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THE SYMBOL CALCULUS FOR THE FUNDAMENTAL SOLUTION OF A DEGENERATE PARABOLIC SYSTEM WITH APPLICATIONS

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Introduction

In the paper [2] S. D. Eidelman has constructed the fundamental solution of a system of partial differential operators which is parabolic in the sense of Petrowski with sufficiently smooth coefficients. A few years later, the assumptions on the smoothness of the coefficients have been weakened to uniform Hölder continuity. The bibliography and bibliographical remarks concerning this topics are found in A. Friedman's book [4]. The applications of the fundamental solution to the study of the Cauchy problem and other related problems are found in the above book and S. D. Eidelman's book [3]. On the other hand, if the coefficients are sufficiently smooth, the recent results of the theory of pseudo-differential operators, especially that of H. Kumano-go [8] and [9], have enabled us to construct a symbol of the fundamental solution of a parabolic operator which may be of degenerate type through only the symbol calculus. (See the paper C. Tsutsumi [18].)

In the present paper we shall, using a method similar to that of [18], construct the fundamental solution of a degenerate parabolic system $L=\partial_t+p(t; X, D_x)$ which has the property (F) (See the Definition 2.2). A system of partial differential operators which is parabolic in the sense of Petrowski with C^∞-coefficients has this property, and so do the operators treated in T. Matsuzawa [11], B. Helffer [6], C. Tsutsumi [18] and M. Miyake [12]. In the papers [11], [6] and [12], a family of parametrices $K_0+K_1+\cdots+K_j$ of the operator L is constructed so that they satisfy the equation $L_{t,x}(\sum_{k=0}^{j}K_k(x, y, t, t'))=\delta(x-y, t-t')$ $+F_j(x, y, t, t',)$, and $K_1, \cdots, K_j$ and F_j are very regular. In [18] and the present paper, however, the fundamental solution is constructed in the class of pseudo-differential operators.

In section 1 we shall give some lemmas on the symbol calculus. In section 2 the matrix $e(t, s; x, \xi)$ of symbols of fundamental solution will be constructed and its asymptotic expansion will be given in a very natural form (See the formula (2.23)). In section 3 the general result of section 2 is applied to a degenerate parabolic operator of higher order of the form

$$L=\partial_t^M+\sum_{k=0}^{l} t^k a_{1,k}(t;X,D_x)\partial_t^{M-1}+\cdots+\sum_{k=0}^{Ml} t^k a_{M,k}(t;X,D_x),$$

where l is a positive integer and $a_{j,k}\in\mathcal{B}_t^0(S_{\rho,\delta}^{(j+k)m/(l+1)})$, $j=1,2,\cdots,M$, $k=1,2,\cdots,jl$, to obtain a result including that of M. Miyake [12]. In section 4 using the symbol of the fundamental solution and following the idea of Y. Kannai [7], we shall give sufficient conditions for the operator L to be hypoelliptic. This gives an example of a hypoelliptic operator with multiple characteristics (See Theorem II.1.1 in F. Treves [17]). It will also be shown that the operator treated by B. Helffer [6] satisfies our hypothesis under some additional restriction.

The results of the present paper have been announced partly in [15] and [16].

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1. Definitions and lemmas

Let $x\in R_x^n$, $\xi\in R^n$ and let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ be a multi-index of non-negative integers. We use the following notation:

$$|x|=(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}, \qquad \langle x\rangle=(1+|x|^2)^{1/2},$$
$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n, \qquad \alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!,$$
$$x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \qquad x\cdot\xi=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n,$$
$$\partial_\xi^\alpha=\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\cdots\partial_{\xi_n}^{\alpha_n} \quad \text{where} \quad \partial_{\xi_j}=\partial/\partial\xi_j$$
$$D_x^\alpha=D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}\cdots D_{x_n}^{\alpha_n} \quad \text{where} \quad D_{x_j}=-i\partial/\partial_j.$$

According to L. Schwartz [14] we use the notation $\mathcal{D}(\Omega)$, $\mathcal{D}=\mathcal{D}(R^n)$, $\mathcal{D}'(\mathcal{D})$, $\mathcal{D}'=\mathcal{D}'(R^n)$, $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{B}=\mathcal{B}(R^n)$ to denote the spaces of M-dimensional vector valued functions and distributions. For an interval J of R_t^1 we denote by $\mathcal{E}_t^k(J;\mathcal{B})$ the space of k times continuously differentiable functions of $t\in J$ with values in $\mathcal{B}$. We set $\mathcal{E}_t(J;\mathcal{B})=\bigcap_m\mathcal{E}_t^m(J;\mathcal{B})$.

Definition 1.1 ([10]). We say that an $M\times M$ matrix $p(x,\xi)$ with components $p_{j,k}(x,\xi)\in C^\infty(R_x^n\times R_\xi^n)$ belongs to $S_{\rho,\delta}^m$, $-\infty<m<\infty$, $0\le\delta<\rho\le1$, when for any α, β there exists a constant $C_{\alpha,\beta}$ such that

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \tag{1.1}$$

where $p_{(\beta)}^{(\alpha)}(x,\xi)=\partial_\xi^\alpha D_x^\beta p(x,\xi)$ and $|p|$ denotes the norm of the matrix p defined by

$$|p|=\sup\{|py|/|y|;\, y\in C^N,\, y\neq0\}.$$

We define the corresponding operator $p(X, D_x)$ by

$$p(X, D_x)u(x) = \int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)đ\xi\,, \qquad u\in\mathcal{S}\,, \tag{1.2}$$

where $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$ and $đ\xi=(2\pi)^{-n}d\xi$. We call a linear map P of $\mathcal{S}$ into $\mathcal{S}$ a pseudodifferential operator with symbol $p(x, \xi)\in S^m_{\rho,\delta}$ if $P=p(X, D_x)$ and we also write $P\in S^m_{\rho,\delta}$. We set $S^{\infty}_{\rho,\delta}=\bigcup_m S^m_{\rho,\delta}$ and $S^{-\infty}=\bigcap_m S^m_{\rho,\delta}$.

N. B. Throughout this paper we assume that ρ and δ satisfy the condition $0\leq\delta<\rho\leq 1$.

DEFINITION 1.2 ([10]). For a $p(x, \xi)\in S^m_{\rho,\delta}$ we define semi-norms $|p|^{(m)}_l$, $l=0, 1, 2, \cdots$, by

$$|p|^{(m)}_l = \max_{|\alpha+\beta|\leq l}\ \sup_{(x,\xi)}\{|p^{(\alpha)}_{(\beta)}(x, \xi)|\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}\}\,. \tag{1.3}$$

Then $S^m_{\rho,\delta}$ makes a Fréchet space with these semi-norms. For a symbol $p(t; x, \xi)$ with a parameter t we write

$$p(t; x, \xi)\in\mathcal{B}^k_t(S^m_{\rho,\delta}) \qquad \text{in} \quad (s, T)\,,$$

when $p(t; x, \xi)$ is a k times continuously differentiable $S^m_{\rho,\delta}$-valued function in $s<t<T$. A subset B of $S^m_{\rho,\delta}$ is said to be a bounded set in $S^m_{\rho,\delta}$ when $\sup\{|p|^{(m)}_l; p\in B\}<\infty$ for $l=0, 1, 2,\cdots$. We write

$$\operatorname*{w-lim}_{t\downarrow s} p(t; x, \xi) = p(s; x, \xi) \qquad \text{in} \quad S^m_{\rho,\delta}\,,$$

when there exists a constant c such that $\{p(t; x, \xi); t\in(s, s+c)\}$ is a bounded set in $S^m_{\rho,\delta}$ and for any and α, β for any compact set $K\subset R^n_\xi$, $p^{(\alpha)}_{(\beta)}(t; x, \xi)$ converges to $p^{(\alpha)}_{(\beta)}(s; x, \xi)$ uniformly in $R^n_x\times K$ as t tends to s.

In order to treat a product of pseudodifferential operators, we introduce the oscillatory integral and multiple symbols.

DEFINITION 1.3 ([10]). We say that an $M\times M$ matrix $a(\eta, y)$ with components $a_{j,k}(\eta, y)\in C^\infty(R^{\nu n}_\eta\times R^{\nu n}_y)$ belong to a class $\mathcal{A}^m_{\delta,\tau}$, $-\infty<m<\infty$, $0\leq\delta<1$, $0\leq\tau$ when for any multi-index α, β, we have

$$|\partial^\alpha_\eta\partial^\beta_y a(\eta, y)|\leq C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^\tau \tag{1.4}$$

for a constant $C_{\alpha,\beta}$. For an $a(\eta, y)\in\mathcal{A}^m_{\delta,\tau}$ we define semi-norms $|a|_l$, $l=0, 1, 2, \cdots$, by

$$|a|_l = \max_{|\alpha+\beta|\leq l}\ \sup_{(\eta,y)}\{|\partial^\alpha_\eta\partial^\beta_y a(\eta, y)|\langle\eta\rangle^{-m-\delta|\beta|}\langle y\rangle^{-\tau}\}\,. \tag{1.5}$$

Then $\mathcal{A}^m_{\delta,\tau}$ makes a Fréchet space. We set $\mathcal{A}=\bigcup_{0\leq\delta<1}\ \bigcup_{-\infty<m<\infty}\ \bigcup_{0\leq\tau}\mathcal{A}^m_{\delta,\tau}$. We say that a subset B of $\mathcal{A}$ is bounded when $B\subset\mathcal{A}^m_{\delta,\tau}$ for some δ, m and τ, and

$\sup\{|a|_l;\ a\in B\}<\infty$ for any l. For an $a(\eta, y)\in\mathcal{A}^m_{\delta,\tau}$ we define the oscillatory integral $Os[e^{-iy\cdot\eta}a]$ by

$$(1.6)\qquad Os[e^{-iy\cdot\eta}a] = Os-\iint e^{-iy\cdot\eta}a(\eta, y)dy\,d\eta$$
$$= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta, \varepsilon y)a(\eta, y)dy\,d\eta\,,$$

where $\chi(\eta, y)\in\mathcal{S}(R^{2\nu n}_{\eta,y})$ such that $\chi(0, 0)=1$.

The following lemma proves the well-definedness of (1.6).

Lemma 1.4 ([8]). *For an $a(\eta, y),\in\mathcal{A}^m_{\delta,\tau}$ let l and l' be positive integers such that*

$$(1.7)\qquad -2l(1-\delta)+m<-\nu n,\quad -2l'+<\tau-\nu n\,.$$

Then we have

$$(1.8)\qquad Os[e^{-iy\cdot\eta}a] = \int e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_y\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_\eta\rangle^{2l}a(\eta, y)\}dy\,d\eta$$

and

$$(1.9)\qquad |Os[e^{-iy\cdot\eta}a]|\leq C|a|_{l+l'}$$

for a constant C which is independent of $a(\eta, y)$.

Lemma 1.5 ([9]). *Let $\{a(t;\eta, y)\}_{0<t\leq 1}$ be a bounded set of $\mathcal{A}$. Suppose that there exists an $a(0;\eta, y)\in\mathcal{A}$ such that $a(t;\eta, y)\to a(0;\eta, y)$ as $t\to 0$ uniformly on any compact set of $R^{2\nu n}_{\eta,y}$. Then we have*

$$(1.10)\qquad \lim_{t\to\infty} Os[e^{-iy\cdot\eta}a(t)] = Os[e^{-iy\cdot\eta}a(0)]\,.$$

DEFINITION 1.6 ([9]). i) We say that an $M\times M$ matrix $p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)$ whose components are C^∞-functions defined in $R^{(2\nu+1)n}$ is a multiple symbol of class $S^{(m_1,m_2,\cdots,m_\nu)}$, when for any $\alpha^1, \cdots, \alpha^\nu, \beta^0, \beta^1, \cdots, \beta^\nu$, there exists a constant $C=C(\alpha^1, \cdots, \alpha^\nu, \beta^0, \beta^1, \cdots, \beta^\nu)$ such that

$$(1.11)\qquad |\partial^{\alpha^1}_{\xi^1}\cdots\partial^{\alpha^\nu}_{\xi^\nu}\partial^{\beta^0}_{x^0}\partial^{\beta^1}_{x^1}\cdots\partial^{\beta^\nu}_{x^\nu}p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)|$$
$$\leqq C\langle\xi^1\rangle^{\delta|\beta^0|}\prod_{j=1}^{\nu}\langle\xi^j\rangle^{m_j-\rho|\alpha^j|}(\langle\xi^j\rangle+\langle\xi^{j+1}\rangle)^{\delta|\beta^j|}\,,$$

where $\xi^{\nu+1}=0$. For a $p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)\in S^{(m_1,m_2,\cdots,m_\nu)}_{\rho,\delta}$ we define semi-norms $|p|^{(m_1,m_2,\cdots,m_\nu)}_{l,l'}$, $l, l'=0, 1, 2, \cdots$, by

$$(1.12)\qquad |p|^{(m_1,m_2,\cdots,m_\nu)}_{l,l'} = \max_{|\alpha|\leq l,|\beta^j|\leq l'}\inf\{C \text{ of } (1.11)\}\,.$$

Then $S^{(m_1,m_2,\cdots,m_\nu)}_{\rho,\delta}$ makes a Fréchet space.

ii) The associated pseudodifferential operator $P=p(X^0, D_{x^1}, X^1, \cdots, D_{x^\nu}, X^\nu)$

with multiple symbol $p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)$ is defined by

$$(1.13)\qquad Pu(x) = Os-\iint e^{-i(y^1\cdot\eta^1+\cdots+y^\nu\cdot\eta^\nu)}p(x^1, \eta^1, x+y^1, \eta^2, \cdots, \eta^\nu, x+y^1+\cdots+y^\nu)u(x+y^1+\cdots+y^\nu)dy^1d\eta^1\cdots dy^\nu d\eta^\nu$$

for $u\in\mathcal{B}$.

REMARK 1.7. i) The pseudodifferential operator P defined by (1.2) is extended to a continuous operator P: $\mathcal{B}\to\mathcal{B}$ by setting $\nu=1$ in (1.13).

ii) When $p_j(x, \xi)\in S^{m_j}_{\rho,\delta}$, $j=1, 2, \cdots, \nu$, we have that $p_1(x^0, \xi^1)p_2(x^1, \xi^2)\cdots \times p_\nu(x^{\nu-1}, \xi^\nu)\in S^{(m_1,\cdots,m_\nu)}_{\rho,\delta}$.

Lemma 1.8 ([10]). *Let $p(t; x, \xi)\in\mathcal{B}^0_t(S^m_{\rho,\delta})$ in (s, T) Then we have*

$$(1.14)\qquad \text{if } \operatorname*{w-lim}_{t\downarrow s} p(t; x, \xi) = p(s; x, \xi) \text{ in } S^m_{\rho,\delta}, \text{ then } \lim_{t\downarrow s} P(t)u = P(s)u \quad \text{for any } u\in\mathcal{B}\,.$$

Lemma 1.9 ([9]). *For a $p(x^0, \xi^1, x^1, \cdots, \xi^\nu)\in S^{(m_1,m_2,\cdots,m_\nu)}_{\rho,\delta}$, set*

$$(1.15)\qquad q(x, \xi) = Os-\iint e^{-i(y^1\cdot\eta^1+\cdots+y^{\nu-1})\cdot\eta^{\nu-1}} \times p(x, \xi+\eta^1, x+y^1, \cdots, \xi+\eta^{\nu-1}, x+y^1+\cdots+y^{\nu-1}, \xi) \times dy^1d\eta^1\cdots dy^{\nu-1}d\eta^{\nu-1}\,.$$

Then we have, for $m=m_1+m_2+\cdots+m_\nu$,

$$(1.16)\qquad q(x, \xi)\in S^m \quad \text{and} \quad q(X, D_x) = p(X^0, D_{x^1}, \cdots, X^\nu, D_{x^\nu})\,.$$

Furthermore, for any l there exists a constant C such that

$$(1.17)\qquad |q|^{(m)}_l\le C^\nu|p|^{(m_1,\cdots,m_\nu)}_{l_0,l_0'},$$

where

$$(1.18)\qquad l_0 = l+2[n/2+1]\,,\quad l_0' = l+2[(n+\sum_{j=1}^{\nu}|m_j|+\rho l+\delta l)/(2(1-\delta))+1]\,.$$

Lemma 1.10 ([10]). *For a $p(x, \xi)\in S^m_{\rho,\delta}$, set*

$$(1.19)\qquad p^*(x, \xi) = Os-\iint e^{-iy\cdot\eta}p(x+y, \xi+\eta)^{(*)}dy\,d\eta\,,$$

where $p^{(*)}$ is the conjugate transpose of the matrix p. Then $P^*=p^*(X, D_x)$ is the formal adjoint of $P=p(X, D_x)$ in the sense that $(Pu, v)=(u, P^*v)$ for any $u, v\in S$. Let $L=\partial_t+p(X, D_x)$. Then the formal adjoint ${}^tL$ of L in $R^1_t\times R^n_x$ is given by

$$(1.20)\qquad {}^tL=-\partial^t+p^*(X, D_x)\,.$$

For $p_j(x, \xi) \in S^{m_j}_{\rho,\delta}$, $j=1, 2, \cdots, \nu$, we denote by $p_1 \circ p_2 \circ \cdots \circ p_\nu(x, \xi)$ the symbol of the product $P_1 P_2 \cdots P_\nu$ of pseudodifferential operators $P_j = p_j(X, D_x)$. By Lemma 1.9 we have

$$(1.21) \qquad p_1 \circ p_2 \circ \cdots \circ p_\nu(x, \xi) = Os-\iint e^{-i(y^1 \cdot \eta^1 + \cdots + y^{\nu-1} \cdot \eta^{\nu-1})} p_1(x, \xi+\eta^1)$$
$$p_2(x+y^1, \xi+\eta^2) \cdots p(x+y^1+\cdots+y^{\nu-1}, \xi) dy^1 d\eta^1 \cdots dy^{\nu-1} d\eta^{\nu-1}.$$

For $k=1, 2, \cdots$, we set

$$(1.22) \quad [p_1 \circ p_2 \circ \cdots \circ p_\nu]_k(x, \xi) = \sum_{|\alpha_2^1+\alpha_3^1+\cdots+\alpha_\nu^{\nu-1}|=k} \frac{1}{\alpha_2^1! \alpha_3^1! \cdots \alpha_\nu^{\nu-1}!} p_1^{(\alpha_2^1+\cdots+\alpha_\nu^1)}(x, \xi)$$
$$\times p_{2,(\alpha_2^1)}^{(\alpha_3^2+\cdots+\alpha_\nu^2)}(x, \xi) \cdots p_{\nu-1,(\alpha_{\nu-1}^1+\cdots+\alpha_{\nu-1}^{\nu-2})}^{(\alpha_\nu^{\nu-1})}(x, \xi) p_{\nu,(\alpha_\nu^1+\cdots+\alpha_\nu^{\nu-1})}(x, \xi),$$

and for $k=0$ we set

$$(1.23) \qquad [p_1 \circ p_2 \circ \cdots \circ p_\nu]_0(x, \xi) = p_1(x, \xi) p_2(x, \xi) \cdots p_\nu(x, \xi).$$

Lemma 1.11 ([13]). *Let $p_j(x, \xi) \in S^{m_j}_{\rho,\delta}$, $j=0, 1, \cdots, \nu$. Thrn we have:*

i) *For any positive integer N,*

$$(1.24) \qquad p_1 \circ p_2 \circ \cdots \circ p_\nu(x, \xi) - \textstyle\sum_{k=0}^{N-1} [p_1 \circ p_2 \circ \cdots \circ p_\nu]_k(x, \xi) \in S^{m-(\rho-\delta)N}_{\rho,\delta},$$

where $m = m_1 + m_2 + \cdots + m_\nu$.

ii) *For $k=0, 1, 2, \cdots$,*

$$(1.25) \qquad [p_0 \circ p_1 \circ p_2 \circ \cdots \circ p_\nu]_k(x, \xi) = \textstyle\sum_{\mu=0}^{k} \sum_{|\alpha|=\mu} \dfrac{1}{\alpha!} p_0^{(\alpha)}(x, \xi)$$
$$\times [p_1 \circ p_2 \circ \cdots \circ p_\nu]_{k-\mu,(\alpha)}(x, \xi).$$

Note. A proof of (1.24) by using Taylor's expansion has been given in [9].

Lemma 1.12. *Let p be an $M \times M$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_M$ be the eigenvalues of the matrix p, and*

$$\lambda = \min_j \operatorname{Re} \lambda_j,$$

where $\operatorname{Re} \lambda_j$ means the real part of λ_j. Then the inequality

$$(1.26) \qquad |\exp[-tp]| \leq \textstyle\sum_{j=0}^{M-1} (2t|p|)^j \exp[-t\lambda]$$

holds for $t \geq 0$.

For the proof, see Gelfand and Shilov [5] Chap. II, §6.

Lemma 1.13. *Let $a > 1$. Then for any $\varepsilon > 0$ we have the following inequality*

$$(1.27) \qquad v - u \leq \varepsilon(v^a - u^a) + \varepsilon^{-1/(a-1)}$$

for any u and v such that $0 \leq u < v$.

Proof. Let $f(t)=t^a$, $t\geq 0$. Since $df(t)/dt$ is an increasing function of t, we have

$$f(v-u)-f(0)\leq f(v)-f(u)\,.$$

Thus we have $(v-u)^a\leq v^a-u^a$, i.e., $v-u\leq(v-u)^{1-a}(v^a-u^a)$. If $(v-u)^{1-a}\leq\varepsilon$, we have $v-u\leq\varepsilon(v^a-u^a)$. If $(v-u)^{1-a}\geq\varepsilon$, we have $v-u\leq\varepsilon^{-1/(a-1)}$

Lemma 1.14. *Let $f(t;\,x,\,\xi)$ be a non-negative continuous function if there exist constants $C>0$ and c such that $1/(l+1)\leq c<1$ and*

$$tf(t;\,x,\,\xi)\leq C(t^{l+1}\langle\xi\rangle^m)^c \qquad \text{for} \quad t\geq 0\,, \tag{1.28}$$

then for any $\varepsilon>0$ there exists a constant C' such that

$$\int_s^t f(\sigma;\,x,\,\xi)d\sigma\leq\varepsilon\int_s^t \sigma^l\langle\xi\rangle^m d\sigma+C' \qquad \text{for} \quad 0\leq s\leq t\,. \tag{1.29}$$

Proof. Since $f(\sigma;\,x,\,\xi)\leq C\sigma^{c(l+1)-1}\langle\xi\rangle^{cm}$, we have

$$\int_s^t f(\sigma;\,x,\,\xi)d\sigma\leq Cc^{-1}(l+1)^{-1}\{t^{c(l+1)}\langle\xi\rangle^{cm}-s^{c(l+1)}\langle\xi\rangle^{cm}\}\,.$$

If we set $v=t^{c(l+1)}\langle\xi\rangle^{cm}$, $u=s^{c(l+1)}\langle\xi\rangle^{cm}$ and $a=1/c$, then we have by Lemma 1.13 that

$$\int_s^t f(\sigma;\,x,\,\xi)d\sigma\leq Cc^{-1}(l+1)^{-1}\{\varepsilon'(t^{l+1}\langle\xi\rangle^m-s^{l+1}\langle\xi\rangle^m)+C''\}\,.$$

This proves the Lemma.

2. Fundamental solution and Cauchy problem

In this section we first consider the fundamental solution of the Cauchy problem

$$Lu(t,\,x)=f(t,\,x) \qquad \text{in} \quad (0,\,T)\,, \tag{2.1}$$

$$u(0,\,x)=u_0(x)\,, \tag{2.2}$$

where $L=\partial_t+p(t;\,X,\,D_x)$ and $p(t;\,x,\,\xi)\in\mathcal{B}_t^0(S^m_{\rho,\delta})$ in $[0,\,T]$ and then apply it to the solution of (2.1)–(2.2).

DEFINITION 2.1. i) By an $M\times M$ matrix $e(t,\,s;\,x,\,\xi)\in\mathcal{B}_t^0(S^0_{\rho,\delta})\cap\mathcal{B}_t^1(S^m_{\rho,\delta})$ with parameters t and s, we denote, the fundamental solution of the Cauchy problem (2.1) and (2.2), that is, $e(t,\,s;\,x,\,\xi)$ is the solution of the system of symbol equations

$$\partial_t e(t,\,s;\,x,\,\xi)+p(t)\circ e(t,\,s)(x,\,\xi)=0\,, \qquad 0\leq s<t<T\,, \tag{2.3}$$

and satisfies the initial condition

$$\text{w-}\lim_{t \downarrow s} e(t, s; x, \xi) = I \quad \text{in} \quad S^0_{\rho,\delta}, \tag{2.4}$$

where I is the identity matrix. We call $e(t, s; x, \xi)$ the Green's matrix of the operator L.

ii) We say that an $M \times M$ matrix $z(t, s; x, \xi) \in \mathcal{B}^0_t(S^m_{\rho,\delta}) \cap \mathcal{B}^1_t(S^{2m}_{\rho,\delta})$ is the resolvent matrix of the operator L, when $z(t, s; x, \xi)$ satisfies

$$\partial_t z(t, s; x, \xi) + p(t; x, \xi) z(t, s; x, \xi) = 0, \qquad 0 \leq s < t < T \tag{2.5}$$

and

$$\text{w-}\lim_{t \downarrow s} z(t, s; x, \xi) = I \quad \text{in} \quad S^m_{\rho,\delta}. \tag{2.6}$$

DEFINITION 2.2. We say that the operator L has the property (F), when for some non-negative continuous function $\lambda(t; x, \xi)$ the following two conditions are satisfied:

i) For any α, β there exists a constant $C_{\alpha,\beta}$ such that

$$\int_s^t |p^{(\alpha)}_{(\beta)}(\sigma; x, \xi)| d\sigma \leq C_{\alpha,\beta} \langle \xi \rangle^{-\rho|\alpha| + \delta|\beta|} \left\{ \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right\} \quad \text{for} \quad 0 \leq s \leq t \leq T. \tag{2.7}$$

ii) There exist constants $d > 0$ and $C > 0$ such that the resolvent matrix $z(t, s; x, \xi)$ of L satisfies

$$|z(t, s; x, \xi)| \leq C \exp\left[-d \int_s^t \lambda(\sigma; x, \xi) d\sigma \right] \qquad \text{for} \quad 0 \leq s \leq t \leq T. \tag{2.8}$$

When L is a system of partial differential operators, it is said to be parabolic in the sense of Petrowski if the real part of each eigenvalue of the matrix $p(t; x, \xi)$ is not less than $d\langle \xi \rangle^m$. In this case, if the coefficients of L are C^∞, the property (F) is satisfied with $\lambda(t; x, \xi) = \langle \xi \rangle^m$ and stable under the small perturbation of the principal part and any lower order terms. The property (F) is also stable in the following sense.

Lemma 2.3. *Let $L = \partial_t + p(t; X, D_x)$ have the property (F) with $\lambda(t; x, \xi)$, d and C, and let for $q(t; x, \xi) \in \mathcal{B}^0_t(S^m_{\rho,\delta})$ there exist constants ε, C' and $C_{\alpha,\beta}$ such that $0 < \varepsilon < d/C$,*

$$\int_s^t |q(\sigma; x, \xi)| d\sigma \leq \varepsilon \int_s^t \lambda(\sigma; x, \xi) d\sigma + C' \tag{2.9}$$

and

$$\int_s^t |q^{(\alpha)}_{(\beta)}(\sigma; x, \xi)| d\sigma \leq C_{\alpha,\beta} \langle \xi \rangle^{-\rho|\alpha| + \delta|\beta|} \left\{ \int_s^t \lambda(\sigma; x, \xi) d\sigma + 1 \right\}. \tag{2.10}$$

Then the operator $L' = \partial_t + p(t; X, D_x) + q(t; X, D_x)$ has the property (F) with $\lambda(t; x, \xi)$, d' and C'', where $d' = d - \varepsilon C$ and $C'' = C \exp[CC']$.

Proof. Let $z'(t, s; x, \xi)$ be the resolvent matrix of L'. Since it follows from (2.9) and (2.10) that i) of the property (F) is satisfied, we have only to prove ii). Since we can write

$$z'(t, s; x, \xi) = z(t, s; x, \xi) - \int_s^t z(t, \sigma; x, \xi) q(\sigma; x, \xi) z'(\sigma, s; x, \xi) d\sigma\,,$$

using (2.8) we have

$$\begin{aligned} |z'(t, s; x, \xi)| \leq & C \exp\Big[-d\int_s^t \lambda(\sigma; x, \xi) d\sigma\Big] \\ & + C\int_s^t \exp\Big[-d\int_\sigma^t \lambda(\sigma'; x, \xi) d\sigma'\Big] |q(\sigma; x, \xi)| \\ & \times |z'(\sigma, s; x, \xi)| d\sigma\,. \end{aligned}$$

Setting $\varphi(t) = |z'(t, s; x, \xi)| \exp\Big[d\int_s^t \lambda(\sigma; x, \xi) d\sigma\Big]$, we have

$$\varphi(t) \leq C + C\int_s^t \varphi(\sigma) |q(\sigma; x, \xi) d\sigma\,. \tag{2.11}$$

Multiplying both sides by $|q(t; x, \xi)| / \Big\{1 + \int_s^t \varphi(\sigma) |q(\sigma; x, \xi)| d\sigma\Big\}$ and integrating them, we have

$$\log\Big[1 + \int_s^t \varphi(\sigma) |q(\sigma; x, \xi)| d\sigma\Big] \leq C\int_s^t |q(\sigma; x, \xi)| d\sigma\,.$$

Thus we have

$$C + C\int_s^t \varphi(\sigma) |q(\sigma; x, \xi)| d\sigma \leq C \exp\Big[C\int_s^t |q(\sigma; x, \xi)| d\sigma\Big]\,.$$

Hence by (2.11) and (2.9) we have

$$|z'(t, s; x, \xi)| \leq C \exp\Big[C\varepsilon\int_s^t \lambda(\sigma; x, \xi) d\sigma + CC' - d\int_s^t \lambda(\sigma; x, \xi) d\sigma\Big]\,.$$

Thus the proof is complete.

When the coefficients of the operator $L = \partial_t + p(X, D_x)$ are independent of t, its resolvent matrix is given by $\exp[-(t-s)p(x, \xi)]$. When the coefficients depend on t, we have the following

Lemma 2.4. *The resolvent matrix z of the operator $L = \partial_t + p(t; X, D_x)$, $p(t; x, \xi) \in \mathscr{B}_t^0(S_{\rho,\delta}^m)$, can be written*

$$z(t, s; x, \xi) = I + \textstyle\sum_{j=1}^\infty (-1)^j \displaystyle\int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} p(s_1; s, \xi) p(s_2; x, \xi) \cdots p(s_j; x, \xi) ds_j\,. \tag{2.12}$$

Moreover we have

$$z(t, \sigma, x, \xi) z(\sigma, s; x, \xi) = z(t, s; x, \xi), \tag{2.13}$$

$$|z(t, s; x, \xi)| \leq C \exp[C' |t-s| \langle \xi \rangle^m], \tag{2.14}$$

and

$$\partial_s z(t, s; x, \xi) - z(t, s; x, \xi) p(s; x, \xi) = 0. \tag{2.15}$$

Proof is omitted.

Now we shall give two propositions which give examples of sufficient conditions under which the operator L has the property (F).

Proposition 2.5. *Let $p(t; x, \xi) \in \mathcal{B}_t^0(S_{\rho,\delta}^m)$ and let $\lambda(t; x, \xi)$ be one of the following functions:*

$$\inf(\operatorname{Re} p_{j,j}(t; x, \xi) - \sum_{k, k \neq j} |p_{j,k}(t; x, \xi)|), \tag{2.16}$$

$$\inf(\operatorname{Re} p_{k,k}(t; x, \xi) - \sum_{j, j \neq k} |p_{j,k}(t; x, \xi)|), \tag{2.17}$$

(2.18) *smallest eigenvalue of* $(p(t; x, \xi)^{(*)} + p(t; x, \xi))/2$.

If $\lambda(t; x, \xi)$ is non-negative in $[0, T]$, *and if* i) *of Definition* 2.2 *holds, then the operator $L = \partial_t + p(t; X, D_x)$ has the property (F).*

Proof. By Theorem 3 of Chapter III in W. A. Coppel's book [1], we have that there exists a constant $C > 0$ such that

$$|z(t, s; x, \xi)| \leq C \exp\left[-\int_s^t \lambda(\sigma; x, \xi) d\sigma\right], \qquad 0 \leq s \leq t \leq T. \tag{2.19}$$

Thus ii) of Definition 2.2 is satisfied with $d = 1$.

Proposition 2.6. *Let $f(t)$ be a non-negative continuous function and $\tilde{p}(t; x, \xi) \in \mathcal{B}_t^0(S_{\rho,\delta}^m)$ in $[0, T]$. If the real part of each eigenvalue of the matrix $\tilde{p}(t; x, \xi)$ is not less than $d\langle \xi \rangle^m$ for a constant $d > 0$, then the operator $L = \partial_t + f(t)\tilde{p}(t; X, D_x)$ has the property (F) with $\lambda(t; x, \xi) = f(t)\langle \xi \rangle^m$.*

Proof. Set $p(t; t'; x, \xi) = f(t)\tilde{p}(t'; x, \xi)$ and $q(t; t'; x, \xi) = f(t)\{\tilde{p}(t; x, \xi) - \tilde{p}(t'; x, \xi)\}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_s^t |q(\sigma; t'; x, \xi)| d\sigma \leq \varepsilon \langle \xi \rangle^m \int_s^t f(\sigma) d\sigma \qquad \text{if} \quad 0 < t-s < \delta \text{ and } t' \in [s, t].$$

Thus by Lemma 2.3 the operator L has the property (F) with $\lambda(t; x, \xi) = f(t)\langle \xi \rangle^m$, if $|t-s|$ is sufficiently small. For arbitrary s and t, $0 \leq s < t \leq T$, deviding $[s, t]$ into sufficiently small intervals, and using (2.13), we can prove that L has the property (F) in $[s, t]$.

From the expansion formula (2.12) of the resolvent matrix $z(t, s; x, \xi)$, we have formally

(2.20) $e(t, s; x, \xi)=I+\sum_{j=1}^{\infty}(-1)^j \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} p(s_1)\circ p(s_2)\circ\cdots\circ p(s_j)(x, \xi)ds_j$.

Thus, if we set

(2.21) $$e_0(t, s; x, \xi) = z(t, s; x, \xi)$$

and for $k\geq 1$

(2.22) $e_k(t, s; x, \xi)=\sum_{j=2}^{\infty}(-1)^j \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} [p(s_1)\circ p(s_2)\circ\cdots\circ p(s_j)]_k(x, \xi)ds_j$,

then we can infer from (1.24) and (2.20) that the Green's matrix $e(t, s; x, \xi)$ has the following asymptotic expansion:

(2.23) $$e(t, s; x, \xi)\sim e_0(t, s; x, \xi)+e_1(t, s; x, \xi)+\cdots .$$

Now we shall prove the main theorem which ensures us the existence of the Green's matrix and its asymptotic expansion (2.23).

Theorem 2.7. *Let L have the property (F). Then for any α, β and $k=0, 1, 2, \cdots$, there exist constants $C_{k,\alpha,\beta}$ and $C'_{k,\alpha,\beta}$ such that*

(2.24) $$|e_{k(\beta)}^{(\alpha)}(t, s; x, \xi)| \leq C_{k,\alpha,\beta}\langle\xi\rangle^{-k(\rho-\delta)-\rho|\alpha|+\delta|\beta|}\left\{\int_s^t \lambda(\sigma; x, \xi)d\sigma+1\right\}^{k+|\alpha|+|\beta|} \times \exp\left[-d\int_s^t \lambda(\sigma; x, \xi)d\sigma\right], \quad 0\leq s\leq t\leq T.$$

(2.24)′ $$|e_{k(\beta)}^{(\alpha)}(t, s; x, \xi)| \leq C'_{k,\alpha,\beta}(t-s)\langle\xi\rangle^{m-k(\rho-\delta)-\rho|\alpha|+\delta|\beta|} \times\left\{\int_s^t \lambda(\sigma; x, \xi)d\sigma+1\right\}^{k+|\alpha|+|\beta|-1}\exp\left[-d\int_s^t \lambda(\sigma; x, \xi)d\sigma\right],$$
$$k+|\alpha|+|\beta|\neq 0, \quad 0\leq s\leq t\leq T.$$

Moreover we can construct the Green's matrix $e(t, s; x, \xi)$ of L such that

(2.25) $$\begin{cases} e(t, s; x, \xi)\in \mathcal{B}_t^0(S_{\rho,\delta}^0)\cap \mathcal{B}_t^1(S_{\rho,\delta}^m) & \text{in } (s, T] \text{ for } 0\leq s\leq t\leq T, \\ \text{w-}\lim_{t\downarrow s} e(t, s; x, \xi) = I & \text{in } S_{\rho,\delta}^0. \end{cases}$$

If we set

(2.26) $$r_N(t, s; x, \xi) = e(t, s; x, \xi)-\sum_{k=0}^{N} e_k(t, s; x, \xi),$$

then we have

(2.27) $$|r_{N(\beta)}^{(\alpha)}(t, s; x, \xi)| \leq C_{\alpha,\beta}(t-s)\langle\xi\rangle^{m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|} \quad \text{for } 0\leq s\leq t\leq T.$$

Proof. For convenience' sake we omit to descirbe the variables x and ξ. Since we have

$$\left|\partial_\xi^\alpha D_x^\beta \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} [p(s_1)\circ p(s_2)\circ\cdots\circ p(s_j)]_k ds_j\right| \leq (j!)^{-1}(t-s)^j C_{\alpha,\beta}^j\langle\xi\rangle^{jm-(\rho-\delta)k-\rho|\alpha|+\delta|\beta|},$$

the sum in the right hand side of (2.22) converges to a matrix with C^∞ components. By Lemma 1.11 we have

$$\partial_t \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_j} [p(s_1)\circ p(s_2)\circ\cdots\circ p(s_{j+1})]_k ds_{j+1}$$
$$= \sum_{\mu=0}^k \sum_{|\alpha|=\mu} \frac{1}{\alpha!} p^{(\alpha)}(t) \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{j-1}} [p(s_1)\circ p(s_2)\circ\cdots\circ p(s_j)]_{k-\mu,(\alpha)} ds_j \,.$$

Thus we have for $k=1, 2, \cdots$

$$(2.28) \qquad \partial_t e_k(t, s) + p(t) e_k(t, s) = -\sum_{\mu=1}^k \sum_{|\alpha|=\mu} \frac{1}{\alpha!} p^{(\alpha)}(t) e_{k-\mu,(\alpha)}(t, s)\,.$$

First we estimate $e_{0}{}^{(\alpha)}_{(\beta)}(t, s)$. Since $\partial_{\xi_j} e_0(t, s)$ satisfies the equation

$$\partial_t \partial_{\xi_j} e_0(t, s) + p(t) \partial_{\xi_j} e_0(t, s) = -(\partial_{\xi_j} p(t)) e_0(t, s)\,,$$

and $\partial_{\xi_j} e_0(s, s)=0$, we have

$$\partial_{\xi_j} e_0(t, s) = -\int_s^t e_0(t, \sigma)(\partial_{\xi_j} p(\sigma)) e_0(\sigma, s) d\sigma\,.$$

Thus by (2.7) and (2.8) we obtain

$$|\partial_{\xi_j} e_0(t, s)| \le C\langle\xi\rangle^{-\rho} \left\{ \int_s^t \lambda(\sigma) d\sigma + 1 \right\} \exp\left[-d \int_s^t \lambda(\sigma) d\sigma \right] \le C_1 \langle\xi\rangle^{-\rho}\,.$$

After this manner we have for every α, β

$$|\partial_\xi^\alpha D_x^\beta e_0(t, s)| \le C_{\alpha,\beta} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\,.$$

Next we estimate $e_{k}{}^{(\alpha)}_{(\beta)}$ by the induction on k. If (2.24) is valid for $e_0(t, s)$, $e_1(t, s), \cdots, e_{k-1}(t, s)$, then with (2.28) we obtain (2.24) for $e_k(t, s)$. We also have (2.24)′ in the same way.

We set

$$(2.29) \qquad f_N(t, s) = \sum_{k=0}^N e_k(t, s)$$

and

$$(2.30) \qquad q_N(t, s) = -\partial_t f_N(t, s) - p(t)\circ f_N(t, s)\,.$$

Since

$$\left\{ p(t)\circ e_k(t, s) - \sum_{|\alpha|\le N-k} \frac{1}{\alpha!} p^{(\alpha)}(t) e_{k,(\alpha)}(t, s) \right\} \in S_{\rho,\delta}^{m-(\rho-\delta)(N+1)}\,,$$

by using (2.28) we obtain

$$|q_{N}{}^{(\alpha)}_{(\beta)}(t, s)| \le C_{\alpha,\beta} \langle\xi\rangle^{m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}\,.$$

Taking N so large as $m-(\rho-\delta)(N+1)\le 0$, we have

$$q_N(t, s_1; x, \xi^1)q_N(s_1, s_2; x^1, \xi^2)\cdots q_N(s_{j-1}, s; x^{j-1}, \xi)\in S_{\rho,\delta}^{(m-(\rho-\delta)(N+1),0,\cdots,0)}.$$

Hence by Lemma 1.9 we have that for any α, β there exists a constant $A_{\alpha,\beta}$ which is independent of j such that

$$|\partial_\xi^\alpha D_x^\beta\{q_N(t, s_1)\circ q_N(s_1, s_2)\circ\cdots\circ q_N(s_{j-1}, s)(x, \xi)\}| \\ \leq (A_{\alpha,\beta})^j\langle\xi\rangle^{m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}.$$

We set

$$\varphi_1(t, s) = q_N(t, s)$$

and

$$\varphi_j(t, s)=\int_s^t ds_1\int_s^{s_1} ds_2\cdots\int_s^{s_{j-2}} q_N(t, s_1)\circ q_N(s_1, s_2)\circ\cdots\circ q_N(s_{j-1}, s)ds_{j-1}, \\ j = 2, 3, \cdots.$$

Then we have

$$|\varphi_{j(\beta)}^{(\alpha)}(t, s)| \leq (A_{\alpha,\beta})^j\frac{(t-s)^{j-1}}{(j-1)!}\langle\xi\rangle^{m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}.$$

Thus we can define $\varphi(t, s)$ by

$$\varphi(t, s) = \textstyle\sum_{j=1}^\infty\varphi_j(t, s) \tag{2.31}$$

and we have

$$|\varphi_{(\beta)}^{(\alpha)}(t, s)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}.$$

We set

$$r_N(t, s) = \int_s^t f_N(t, \sigma)\circ\varphi(\sigma, s)d\sigma. \tag{2.32}$$

Then by (2.24) and above estimates we have (2.27). Since $\varphi(t, s)$ satisfies the following integral equation

$$\varphi(t, s) = q_N(y, s)+\int_s^t q_N(t, \sigma)\circ\varphi(\sigma, s)d\sigma, \tag{2.33}$$

if we set $e(t, s)=f_N(r, s)+r_N(t, s)$, we have

$$\begin{aligned}\partial_t e(t, s) &= \partial_t f_N(t, s)+\partial_t\int_s^t f_N(t, \sigma)\circ\varphi(\sigma, s)d\sigma \\ &= \partial_t f_N(t, s)+\varphi(t, s)+\int_s^t \partial_t f_N(t, \sigma)\circ\varphi(\sigma, s)d\sigma.\end{aligned}$$

By using (2.33) and (2.30) we have

$$\begin{aligned}\partial_t e(t, s) &= -p(t)\circ f_N(t, s)-\int_s^t p(t)\circ f_N(t, \sigma)\circ\varphi(\sigma, s)d\sigma \\ &= -p(t)\circ e(t, s).\end{aligned}$$

We also have

$$\text{w-}\lim_{t\downarrow s} e(t, s) = \text{w-}\lim_{t\downarrow s} \{f_N(t, s)+r_N(t, s)\} = I \qquad \text{in } S^0_{\rho,\delta}\,.$$

Thus $e(t, s)$ is the Green's matrix of L. The restriction that N is sufficiently large is removed as follows. For any N, we set $r_N=e_{N+1}+e_{N+2}+\cdots+e_{N'}+r_{N'}$, where N' is sufficiently large, then we have (2.27) by (2.24)$'$. The first half of (2.25) follows from $\partial_t e(t, s)=-p(t)\circ e(t, s)$. Thus the proof is complete.

As an application we shall give a representation of the solution for the Cauchy problem $Lu=f$, $u(0, x)=u_0(x)$ which provides the existence and uniqueness of the solution for the problem.

Lemma 2.8. *Let $L=\partial_t+p(t; X, D_x)$, $p(t; x, \xi)\in\mathcal{B}^0_t(S^m_{\rho,\delta})$ has the property (F). Then the Green's matrix $e(t, s; x, \xi)$ of L that is constructed in Theorem 2.7 satisfies the following equations*

$$(2.34)\qquad \partial_s e(t, s; x, \xi)-e(t, s)\circ p(s)(x, \xi) = 0\,, \qquad 0\le s<t\le T\,,$$

and

$$(2.35)\qquad e(t, \sigma)\circ e(\sigma, s)(x, \xi) = e(t, s; x, \xi)\,, \qquad 0\le s\le \sigma\le t\le T\,,$$

here we define $e(\sigma, \sigma; x, \xi)=I$.

The symbol $e^*(t, s; x, \xi)$ of the formal adjoint of $e(t, s; X, D_x)$ satisfies

$$(2.36)\qquad \partial_s e^*(t, s; x, \xi)-p^*(s)\circ e^*(t, s)(x, \xi) = 0\,, \qquad 0\le s<t\le T\,.$$

Proof. We omit to describe the variables x and ξ. Let $e_0(t, s)$, $e_1(t, s)$, $\cdots$, $e_N(t, s)$, $f_N(t, s)$ be the symbols defined by (2.21), (2.22) and (2.29) respectively. Then we have as in the proof of Theorem 2.7

$$\partial_s e_0(t, s)-e_0(t, s)p(s) = 0\,,$$
$$\partial_s e_k(t, s)-e_k(t, s)p(s) = \textstyle\sum_{\mu=1}^k\sum_{|\alpha|=\mu} \dfrac{1}{\alpha!} e^{(\alpha)}_{k-\mu}(t, s)p_{(\alpha)}(s)\,, \qquad k = 1, 2, \cdots\,.$$

Thus we can construct a symbol $\tilde{e}(t, s)\in\mathcal{B}^0_s(S^0_{\rho,\delta})$ such that

$$(2.37)\qquad \partial_s\tilde{e}(t, s)-\tilde{e}(t, s)\circ p(s) = 0 \qquad 0\le s<t\le T\,,$$

$$(2.38)\qquad \text{w-}\lim_{s\downarrow t}\tilde{e}(t, s) = I \qquad \text{in } S^0_{\rho,\delta}\,.$$

We define $\tilde{e}(t, t)=I$. For $f, g\in\mathcal{S}(R^n_x)$ we set

$$h(\sigma) = (e(\sigma, s; X, D_x)f(x),\ \tilde{e}^*(t, \sigma; X, D_x)g(x))\,.$$

Then we have by (2.37)

$$\begin{aligned}\frac{d}{d\sigma}h(\sigma) &= -(p(\sigma)\circ e(\sigma, s)(X, D_x)f(x),\ \tilde{e}^*(t, \sigma; X, D_x)g(x)\\ &\qquad +(e(\sigma, s; X, D_x)f(x),\ p^*(\sigma)\circ\tilde{e}^*(t, \sigma)(X, D_x)g(x))\\ &= 0\,.\end{aligned}$$

Thus $h(\sigma)$ is independent of σ. Letting $\sigma \downarrow s$ and $\sigma \uparrow t$, we have $\tilde{e}(t, s)=e(t, s)$. By (2.37) we have (2.34). Now using (2.34) we have

$$\begin{aligned}\partial_\sigma\{e(t, \sigma)\circ e(\sigma, s)\} &= e(t, \sigma)\circ p(\sigma)\circ e(\sigma, s)-e(t, \sigma)\circ p(\sigma)\circ e(\sigma, s)\\ &=0\,.\end{aligned}$$

Thus $e(t, \sigma)\circ e(\sigma, s)=\lim\limits_{\sigma\downarrow s} e(t, \sigma)\circ e(\sigma, s)=e(t, s)$. We obtain (2.36) taking the symbol of the formal adjoint of operators defined by both sides of (2.34).

Theorem 2.9. *Let $p(t; x, \xi)\in\mathcal{B}_t^0(S^m_{\rho,\delta})$ in $[0, T]$ and let $L=\partial_t+p(t; X, D_x)$ have the property (F). Then the Cauchy problem*

$$(2.39)\qquad \begin{cases} Lu(t, x)=f(t, x) & 0<t<T\,,\\ \lim\limits_{t\downarrow 0} u(t, x)=u_0(x) & \end{cases}$$

has a unique solution $u(t, x)$ in $\mathcal{E}_t^1([0, T]; \mathcal{B})$ for any $f(t, x)\in\mathcal{E}_t^0([0, T]; \mathcal{B})$ and any $u_0(x)\in\mathcal{B}$. This solution $u(t, x)$ is given by

$$(2.40)\qquad u(t, x)=\int_0^t e(t, \sigma; X, D_x)f(\sigma, x)d\sigma+e(t, 0; X, D_x)u_0(x)\,.$$

Proof. If $f(t, x)\in\mathcal{E}_t^0([0, T]; \mathcal{B})$ and $u_0(x)\in\mathcal{B}$, then by using Lemma 1.8 and (2.25) we have that $u(t, x)$ of (2.40) belongs to $\mathcal{E}_t^1([0, T]; \mathcal{B})$ and

$$\begin{aligned}\partial_t u(t, x) &= f(t, x)-\int_0^t p(t)\circ e(t, \sigma)(X, D_x)f(\sigma, x)d\sigma-p(t)\circ e(t, 0)(X, D_x)u_0(x)\\ &=f(t, x)-p(t; X, D_x)u(t, x)\,.\end{aligned}$$

We have also that $\lim\limits_{t\downarrow 0} u(t, x)=u_0(x)$. Thus $u(t, x)$ of (2.40) is the solution of the Cauchy problem (2.39).

Conversely if we let $u(t, x)\in\mathcal{E}_t^1([0, T]; \mathcal{B})$ be a solution of (2.39), then by (2.34) we have

$$\begin{aligned}e(t, \sigma; X, D_x)f(\sigma, x) &= e(t, \sigma; X, D_x)\{\partial_\sigma u(\sigma, x)+p(\sigma; X, D_x)u(\sigma, x)\}\\ &=\partial_\sigma\{e(t, \sigma; X, D_x)u(\sigma, x)\}\,.\end{aligned}$$

Hence we have by Lemma 1.8

$$\int_0^t e(t, \sigma; X, D_x)f(\sigma, x)d\sigma=u(t, x)-e(t, 0; X, D_x)u_0(x)\,.$$

Thus $u(t, x)$ coincides with the one given by (2.40). Thus the proof is completed.

3. A degenerate praabolic operator of higher order

In this section we shall construct the fundamental solution of the Cauchy problem for a single operator

$$L=\partial_t^M+a_1(t;X,D_x)\partial_t^{M-1}+\cdots+a_M(t;X,D_x)\,, \tag{3.1}$$

where for a positive integer l

$$a_j(t;x,\xi)=\sum_{k=0}^{jl}t^k a_{j,k}(t;x,\xi)\,,\qquad j=1,2,\cdots,M\,,$$

when the following two conditions are satisfied:

a) For $j=1,2,\cdots,M$; $k=0,1,\cdots,j$,

$$a_{j,k}(t;x,\xi)\in\mathcal{B}_t^0(S_{\rho,\delta}^{(j+k)m/(l+1)})\qquad \text{in } [0,T]\,. \tag{3.2}$$

b) There exists a positive constant d such that the roots $\tau_j(t;x,\xi)$ of the equation

$$\tau^M+a_{1,l}(t;x,\xi)\tau^{M-1}+a_{2,2l}(t;x,\xi)\tau^{M-2}+\cdots+a_{M,Ml}(t;x,\xi)=0 \tag{3.3}$$

satisfy

$$\operatorname{Re}\tau_j(t;x,\xi)\le -d\langle\xi\rangle^m\,,\qquad j=1,2,\cdots,M;\ 0\le t\le T\,. \tag{3.4}$$

In doing so, we shall use a function $h(t;\xi)$ defined by

$$h(t;\xi)=t^l\langle\xi\rangle^m+\langle\xi\rangle^{m/(l+1)}\,, \tag{3.5}$$

and reduce the operator L to a system which has the property (F).

Lemma 3.1. *The function $h(t;\xi)$ defined by* (3.5) *satisfies*

$$|\partial_\xi^\alpha\{h(t;\xi)^{-(r+1)}\partial_t^r h(t;\xi)\}|\le C_\alpha\langle\xi\rangle^{-|\alpha|}\,,\qquad r=0,1,\cdots, \tag{3.6}$$

$$|\partial_\xi^\alpha\{th(t;\xi)^{-r}\partial_t^r h(t;\xi)\}|\le C_\alpha\langle\xi\rangle^{-|\alpha|}\sum_{k=1}^{l}(t^{l+1}\langle\xi\rangle^m)^{k/(l+1)}\,,\qquad r=1,2,\cdots,l\,. \tag{3.7}$$

and

$$h(t;\xi)^{-1}\in\mathcal{B}_t^0(S_{1,0}^{-m/(l+1)})\cap\mathcal{B}_t^1(S_{1,0}^0)\cap\mathcal{B}_t^2(S_{1,0}^m)\cap\cdots\,. \tag{3.8}$$

Proof. First we note that $\partial_t^r h(t;\xi)=C_r t^{l-r}\langle\xi\rangle^m$ for $0<r\le l$, and $\partial_t^r h(t;\xi)=0$ for $r>l$. Since we have $t^{r+1}h(t;\xi)^{r+1}=\{t^{l+1}\langle\xi\rangle^m+(t^{l+1}\langle\xi\rangle^{m})^{1/(l+1)}\}^{r+1}$, we have

$$t^{l-r}\langle\xi\rangle^m/h(t;\xi)^{r+1}\le(t^{l+1}\langle\xi\rangle^m)^{1-(r+1)/(l+1)}$$

and

$$t^{l-r}\langle\xi\rangle^m/h(t;\xi)^{r+1}\le 2/\{1+(t^{l+1}\langle\xi\rangle^m)^r\}$$

for $r=0,1,\cdots,l$. Thus we have (3.6). The estimate (3.7) follows from

$$\begin{aligned} th(t;\xi)^{-r}\partial_t^r h(t;\xi)&=C_r t^{l+1}\langle\xi\rangle^m/\{t^{l+1}\langle\xi\rangle^m+t\langle\xi\rangle^{m/(l+1)}\}^r\\ &\le C_r(t^{l+1}\langle\xi\rangle^m)^{1-r/(l+1)}\qquad \text{for } r=1,2,\cdots,l\,.\end{aligned}$$

The proof of (3.8) follows from (3.6) and

$$\partial_t^k \frac{1}{h(t;\xi)} = h(t;\xi)^{k-1} \sum_{(r_1,\cdots,r_k)} C_{r_1,\cdots,r_k} \times \left(\frac{\partial_t h(t;\xi)}{h(t;\xi)^2}\right)^{r_1} \left(\frac{\partial_t^2 h(t;\xi)}{h(t;\xi)^3}\right)^{r_2} \cdots \left(\frac{\partial_t^k h(t;\xi)}{h(t;\xi)^{k-1}}\right)^{r_k}. \tag{3.9}$$

Here and in what follows by the notation $\sum_{(r_1,r_2,\cdots,r_k)}$ we mean to take the summation for all the sets $(r_1, r_2, \cdots, r_k)$ of non-negative integers r_ν which satisfy $r_1+2r_2+\cdots+kr_k=k$.

Theorem 3.2. *Let the operator* L *which is defined by* (3.1) *satisfy the conditions* a) *and* b). *Then there exist pseudodifferential symbols* $g_j(t, s; x, \xi)$, $j=0, 1, \cdots, M-1$, *such that* $h(t;\xi)^j g_j(t, s; x, \xi) \in \mathcal{B}_t^0(S^0_{\rho,\delta})$ *in* $(s, T]$, *and for any* $\psi_j(x) \in \mathcal{B}(R^n_x)$ *if we set*

$$v(t, x) = Os-\iint e^{-iy\cdot\eta} \textstyle\sum_{j=0}^{M-1} g_j(t, 0; x, \eta)\psi_j(x+y)dy\, d\eta\,, \tag{3.10}$$

then $v(t, x)$ *satisfies*

$$Lv(t, v) = 0 \tag{3.11}$$

and

$$\lim_{t\downarrow 0} \partial_t^j v(t, x) = \psi_j(x)\,, \qquad j = 0, 1, \cdots, M-1\,. \tag{3.12}$$

Proof. We set

$$p(t; x, \xi) = t^l \tilde{p}(t; x, \xi)$$

$$t^l = \begin{pmatrix} 0 & , & -\langle\xi\rangle^m & & \\ & & & & \\ & & & 0 & , -\langle\xi\rangle^m \\ a_{M,Ml}(t; x, \xi)\langle\xi\rangle^{-(M-1)m}, & \cdots, & a_{2,2l}(t; x, \xi)\langle\xi\rangle^{-m}, & a_{1,l}(t; x, \xi) & \end{pmatrix},$$

and

$$q(t; x, \xi) = \begin{pmatrix} 0 & , & -\langle\xi\rangle^{m/(l+1)} & \\ & & & \\ & & 0 & , -\langle\xi\rangle^{m/(l+1)} \\ q_M(t; x, \xi), & \cdots, & q_2(t; x, \xi), & q_1(t; x, \xi) \end{pmatrix}$$

where $q_1(t; x, \xi), \cdots, q_M(t; x, \xi)$ will be determined later.

We define symbols $b_{j,k}(t;\xi)$, $j=1, \cdots, M$; $k=1, \cdots, M$ by

$$\begin{cases} b_{1,1}(t;\xi) = h(t;\xi)^{M-1}\,, \\ b_{k,j}(t;\xi) = h(t;\xi)^{-1} b_{j-1,k-1}(t;\xi) + h(t;\xi)^{-1}\partial_t b_{j-1,k}(t;\xi)\,, \\ \qquad\qquad j = 2, \cdots, M;\ k = 1, \cdots, M\,, \\ b_{j,k}(t;\xi) = 0 \qquad \text{when } j>k \text{ or } k = 0\,, \end{cases} \tag{3.13}$$

and set

$$b(t: \xi) = (b_{j,k}(t; \xi)) . \tag{3.14}$$

Since $b(t; \xi)$ is a triangular matrix with diagonal elements $b_{j,j}(t; \xi)=h(t; \xi)^{M-j} \neq 0$, $b(t;\xi)$ is non-singular. We set

$$r(t; \xi) = b(t; \xi)^{-1} . \tag{3.15}$$

By $r_{j,k}(t; \xi)$ we denote the component of $r(t; \xi)$ in the j-th raw and the k-th column. Then we have $r_{j,j}(t; \xi)=h(t; \xi)^{j-M}$ and $r_{j,k}(t; \xi)=0$ for $j<k$. We set

$$w(t, x) = {}^t(v(t, x), \partial_t v(t; x), \cdots, \partial_t^{M-1} v(t, x)) .$$

If $u(t, x)={}^t(u_1(t, x), u_2(t, x),\cdots, u_M(t, x))=b(t; D_x)w(t, x)$, then by (3.13) we have

$$\begin{cases} u_1(t, x) = h(t; D_x)^{M-1} v(t, x) \\ h(t; D_x)u_2(t, x) = \partial_t u_1(t, x) \\ \cdots \\ h(t; D_x)u_M(t, x) = \partial_t u_{M-1}(t, x) . \end{cases} \tag{3.16}$$

Conversely if $u(t, x)$ satisfies (3.16), then because of (3.15) we have

$$w(t, x) = r(t; D_x)u(t, x) . \tag{3.17}$$

Hence we have

$$\begin{cases} r_{j,j}(t; \xi) = h(t; \xi)^{j-M} , \qquad j = 1, 2, \cdots, M , \\ r_{j,k}(t; \xi) = \partial_t r_{j-1,k}(t; \xi)+h(t; \xi)r_{j-1,k-1}(t; \xi) \\ \qquad\qquad j=2, 3, \cdots, M;\ k = 1, 2, \cdots, M , \\ r_{j,k}(t; \xi) = 0 \qquad \text{when } j<k \text{ or } k=0 . \end{cases} \tag{3.18}$$

From (3.13) we have

$$b_{j,j-k}(t; \xi) = h(t; \xi)^{M-j+k} \sum_{(r_1,\cdots,r_k)} C_{j,r_1\cdots,r_k} \times \left(\frac{\partial_t h(t; \xi)}{h(t; \xi)^2}\right)^{r_1} \left(\frac{\partial_t^2 h(t; \xi)}{h(t; \xi)^3}\right)^{r_2} \cdots \left(\frac{\partial_t^k h(t; \xi)}{h(t; \xi)^{k+1}}\right)^{r_k} . \tag{3.19}$$

From (3.18) we have

$$r_{j,j-k}(t; \xi) = h(t; \xi)^{j-M} \sum_{(r_1,\cdots,r_k)} C'_{j,r_1\cdots,r_k} \times \left(\frac{\partial_t h(t; \xi)}{h(t; \xi)^2}\right)^{r_1} \left(\frac{\partial_t^2 h(t; \xi)}{h(t; \xi)^3}\right)^{r_2} \cdots \left(\frac{\partial_t^k h(t; \xi)}{h(t; \xi)^{k+1}}\right)^{r_k} . \tag{3.20}$$

By Lemma 3.1 and (3.20) we have that there exists for any α a constant $C_\alpha>0$ such that

$$(3.21)\qquad |\partial_\xi^\alpha\{th(t;\xi)^{M-j+1}r_{j,j-k}(t;\xi)\}|\leq C_\alpha\langle\xi\rangle^{-|\alpha|}\textstyle\sum_{k=1}^{l}(t^{l+1}\langle\xi\rangle^m)^{k/(l+1)}$$

and

$$(3.22)\qquad |\partial_\xi^\alpha t\partial_t\{h(t;\xi)^{M-j}r_{j,j-k}(t;\xi)\}|\leq C_\alpha\langle\xi\rangle^{-|\alpha|}\textstyle\sum_{k=1}^{l}(t^{l+1}\langle\xi\rangle^m)^{k/(l+1)}\,.$$

We also have

$$(3.23)\qquad h(t;\xi)^{M-j}r_{j,j-k}(t;\xi)\in\mathcal{B}_t^0(S_{1,0}^0)\,,$$

and by (3.19) we have

$$(3.24)\qquad h(t;\xi)^{j-k-M}b_{j,j-k}(t;\xi)\in\mathcal{B}_t^0(S_{1,0}^0)\,.$$

Now we shall determine $q_1(t;x,\xi),\cdots,q_M(t;x,\xi)$ so that if $u(t,x)$ is a solution of

$$\{\partial_t+p(t;X,D_x)+q(t;X,D_x)\}u(t,x)=0\,,$$

then $v(t,x)=h(t;D_x)^{1-M}u_1(t,x)$ satisfies $Lv(t,x)=0$. By (3.16) and (3.17) we have

$$\partial_t^{M-1}v(t,x)=\textstyle\sum_{k=1}^{M}r_{M,k}(t;X_x)u_k(t,x)\,.$$

Thus we have

$$\begin{aligned}\partial_t^M v(t,x)&=\textstyle\sum_{k=1}^{M}\{(\partial_t r_{M,k}(t;D_x))u_k(t,x)+r_{M,k}(t;D_x)\partial_t u_k(t,x)\}\\&=\textstyle\sum_{k=1}^{M}\{\partial_t r_{M,k}(t;D_x)+r_{M,k-1}(t;D_x)h(t;D_x)\}u_k(t,x)+\partial_t u_M(t,x)\,.\end{aligned}$$

Hence we have

$$(3.25)\qquad \begin{aligned}q_j(t;x,\xi)=&-t^l a_{j,jl}(t;x,\xi)\langle\xi\rangle^{-(j-1)m}\\&+\textstyle\sum_{k=1}^{j}a_k(t;x,\xi)r_{M-k+1,M-j+1}(t;\xi)\\&+\partial_t r_{M,M-j+1}(t;x,\xi)+h(t;\xi)r_{M,M-j}(t;x,\xi)\,,\\&\qquad j=1,2,\cdots,M\,.\end{aligned}$$

We shall show that the operator $\partial_t+p(t;X,D_x)+q(t;X,D_x)$ has the property (F) with

$$(3.26)\qquad \lambda(t;x,\xi)=t^l\langle\xi\rangle^m\,.$$

We write

$$\begin{aligned}q_j(t;x,\xi)=\{&-t^l a_{j,jl}(t;x,\xi)\langle\xi\rangle^{-(j-1)m}+t^{jl}a_{j,jl}(t;x,\xi)r_{M-j+1,M-j+1}(t;\xi)\}\\&+\text{the rest}=I_1+I_2\,.\end{aligned}$$

By (3.2) and (3.18) we have

$$\begin{aligned}|I_1|&=|a_{j,jl}(t;x,\xi)\langle\xi\rangle^{-jm}\{-t^l\langle\xi\rangle^m+t^{jl}\langle\xi\rangle^{jm}h(t;\xi)^{1-j}\}|\\&\leq Ch(t;\xi)^{1-j}t^l\langle\xi\rangle^m\textstyle\sum_{k=0}^{j-2}\binom{j-1}{k}t^{kl}\langle\xi\rangle^{km}\langle\xi\rangle^{(j-1-k)m/(l+1)}\end{aligned}$$

$$= Ch(t;\,\xi)^{-1}t^l\langle\xi\rangle^m\langle\xi\rangle^{m/(l+1)}h(t;\,\xi)^{2-j}$$
$$\times\sum_{k=0}^{j-2}\binom{j-1}{k}t^{kl}\langle\xi\rangle^{km}\langle\xi\rangle^{(j-2-k)m/(l+1)}$$
$$\leq C'h(t;\,\xi)^{-1}t^l\langle\xi\rangle^m\langle\xi\rangle^{m/(l+1)}\leq C't^{l/2}\langle\xi\rangle^{m(l+2)/(2l+2)}\,,$$

here we used the inequality $ab/(a+b)\leq\sqrt{ab}$ for $a>0$, $b>0$. Thus we have

$$|tI_1|\leq Ct^{(l+2)/2}\langle\xi\rangle^{m(l+2)/(2l+2)}=C\{t^{l+1}\langle\xi\rangle^m\}^{(l+2)/(2l+2)}\,. \tag{3.27}$$

Next we estimate I_2. Since $h(t;\,\xi)^j\geq Ct^k\langle\xi\rangle^{(j+k)m/(l+1)}$ for $0\leq k\leq jl$, we have by (3.2) that $|t^k a_{j,k}(t;\,x,\,\xi)h(t;\,\xi)^{-j}|$ is bounded. Thus by (3.21) and (3.22) we have

$$|tI_2|\leq C\sum_{k=1}^{l}(t^{l+1}\langle\xi\rangle^m)^{k/(l+1)}\,. \tag{3.28}$$

In view of Lemma 1.14 with (3.27) and (3.28), we have that $q(t;\,x,\,\xi)$ satisfies (2.9) of Lemma 2.3. It can be proved similarly that $q(t;\,x,\,\xi)$ also satisfies (2.10). Since the characteristic polynomial of the matrix $-\tilde{p}(t;\,x,\,\xi)$ coincides with the left hand side of (3.3), we have that the real part of each eigenvalue of the matrix $\tilde{p}(t;\,x,\,\xi)$ is not less than $d\langle\xi\rangle^m$. Thus by Proposition 2.6. it follows that $\partial_t+p(t;\,X,\,D_x)$ has the property (F). Hence by Lemma 2.3 it follows that the operator $\partial_t+p(t;\,X,\,D_x)+q(t;\,X,\,D_x)$ also has the property (F).

Let $e(t,\,s;\,x,\,\xi)$ be the Green's matrix of $\partial_t+p(t;\,X,\,D_x)+q(t;\,X,\,D_x)$. Then by (3.23), (3.24) and (2.25) we have

$$r(t)\circ e(t,\,s)\circ b(s)(x,\,\xi)\in\mathcal{B}_t^0(S^0_{\rho,\delta})$$

and

$$\text{w-}\lim_{t\downarrow s}r(t)\circ e(t,\,s)\circ b(s)(x,\,\xi)=I\qquad\text{in }\ S^0_{\rho,\delta}\,.$$

Hence we have

$$g_j(t,\,s;\,x,\,\xi)=h(t)^{1-M}\circ\sum_{k=1}^{M}e_{1,k}(t,\,s)\circ b_{k,j+1}(s)(x,\,\xi)\,, \tag{3.29}$$

where $e_{j,k}(t,\,s;\,x,\,\xi)$ is the component of $e(t,\,s;\,x,\,\xi)$ in the j-th raw and the k-th column. By (3.24) we have $h(t;\,\xi)^j g_j(t,\,s;\,x,\,\xi)\in\mathcal{B}_t^0(S^0_{\rho,\delta})$. Hence the proof is complete.

4. Hypoellipticity

Definition 4.1. We say that a linear operator T: $C_0^\infty(\Omega)\to C^\infty(\Omega)$ is properly supported in an open set Ω when for any compact set $K\subset\Omega$ there exists another compact set $K'\subset\Omega$ such that

$$\operatorname{supp}Tu\subset K'\qquad\text{if }\operatorname{supp}u\subset K$$

and

$$Tu=0\ \text{ on }K\text{ if }u=0\ \text{ on }K'\,.$$

Lemma 4.2 ([10]). *If $P\in S^m_{\rho,\delta}$ is properly supported in Ω, then so is its formal adjoint P^*.*

DEFINITION 4.3. i) For $P\in S^m_{\rho,\delta}$ we define $P\colon \mathcal{S}'\to\mathcal{S}'$, $(u\to Pu)$ by

$$(Pu, v)=(u, P^*v) \qquad \text{for } v\in\mathcal{S}\,, \tag{4.1}$$

where $(u, v)=\langle u, \bar{v}\rangle$ and $\langle u, v\rangle$ means the value of u at v.

ii) If $P\in S^m_{\rho,\delta}$ is properly supported in Ω, we define $P\colon \mathcal{D}'(\Omega)\to\mathcal{D}'(\Omega)$, $(u\to Pu)$ by

$$(Pu, v)=(u, P^*v) \qquad \text{for } v\in\mathcal{D}(\Omega)\,. \tag{4.2}$$

DEFINITION 4.4. An operator $P\in S^m_{\rho,\delta}$ which is properly supported in Ω is said to be hypoelliptic in Ω, if

$$\text{sing supp}\, u=\text{sing supp}\, Pu, \qquad u\in\mathcal{D}'(\Omega)\,, \tag{4.3}$$

where the singular support of a distribution u(sing supp u) is the smallest closed set outside which it is a C^∞-function.

Now we shall study the hypoellipticity of a system of operators which degenerates at $t=0$. When the function $\lambda_0(t; x, \xi)$ which fills the role of the scale of degeneration does not change the sign at $t=0$, $L=\partial_t+p(t; X, D_x)$ and its formal adjoint ${}^tL=-\partial_t+p^*(t; X, D_x)$ are hypoelliptic at the origin under appropriate conditions. When $\lambda_0(t; x, \xi)\gtrless 0$ for $t\gtrless 0$, only ${}^tL$ is proved to be hypoelliptic.

EXAMPLE 4.5. i) Let $L=\partial_t+t^2D_x^2$. Then $\lambda_0(t; x, \xi)=t^2\xi^2$ and $e(t, s; x, \xi)=\exp[-(1/3)(t^3-s^3)\xi^2]$. Thus $e(t, s; x, \xi)\in S^{-\infty}$ for any s, t such that $s<t$.

ii) (Y. Kannai [7]) Let $L=\partial_t+tD_x^2$. Then $\lambda_0(t; x, \xi)=t\xi^2$ and $e(t, s; x, \xi)=\exp[-(1/2)(t^2-s^2)\xi^2]$. Thus the Cauchy problem with a data on $t=0$ has a solution not only towards the future but also towards the past. This means that L is not hypoelliptic at the origin. The solution, if there exist, of the Cauchy problem ${}^tLu=f$, $u(-T, x)=u_T(x)$, $T>0$, has an explicit representation in $(0, T)$, and so does that of ${}^tLu=f$, $u(-T, x)=u_T(x)$ in $(-T, 0)$. These two solutions match up smoothly at $t=0$.

Theorem 4.6. *Let $p(t; x, \xi)\in\mathcal{B}_t(S^m_{\rho,\delta})$ in $[-T, T]$ be properly supported in $(-T, T)\times R^n_x$, and let $C(t)$ be an integrable function such that $C(t)>0$ when $t\neq 0$.*

i) *If there exists a continuous function $\lambda_0(t; x, \xi)$ such that $\lambda_0(t; x, \xi)\geq C(t)\langle\xi\rangle^c$ in $[-T, T]$ for a constant $c>0$ and $L=\partial_t+p(t; X, D_x)$ has the property (F) with $\lambda(t; x, \xi)=\lambda_0(t; x, \xi)$, then L is hypoelliptic in $(-T, T)\times R^n_x$.*

ii) *If there exists a continuous function $\lambda_0(t; x, \xi)$ such that*

$$\begin{cases} \lambda_0(t; x, \xi)\geq C(t)\langle\xi\rangle^c & \text{in } [0, T] \\ \lambda_0(t; x, \xi)\leq -C(t)\langle\xi\rangle^c & \text{in } [-T, 0] \end{cases} \tag{4.5}$$

and the resolvent matrix $z(t, s; x, \xi)$ *of* $L=\partial_t+p(t; X, D_x)$ *satisfies*

$$(4.6)\qquad |z(t, s; x, \xi)|\leq C\exp\left[-d\int_s^t \lambda_0(\sigma; x, \xi)d\sigma\right]$$

for $0\leq s\leq t\leq T$ *or for* $-T\leq t\leq s\leq 0$, *and* $p(t; x, \xi)$ *satisfies for any* α, β

$$(4.7)\qquad \int_s^t |p_{(\beta)}^{(\alpha)}(\sigma; x, \xi)|d\sigma\leq C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\left\{\int_s^t \lambda_0(\sigma; x, \xi)d\sigma+1\right\}$$

for $-T\leq s\leq t\leq T$, *then* ${}^tL=-\partial_t+p^*(t; X, D_x)$ *is hypoelliptic in* $(-T, T)\times R_x^n$.

Before the proof we note that the properly supportedness of operator $p(t; X, D_x)$ permits us to extend

$$p(t; X, D_x):\ \mathcal{E}_t^0((-T, T); \mathcal{S})\rightarrow\mathcal{E}_t^0((-T, T); \mathcal{S})$$

uniquely to

$$p(t; X, D_x):\ \mathcal{D}'((-T, T)\times R_x^n)\rightarrow\mathcal{D}'((-T, T)\times R_x^n)\,.$$

Let $H_\mu(\infty-<\mu<\infty)$ be the usual Sobolev space and let $H_{\nu,\mu}(\infty<\nu, \mu<\infty)$ be the Sobolev space defined by

$$(4.8)\qquad H_{\nu,\mu}=\left\{w\in\mathcal{D}'(R_t^1\times R_x^n);\ \iint|\hat{w}(\tau, \xi)|^2\langle\tau\rangle^{2\nu}\langle\xi\rangle^{2\mu}d\tau\, d\xi<\infty\right\}.$$

For an open set Ω such that $\bar{\Omega}$ is a compact subset of R_x^n, $H_{\nu,\mu}^{loc}(\Omega)$ denotes the Fréchet space

$$(4.9)\qquad \{v\in\mathcal{D}'(\Omega);\ \varphi u\in H_\mu \quad \text{for any } \varphi\in C_0^\infty(\Omega)\}\,,$$

and for a rectangular domain $W=(a, b)\times\Omega$ of $(-T, T)\times R_x^n$, $H_\mu^{loc}(W)$ denotes the Fréchet space

$$(4.10)\qquad \{w\in\mathcal{D}'(W);\ \psi u\in H_{\nu,\mu} \quad \text{for any } \psi\in C_0^\infty(W)\}\,.$$

Lemma 4.7. *Let* $p(t; x, \xi)\in\mathcal{B}_t(S_{\rho,\delta}^m)$ *in* $[-T, T]$ *and let* $u\in\mathcal{D}'((-T, T)\times R_x^n)$ *be a solution of the equation*

$$(4.11)\qquad \partial_t u+p(t; X, D_x)u=f\,,\quad f\in\mathcal{D}'((-T, T)\times R_x^n)\,.$$

If f *is infinitely differentiable in* W, *then there exists a real number* ν_0 *such that*

$$(4.12)\qquad u\in\bigcap_{k=0}^\infty\mathcal{E}_t^k((a, b);\ H_{\nu_0-km}^{loc}(\Omega))\,.$$

Proof. We write

$$\partial_t u=-p(t; X, D_x)u+f$$

and differentiate both sides l times with respect to t. Then we have

$$\partial_t^{l+1} u = -\sum_{j=0}^{l} \binom{l}{j} \frac{d^{l-j}}{dt^{l-j}} p(t; X, D_x)(\partial_t^j u) + \partial_t^l f. \tag{4.13}$$

Now, since $\bar{W}$ is compact, there exists a real number ν_1 such that $u \in H^{loc}_{\nu_1, \nu_1 - m}(W)$. Then from (4.11) and $f \in C^\infty(W)$ we see that

$$\partial_t u \in H^{loc}_{\nu_1, \nu_1 - m}(W),$$

and using (4.13) inductively for $l = 1, 2, \cdots$, we see that

$$\partial_t^l u \in H^{loc}_{\nu_1, \nu_1 - lm}(W), \qquad l = 0, 1, 2, \cdots.$$

Then, using Sobolev's lemma with respect to t we have

$$u \in \mathcal{E}_t^{\nu_1 + l - 1}((a, b); H^{loc}_{\nu_1 - lm}(W)), \qquad l \geq -\nu_1 + 1. \tag{4.14}$$

Hence setting $\nu_0 = \nu_1 - (1 - \nu_1)m$, we set (4.12) and the lemma is proved.

Lemma 4.8. *If* $v \in \bigcap_{k=0}^{\infty} \mathcal{E}_t((a, b); H_{\nu_0 - km}) \cap H_{\nu_1, \infty} \cap \mathcal{E}'((a, b) \times R^n_x)$, *then* $v \in C_0^\infty((a, b) \times R^n_x)$.

Proof. We shall show that there exists a constant M_j such that

$$(\langle\tau\rangle + \langle\xi\rangle)^j \leq 2^j (\langle\tau\rangle^{j+1} \langle\xi\rangle^{\nu_0 - (j+1)m} + \langle\tau\rangle^{\nu_1} \langle\xi\rangle^{M_j}) \qquad \text{for } j = 0, 1, 2, \cdots. \tag{4.15}$$

Then by hypothesis on v we have

$$\iint |(\langle\tau\rangle + \langle\xi\rangle)^j \hat{v}(\tau, \xi)|^2 d\tau\, d\xi < \infty \qquad \text{for } j = 0, 1, 2, \cdots.$$

Hence we have $v \in C_0^\infty((a, b) \times R^n_x)$.

We divide the proof of (4.15) into two cases: I) $\langle\tau\rangle \geq \langle\xi\rangle^{N_j}$ and II) $\langle\tau\rangle < \langle\xi\rangle^{N_j}$, where $N_j = \max\{(j+1)m - \nu_0, 1\}$.

When I) holds, we have

$$\begin{aligned}(\langle\tau\rangle + \langle\xi\rangle)^j &\leq 2^j \langle\tau\rangle^{j+1} \langle\tau\rangle^{-1} \\ &\leq 2^j \langle\tau\rangle^{j+1} \langle\xi\rangle^{-N_j} \\ &\leq 2^j \langle\tau\rangle^{j+1} \langle\xi\rangle^{\nu_0 - (j+1)m}.\end{aligned}$$

When II) holds, we have

$$\begin{aligned}(\langle\tau\rangle + \langle\xi\rangle)^j &\leq 2^j \langle\xi\rangle^{jN_j} \\ &\leq 2^j \langle\xi\rangle^{-N_j|\nu_1|} \langle\xi\rangle^{jN_j + N_j|\nu_1|} \\ &\leq 2^j \langle\tau\rangle^{\nu_1} \langle\xi\rangle^{jN_j + N_j|\nu_1|}.\end{aligned}$$

Thus setting $M_j = (j + |\nu_1|)N_j$ we have (4.15).

If L satisfies the hypothesis of i) of Theorem 4.6, then it is shown as in the proof of Theorem 2.7 that there exists the Green's matrix $e(t, s; x, \xi)$ of L such that

$$(4.16) \qquad e(t, s; x, \xi) \in \mathcal{B}_t(S^{-\infty}) \qquad \text{in } [s+\varepsilon_0, T] \text{ for any } \varepsilon_0 > 0,$$

and

$$(4.17) \qquad \text{w-lim}\, \partial_t^j e(t, s; x, \xi) \text{ exists in } S^{jm}_{\rho,\delta} \qquad \text{for } j = 0, 1, 2, \cdots.$$

Moreover we have the following

Lemma 4.9. *Let $g_j(t) \in \mathcal{E}_t^0((a, b); H_\mu)$ and let* supp $g_j \subset (a, b)$ *for $j=1, 2$.*

i) *If there exists a constant $\varepsilon > 0$ such that $g_1 \in C^\infty((-2\varepsilon, 2\varepsilon) \times R^n_x)$, then*

$$(4.18) \qquad v_1 = \int_{-T}^{t} e(t, \sigma; X, D_x) g_1(\sigma) d\sigma \in C^\infty((-\varepsilon, \varepsilon) \times R^n_x).$$

ii) *Let $\gamma_2 \subset \subset \gamma_1$ in Ω. Then*

$$(4.19) \qquad v_2 = \gamma_2(x) \int_{-T}^{t} e(t, \sigma; X, D_x)(1-\gamma_1(X')) g_2(\sigma) d\sigma \in C^\infty(W).$$

Here and in what follows by "$\gamma_2 \subset \subset \gamma_1$ in Ω" we mean that $\gamma_j \in C_0^\infty(\Omega)$ for $j=1, 2$ and $\gamma_1 = 1$ in a neighborhood of supp γ_2.

Proof. i) By (4.16), (4.17) and hypothesis on g_1 we have that v_1 is infinitely differentiable with respect to x when $|t| > \varepsilon$. Differentiatin v_1 with respect to t we have by (2.3) and (2.4)

$$\partial_t v_1(t) = g_1(t) + p(t; X, D_x) v_1(t)$$

This shows that $\partial_t v_1(t)$ is also infinitely differentiable with respect to x. Interation of this procedure proves (4.18).

ii) Since $\gamma_2(X) e(t, \sigma; X, D_x)(1-\gamma_1(X')) \in \mathcal{B}_t(S^{-\infty})$ in $(a, T]$, we have that v_2 is infinitely differen tiable wih respectt to x. By (2.3) and (2.4) we have

$$\partial_t v_2(t) = \int_{-T}^{t} \gamma_2(X) p(t; X, D_x) e(t, \sigma; X', D_{x'})(1-\gamma_1(X'')) g_2(\sigma) d\sigma$$

and $\partial_t v_2(t)$ is infinitely differentiable with respect to x. Iteration of this procedure proves (4.19) and the lemma is proved.

Proof of i) of Theorem 4.6. Let Ω be an open bounded set of R^n_x and let $W = (a, b) \times \Omega \ni (0, 0)$. Let $u \in \mathcal{D}'((-T, T) \times R^n_x)$ be a solution of

$$(4.20) \qquad Lu = f, \qquad f \in \mathcal{D}'((-T, T) \times R^n_x) \cap C^\infty(W).$$

We shall prove that u is infinitely differentiable in a neighborhood of the origin. Take $\psi(t, x) = \chi(t)\gamma(x) \in C_0^\infty(W)$ such that $\psi = 1$ in a neighborhood of the origin. Then by Lemma 4.7 we have

$$(4.21)\qquad \psi u\in\bigcap_{k=0}^{\infty}\mathcal{E}_t^k((a, b); H_{\nu_0-km}),$$

and setting $g=L(\psi u)$ we have

$$(4.22)\qquad g(t)\in\bigcap_{k=0}^{\infty}\mathcal{E}_t^k((a, b); H_{\nu_0-km-m}).$$

Now we write as in the proof of Theorem 2.9

$$\begin{aligned} e(t, \sigma; X, D_x)g(\sigma) &= e(t, \sigma; X, D_x)\{\partial_\sigma(\psi u)+p(\sigma; X', D_{x'})(\psi u)\} \\ &= \partial_\sigma\{e(t, \sigma; X, D_x)(\psi u)(\sigma)\}, \end{aligned}$$

then we have

$$(\psi u)(t) = e(t, s; X, D_x)(\psi u)(s)+\int_s^t e(t, \sigma; X, D_x)g(\sigma)d\sigma$$

for $a<s<t<b$. Noting $\psi u=0$ in $(-T, a+\varepsilon_1)$ for a fixed positive constant ε_1 we have

$$(4.23)\qquad (\psi u)(t)=\int_{-T}^t e(t, \sigma; X, D_x)g(\sigma)d\sigma \qquad \text{for } a<t<b.$$

Now take γ_1 such that $\gamma_1\subset\subset\gamma$ in Ω and $\gamma_1=1$ in a neighborhood of the origin, and set

$$\begin{cases} g_1 = \left(\dfrac{d}{dt}\chi(t)\right)\gamma(x)u+\gamma_1(x)\chi(t)[P, \gamma(x)]u+\psi f, \\ g_2 = \chi(t)[P, \gamma(x)]u. \end{cases}$$

Then we have

$$(4.24)\qquad g_2 = L(\psi u)-\psi f-\chi'(t)\gamma(x)u\in\bigcap_{k=0}^{\infty}\mathcal{E}_t^k((a, b); H_{\nu_0-km-m}),$$

and

$$(4.25)\qquad g = g_1+(1-\gamma_1)g_2.$$

We claim that there exists a constant $\varepsilon>0$ such that $\gamma_1\chi[P, \gamma]u$ is infinitely differentiable with respect to t and x for $|t|<2\varepsilon$. Set $\psi_1(t, x)=\chi_1(t)\gamma_1(x)$ where $\chi_1\subset\subset\chi$ in (a, b) and $\chi_1=1$ for $|t|<2\varepsilon$. Then we have $\psi_1\chi[P, \gamma]u= -\psi_1P\{(1-\psi)u\}$. Since $P(t; X, D_x)(1-\psi(t, X'))$ is properly supported, there exists $\psi_2\in C_0^\infty((-T, T)\times R_x^n)$ such that $\psi_2=1$ in a neighborhood of the origin and $P\psi_1\{(1-\psi)(1-\psi_2)u\}=0$. Hence we have $\psi_1\chi[P, \gamma]u=-\psi_1P\{(1-\psi)\psi_2u\}$. Since $\psi_1(t, X)P(t; X, D_x)(1-\psi(t, X'))\in\mathcal{B}_t(S^{-\infty})$ in (a, b) and there exists a constant ν_1 such that $\psi_2u\in H_{\nu_1,\nu_1}((a, b)\times R_x^n)$, we have $\psi_1\chi[P, \gamma]u\in H_{\nu_1,\infty}\cap\mathcal{E}'(W)$. So by Lemma 4.8 we have $\gamma_1\chi[P, \gamma]u\in C^\infty((-2\varepsilon, 2\varepsilon)\times R_x^n)$. Thus we can apply i) of Lemma 4.9 to g_1 and we have

$$(4.26)\qquad \int_{T-}^t e(t, \sigma; X, D_x)g_1(\sigma)d\sigma\in C^\infty((-\varepsilon, \varepsilon)\times R_x^n).$$

If we choose γ_2 such that $\gamma_2\subset\subset\gamma_1$ in Ω and $\gamma_2=1$ in a neighborhood of Ω_0 the origin, then by (4.24) we can apply ii) of Lemma 4.9 and we have

$$\gamma_2(x)\int_{-T}^{t} e(t, \sigma; X, D_x)(1-\gamma_1(X'))g_2(\sigma)d\sigma\in C^\infty(W)\,. \tag{4.27}$$

By (4.23) and (4.25)~(4.27) we have that $\psi u\in C^\infty((-\varepsilon, \varepsilon)\times\Omega_0)$ and i) of Theorem 4.6 is proved.

Proof of ii) of Theorem 4.6. Let $u\in\mathcal{D}'((-T, T)\times R_x^n)$ be a solution of the equation

$$^tLu=f \qquad f\in\mathcal{D}'((-T, T)\times R_x^n)\cap C^\infty(W)\,. \tag{4.28}$$

As in the proof of Theorem 2.7 we can construct the Green's matrix $e(\sigma, t; x, \xi)$ of $L_{\sigma,x}$ for $-T\leq\sigma\leq t\leq 0$ which belongs to $\mathcal{B}_t(S^{-\infty})$ in $[\sigma+\varepsilon_0, 0]$ for any $\varepsilon_0>0$. Since $\partial_\sigma e^*(\sigma, t; x, \xi)+e^*(\sigma, t)\circ p^*(\sigma)(x, \xi)=0$, we have as in the proof of i)

$$(\psi u)(t, x) = -\int_{-T}^{t} e^*(\sigma, t; X, D_x)g(\sigma)d\sigma \qquad \text{for } a<t\leq 0\,,$$

where $g={}^tL(\psi u)=\psi f+[{}^tL, \psi]u\in\bigcap_{k=0}^{\infty}\mathcal{E}_t^k((a, b); H_{\nu_0-km-m})$. Let $g_1=-\chi'\gamma u+\gamma_1\chi[P^*, \gamma]u+\psi f$ and $g_2=\chi[P^*, \gamma]u$, where $\psi=\chi\gamma$ and $\gamma_1\subset\subset\gamma$ in Ω are the fuctions given in the proof of i). Then we have $g_1\in C^\infty((-2\varepsilon, 2\varepsilon)\times R_x^n)$, $g_j\in\bigcap_{k=0}^{\infty}\mathcal{E}_k^t((a, b); H_{\nu_0-km-m})$ and $\operatorname{supp} g_j\subset(a, b)$ for $j=1, 2$. Since $e^*(\sigma, t; x, \xi)\in\mathcal{B}_t(S^{-\infty})$ in $[\sigma+\varepsilon_0, 0]$, $\partial_t e^*(\sigma, t; x, \xi)=p^*(t)\circ e^*(\sigma, t)(x, \xi)$ and $\operatorname{w-lim}_{\sigma\uparrow t}\partial_t^j e^*(\sigma, t; x, \xi)$ exists in $S_{\rho,\delta}^{jm}$ for $-T<t\leq 0$, we have as in the proof of Lemma 4.9 that both $v_1(t)=-\int_{-T}^{t} e^*(\sigma, t; X, D_x)g_1(\sigma)d\sigma$ and $v_2(t)=-\gamma_2(x)\int_{-T}^{t} e^*(\sigma, t; X, D_x)\times(1-\gamma_1(X'))g_2(\sigma)d\sigma$ belong to $C^\infty((-\varepsilon, 0)\times\Omega_0)$ and every derivative of v_j, $j=1, 2$ is uniformly bounded in Ω_0 as $t\uparrow 0$. In a similar fashion we have that

$$(\psi u)(t, x) = \int_{t}^{T} e^*(\sigma, t; X, D_x)g(\sigma)d\sigma \qquad \text{for } 0\leq t<b\,.$$

We also have that ψu is infinitely differentiable in $(0, \varepsilon)\times\Omega_0$ and its every derivative is uniformly bounded in Ω_0 as $t\downarrow 0$. Thus $u(t, x)$ is infinitely differentiable in $(-\varepsilon, -0]\times\Omega_0\cup[+0, \varepsilon)\times\Omega_0$.

Now we shall show according to Y. Kannani [7] that the boundary values of $u(t, x)$ and its derivatives as t tends to zero from the right and from the left actually match up and that u has no singular part with support on the line $t=0$. That is, we shall prove

$$\lim_{t\to 0}[(\partial_t^j D_x^\alpha u(t, x)-(\partial_t^j D_x^\alpha u)(-t, x)] = 0 \qquad \text{for all } j \text{ and } \alpha\,, \tag{4.29}$$

and that the distribution $v\in\mathcal{D}'(w_0)$ defined by

$$(4.30)\qquad \langle v, \varphi\rangle = \langle u, \varphi\rangle - \lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon} + \int_{\infty}^{\varepsilon}\right]\int (u\varphi)(t, x)\,dx\,dt\,, \qquad \varphi\in C_0^\infty(w_0)\,,$$

is the zero distribution. The functional $\langle v, \varphi\rangle$ is well defined because of the existence of boundary values for $u(t, x)$ as $t\uparrow 0$ and $t\downarrow 0$, and obviously $\operatorname{supp} v\subset\{(t, x);\, t=0\}$. From (4.30) we have

$$\langle {}^tLv, \varphi\rangle = \langle f, \varphi\rangle - \lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right]\int ({}^tLu)\varphi\,dx\,dt$$
$$-\lim_{\varepsilon\downarrow 0}\int [(u\varphi)(-\varepsilon, x) - (u\varphi)(\varepsilon, x)]\,dx\,.$$

Since $f\in C^\infty(w)$, we have

$$\langle {}^tLv, \varphi\rangle = \lim_{\varepsilon\downarrow 0}\int [(u\varphi)(\varepsilon, x) - (u\varphi)(-\varepsilon, x)]\,dx\,,$$

that is

$$(4.31)\qquad {}^tLv = \delta_t\otimes\{u(+0, x) - u(-0, x)\}\,.$$

According to L. Schwartz [14], we may write locally

$$v = \sum_{j=0}^{N} Ev_j\,\partial_t^j\,,$$

where $v_j\in\mathcal{D}'(R_x^n\cap w_0)$ and E is the natural inclusion map $E\colon \mathcal{D}'(R_x^n\cap w_0)\to \mathcal{D}'(w_0)$. Thus we find by (4.31) that $Ev_N\,\partial_t^{N+1}$ have to vanish, and therefore all the v_j have to vanish, so that $v=0$. Hence we have by (4.31)

$$u(+0, x) - u(-0, x) = 0\,.$$

Differentiating the equation ${}^tLu=f$ and iterating the same argument we have (4.29).

Corollary 4.9. *Let L be the operator which satisfies the condition of Theorem 3.1, and let $a_{j,k}(t;\, x, \xi)\in\mathcal{B}_t(S_{\rho,\delta}^{(j+k)m/(l+1)})$ in $[-T, T]$ be properly supported. Then we have* i) *if l is even, then both L and ${}^tL$ are hypoelliptic, and* ii) *if l is odd, then ${}^tL$ is hypoelliptic.*

EXAMPLE 4.10. B. Helffer [6] proved the hypoellipticity of the operator

$$L = \partial_t - a_{2m}(t;\, X, D_x) + \sum_{j=0}^{2m-1} a_j(t;\, X, D_x)\,,$$

where $a_j(t;\, x, \xi)$ $j=0, 1,\cdots, 2m$ is a polynomial of homogeneous order j in ξ with C^∞-coefficients, under the following three conditions:

[H.1] $\operatorname{Re}\int_s^t a_{2m}(\sigma;\, x, \xi)\,d\sigma \geq C\,|t-s|^{k+1}|\xi|^{2m}\,.$

[H.2] There exist real constants θ and τ such that for any α, β and j which satisfy

$$|\alpha|+|\beta|+j>0, \quad (|\alpha|+j)\theta+(|\beta|+j)\tau\leq 1,$$

we have with some constant $C_{j,\alpha,\beta}$

$$|a_{2m-j}{}_{(\beta)}^{(\alpha)}(t; x, \xi)|\leq C_{j,\alpha,\beta}|\mathrm{Re}\, a_{2m}|^{1-(|\alpha|+j)\theta-(|\beta|+j)\tau}|\xi|^{2m[(|\alpha|+j)\theta+(|\beta|+j)\tau]-|\alpha|-j}.$$

[H.3] $2mk(\tau+\theta)/(k+1)<1$.

If we assume that $2mk\tau/(k+1)<1$ when $\theta<0$, then the above L satisfies the condition ii) of Theorem 4.6 with $\lambda_0(t; x, \xi)=\mathrm{Re}\, a_{2m}(t; x, \xi)$, $\rho=\min\{1, 1-\nu\theta\}$ and $\delta=\max\{0, \nu\tau\}$, where $\nu=2mk/(k+1)$.

Proof. Set $\mu=(|\alpha|+j)\theta+(|\beta|+j)\tau$. When $\mu<0$, since $|\mathrm{Re}\, a_{2m}|^{-\mu}|\xi|^{2m\mu}$ is bounded, we have by [H.2]

$$\begin{aligned}|a_{2m-j}{}_{(\beta)}^{(\alpha)}(t; x, \xi)| &\leq C_{j,\alpha,\beta}|\mathrm{Re}\, a_{2m}|^{1-\mu}|\xi|^{2m\mu-|\alpha|-j}\\ &\leq C_{j,\alpha,\beta}|\mathrm{Re}\, a_{2m}|\,|\xi|^{-|\alpha|}.\end{aligned}$$

Thus we have

$$\int_s^t|a_{2m-j}{}_{(\beta)}^{(\alpha)}(\sigma; x, \xi)|d\sigma\leq C_{j,\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\left\{\int_s^t\lambda(\sigma; x, \xi)d\sigma+1\right\}. \tag{4.32}$$

When $0\leq\mu\leq 1$, we have by $\int_s^t\lambda(\sigma)^{1-\mu}d\sigma\leq\left\{\int_s^t\lambda(\sigma)d\sigma\right\}^{1-\mu}(t-s)^{\mu}$ and [H.2] that

$$\int_s^t|a_{2m-j}{}_{(\beta)}^{(\alpha)}(\sigma; x, \xi)|d\sigma\leq C_{j,\alpha,\beta}\left\{\int_s^t\lambda(\sigma; x, \xi)d\sigma\right\}^{1-\mu}(t-s)^{\mu}|\xi|^{2m-|\alpha|-j}.$$

By [H.1] we have

$$\begin{aligned}(t-s)^{\mu}|\xi|^{2m\mu-1} &= \{(t-s)^{k+1}|\xi|^{2m}\}^{\mu/(k+1)}|\xi|^{\mu\nu-j}\\ &\leq\left\{\int_s^t\lambda(\sigma; x, \xi)d\sigma\right\}^{\mu/(k+1)}|\xi|^{\mu\nu-j-|\alpha|}.\end{aligned}$$

Thus we have

$$\int_s^t|a_{2m-j}{}_{(\beta)}^{(\alpha)}(\sigma; x, \xi)|d\sigma\leq C_{j,\alpha,\beta}\left\{\int_s^t\lambda(\sigma; x, \xi)d\sigma\right\}^{1-k\mu/(k+1)}|\xi|^{\mu\nu-j-|\alpha|}.$$

By [H.3] we have

$$\begin{aligned}\mu\nu-j &= \nu\{|\alpha|\theta+|\beta|\tau\}+j\{\nu(\theta+\tau)-1\}\\ &\leq \nu\{|\alpha|\theta+|\beta|\tau\}.\end{aligned}$$

Hence we have

$$\int_s^t |a_{2m-j}{}^{(\alpha)}_{(\beta)}(\sigma;x,\xi)|d\sigma \leq C_{j,\alpha,\beta}\langle\xi\rangle^{-(1-\nu\theta)|\alpha|+\nu\tau|\beta|} \times\left\{\int_s^t \lambda(\sigma;x,\xi)d\sigma+1\right\}. \tag{4.33}$$

When $\mu>1$, by [H.3] we have

$$\begin{aligned}2m-j-|\alpha| &= -(1-\nu\theta)|\alpha|+\nu\tau|\beta|+2m-j-\nu(\theta|\alpha|+\tau|\beta|)\\ &=2m/(k+1)-(1-\nu\theta)|\alpha|+\nu\tau|\beta|+\nu(1-\mu)\\ &\qquad +j\{\nu(\theta+\tau)-1\}\\ &\leq 2m/(k+1)-(1-\nu\theta)|\alpha|+\nu\tau|\beta|\ .\end{aligned}$$

Thus we have

$$\begin{aligned}|a_{2m-j}{}^{(\alpha)}_{(\beta)}(t;x,\xi)| &\leq C_{j,\alpha,\beta}\langle\xi\rangle^{2m-j-|\alpha|}\\ &\leq C_{j,\alpha,\beta}\langle\xi\rangle^{-(1-\nu\theta)|\alpha|+\nu\tau|\beta|}\langle\xi\rangle^{2m/(k+1)}\ .\end{aligned}$$

Hence by using Lemma 1.13 we have

$$\int_s^t |a_{2m-j}{}^{(\alpha)}_{(\mu)}(\sigma;x,\xi)|d\sigma \leq C_{j,\alpha,\beta}\langle\xi\rangle^{-(1-\nu\theta)|\alpha|+\nu\tau|\beta|} \times\left\{\int_s^t \lambda(\sigma;x,\xi)d\sigma+1\right\}. \tag{4.34}$$

By (4.32), (4.33) and (4.34) we have that L satisfies (2.7). Since L is a single equation, (2.8) is easily verified. Thus the proof is complete.

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