemma.

2. Fundamental solution and Cauchy problem

In this section we first consider the fundamental solution of the Cauchy problem
\[
(2.1) \quad Lu(t, x) = f(t, x) \quad \text{in} \quad (0, T),
\]
\[
(2.2) \quad u(0, x) = u_0(x),
\]
where \( L=\partial_t + p(t; X, D) \) and \( p(t; x, \xi) \in D_i(S^m_{p,s}) \) in \([0, T]\) and then apply it to the solution of (2.1)-(2.2).

Definition 2.1. i) By an \( M \times M \) matrix \( e(t, s; x, \xi) \in D_i(S^m_{p,s}) \cap D_i(S^m_{p,s}) \) with parameters \( t \) and \( s \), we denote, the fundamental solution of the Cauchy problem (2.1) and (2.2), that is, \( e(t, s; x, \xi) \) is the solution of the system of symbol equations
\[
(2.3) \quad \partial_t e(t, s; x, \xi) + p(t) \circ e(t, s)(x, \xi) = 0, \quad 0 \leq s < t < T,
\]
and satisfies the initial condition
(2.4) \[ \lim_{s \to t^+} e(t, s; x, \xi) = I \quad \text{in} \quad S^0_{\rho, \delta}, \]

where \( I \) is the identity matrix. We call \( e(t, s; x, \xi) \) the Green's matrix of the operator \( L \).

ii) We say that an \( M \times M \) matrix \( z(t, s; x, \xi) \in \mathcal{B}_1^0(S^m_{\rho, \delta}) \cap \mathcal{B}_1^0(S^2_{\rho, \delta}) \) is the resolvent matrix of the operator \( L \), when \( z(t, s; x, \xi) \) satisfies

\begin{align*}
\partial_s z(t, s; x, \xi) + p(t; x, \xi)z(t, s; x, \xi) &= 0, \quad 0 < s < t < T \\
\lim_{s \to t^+} z(t, s, x, \xi) &= I \quad \text{in} \quad S^0_{\rho, \delta}.
\end{align*}

DEFINITION 2.2. We say that the operator \( L \) has the property (F), when for some non-negative continuous function \( \lambda(t; x, \xi) \) the following two conditions are satisfied:

i) For any \( \alpha, \beta \) there exists a constant \( C_{\alpha, \beta} \) such that

\begin{equation}
\int_s^t |\rho(\sigma; x, \xi)| d\sigma \leq C_{\alpha, \beta} \left\{ \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right\}
\end{equation}

for \( 0 < s < t < T \).

ii) There exist constants \( d > 0 \) and \( C > 0 \) such that the resolvent matrix \( z(t, s; x, \xi) \) of \( L \) satisfies

\begin{equation}
|z(t, s; x, \xi)| \leq C \exp \left[ -\frac{1}{d} \int_s^t \lambda(\sigma; x, \xi) d\sigma \right]
\end{equation}

for \( 0 < s < t < T \).

When \( L \) is a system of partial differential operators, it is said to be parabolic in the sense of Petrovski if the real part of each eigenvalue of the matrix \( p(t; x, \xi) \) is not less than \( \lambda \). In this case, if the coefficients of \( L \) are \( C^\infty \), the property (F) is satisfied with \( \lambda(t; x, \xi) = \langle \xi \rangle^m \) and stable under the small perturbation of the principal part and any lower order terms. The property (F) is also stable in the following sense.

Lemma 2.3. Let \( L = \partial_t + p(t; X, D_x) \) have the property (F) with \( \lambda(t; x, \xi) \), \( d \) and \( C \), and let for \( q(t; x, \xi) \in \mathcal{B}_1^0(S^m_{\rho, \delta}) \) there exist constants \( \varepsilon, C' \) and \( C_{\alpha, \beta} \) such that \( 0 < \varepsilon < d/C \),

\begin{align*}
\int_s^t |q(\sigma; x, \xi)| d\sigma &\leq \varepsilon \int_s^t \lambda(\sigma; x, \xi) d\sigma + C' \\
\int_s^t |\rho(\sigma; x, \xi)| d\sigma &\leq C_{\alpha, \beta} \left\{ \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right\}.
\end{align*}

Then the operator \( L' = \partial_t + p(t; X, D_x) + q(t; X, D_x) \) has the property (F) with \( \lambda(t; x, \xi), d' \) and \( C'' \), where \( d' = d - \varepsilon C \) and \( C'' = C \exp [CC'] \).
Proof. Let $z'(t, s; x, \xi)$ be the resolvent matrix of $L'$. Since it follows from (2.9) and (2.10) that i) of the property $(F)$ is satisfied, we have only to prove ii). Since we can write

$$z'(t, s; x, \xi) = z(t, s; x, \xi) - \int_s^t z(t, \sigma; x, \xi) q(\sigma; x, \xi) z'(\sigma, s; x, \xi) d\sigma,$$

using (2.8) we have

$$|z'(t, s; x, \xi)| \leq C \exp \left[ -d \int_s^t \lambda(\sigma; x, \xi) d\sigma \right] + C \int_s^t \exp \left[ -d \int_\sigma^t \lambda(\sigma'; x, \xi) d\sigma' \right] q(\sigma; x, \xi)|z'(\sigma, s; x, \xi)| d\sigma.$$

Setting $\varphi(t) = |z'(t, s; x, \xi)| \exp \left[ d \int_s^t \lambda(\sigma; x, \xi) d\sigma \right]$, we have

$$\varphi(t) \leq C + C \int_s^t \varphi(\sigma) q(\sigma; x, \xi) d\sigma. \tag{2.11}$$

Multiplying both sides by $|q(t; x, \xi)| \left| \left\{ 1 + \int_s^t \varphi(\sigma) q(\sigma; x, \xi) d\sigma \right\} \right|$ and integrating them, we have

$$\log \left[ 1 + \int_s^t \varphi(\sigma) q(\sigma; x, \xi) d\sigma \right] \leq C \int_s^t |q(\sigma; x, \xi)| d\sigma.$$

Thus we have

$$C + C \int_s^t \varphi(\sigma) q(\sigma; x, \xi) d\sigma \leq C \exp \left[ C \int_s^t |q(\sigma; x, \xi)| d\sigma \right].$$

Hence by (2.11) and (2.9) we have

$$|z'(t, s; x, \xi)| \leq C \exp \left[ CE \int_s^t \lambda(\sigma; x, \xi) d\sigma + CC' - d \int_s^t \lambda(\sigma; x, \xi) d\sigma \right].$$

Thus the proof is complete.

When the coefficients of the operator $L = \partial_t + p(X, D_x)$ are independent of $t$, its resolvent matrix is given by $\exp \left[ -(t-s)p(x, \xi) \right]$. When the coefficients depend on $t$, we have the following

**Lemma 2.4.** The resolvent matrix $z$ of the operator $L = \partial_t + p(t; X, D_x)$, $p(t; x, \xi) \in \mathcal{B}(S_p^n)$, can be written

$$z(t, s; x, \xi) = I + \sum_{j=1}^\infty (-1)^j \int_s^t ds_1 \int_s^{s_1} ds_2 \ldots \int_s^{s_{j-1}} p(s_j; s, \xi) p(s; x, \xi) \cdots p(s_j; x, \xi) ds_j. \tag{2.12}$$

Moreover we have
\[ z(t, \sigma, x, \xi)z(\sigma, s; x, \xi) = z(t, s; x, \xi), \tag{2.13} \]

\[ |z(t, s; x, \xi)| \leq C \exp \left[ C' |t-s| \langle \xi \rangle^m \right], \tag{2.14} \]

and

\[ \partial_z z(t, s; x, \xi) - z(t, s; x, \xi)p(s; x, \xi) = 0. \tag{2.15} \]

Proof is omitted.

Now we shall give two propositions which give examples of sufficient conditions under which the operator \( L \) has the property \((F)\).

**Proposition 2.5.** Let \( p(t, x, \xi) \in \mathcal{B}_0^1(S_{n,n}^m) \) and let \( \lambda(t; x, \xi) \) be one of the following functions:

\[ \inf \left( \text{Re } p_{j,k}(t; x, \xi) - \sum_{k,j} |p_{j,k}(t; x, \xi)| \right), \tag{2.16} \]

\[ \inf \left( \text{Re } p_{k,j}(t; x, \xi) - \sum_{j,k} |p_{j,k}(t; x, \xi)| \right), \tag{2.17} \]

\[ \text{smallest eigenvalue of } (p(t; x, \xi) + p(t; x, \xi))/2. \tag{2.18} \]

If \( \lambda(t; x, \xi) \) is non-negative in \([0, T]\), and if ii) of Definition 2.2 holds, then the operator \( L = \partial_t + p(t; X, D_x) \) has the property \((F)\).

Proof. By Theorem 3 of Chapter III in W. A. Coppel's book [1], we have that there exists a constant \( C>0 \) such that

\[ |z(t, s; x, \xi)| \leq C \exp \left[ -\int_s^t \lambda(\sigma; x, \xi)d\sigma \right], \tag{2.19} \]

Thus ii) of Definition 2.2 is satisfied with \( d=1 \).

**Proposition 2.6.** Let \( f(t) \) be a non-negative continuous function and \( \bar{p}(t; x, \xi) \in \mathcal{B}_0^0(S_{n,n}^m) \) in \([0, T]\). If the real part of each eigenvalue of the matrix \( \bar{p}(t; x, \xi) \) is not less than \( d \langle \xi \rangle^m \) for a constant \( d>0 \), then the operator \( L = \partial_t + f(t)p(t; X, D_x) \) has the property \((F)\) with \( \lambda(t; x, \xi) = f(t)\).

Proof. Set \( p(t; t'; x, \xi) = f(t)\bar{p}(t'; x, \xi) \) and \( q(t; t'; x, \xi) = f(t)\{\bar{p}(t; x, \xi) - \bar{p}(t'; x, \xi)\} \). Then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[ \int_s^t |q(\sigma; t'; x, \xi)|d\sigma \leq \varepsilon \langle \xi \rangle^m \int_s^t f(\sigma)d\sigma \quad \text{if } 0<s<t<\delta \text{ and } t' \in [s, t]. \]

Thus by Lemma 2.3 the operator \( L \) has the property \((F)\) with \( \lambda(t; x, \xi) = f(t)\langle \xi \rangle^m \), if \( |t-s| \) is sufficiently small. For arbitrary \( s \) and \( t \), \( 0 \leq s < t \leq T \), dividing \([s, t]\) into sufficiently small intervals, and using (2.13), we can prove that \( L \) has the property \((F)\) in \([s, t]\).

From the expansion formula (2.12) of the resolvent matrix \( z(t, s; x, \xi) \), we have formally
\( (2.20) \quad e(t, s; x, \xi) = 1 + \sum_{j=0}^{\infty} (-1)^j \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} p(s_1) \circ p(s_2) \circ \cdots \circ p(s_j)(x, \xi) ds_j. \)

Thus, if we set
\( (2.21) \quad e_0(t, s; x, \xi) = z(t, s; x, \xi) \)
and for \( k \geq 1 \)
\( (2.22) \quad e_k(t, s; x, \xi) = \sum_{j=0}^{\infty} (-1)^j \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} [p(s_1) \circ p(s_2) \circ \cdots \circ p(s_j)]_k(x, \xi) ds_j, \)
then we can infer from (1.24) and (2.20) that the Green's matrix \( e(t, s; x, \xi) \) has the following asymptotic expansion:
\( (2.23) \quad e(t, s; x, \xi) \sim e_0(t, s; x, \xi) + e_1(t, s; x, \xi) + \cdots. \)

Now we shall prove the main theorem which ensures us the existence of the Green's matrix and its asymptotic expansion (2.23).

**Theorem 2.7.** Let \( L \) have the property \((F)\). Then for any \( \alpha, \beta \) and \( k=0, 1, 2, \ldots \), there exist constants \( C_{k, \alpha, \beta} \) and \( C'_{k, \alpha, \beta} \) such that
\( (2.24) \quad |e_k(t, s; x, \xi)| \leq C_{k, \alpha, \beta} (t-s) \leq k^{-(p-\beta)^{-p}(\alpha+\beta)} \left( \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right)^{k+1+|\alpha|+|\beta|} \times \exp \left[ -d \int_s^t \lambda(\sigma; x, \xi) d\sigma \right], \quad 0 \leq s < t \leq T. \)

Moreover we can construct the Green's matrix \( e(t, s; x, \xi) \) of \( L \) such that
\( (2.25) \quad \left\{ \begin{array}{l} e(t, s; x, \xi) \in B_1^p(S_{p, \delta}^0) \cap B_1^p(S_{p, \alpha}^0) \quad \text{in} \quad (s, T] \quad \text{for} \quad 0 \leq s < t \leq T, \\
\quad \lim_{t \to s^+} e(t, s; x, \xi) = I \quad \text{in} \quad S_{p, \delta}^0. \end{array} \right. \)

If we set
\( (2.26) \quad r_N(t, s; x, \xi) = e(t, s; x, \xi) - \sum_{k=0}^{N} e_k(t, s; x, \xi), \)
then we have
\( (2.27) \quad |r_N(t, s; x, \xi)| \leq C_{\alpha, \beta} (t-s)^{\xi} \leq k^{-(p-\beta)^{-p}(\alpha+\beta)} \left( \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right)^{k+1+|\alpha|+|\beta|} \times \exp \left[ -d \int_s^t \lambda(\sigma; x, \xi) d\sigma \right], \quad 0 \leq s < t \leq T. \)

Proof. For convenience sake we omit to describe the variables \( x \) and \( \xi \). Since we have
\[ |\partial_x^k D^j \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} [p(s_1) \circ p(s_2) \circ \cdots \circ p(s_j)] ds_j| \leq (j!)^{-1} (t-s)^{k} C_{\alpha, \beta} \xi \leq k^{-(p-\beta)^{-p}(\alpha+\beta)} \left( \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right)^{k+1+|\alpha|+|\beta|}, \]
the sum in the right hand side of (2.22) converges to a matrix with $C^\infty$ components. By Lemma 1.11 we have

$$
\partial_t \int_s^{s_1} ds_1 \int_s^{s_2} ds_2 \cdots \int_s^{s_j} [p(s_1) \circ p(s_2) \cdots \circ p(s_j)] \Lambda ds_{j+1} = \sum_{k=0}^n \sum_{|\alpha|=n-k} \frac{1}{\alpha!} p^{(\alpha)}(t) \int_s^{s_1} ds_1 \int_s^{s_2} ds_2 \cdots \int_s^{s_{j-1}} [p(s_1) \circ p(s_2) \cdots \circ p(s_j)] \Lambda_{k-n, \alpha} ds_j.
$$

Thus we have for $k=1, 2, \ldots$

$$(2.28) \quad \partial_t \epsilon_k(t, s) + p(t) \epsilon_k(t, s) = -\sum_{k=1}^n \sum_{|\alpha|=n-k} \frac{1}{\alpha!} p^{(\alpha)}(t) \epsilon_{k-n, \alpha}(t, s).$$

First we estimate $e_0(t, s)$. Since $\partial_t \epsilon_0(t, s)$ satisfies the equation

$$
\partial_t \partial_t \epsilon_0(t, s) + p(t) \partial_t \epsilon_0(t, s) = -\partial_t(p(t)) e_0(t, s),
$$

and $\partial_t \epsilon_0(t, s) = 0$, we have

$$
\partial_t \epsilon_0(t, s) = -\int_s^t e_0(t, \sigma)(\partial_t p(\sigma)) e_0(\sigma, s) d\sigma.
$$

Thus by (2.7) and (2.8) we obtain

$$
|\partial_t \epsilon_0(t, s)| \leq C <\xi>^{-\rho} \left\{ \int_s^t \lambda(\sigma) d\sigma + 1 \right\} \exp \left[ -d \int_s^t \lambda(\sigma) d\sigma \right] \leq C <\xi>^{-\rho}.
$$

After this manner we have for every $\alpha, \beta$

$$
|\partial_t^\alpha \partial_s^\beta \epsilon_0(t, s)| \leq C_{\alpha, \beta} <\xi>^{-p|\alpha| - \rho|\beta|}.
$$

Next we estimate $e_k^{(\alpha)}(t, s)$ by the induction on $k$. If (2.24) is valid for $e_0(t, s)$, $e_1(t, s)$, $\cdots$, $e_{k-1}(t, s)$, then with (2.28) we obtain (2.24) for $e_k(t, s)$. We also have (2.24)' in the same way.

We set

$$(2.29) \quad f_N(t, s) = \sum_{k=0}^N \epsilon_k(t, s)
$$

and

$$(2.30) \quad q_N(t, s) = -\partial_t f_N(t, s) - p(t) \circ f_N(t, s).$$

Since

$$
\{ p(t) \circ e_k(t, s) - \sum_{|\alpha|=N-k} \frac{1}{\alpha!} p^{(\alpha)}(t) e_{k-\alpha}(t, s) \} \in S_{n, \delta}^{m-(\rho-\delta)(N+1)},
$$

by using (2.28) we obtain

$$
|q_N^{(\alpha)}(t, s)| \leq C_{\alpha, \beta} <\xi>^{m-(\rho-\delta)(N+1) - \rho|\beta|}.
$$

Taking $N$ so large as $m-(\rho-\delta)(N+1) \leq 0$, we have
Hence by Lemma 1.9 we have that for any \( \alpha, \beta \) there exists a constant \( A_{\alpha, \beta} \) which is independent of \( j \) such that

\[
| \partial_\xi^\alpha D_\xi^\beta \{ q_N(t, s_1) \circ q_N(s_2, s_2) \circ \cdots \circ q_N(s_{j-1}, s) \} | \\
\leq (A_{\alpha, \beta})^j \langle \xi \rangle^{m-(p-\delta)(N+1)-p|\alpha|+\delta|\beta|}.
\]

We set

\[
\varphi_j(t, s) = q_N(t, s)
\]

and

\[
\varphi_j(t, s) = \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-2}} ds_j \cdot q_N(t, s_1) \circ q_N(s_2, s_2) \circ \cdots \circ q_N(s_{j-1}, s) ds_{j-1},
\]

Then we have

\[
| \varphi_j(t, s) | \leq (A_{\alpha, \beta})^j \langle \xi \rangle^{m-(p-\delta)(N+1)-p|\alpha|+\delta|\beta|}.
\]

Thus we can define \( \varphi(t, s) \) by

\begin{equation}
\varphi(t, s) = \sum_{j=1}^\infty \varphi_j(t, s)
\end{equation}

and we have

\[
| \varphi(t, s) | \leq C_{\alpha, \beta} \langle \xi \rangle^{m-(p-\delta)(N+1)-p|\alpha|+\delta|\beta|}.
\]

We set

\begin{equation}
R_N(t, s) = \int_s^t f_N(t, \sigma) \circ \varphi(\sigma, s) d\sigma.
\end{equation}

Then by (2.24) and above estimates we have (2.27). Since \( \varphi(t, s) \) satisfies the following integral equation

\begin{equation}
\varphi(t, s) = q_N(y, s) + \int_s^t q_N(t, \sigma) \circ \varphi(\sigma, s) d\sigma,
\end{equation}

if we set \( e(t, s) = f_N(r, s) + R_N(t, s) \), we have

\[
\partial_r e(t, s) = \partial_r f_N(t, s) + \varphi(t, s) + \int_s^t \partial_r f_N(t, \sigma) \circ \varphi(\sigma, s) d\sigma.
\]

By using (2.33) and (2.30) we have

\[
\partial_r e(t, s) = -p(t) \circ f_N(t, s) - \int_s^t p(t) \circ f_N(t, \sigma) \circ \varphi(\sigma, s) d\sigma
\]

\[
= -p(t) \circ e(t, s).
\]
We also have
\[ w\text{-lim}_{t \to s} e(t, s) = w\text{-lim}_{t \to s} \{f_n(t, s) + r_n(t, s)\} = I \quad \text{in } S^0_{\rho, \delta}. \]
Thus \( e(t, s) \) is the Green's matrix of \( L \). The restriction that \( N \) is sufficiently large is removed as follows. For any \( N \), we set \( r_N = e_{N+1} + e_{N+2} + \cdots + e_{N'} + r_{N'} \), where \( N' \) is sufficiently large, then we have (2.27) by (2.24)' . The first half of (2.25) follows from \( \partial_t e(t, s) = -p(t) e(t, s) \). Thus the proof is complete.

As an application we shall give a representation of the solution for the Cauchy problem \( Lu = f, \ u(0, x) = u_0(x) \) which provides the existence and uniqueness of the solution for the problem.

Lemma 2.8. Let \( L = \partial_t + p(t; X, D_x), p(t; x, \xi) \in \mathcal{B}_0^0(S^0_{\rho, \delta}) \) has the property (F). Then the Green's matrix \( e(t, s; x, \xi) \) of \( L \) that is constructed in Theorem 2.7 satisfies the following equations

\[ \begin{align*}
\partial_t e(t, s; x, \xi) - e(t, s) \circ p(s)(x, \xi) &= 0, & 0 \leq s < t \leq T, \\
\partial_t e^*(t, s; x, \xi) - p^*(s) \circ e^*(t, s)(x, \xi) &= 0, & 0 \leq s < t \leq T,
\end{align*} \]

where we define \( e(s, \sigma; x, \xi) = I \).

The symbol \( e^*(t, s; x, \xi) \) of the formal adjoint of \( e(t, s; X, D_x) \) satisfies

\[ \begin{align*}
\partial_t e^*(t, s; x, \xi) - p^*(s) \circ e^*(t, s)(x, \xi) &= 0, & 0 \leq s < t \leq T.
\end{align*} \]

Proof. We omit to describe the variables \( x \) and \( \xi \). Let \( e_0(t, s), e_1(t, s), \cdots, e_n(t, s), f_n(t, s) \) be the symbols defined by (2.21), (2.22) and (2.29) respectively. Then we have as in the proof of Theorem 2.7

\[ \begin{align*}
\partial_t e_0(t, s) - e_0(t, s) \circ p(s) &= 0, \\
\partial_t e_k(t, s) - e_k(t, s) \circ p(s) &= \sum_{k=1}^{k-1} \sum_{|\alpha|=1}^k \frac{1}{\alpha!} e_{\alpha}(t, s) p_\alpha(s), \quad k = 1, 2, \cdots.
\end{align*} \]

Thus we can construct a symbol \( e(t, s) \in S^0_{\rho, \delta}(S^0_{\rho, \delta}) \) such that

\[ \begin{align*}
\partial_t e(t, s) - e(t, s) \circ p(s) &= 0, & 0 \leq s < t \leq T, \\
w\text{-lim}_{t \to s} e(t, s) &= I \quad \text{in } S^0_{\rho, \delta}.
\end{align*} \]

We define \( e(t, t) = I \). For \( f, g \in S(R_+^*) \) we set

\[ h(\sigma) = (e(\sigma, s; X, D_x)f(x), e^*(t, \sigma; X, D_x)g(x)). \]

Then we have by (2.37)

\[ \frac{d}{d\sigma} h(\sigma) = -(p(\sigma) \circ e(\sigma, s; X, D_x)f(x), e^*(t, \sigma; X, D_x)g(x)) \]

\[ + (e(\sigma, s; X, D_x)f(x), p^*(\sigma) \circ e^*(t, \sigma; X, D_x)g(x)) = 0. \]
Thus \( h(\sigma) \) is independent of \( \sigma \). Letting \( \sigma \downarrow s \) and \( \sigma \uparrow t \), we have \( \bar{e}(t, s) = e(t, s) \). By (2.37) we have (2.34).

By (2.34) we have
\[
\begin{align*}
\partial_\sigma \{e(t, \sigma) & \circ e(\sigma, s)\} = e(t, \sigma) \circ p(\sigma) \circ e(\sigma, s) - e(t, \sigma) \circ p(\sigma) \circ e(\sigma, s) \\
& = 0.
\end{align*}
\]

Thus \( e(t, \sigma) \circ e(\sigma, s) = \lim_{\sigma \downarrow s} e(t, \sigma) \circ e(\sigma, s) = e(t, s) \). We obtain (2.36) taking the symbol of the formal adjoint of operators defined by both sides of (2.34).

**Theorem 2.9.** Let \( p(t; x, \xi) \in \mathcal{B}(S_{\eta}^{\infty}) \) in \([0, T]\) and let \( L = \partial_t + p(t; X, D_x) \) have the property (F). Then the Cauchy problem

\[
\begin{align*}
Lu(t, x) &= f(t, x) \\
\lim_{t \downarrow 0} u(t, x) &= u_0(x)
\end{align*}
\]

has a unique solution \( u(t, x) \) in \( \mathcal{E}_1([0, T]; \mathcal{B}) \) for any \( f(t, x) \in \mathcal{C}_1([0, T]; \mathcal{B}) \) and any \( u_0(x) \in \mathcal{B} \). This solution \( u(t, x) \) is given by

\[
\begin{align*}
u(t, x) &= \int_0^t \left( e(t, \sigma; X, D_x)f(\sigma, x) + e(t, 0; X, D_x)u_0(x) \right) d\sigma.
\end{align*}
\]

**Proof.** If \( f(t, x) \in \mathcal{C}_1([0, T]; \mathcal{B}) \) and \( u_0(x) \in \mathcal{B} \), then by using Lemma 1.8 and (2.25) we have that \( u(t, x) \) of (2.40) belongs to \( \mathcal{C}_1([0, T]; \mathcal{B}) \) and

\[
\begin{align*}
\partial_t u(t, x) &= f(t, x) - \int_0^t p(t) \circ e(t, \sigma)(X, D_x)f(\sigma, x) d\sigma - p(t) \circ e(t, 0)(X, D_x)u_0(x) \\
& = f(t, x) - p(t; X, D_x)u(t, x).
\end{align*}
\]

We have also that \( \lim_{t \downarrow 0} u(t, x) = u_0(x) \). Thus \( u(t, x) \) of (2.40) is the solution of the Cauchy problem (2.39).

Conversely if we let \( u(t, x) \in \mathcal{C}_1([0, T]; \mathcal{B}) \) be a solution of (2.39), then by (2.34) we have

\[
\begin{align*}
e(t, \sigma; X, D_x)f(\sigma, x) &= e(t, \sigma; X, D_x)\{ \partial_\sigma u(\sigma, x) + p(\sigma; X, D_x)u(\sigma, x) \} \\
& = \partial_\sigma \{e(t, \sigma; X, D_x)u(\sigma, x)\}.
\end{align*}
\]

Hence we have by Lemma 1.8

\[
\int_0^t e(t, \sigma; X, D_x)f(\sigma, x) d\sigma = u(t, x) - e(t, 0; X, D_x)u_0(x).
\]

Thus \( u(t, x) \) coincides with the one given by (2.40). Thus the proof is completed.

### 3. A degenerate parabolic operator of higher order

In this section we shall construct the fundamental solution of the Cauchy problem for a single operator.
\[ L = \partial_t^M + a_1(t; X, D_x)\partial_t^{M-1} + \cdots + a_M(t; X, D_x), \]

where for a positive integer \( l \)

\[ a_j(t; x, \xi) = \sum_{k=0}^{l-j} t^k a_{j,k}(t; x, \xi), \quad j = 1, 2, \ldots, M, \]

when the following two conditions are satisfied:

- a) For \( j=1, 2, \ldots, M; k=0, 1, \ldots, j, \)

\[ a_{j,k}(t; x, \xi) \in B_{l}^{0}(S^{((j+k)\max(l+1))}_{x, \xi}) \quad \text{in } [0, T]. \]

- b) There exists a positive constant \( d \) such that the roots \( \tau_j(t; x, \xi) \) of the equation

\[ \tau^M + a_1(t; x, \xi)\tau^{M-1} + a_2(t; x, \xi)\tau^{M-2} + \cdots + a_M(t; x, \xi) = 0 \]

satisfy

\[ \text{Re } \tau_j(t; x, \xi) \leq -d\langle \xi \rangle^m, \quad j=1, 2, \ldots, M; 0 \leq t \leq T. \]

In doing so, we shall use a function \( h(t; \xi) \) defined by

\[ h(t; \xi) = t^{\langle \xi \rangle^m} + \langle \xi \rangle^{m/(l+1)}, \]

and reduce the operator \( L \) to a system which has the property \((F)\).

**Lemma 3.1.** The function \( h(t; \xi) \) defined by (3.5) satisfies

\[ |\partial^r_t\{h(t; \xi)^{-(r+1)} \partial_t^r h(t; \xi)\}| \leq C_\alpha \langle \xi \rangle^{-m}, \quad r = 0, 1, \ldots, \]

and

(3.7) \[ |\partial_{\xi}^r\{\partial_t h(t; \xi)^{-r} \partial_t h(t; \xi)\}| \leq C_\alpha \langle \xi \rangle^{-m} \sum_{k=0}^{l-1} (t^k \langle \xi \rangle^m)^{(k+l+1)}, \quad r = 1, 2, \ldots, l. \]

Proof. First we note that \( \partial_t h(t; \xi) = C_{r} t^{l-r} \langle \xi \rangle^m \) for \( 0 < r \leq l \), and \( \partial_t h(t; \xi) = 0 \) for \( r > l \). Since we have \( t^{r+1} h(t; \xi)^{r+1} = t^{l+1} \langle \xi \rangle^m + (t^{l+1} \langle \xi \rangle^m + t^{l+1} \langle \xi \rangle^m) r^1 \), we have

\[ t^{l-r} \langle \xi \rangle^m / h(t; \xi)^{r+1} \leq (t^{l+1} \langle \xi \rangle^m)^{(l+1)/(l+1)} \]

and

\[ t^{l-r} \langle \xi \rangle^m / h(t; \xi)^{r+1} \leq 2/\{(l+1)(t^{l+1} \langle \xi \rangle^m)^r\} \]

for \( r = 0, 1, \ldots, l \). Thus we have (3.6). The estimate (3.7) follows from

\[ th(t; \xi)^{-r} \partial_t^r h(t; \xi) = C_\alpha t^{l+1} \langle \xi \rangle^m / \{t^{l+1} \langle \xi \rangle^m + t^{l+1} \langle \xi \rangle^m\}^r \]

\[ \leq C_\alpha (t^{l+1} \langle \xi \rangle^m)^{1-r/(l+1)} \quad \text{for } r = 1, 2, \ldots, l. \]

The proof of (3.8) follows from (3.6) and
Here and in what follows by the notation \( \sum_{r_1, r_2, \ldots, r_k} \) we mean to take the summation for all the sets \((r_1, r_2, \ldots, r_k)\) of non-negative integers \(r\), which satisfy \(r_1 + 2r_2 + \cdots + kr_k = k\).

**Theorem 3.2.** Let the operator \( L \) which is defined by (3.1) satisfy the conditions a) and b). Then there exist pseudodifferential symbols \( g_j(t, s; x, \xi) \), \( j=0, 1, \ldots, M-1 \), such that \( h(t; \xi)g_j(t, s; x, \xi) \in \mathcal{B}^0(S^0, s) \) in \((s, T)\), and for any \( \psi_j(x) \in \mathcal{B}(R^n) \) if we set

\[
(3.10) \quad v(t, x) = Os - \int e^{-i\xi \cdot \eta} \sum_{j=0}^{M-1} g_j(t, 0; x, \eta) \psi_j(x+y) dy \, d\eta,
\]

then \( v(t, x) \) satisfies

\[
(3.11) \quadLv(t, v) = 0
\]

and

\[
(3.12) \quad \lim_{t \to 0} \partial_t^j v(t, x) = \psi_j(x), \quad j = 0, 1, \ldots, M-1.
\]

Proof. We set

\[
p(t; x, \xi) = t^i \bar{p}(t; x, \xi)
\]

\[
t^i = \begin{pmatrix}
0 & -\langle \xi \rangle^m
\end{pmatrix}
\]

\[
a_{M, M}(t; x, \xi) \left\langle \xi \right\rangle^{-(M-1)m}, \ldots, a_{2, 2}(t; x, \xi) \left\langle \xi \right\rangle^{-m}, a_{1, 1}(t; x, \xi),
\]

and

\[
q(t; x, \xi) = \begin{pmatrix}
0 & -\langle \xi \rangle^{m(l+1)}
\end{pmatrix}
\]

\[
q_0(t; x, \xi), \ldots, q_2(t; x, \xi), q_1(t; x, \xi)
\]

where \( q_0(t; x, \xi), \ldots, q_M(t; x, \xi) \) will be determined later.

We define symbols \( b_{j,k}(t; \xi), \quad j=1, \ldots, M; \quad k=1, \ldots, M \) by

\[
(3.13) \begin{cases}
 b_{1,1}(t; \xi) = h(t; \xi)^{M-1}, \\
 b_{j,k}(t; \xi) = h(t; \xi)^{-1} b_{j-1,k}(t; \xi) + h(t; \xi)^{-1} \partial b_{j-1,k}(t; \xi), \\
 b_{j,k}(t; \xi) = 0 \quad \text{when} \ j > k \text{ or } k = 0,
\end{cases}
\]
and set

\[(3.14)\]

\[b(t; \xi) = (b_{j,k}(t; \xi)).\]

Since \(b(t; \xi)\) is a triangular matrix with diagonal elements \(b_{j,j}(t; \xi) = h(t; \xi)^{M-j}\) \(\neq 0\), \(b(t; \xi)\) is non-singular. We set

\[(3.15)\]

\[r(t; \xi) = b(t; \xi)^{-1}.\]

By \(r_{j,k}(t; \xi)\) we denote the component of \(r(t; \xi)\) in the \(j\)-th raw and the \(k\)-th column. Then we have \(r_{j,j}(t; \xi) = h(t; \xi)^{M-j}\) and \(r_{j,k}(t; \xi) = 0\) for \(j < k\). We set

\[\omega(t, x) = \langle (v(t, x), \partial_t v(t, x), \ldots, \partial_t^{M-1} v(t, x)) \rangle.\]

If \(u(t, x) = \langle u_1(t, x), u_2(t, x), \ldots, u_M(t, x) \rangle = b(t; D_x)\omega(t, x)\), then by (3.13) we have

\[(3.16)\]

\[
\begin{aligned}
&u_1(t, x) = h(t; D_x)\omega(t, x) \\
&h(t; D_x)u_2(t, x) = \partial_t u_1(t, x) \\
&\vdots \\
&h(t; D_x)u_M(t, x) = \partial_t u_{M-1}(t, x).
\end{aligned}
\]

Conversely if \(u(t, x)\) satisfies (3.16), then because of (3.15) we have

\[(3.17)\]

\[\omega(t, x) = r(t; D_x)u(t, x).\]

Hence we have

\[(3.18)\]

\[
\begin{aligned}
r_{j,j}(t; \xi) = h(t; \xi)^{M-j}, & \quad j = 1, 2, \ldots, M, \\
r_{j,k}(t; \xi) = \partial_t r_{j-1,k}(t; \xi) + h(t; \xi)r_{j-1,k-1}(t; \xi) & \quad j = 2, 3, \ldots, M; k = 1, 2, \ldots, M, \\
r_{j,k}(t; \xi) = 0 & \quad \text{when } j < k \text{ or } k = 0.
\end{aligned}
\]

From (3.13) we have

\[(3.19)\]

\[b_{j-k}(t; \xi) = h(t; \xi)^{M-j-k} \sum_{\langle r_1, \ldots, r_k \rangle} C_{j, r_1 \ldots r_k} \left(\frac{\partial h(t; \xi)}{h(t; \xi)}\right)^{r_1} \left(\frac{\partial^2 h(t; \xi)}{h(t; \xi)}\right)^{r_2} \cdots \left(\frac{\partial^k h(t; \xi)}{h(t; \xi)}\right)^{r_k}.
\]

From (3.18) we have

\[(3.20)\]

\[r_{j-k}(t; \xi) = h(t; \xi)^{M-j} \sum_{\langle r_1, \ldots, r_k \rangle} C_{r_1 \ldots r_k} \left(\frac{\partial h(t; \xi)}{h(t; \xi)}\right)^{r_1} \left(\frac{\partial^2 h(t; \xi)}{h(t; \xi)}\right)^{r_2} \cdots \left(\frac{\partial^k h(t; \xi)}{h(t; \xi)}\right)^{r_k}.
\]

By Lemma 3.1 and (3.20) we have that there exists for any \(\alpha\) a constant \(C_\alpha > 0\) such that
(3.21) \[ |\partial_t^* \{th(t; \xi)^{M-1} r_{j,j}(t; \xi)\}| \leq C_0 <\xi>-^{(j-1)} \sum_{l=1}^R (t^{l+1} <\xi>-^m)^{k/(l+1)} \]
and
(3.22) \[ |\partial_t^* \partial_1 \{h(t; \xi)^{M-1} r_{j,j}(t; \xi)\}| \leq C_0 <\xi>-^{(j-1)} \sum_{l=1}^R (t^{l+1} <\xi>-^m)^{k/(l+1)}. \]
We also have

(3.23) \[ h(t; \xi)^{M-1} r_{j,j}(t; \xi) \in B_0^p(S_1,0), \]
and by (3.19) we have

(3.24) \[ h(t; \xi)^{-k-M} b_{j,j}(t; \xi) \in B_0^p(S_1,0). \]
Now we shall determine \( q_j(t; x, \xi), \ldots, q_M(t; x, \xi) \) so that if \( u(t, x) \) is a solution of

\[ \{ \partial_t + \rho(t; X, D_x) + q(t; X, D_x) \} u(t, x) = 0, \]
then \( v(t, x) = h(t; D_x)^{-M} u(t, x) \) satisfies \( Lv(t, x) = 0 \). By (3.16) and (3.17) we have

\[ \partial_t^* v(t, x) = \sum_{k=1}^M r_{j,k}(t; X_x) u_k(t, x). \]
Thus we have

\[ \partial_t^* v(t, x) = \sum_{k=1}^M \{ (\partial_r r_{j,k}(t; D_x)) u_k(t, x) + r_{j,k}(t; D_x) \partial_i u_k(t, x) \} \]
\[ = \sum_{k=1}^M \{ (\partial_r r_{j,k}(t; D_x) + r_{j,k}(t; D_x) h(t; D_x)) u_k(t, x) + \partial_i u_k(t, x) \}. \]
Hence we have

(3.25) \[ q_j(t; x, \xi) = -t^j a_j(t; x, \xi) <\xi>-^{(j-1)} \]
\[ + \sum_{l=1}^L a_l(t; x, \xi) r_{j,k} r_{j,k+1}(t; \xi) \]
\[ + \partial_r r_{j,k+1}(t; x, \xi) + h(t; \xi) r_{j,k}(t; x, \xi), \]
\[ j = 1, 2, \ldots, M. \]
We shall show that the operator \( \partial_t + \rho(t; X, D_x) + q(t; X, D_x) \) has the property (F) with

(3.26) \[ \lambda(t; x, \xi) = t^j <\xi>-^m. \]
We write

\[ q_j(t; x, \xi) = \{-t^j a_j(t; x, \xi) <\xi>-^{(j-1)} + t^j a_j(t; x, \xi) r_{j,k+1}(t; \xi) \}
\[ + \text{the rest} = I_1 + I_2. \]
By (3.2) and (3.18) we have

\[ |I_1| = |a_j(t; x, \xi) <\xi>-^m \{-t^j <\xi>-^m + t^j <\xi>-^m h(t; \xi)^{-1} \}| \]
\[ \leq Ch(t; \xi)^{-j} t^j <\xi>-^m \sum_{k=0}^{j-1} \left( \frac{j-1}{k} \right) t^k <\xi>-^m <\xi>-^{(j-1-k)m/(l+1)} \]
\[ \begin{align*}
&= C h(t; \xi)^{-1} t^{\langle \xi \rangle m/((l+1))} h(t; \xi)^{2-j} \\
&\quad \times \sum_{i=0}^{j-1} \binom{j-1}{k} t^{k \langle \xi \rangle m(j-2-k) m/((l+1))} \\
&\leq C' h(t; \xi)^{-1} t^{\langle \xi \rangle m/((l+1))} \leq C' t^{l/2 \langle \xi \rangle m/(l+2)}.
\end{align*} \]

Here we used the inequality \( ab/(a+b) \leq \sqrt{ab} \) for \( a > 0, b > 0 \). Thus we have

\[(3.27) \quad |t I_1| \leq C t^{(l+2)/2 \langle \xi \rangle m/((2l+2))} = C \{ t^{l+1 \langle \xi \rangle m} \}^{(l+2)/(2l+2)}.
\]

Next we estimate \( I_2 \). Since \( h(t; \xi) \geq C t^k \langle \xi \rangle^{(j+k) m/((l+1))} \) for \( 0 \leq k \leq j \), we have by (3.2) that \( |t^a \partial_{j,k}(t; x, \xi) h(t; \xi)^{-j} | \) is bounded. Thus by (3.21) and (3.22) we have

\[(3.28) \quad |t I_2| \leq C \sum_{i=1}^{l+1} (t^{l+1 \langle \xi \rangle m})^{b(l+1)}.
\]

In view of Lemma 1.14 with (3.27) and (3.28), we have that \( q(t; x, \xi) \) satisfies (2.9) of Lemma 2.3. It can be proved similarly that \( q(t; x, \xi) \) also satisfies (2.10). Since the characteristic polynomial of the matrix \( \tilde{p}(t; x, \xi) \) coincides with the left hand side of (3.3), we have that the real part of each eigenvalue of the matrix \( \tilde{p}(t; x, \xi) \) is not less than \( \delta(\xi, \mu) \). Thus by Proposition 2.6. it follows that \( \partial_t + p(t; X, D_x) \) has the property (F). Hence by Lemma 2.3 it follows that the operator \( \partial_t + p(t; X, D_x) + q(t; X, D_x) \) also has the property (F).

Let \( e(t, s; x, \xi) \) be the Green's matrix of \( \partial_t + p(t; X, D_x) + q(t; X, D_x) \). Then by (3.23), (3.24) and (2.25) we have

\[ r(t) = e(t, s) b(s)(x, \xi) \in B^0_0(S^0_{r, \delta}) \]

and

\[ \text{w-lim}_{t \rightarrow s} r(t) = b(s)(x, \xi) = I \quad \text{in} \quad S^0_{r, \delta}. \]

Hence we have

\[(3.29) \quad g_j(t, s; x, \xi) = h(t)^{l-M} \sum_{k=1}^{M} e_{j,k}(t, s) b_{k,j+1}(s)(x, \xi), \]

where \( e_{j,k}(t, s; x, \xi) \) is the component of \( e(t, s; x, \xi) \) in the \( j \)-th raw and the \( k \)-th column. By (3.24) we have \( h(t; \xi) g_j(t, s; x, \xi) \in B^0_0(S^0_{r, \delta}) \). Hence the proof is complete.

4. Hypoellipticity

**Definition 4.1.** We say that a linear operator \( T: C^r(\Omega) \rightarrow C^m(\Omega) \) is properly supported in an open set \( \Omega \) when for any compact set \( K \subset \Omega \) there exists another compact set \( K' \subset \Omega \) such that

\[ \text{supp} \ Tu \subset K' \quad \text{if} \quad \text{supp} \ u \subset K \]

and

\[ Tu = 0 \ \text{on} \ K \quad \text{if} \ u = 0 \ \text{on} \ K'. \]
Lemma 4.2 ([10]). If $P \in S^m_{\alpha, \beta}$ is properly supported in $\Omega$, then so is its formal adjoint $P^\ast$.

**Definition 4.3.** i) For $P \in S^m_{\alpha, \beta}$ we define $P: S' \to S'$, $(u \to Pu)$ by

$$
(Pu, v) = (u, P^*v)
$$

for $v \in S'$, where $(u, v) = \langle u, v \rangle$ and $\langle u, v \rangle$ means the value of $u$ at $v$.

ii) If $P \in S^m_{\alpha, \beta}$ is properly supported in $\Omega$, we define $P: \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$, $(u \to Pu)$ by

$$
(Pu, v) = (u, P^*v)
$$

for $v \in \mathcal{D}(\Omega)$.

**Definition 4.4.** An operator $P \in S^m_{\alpha, \beta}$ which is properly supported in $\Omega$ is said to be hypoelliptic in $\Omega$, if

$$
\text{sing supp } u = \text{sing supp } Pu, \quad u \in \mathcal{D}'(\Omega),
$$

where the singular support of a distribution $u$(sing supp $u$) is the smallest closed set outside which it is a $C^\infty$-function.

Now we shall study the hypoellipticity of a system of operators which degenerates at $t=0$. When the function $\lambda_0(t; x, \xi)$ which fills the role of the scale of degeneration does not change the sign at $t=0$, $L=\partial_t + p(t; X, D_x)$ and its formal adjoint $L^\ast = -\partial_t + p^\ast(t; X, D_x)$ are hypoelliptic at the origin under appropriate conditions. When $\lambda_0(t; x, \xi) \geq 0$ for $t \geq 0$, only $L^\ast$ is proved to be hypoelliptic.

**Example 4.5.** i) Let $L=\partial_t + t^2 D_x^2$. Then $\lambda_0(t; x, \xi) = t^2 \xi^2$ and $e(t, s; x, \xi) = \exp\left[-\frac{1}{3}(t^3 - s^3)\xi^2\right]$. Thus $e(t, s; x, \xi) \in S^{-\infty}$ for any $s, t$ such that $s < t$.

ii) (Y. Kannai [7]) Let $L=\partial_t + tD_x^2$. Then $\lambda_0(t; x, \xi) = t \xi^2$ and $e(t, s; x, \xi) = \exp\left[-\frac{1}{2}(t^2 - s^2)\xi^2\right]$. Thus the Cauchy problem with a data on $t=0$ has a solution not only towards the future but also towards the past. This means that $L$ is not hypoelliptic at the origin. The solution, if there exist, of the Cauchy problem $Lu=f$, $u(-T, x)=u_T(x)$, $T>0$, has an explicit representation in $(0, T)$, and so does that of $L^\ast u=f$, $u(-T, x)=u_T(x)$ in $(-T, 0)$. These two solutions match up smoothly at $t=0$.

**Theorem 4.6.** Let $p(t; x, \xi) \in \mathcal{B}_2(S^m_{\alpha, \beta})$ in $[-T, T] \times R^*_n$, and let $C(t)$ be an integrable function such that $C(t)>0$ when $t \neq 0$.

i) If there exists a continuous function $\lambda_0(t; x, \xi)$ such that $\lambda_0(t; x, \xi) \geq C(t)\xi^c$ in $[-T, T]$ for a constant $c>0$ and $L=\partial_t + p(t; X, D_x)$ has the property (F) with $\lambda(t; x, \xi) = \lambda_0(t; x, \xi)$, then $L$ is hypoelliptic in $(-T, T) \times R^*_n$.

ii) If there exists a continuous function $\lambda_0(t; x, \xi)$ such that

$$
\begin{align*}
\lambda_0(t; x, \xi) \geq C(t)\xi^c & \quad \text{in } [0, T] \\
\lambda_0(t; x, \xi) \leq -C(t)\xi^c & \quad \text{in } [-T, 0]
\end{align*}
$$

Then $L$ is hypoelliptic in $(-T, T) \times R^*_n$. 

Theorem 4.6.
and the resolvent matrix \( z(t, s; x, \xi) \) of \( L = \partial_t + p(t; X, D_x) \) satisfies

\[
|z(t, s; x, \xi)| \leq C \exp \left[ -d \int_s^t \lambda_d(\sigma; x, \xi) d\sigma \right]
\]

for \( 0 \leq s \leq t \leq T \) or for \( -T \leq t \leq s \leq 0 \), and \( p(t; x, \xi) \) satisfies for any \( \alpha, \beta \)

\[
\int_s^t |p_{(\alpha)}(\sigma; x, \xi)| \, d\sigma \leq C_{\alpha, \beta}^{[\alpha]+[\beta]+1} \int_s^t \lambda_d(\sigma; x, \xi) d\sigma + 1
\]

for \( -T \leq s \leq t \leq T \), then \( \mathcal{L} = -\partial_t + p^*(t; X, D_x) \) is hypoelliptic in \( (-T, T) \times \mathbb{R}^n \).

Before the proof we note that the properly supportedness of operator \( p(t; X, D_x) \) permits us to extend

\[
p(t; X, D_x): \mathcal{E}^0_\alpha((-T, T); \mathcal{S}) \to \mathcal{E}^0_\beta((-T, T); \mathcal{S})
\]

uniquely to

\[
p(t; X, D_x): \mathcal{D}'((-T, T) \times \mathbb{R}^n_+) \to \mathcal{D}'((-T, T) \times \mathbb{R}^n_+).
\]

Let \( H_{\mu}(\infty - \mu < \infty) \) be the usual Sobolev space and let \( H_{\nu, \mu}(\infty < \nu, \mu < \infty) \) be the Sobolev space defined by

\[
H_{\nu, \mu} = \left\{ w \in \mathcal{D}'(R_+^n \times R^n_+); \left\{ \int \int |\partial \tau(\tau, \xi)| \langle \tau \rangle^{2\nu} \langle \xi \rangle^{2\mu} d\tau d\xi < \infty \right\} \right\}.
\]

For an open set \( \Omega \) such that \( \bar{\Omega} \) is a compact subset of \( \mathbb{R}^n \), \( H_{\nu, \mu}^{loc}(\Omega) \) denotes the Fréchet space

\[
\{ v \in \mathcal{D}'(\Omega); \varphi u \in H_{\mu} \quad \text{for any } \varphi \in C_0^\infty(\Omega) \},
\]

and for a rectangular domain \( W = (a, b) \times \Omega \) of \( (-T, T) \times \mathbb{R}^n \), \( H_{\nu, \mu}^{loc}(W) \) denotes the Fréchet space

\[
\{ w \in \mathcal{D}'(W); \psi u \in H_{\nu, \mu} \quad \text{for any } \psi \in C_0^\infty(W) \}.
\]

**Lemma 4.7.** Let \( p(t; x, \xi) \in \mathcal{D}'(S^m_{\nu, \mu}) \) in \([-T, T]\) and let \( u \in \mathcal{D}'((-T, T) \times \mathbb{R}^n_+) \) be a solution of the equation

\[
\partial_t u + p(t; X, D_x)u = f, \quad f \in \mathcal{D}'((-T, T) \times \mathbb{R}^n_+).
\]

If \( f \) is infinitely differentiable in \( W \), then there exists a real number \( \nu_0 \) such that

\[
u_0 \in \cap_{k=0}^m \mathcal{E}^k((-a, b); H_{\nu_0, \nu_0}^{loc}(\Omega)).
\]

**Proof.** We write

\[
\partial_t u = -p(t; X, D_x)u + f
\]

and differentiate both sides \( l \) times with respect to \( t \). Then we have
(4.13) \[ \partial_t^{l+1}u = -\sum_{j=0}^{l} \binom{l}{j} \frac{d^{l-j}}{dt^{l-j}} p(t; X, D_2)(\partial^j_t u) + \partial^l_t f. \]

Now, since \( W \) is compact, there exists a real number \( \nu \) such that \( u \in H^{l+\nu}_{1,1}-m(W) \). Then from (4.11) and \( f \in C^\infty(W) \) we see that
\[ \partial_t u \in H^{l+\nu}_{1,1}-m(W), \]
and using (4.13) inductively for \( l=1, 2, \ldots \), we see that
\[ \partial_t^l u \in H^{l+\nu}_{1,1}-m(W), \quad l = 0, 1, 2, \ldots. \]

Then, using Sobolev's lemma with respect to \( t \) we have
\[ (4.14) \quad u \in \mathcal{E}^{\nu_1+\nu_1}((a, b); H^{l+\nu}_{1,1}-m(W)), \quad l \geq -\nu_1+1. \]

Hence setting \( \nu_0 = \nu_1 - (1-\nu_1)m \), we set (4.12) and the lemma is proved.

**Lemma 4.8.** If \( v \in \bigcap_{k=0}^\infty \mathcal{E}_k((a, b); H_{v_0-km} \cap H_{v_1-\nu_1} \cap \mathcal{E}'((a, b) \times R^*_2), \) then \( v \in C^\nu_0((a, b) \times R^*_2). \)

**Proof.** We shall show that there exists a constant \( M \) such that
\[ (4.15) \quad \langle \tau \rangle^{j+\nu_1} \langle \xi \rangle^j \beta(\tau, \xi) \leq C^{\nu_1}((a, b) \times R^*_2). \]

Then by hypothesis on \( v \) we have
\[ \int \int |(\langle \tau \rangle^{j+\nu_1} \langle \xi \rangle^j \beta(\tau, \xi)|d\tau d\xi < \infty \quad \text{for } j = 0, 1, 2, \ldots. \]

Hence we have \( v \in C^\nu_0((a, b) \times R^*_2). \)

We divide the proof of (4.15) into two cases: I) \( \langle \tau \rangle \geq \langle \xi \rangle^{N_1} \) and II) \( \langle \tau \rangle < \langle \xi \rangle^{N_1} \), where \( N_1 = \max \{(j+1)m - \nu_0, 1\}. \)

When I) holds, we have
\[ (\langle \tau \rangle^{j+\nu_1} \langle \xi \rangle^j \leq 2 \langle \tau \rangle^{j+1} \langle \xi \rangle^{-N_1} \]
\[ \leq 2 \langle \tau \rangle^{j+1} \langle \xi \rangle^{-N_1} \]
\[ \leq 2 \langle \tau \rangle^{j+1} \langle \xi \rangle^{-m}. \]

When II) holds, we have
\[ (\langle \tau \rangle^{j+\nu_1} \langle \xi \rangle^j \leq 2 \langle \xi \rangle^{jN_1} \]
\[ \leq 2 \langle \xi \rangle^{-m} \langle \xi \rangle^{jN_1+m} \]
\[ \leq 2 \langle \xi \rangle^{jN_1} \langle \xi \rangle^{-m} \cdot \]

Thus setting \( M_j = (j+1)|\nu_1|N_1 \) we have (4.15).

If \( L \) satisfies the hypothesis of i) of Theorem 4.6, then it is shown as in the
proof of Theorem 2.7 that there exists the Green's matrix $e(t, s; x, ξ)$ of $L$ such that

\begin{equation}
(4.16) \quad e(t, s; x, ξ) \in \mathcal{B}_i(S^{-\infty}) \quad \text{in } [s+\varepsilon_0, T] \text{ for any } \varepsilon_0 > 0,
\end{equation}

and

\begin{equation}
(4.17) \quad \text{w-lim } \partial_t^j e(t, s; x, ξ) \text{ exists in } S_{ij}^\infty \quad \text{for } j = 0, 1, 2, \ldots.
\end{equation}

Moreover we have the following

**Lemma 4.9.** Let $g_j(t) \in C_0^\infty((a, b); H_m)$ and let $\text{supp } g_j \subset (a, b)$ for $j=1, 2$.

i) If there exists a constant $\varepsilon > 0$ such that $g_j \in C^\infty((-2\varepsilon, 2\varepsilon) \times R^n)$, then

\begin{equation}
(4.18) \quad v_1(x) = \int_{-\varepsilon}^t e(t, s; X, D_x)g_1(\sigma)d\sigma \in C^\infty((-\varepsilon, \varepsilon) \times R^n).
\end{equation}

ii) Let $\gamma_2 \subset \gamma_1$ in $\Omega$. Then

\begin{equation}
(4.19) \quad v_2(x) = \gamma_2(\tau)\int_{-\varepsilon}^t e(t, s; X, D_x)(1-\gamma_1(X'))g_2(\sigma)d\sigma \in C^\infty(W).
\end{equation}

Here and in what follows by "$\gamma_2 \subset \gamma_1$ in $\Omega$" we mean that $\gamma_j \in C^\infty(\Omega)$ for $j=1, 2$ and $\gamma_1=1$ in a neighborhood of supp $\gamma_2$.

Proof. i) By (4.16), (4.17) and hypothesis on $g_1$ we have that $v_1$ is infinitely differentiable with respect to $x$ when $|t| > \varepsilon$. Differentiating $v_1$ with respect to $t$ we have by (2.3) and (2.4)

\begin{equation}
\partial_t v_1(t) = g_1(t) + p(t; X, D_x)v_1(t)
\end{equation}

This shows that $\partial_t v_1(t)$ is also infinitely differentiable with respect to $x$. Iteration of this procedure proves (4.18).

ii) Since $\gamma_j(X)e(t, s; X, D_x)(1-\gamma_1(X')) \in \mathcal{B}_i(S^{-\infty})$ in $(a, T)$, we have that $v_2$ is infinitely differentiable with respect to $x$. By (2.3) and (2.4) we have

\begin{equation}
\partial_t v_2(t) = \int_{-\varepsilon}^t \gamma_2(\tau)p(t; X, D_x)e(t, s; X', D_x')(1-\gamma_1(X'))g_2(\sigma)d\sigma
\end{equation}

and $\partial_t v_2(t)$ is infinitely differentiable with respect to $x$. Iteration of this procedure proves (4.19) and the lemma is proved.

Proof of i) of Theorem 4.6. Let $\Omega$ be an open bounded set of $R^n$ and let $W=(a, b) \times \Omega \ni (0, 0)$. Let $u \in \mathcal{D}'((-T, T) \times R^n)$ be a solution of

\begin{equation}
(4.20) \quad Lu = f, \quad f \in \mathcal{D}'((-T, T) \times R^n) \cap C^\infty(W).
\end{equation}

We shall prove that $u$ is infinitely differentiable in a neighborhood of the origin. Take $\psi(t, x) = X(t)\gamma(x) \in C^\infty(W)$ such that $\psi = 1$ in a neighborhood of the origin. Then by Lemma 4.7 we have
and setting $g = L(\psi u)$ we have
\[(4.22)\]
and setting $g = L(\psi u)$ we have
\[
g(t) \in \bigcap_{k=0}^\infty C^k((a, b); H_{\nu_0-km-\nu}).
\]

Now we write as in the proof of Theorem 2.9
\[
e(t, \sigma; X, D_x) g(\sigma) = e(t, \sigma; X, D_x) \{ \partial_\sigma (\psi u) + p(\sigma; X', D_x)(\psi u) \}
\]
then we have
\[
\psi u \in C^\infty((a, b); H_{\nu_0-km}),
\]
and setting $g = L(\psi u)$ we have
\[(4.22)\]

Now we write as in the proof of Theorem 2.9
\[
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then we have
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\]
then we have
\[
\psi u \in C^\infty((a, b); H_{\nu_0-km}),
\]
and setting $g = L(\psi u)$ we have
\[(4.22)\]

We claim that there exists a constant $\varepsilon > 0$ such that $\gamma \chi[P, \gamma]u$ is infinitely differentiable with respect to $t$ and $x$ for $|t| < 2\varepsilon$. Set $\psi_1(t, x) = \chi(t) \chi(x)$ where $\chi_1 \subset \subset \chi$ in $(a, b)$ and $\chi_1 = 1$ for $|t| < 2\varepsilon$. Then we have $\psi_1 \chi[P, \gamma]u = \psi_2 P((1 - \psi)u)$. Since $P(t; X, D_x)(1 - \psi(t, x'))$ is properly supported, there exists $\psi_2 \in C^\infty((-T, T) \times R^n)$ such that $\psi_2 = 1$ in a neighborhood of the origin and $P\chi_1((1 - \psi)(1 - \psi_2)u) = 0$. Hence we have $\psi_1 \chi[P, \gamma]u = -\psi_2 P((1 - \psi)u)$. Since $\psi_1(t, x) P(t; X, D_x)(1 - \psi(t, x')) \in B_{C^\infty}(S^{-\nu})$ in $(a, b)$ and there exists a constant $\nu_1$ such that $\psi_1 u \in H_{\nu_1, \nu_1}(a, b) \times R^n)$, we have $\psi_1 \chi[P, \gamma]u \in H_{\nu_1, \nu_1} \cap C'(W)$. So by Lemma 4.8 we have $\gamma \chi[P, \gamma]u \in C^\infty((-\varepsilon, \varepsilon) \times R^n)$. Thus we can apply i) of Lemma 4.9 to $g_1$ and we have
\[(4.26)\]
If we choose $\gamma_2$ such that $\gamma_2 \subset \gamma_1$ in $\Omega$ and $\gamma_2=1$ in a neighborhood of $\Omega_0$ the origin, then by (4.24) we can apply ii) of Lemma 4.9 and we have

\begin{equation}
(4.27) \quad \gamma_2(x) \int_{-T}^{T} e(t, \sigma; X, D_s)(1-\gamma_1(X')) g_2(\sigma) d\sigma \in C^\infty(W).
\end{equation}

By (4.23) and (4.25) we have that $\psi u \in C^\infty((-\epsilon, \epsilon) \times \Omega_0)$ and i) of Theorem 4.6 is proved.

Proof of ii) of Theorem 4.6. Let $u \in D'((T, T) \times R^s)$ be a solution of the equation

\begin{equation}
(4.28) \quad 'Lu = f \quad f \in D'((T, T) \times R^s) \cap C^\infty(W).
\end{equation}

As in the proof of Theorem 2.7 we can construct the Green’s matrix $e(\sigma, t; x, \xi)$ of $L_{\sigma, t}$ for $-T < \sigma < t < 0$ which belongs to $B_{\sigma}(S^{-\infty})$ in $[\sigma + \epsilon_0, 0]$ for any $\epsilon_0 > 0$. Since $\partial_\sigma e^*(\sigma, t; x, \xi) + e^*(\sigma, t) \circ p^*(\sigma)(x, \xi) = 0$, we have as in the proof of i)

\begin{equation}
(\psi u)(t, x) = -\int_{-T}^{T} e^*(\sigma, t; X, D_s) g(\sigma) d\sigma \quad \text{for } a < t \leq 0,
\end{equation}

where $g = 'L(\psi u) = \psi f + ['L, \psi] u \in \mathfrak{h}_{k+1}(a, b); H_{\sigma_0-km-m})$. Let $g_1 = -x' \gamma u + 
\gamma_1 \chi[P^*, \gamma] u + \psi f$ and $g_2 = \chi[P^*, \gamma] u$, where $\psi = \chi \gamma$ and $\gamma_1 \subset \gamma$ in $\Omega$ are the functions given in the proof of i). Then we have $g_1 \in C^\infty((-2\epsilon, 2\epsilon) \times R^s)$, $g_j \in \mathfrak{h}_{k+1}(a, b); H_{\sigma_0-km-m})$ and supp $g_j \subset (a, b)$ for $j = 1, 2$. Since $e^*(\sigma, t; x, \xi) \in B_{\sigma}(S^{-\infty})$ in $[\sigma + \epsilon_0, 0)$, $\partial_\sigma e^*(\sigma, t; x, \xi) = p^*(t) \circ e^*(\sigma, t)(x, \xi)$ and $w-lim \partial_\sigma e^*(\sigma, t; x, \xi)$ exists in $S_{\sigma}^{\infty}$ for $-T < t \leq 0$, we have as in the proof of Lemma 4.9 that both $v_j(t) = -\int_{-T}^{T} e^*(\sigma, t; X, D_s) g_j(\sigma) d\sigma$ and $v_k(t) = \gamma_2(x) \int_{-T}^{T} e^*(\sigma, t; X, D_s) \times (1-\gamma_1(X')) g_2(\sigma) d\sigma$ belong to $C^\infty((-\epsilon, 0) \times \Omega_0)$ and every derivative of $v_j$, $j = 1, 2$ is uniformly bounded in $\Omega_0$ as $t \uparrow 0$. In a similar fashion we have that

\begin{equation}
(\psi u)(t, x) = \int_{-T}^{T} e^*(\sigma, t; X, D_s) g(\sigma) d\sigma \quad \text{for } 0 \leq t < b.
\end{equation}

We also have that $\psi u$ is infinitely differentiable in $(0, \epsilon) \times \Omega_0$ and its every derivative is uniformly bounded in $\Omega_0$ as $t \downarrow 0$. Thus $u(t, x)$ is infinitely differentiable in $(-\epsilon, 0] \times \Omega_0 \cup [0, \epsilon) \times \Omega_0$.

Now we shall show according to Y. Kannani [7] that the boundary values of $u(t, x)$ and its derivatives as $t$ tends to zero from the right and from the left actually match up and that $u$ has no singular part with support on the line $t = 0$.

That is, we shall prove

\begin{equation}
(4.29) \quad \lim_{t \to 0} (\partial_\sigma D_s u(t, x) - (\partial_\sigma D_s u)(-t, x)) = 0 \quad \text{for all } j \text{ and } \alpha,
\end{equation}
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and that the distribution \( v \in \mathcal{D}'(w_0) \) defined by

\[
\langle v, \varphi \rangle = \langle u, \varphi \rangle - \lim_{\varepsilon \to 0} \left[ \int_{-\varepsilon}^{\varepsilon} \int \left( u \varphi \right)(t, x) dx dt , \ v(t, x) \right]
\]

is the zero distribution. The functional \( \langle v, \varphi \rangle \) is well defined because of the existence of boundary values for \( u(t, x) \) as \( t \uparrow 0 \) and \( t \downarrow 0 \), and obviously \( \text{supp } v \subset \{(t, x); t=0\} \). From (4.30) we have

\[
\langle Lv, \varphi \rangle = \langle f, \varphi \rangle - \lim_{\varepsilon \to 0} \left[ \int_{-\varepsilon}^{\varepsilon} \int (Lu) \varphi dx dt \right]
\]

Since \( f \in C^\infty(w) \), we have

\[
\langle Lv, \varphi \rangle = \lim_{\varepsilon \to 0} \int ([u \varphi](-\varepsilon, x) - [u \varphi](\varepsilon, x)) dx
\]

that is

\[
(4.31) \quad Lv = \delta \otimes \{u(0, x) - u(-0, x)\}
\]

According to L. Schwartz [14], we may write locally

\[
v = \sum_{j=1}^{N} Ev_j \partial_t^j
\]

where \( v_j \in \mathcal{D}'(R^*_t \cap w_0) \) and \( E \) is the natural inclusion map \( E: \mathcal{D}'(R^*_t \cap w_0) \to \mathcal{D}'(w_0) \). Thus we find by (4.31) that \( E v_n \partial_t^{n+1} \) have to vanish, and therefore all the \( v_j \) have to vanish, so that \( v = 0 \). Hence we have by (4.31)

\[
u(0, x) - u(-0, x) = 0.
\]

Differentiating the equation \( 'Lu = f \) and iterating the same argument we have (4.29).

**Corollary 4.9.** Let \( L \) be the operator which satisfies the condition of Theorem 3.1, and let \( a_{j,h}(t; x, \xi) \in \mathcal{B}(S^{(i+k)m/(i+l+1)} \mid -T, T) \) be properly supported. Then we have i) if \( l \) is even, then both \( L \) and \( 'L \) are hypoelliptic, and ii) if \( l \) is odd, then \( 'L \) is hypoelliptic.

**Example 4.10.** B. Helffer [6] proved the hypoellipticity of the operator

\[
L = \partial_t - a_{2m}(t; X, D_x) + \sum_{j=0}^{2m-1} a_{j}(t; X, D_x),
\]

where \( a_{j}(t; x, \xi) j=0, 1, \cdots, 2m \) is a polynomial of homogeneous order \( j \) in \( \xi \) with \( C^\infty \)-coefficients, under the following three conditions:

\[
[H.1] \quad \text{Re} \int_{s}^{t} a_{2m}(\sigma; x, \xi) d\sigma \geq C |t-s|^{k+1} |\xi|^{2m}.
\]
There exist real constants $\theta$ and $\tau$ such that for any $\alpha$, $\beta$ and $j$ which satisfy

$$|\alpha| + |\beta| + j > 0, \quad (|\alpha| + j)\theta + (|\beta| + j)\tau \leq 1,$$

we have with some constant $C_{j, \alpha, \beta}$

$$|a_{2m-j}^{(n)}(t; x, \xi)| \leq C_{j, \alpha, \beta} |\Re a_{2m}^{(n)}| |\xi|^{2m}.$$

If we assume that $2m\kappa/(k+1) < 1$, then the above $L$ satisfies the condition ii) of Theorem 4.6 with $\lambda_0(t; x, \xi) = \Re a_{2m}(t; x, \xi)$, $\kappa = \min\{1, 1 - \nu\theta\}$ and $\delta = \max\{0, \nu\tau\}$, where $\nu = 2m\kappa/(k+1)$.

Proof. Set $\mu = (|\alpha| + j)\theta + (|\beta| + j)\tau$. When $\mu < 0$, since $|\Re a_{2m}^{(n)}| |\xi|^{2m\mu}$ is bounded, we have by [H.2]

$$|a_{2m-j}^{(n)}(t; x, \xi)| \leq C_{j, \alpha, \beta} |\Re a_{2m}^{(n)}| |\xi|^{2m\mu - j}$$

Thus we have

$$(4.32) \quad \int_s^t |a_{2m-j}^{(n)}(\sigma; x, \xi)| d\sigma \leq C_{j, \alpha, \beta} \left\{ \int_s^t \lambda(\sigma; x, \xi)d\sigma + 1 \right\}.$$
When $\mu > 1$, by [H.3] we have
\[
2m - j - |\alpha| = -(1 - \nu \theta) |\alpha| + \nu \tau |\beta| + 2m - j - \nu (\theta |\alpha| + \tau |\beta|)
\]
\[
= 2m/(k+1) - (1 - \nu \theta) |\alpha| + \nu \tau |\beta| + \nu (1 - \mu)
\]
\[
+ j \{\nu(\theta + \tau) - 1\}
\]
\[
\leq 2m/(k+1) - (1 - \nu \theta) |\alpha| + \nu \tau |\beta|.
\]
Thus we have
\[
|a_{2m-j}[\sigma(t; x, \xi)]| \leq C_{j,a,b} c \xi^{-2m-j-|\sigma|}
\]
\[
\leq C_{j,a,b} c \xi^{-(1-\nu \theta)|\alpha| + \nu \tau |\beta| - 2m/(k+1)}.
\]

Hence by using Lemma 1.13 we have
\[
(4.34) \int |a_{2m-j}[\sigma(x, \xi)]| d\sigma \leq C_{j,a,b} c \xi^{-(1-\nu \theta)|\alpha| + \nu \tau |\beta| - 2m/(k+1)}
\]
\[
\times \left\{ \int \lambda(\sigma; x, \xi) d\sigma + 1 \right\}.
\]

By (4.32), (4.33) and (4.34) we have that $L$ satisfies (2.7). Since $L$ is a single equation, (2.8) is easily verified. Thus the proof is complete.

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References


