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The Fundamental Solution of the Parabolic Equation in a Differentiable Manifold, II

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§ 0. Introduction (and supplements to the previous paper). Recently we have shown the existence of the fundamental solution of parabolic differential equations in a differentiable manifold (under some assumptions) in a previous paper[1] which will be quoted here as [FS]. We have set no boundary condition in [FS], while we shall here show the existence of the fundamental solution of parabolic differential equations with some boundary conditions in a compact subdomain of a differentiable manifold.

We shall first add the following supplements 1°) and 2°) to [FS], as we shall quote not only the results obtained in the paper but also the procedures used in it:

1°) CORRECTIONS. Throughout the paper [FS]

for $\exp\{M_1(t-s)^{\frac{1}{2}}\}$, read $\exp\{M_1(t-s)\}$;
for $\exp\{2M_1(t-s)^{\frac{1}{2}}\}$, read $\exp\{2M_1(t-s)\}$.

In the inequality (3.4),

for $(t-s)^{-(\frac{m}{2}+1)}$, read $(t-s)^{-\frac{m+1}{2}}$

2°) The proof of Theorem 4 in [FS, § 4] is available only for the case: $t_0=\infty$. Instead of completing the proof, we are enough to establish a slightly ameliorated theorem as follows:

Theorem 4. i) *The function* $u(t, x; s, y)$ *is non-negative, and* $\int_M u(t, x; s, y)d_a y \leq \exp\{\lambda(t-s)\}$ *where* $\lambda=\sup_{t,x} c(t, x)$; ii) *if especially* $c(t, x)\equiv 0$, *then* $\int_M u(t, x; s, y)d_a y=1$.

We see that $|\lambda| \leq K(<\infty)$ by virtue of the assumption II) in [FS, p. 76]. To prove this theorem, we consider the functions

$$f_s(t, x) = \int_M u(t, x; s, y)f(y)d_a y \tag{0.1}$$

and

1) S. Itô: The fundamental solution of the parabolic equation in a differentiable manifold, Osaka Math. J. 5 (1953) 75–92.

(0. 2) $$g_s^{(\tau, n)}(t, x) = f_s(t, x) \exp\left\{-\left(\frac{t-s}{\tau-s}\right)^n\right\}$$

where $f(x)$ is an arbitrary function continuous on $\boldsymbol{M}$, with a compact support $\subset \boldsymbol{M}$ and satisfying $0 \leqq f(x) \leqq 1$, and n is a natural number $\geqq 2$ and $s < \tau < t_0$. Then $g_s^{(\tau, n)}(t, x)$ is continuous in $[s, t_0) \times \boldsymbol{M}$ and

(0. 3) $$g_s^{(\tau, n)}(s, x) \equiv f(x)\,, \quad \text{consequently} \quad 0 \leqq g_s^{(\tau, n)}(s, x) \leqq 1\,.$$

By virtue of [FS, (3. 10)] and the correction 1°) stated just above, we have

(0. 4) $$|f_s(t, x)| \leqq M \exp\{M(t-s)\}$$

for a suitable constant $M > 0$.

Lemma A. *If $c(t,x) \leqq 0$, then the function $g_s^{(\tau, n)}(t, x)$ takes neither positive maximum nor negative minimum at any point in $(s, t_0) \times \boldsymbol{M}$.*

The proof may be achieved by the well known method and so will be omitted.

Lemma B. *If $c(t, x) \leqq 0$, then $u(t, x; s, y) \geqq 0$ and $\int_M u(t, x; s, y) d_a y \leqq 1$.*

Proof. By virtue of the continuity of $u(t, x; s, y)$ (see [FS, Theorem 1]), it is sufficient to prove that $0 \leqq f_s(t, x) \leqq 1$ for any function $f(x)$ satisfying the above stated conditions (see (0. 1)).

Suppose that $f_{s_1}(t_1, x_1) > 1$ for some $t_1 > s_1$ and x_1. Then, if we take τ and τ' such that $t_1 < \tau < \tau' < t_0$ and sufficiently large n, we have

$$g_{s_1}^{(\tau, n)}(t_1, x_1) > 1$$

and

$$|g_{s_1}^{(\tau, n)}(t_1, x_1)| > |g_{s_1}^{(\tau, n)}(t, x)| \quad \text{for any } t > \tau' \text{ and } x \in \boldsymbol{M}$$

by virtue of (0. 2) and (0. 4). From this fact and (0. 3), it follows that $g_{s_1}^{(\tau, n)}(t, x)$ takes the positive maximum at some point in $(s, t_0) \times \boldsymbol{M}$; this contradicts Lemma A. Hence we have $f_s(t, x) \leqq 1$.

Similar argument shows that, if $f_{s_1}(t_1, x_1) < 0$ for some $t_1 < s_1$ and x_1, there exist τ and n such that $g_{s_1}^{(\tau, n)}(t, x)$ takes the negative minimum at some point in $(s, t_0) \times \boldsymbol{M}$ contradictly to Lemma A. Hence we get $f_s(t, x) \geqq 0$, q.e.d.

Proof of Theorem 4. Let $u(t, x; s, y)$ be the fundamental solution of the equation $Lf = 0$. Then we may easily prove that the function

$$u_\lambda(t, x; s, y) = e^{-\lambda(t-s)} u(t, x; s, y)$$

is the fundamental solution of the equation $(L-\lambda)f=0$. Since $c(t, x)-\lambda\leqq 0$, we have

$$u_\lambda(t, x; s, y)\geqq 0 \quad \text{and} \quad \int_M u_\lambda(t, x; s, y)d_a y\leqq 1$$

by Lemma B, and hence

$$u(t, x; s, y)\geqq 0 \quad \text{and} \quad \int_M u(t, x; s, y)d_a y\leqq e^{\lambda(t-s)}.$$

Finally, if $c(t, x)\equiv 0$, we may apply Theorem 2 in [FS] to the function $f(t, x)\equiv 1$ and we get

$$\int_M u(t, x; s, y)d_a y=1, \qquad \text{q.e.d.}$$

§ 1. **Fundametal notions and main results.** We shall say, by definition, that a function $f(x)$ defined on a subset E of the Euclidean m-space R^m satisfies the *generalized Lipschitz condition* in E if, for any $x\in E$, there exist positive numbers N, δ and γ (each of them may depend on x) such that $|f(x)-f(y)|\leqq N\sum_i|x^i-y^i|^\gamma$ whenever $y\in E$ and $|x^i-y^i|\leqq\delta(i=1, \dots, m)$, where (x^i) and (y^i) denote the coordinates of x and y respectively[2].

A function $f(x)$ defined on a domain $G\subset R^m$ is said to be *of* $C^{k,L}$-*class* if $f(x)$ is of C^k-class in the usual sense and each partial derivative of k-th order of $f(x)$ satisfies the generalized Lipschitz condition in G. A *manifold of* $C^{k,L}$-*class*, a *hypersurface of* $C^{k,L}$-*class*, etc. should be understood analogously.

Let $\boldsymbol{M}$ be an m-dimensional manifold of $C^{4,L}$-class, and $\boldsymbol{G}$ be a domain in $\boldsymbol{M}$ such that the closure $\bar{\boldsymbol{G}}$ is compact and the boundary $\boldsymbol{B}=\bar{\boldsymbol{G}}-\boldsymbol{G}$ consists of a finite number of hypersurfaces of m-1 dimension and of $C^{4,L}$-class.

Under a *canonical coordinate around* $x\in\boldsymbol{M}$, we understand any local coordinate which maps a neighbourhood of x onto the interier of the unit sphere in R^m and especially transforms x to the centre of the sphere. For each $x\in\boldsymbol{M}$ and any fixed canonical coordinate around x, we denote by $U_\varepsilon(x)$ the neighbourhood of x of the form $\{y\in\boldsymbol{M};\ \sum(y^i-x^i)^2<\varepsilon\}$ where $0<\varepsilon\leqq 1$.

We understand the partial derivatives of a function $f(x)$ (defined on $\bar{\boldsymbol{G}}$) at $\xi\in\boldsymbol{B}$ as follows: $\partial f(\xi)/\partial x^i=\alpha_i$ $(\xi\in\boldsymbol{B})$, $i=1, \dots, m$, means that

$$f(x)=f(\xi)+\alpha_i(x^i-\xi^i)+o\left(\textstyle\sum_i|x^i-\xi^i|\right) \quad \text{for any} \quad x\in U(\xi)\cap\bar{\boldsymbol{G}}$$

where $U(\xi)$ is a coordinate neighbourhood of ξ.

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2) Cf. Footnote 1) in [FS].

We fix s_0 and t_0 such that $-\infty < s_0 < t_0 < \infty$ and consider the parabolic differential operator L:

$$L \equiv L_{tx} = A_{tx} - \frac{\partial}{\partial t}, \qquad (x \in \bar{\boldsymbol{G}},\ s_0 < t < t_0) \tag{1.1}$$

where

$$A \equiv A_{tx} = a^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x)\frac{\partial}{\partial x^i} + c(t, x) \tag{1.2}$$

and $\| a^{ij}(t, x) \|$ is a strictly positive-definite symmetric matrix for each $< t, x > \in (s_0, t_0) \times \bar{\boldsymbol{G}}$; $a^{ij}(t, x)$ and $b^i(t, x)$ are transformed between any two local coordinates by means of (1.3) and (1.4) in [FS]. We assume that

(A.1) *the functions*

$$\frac{\partial a^{ij}(t, x)}{\partial t}, \ \frac{\partial^3 a^{ij}(t, x)}{\partial x^h \partial x^k \partial x^l}, \ \frac{\partial b^i(t, x)}{\partial x^k} \quad (i, j, h, k, l = 1, \ldots, m)$$

and $c(t, x)$

satisfy the generalized Lipschitz condition in $[s_0, t_0] \times \bar{\boldsymbol{G}}$.

We define the *partial derivative* $\partial f(\xi)/\partial \boldsymbol{n}_{t\xi}$ *to the outer transversal direction* $\boldsymbol{n}_{t\xi}$ as follows: when $\boldsymbol{B}$ is represented by $\psi(x) \equiv \psi(x^1, \ldots, x^m) = 0$ with respect to a local coordinate around ξ and $\psi(x) > 0$ in $\boldsymbol{G}$, we set

$$\frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t\xi}} = -\frac{\partial f(\xi)}{\partial x^i} \cdot \frac{\partial \psi(\xi)}{\partial x^j} a^{ij}(t, \xi); \tag{1.3}$$

this notion is independent of the special choice of the local coordinate around ξ by virtue of the transformation rule for $a^{ij}(t, x)$ (see [FS. (1.3)]). If we take a local coordinate with respect to which $a^{ij}(t, \xi) = \delta^{ij}$ i.e. $a^{ij}(t, \xi)\frac{\partial^2}{\partial x^i \partial x^j}$ = *Laplacian* at the point $< t, \xi >$ (fixed), then $\partial f(\xi)/\partial \boldsymbol{n}_{t\xi}$ means the partial derivative to the outer normal direction to $\boldsymbol{B}$. We consider the boundary condition:

$$\alpha(t, \xi) f(\xi) + \{1 - \alpha(t, \xi)\} \frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \quad (\xi \in \boldsymbol{B}) \tag{$B_{\alpha(t)}$}$$

for each t, where $\alpha(t, \xi)$ is a function on $[s_0, t_0] \times \boldsymbol{B}$, of C^1-class in t and of $C^{2,r}$-class in ξ and $0 \leqq \alpha(t, \xi) \leqq 1$. We shall say that a function $f(t, x)$ on $(s_0, t_0) \times \bar{\boldsymbol{G}}$ satisfies the boundary condition (B_α) if it satisfies $(B_{\alpha(t)})$ for any $t \in (s_0, t_0)$.

We define the metric tensor $a_{ij}(x)$, as stated in [FS, p. 79], and consider the measure $d_a x = \sqrt{a(x)} dx^1 \cdots dx^m$ ($a(x) = \det \| a_{ij}(x) \|$) and define the adjoint operator L^* resp. A^* of L resp. A with respect to this measure. If $\boldsymbol{M}$ is an orientable Riemannian manifold with a metric tensor $g_{ij}(x)$ *a priori*, then it is natural to take the measure $d_g x = \sqrt{g(x)}dx^1 \cdots dx^m$ $(g(x) = \det \| g_{ij}(x) \|)$ in place of $d_a x$; in this case, it is sufficient only to replace $a(x)$ by $g(x)$ throughout the course of the present paper, while $a_{ij}(x)$ should not be replaced by $g_{ij}(x)$.

We assume further that:

(A. 2) *the following relations hold on the set*

$$\{\langle t, \xi \rangle;\ \alpha(t, \xi) \neq 1\}\ (\subset [s_0, t_0] \times \boldsymbol{B}):$$

$$\frac{\partial a^{ij}(t, \xi)}{\partial \boldsymbol{n}_{t\xi}} = 0^{3)} \quad (i, j = 1, \dots, m) \quad \textit{and} \tag{1.4}$$

$$b^i(t, \xi) = \frac{1}{\sqrt{a(\xi)}} \cdot \frac{\partial}{\partial x^j}[\sqrt{a(\xi)}a^{ij}(t, \xi)] \ (i = 1, \dots, m). \tag{1.5}$$

Under the above stated conditions (A. 1) and (A. 2), we shall consider the parabolic differential equations $Lf = 0$ and $L^*f^* = 0$ in the domain $\boldsymbol{G}$ with the boundary condition (B_α).

By definition, a function $u(t, x; s, y)$, $s_0 < s < t < t_0$; $x, y \in \bar{\boldsymbol{G}}$, is called a *fundamental solution of the parabolic equation $Lf = 0$ with the boundary condition* (B_α) if, for any s and any function $f(x)$ which is continuous in $\bar{\boldsymbol{G}}$ and satisfies the condition $(B_{\alpha(s)})$, the function

$$f(t, x) = \int_G u(t, x; s, y) f(y) d_a y \quad (t < s) \tag{1.6}$$

satisfies the conditions[4]:

$$\begin{cases} f(t, x) \text{ is of } C^1\text{–class in } t \text{ and of } C^2\text{–class in } x, \text{ and satisfies the} \\ \text{equation } Lf = 0 \text{ as well as the boundary condition } (B_\alpha) \end{cases} \tag{1.7}$$

and

$$\lim_{t \downarrow s} f(t, x) = f(x) \qquad \text{uniformly on } \bar{\boldsymbol{G}}. \tag{1.8}$$

A function $u^*(s, y; t, x)$, $s_0 < s < t < t_0$; $x, y \in \bar{\boldsymbol{G}}$, is called a *fundamental solution of the adjoint equation* $L^*f^* = 0$ (of the equation $Lf = 0$) *with the boundary condition* (B_α) if, for any t and any continuous function

3) It is true that $\partial a^{ij}/\partial \boldsymbol{n}_t$ depends on the local coodinate, but the condition (1.4) is independent of it, because, if $\| a^{ij} \|$ is changed into $\| \bar{a}^{ij} \|$ by means of the coodinate transformation $(x^i) \to (\bar{x}^i)$, then we get $\dfrac{\partial \bar{a}^{ij}}{\partial \boldsymbol{n}_t} = \dfrac{\partial \bar{x}^i}{\partial x^k} \cdot \dfrac{\partial \bar{x}^j}{\partial x^l} \cdot \dfrac{\partial a^{kl}}{\partial \boldsymbol{n}_t}$ by virtue of [FS, (1.3)].

4), 5) Cf. [FS, Definition 2]. The conditions corresponding to (1.9) and (1.9*) in [FS] follow from (1.7) and (1.7*) respectively in the case where $\bar{\boldsymbol{G}}$ is compact.

$f(x)$ on G, the function

$$f^*(s,y) = \int_G u^*(s,y\,;\,t,x)f(x)d_a x \quad (s<t) \tag{1.6*}$$

satisfies the conditions[5]:

$$\begin{cases} f^*(s,y) \text{ is of } C^1\text{-class in } s \text{ and of } C^2\text{-class in } y, \text{ and satisfies} \\ \text{the equation } L^*f^*=0 \text{ as well as the boundary condition } (B_\alpha) \end{cases} \tag{1.7*}$$

and

$$\lim_{s\uparrow t} f^*(s,y) = f(y) \tag{1.8*}$$

pointwisely in G and also strongly in $L^1(G)$.

The purpose of the present paper is to prove the following theorems, which are literally the same as those in [FS][6] except the statements concerning the boundary condition.

Theorem 1. *There exists a function $u(t,x,s,y)$ of C^1-class in t and $s (s_0<s<t<t_0)$ and of C^2-class in x and y $(x,y\in\bar{G})$, with the following properties:*

i) *$u(t,x\,;\,s,y)$ is a fundamental solution of the equation $Lf=0$ with the boundary condition (B_α),*

ii) *$u^*(s,y\,;\,t,x)=u(t,x\,;\,s,y)$ is a fundamental solution of the adjoint equation $L^*f^*=0$ with the boundary condition (B_α),*

iii) *$L_{tx}u(t,x\,;\,s,y)=0$, $L^*_{sy}u(t,x\,;\,s,y)=0$ and $u(t,x\,;\,s,y)$ satisfies the boundary condition (B_α) as a function of $\langle t,x\rangle$ and also as a function of $\langle s,y\rangle$,*

iv) *$\int_G u(t,x,\tau,z)u(\tau,z\,;\,s,y)d_a z = u(t,x\,;\,s,y)$, $s<\tau<t$.*

Theorem 2. *Let $u(t,x\,;\,s,y)$ and $u^*(s,y\,;\,t,x)$ be the functions stated in Theorem 1.*

i) *If a function $f(t,x)$ on $(s,t_0)\times\bar{G}$ satisfies (1.7) and (1.8) where $f(x)$ is continuous in $\bar{G}$ and satisfies (B_α), then it is expressible by (1.6).*

ii) *If a function $f^*(s,y)$ on $(s_0,t)\times\bar{G}$ satisfies (1.7*) and (1.8*) where $f(x)$ is a continuous function on $\bar{G}$, then it is expressible by (1.6*).*

Theorem 3. *If a function $v(t,x\,;\,s,y)$ is continuous in the region: $s_0<s<t<t_0\,;\,x,y\in\bar{G}$, and fatisfies the condition i) or ii) in Theorem 1, then it is identical with $u(t,x\,;\,s,y)$ stated in Theorem 1.*

Theorem 4. i) *$u(t,x\,;\,s,y)\geqq 0$ and $\int_G u(t,x\,;\,s,y)d_a y \leqq e^{\lambda(t-s)}$ where*

6) As for Theorem 4, see the supplement to [FS] in §0 of the present paper.

$\lambda = \sup_{t,x} c(t, x)$; ii) *if* $c(t, x) \equiv 0$ *in the differential operator* A_{tx} *and if* $\alpha(t, \xi) \equiv 1$ *in the boundary condition* (B_α), *then* $\int_G u(t, x; s, y) d_a y = 1$.

We shall show, in another paper[7], the existence of the fundamental solution of the parabolic differential equation with a boundary condition considered in a domain whose closure is not compact.

§ 2. **Preliminaries.** The following lemma may be proved by means of Lebesgue's convergence theorem, and will be useful throughout the present paper:

Lemma 1. *Let* (X, μ) *be a measure space, and assume that*

i) $f(t, \mathcal{X})$ *is measurable in* $\mathcal{X} \in X$ *for each* $t \in (t_1, t_2)$,

ii) $f(t, \mathcal{X})$ *is differentiable in* t *for a.a.* $\mathcal{X} \in X$ *and*

iii) *there exists a measurable function* $\varphi(\mathcal{X})$ *such that*

$$\left| \frac{\partial f(t, \mathcal{X})}{\partial t} \right| \leq \varphi(\mathcal{X}) \text{ in } (t_1, t_2) \text{ and } \int_X \varphi(\mathcal{X}) d\mu(\mathcal{X}) < \infty .$$

Then

$$\frac{d}{dt} \int_X f(t, \mathcal{X}) d\mu(\mathcal{X}) = \int_X \frac{\partial f(t, \mathcal{X})}{\partial t} d\mu(\mathcal{X}) .$$

Now let $\boldsymbol{G}, \boldsymbol{B}$ and A_{tx} be as stated in §1 and z be any fixed point in $\boldsymbol{B}$. Then, for any canonical coordinate (see §1) around z, $\boldsymbol{B} \frown U_1(z)$ is represented by means of $\psi(x^1, \dots, x^m) = 0$ where ψ is a function of $C^{4,L}$-class. Hence, considering a suitable coordinate transformation in $U_1(z)$, we may show that

Lemma 2. *There exists a canonical coordinate* (x^i) *around* z *such that* $\boldsymbol{B} \smile U_1(z)$ *is expressible by* $x^1 = 0$ *and that* $x^1 > 0$ *in* $\boldsymbol{G} \frown U_1(z)$.

Next we shall prove that

Lemma 3. *Let* (x^i) *be a canonical coordinate as stated in Lemma* 2, *and consider the coordinate transformation:* $(x^i) \to (x^i_t)$, *for each* $t (s_0 \leqq t \leqq t_0)$, *defined by*

$$(2.1) \qquad \begin{cases} x^1_t = \varphi^1(t, x) \equiv \gamma x^1 \\ x^j_t = \varphi^j(t, x) \equiv \gamma \left\{ -\dfrac{a^{1j}(t, \xi_x)}{a^{11}(t, \xi_x)} x^1 + x^j \right\}, \quad j = 2, \dots, m, \end{cases}$$

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7) See the author's paper: Fundamental solutions of parabolic differential equations and eigenfunction expensions for elliptic differential equations, forthcoming to Nagoya Mathematical Journal.

where $\xi_x = \langle 0, x^2, \dots, x^m \rangle (\in \boldsymbol{B})$ *for* $x = \langle x^1, \dots, x^m \rangle \in U_1(z)$ *and* γ *is a suitable positive constant. Then there exists* $\delta = \delta_z > 0$ *such that* i) $U_\delta(z) \subset U_1^t(z) \subset U_1(z)$ *and* $U_{\delta/3}(z) \subset U_{1/3}^t(z)$ *for any* t, ii) $\boldsymbol{B}$ *is represented by* $x_t^1 = 0$ *in* $U_1^t(z)$ *and* iii) *if* $a^{ij}(t, x)$ *is changed into* $a_\varphi^{ij}(t, x)$ *by means of this transformation* $(i, j = 1, \dots, m)$, *then*

$$a_\varphi^{1j}(t, \xi) = a_\varphi^{j1}(t, \xi) = 0 \quad \textit{and} \quad a_{1j}^\varphi(t, \xi) = a_{j1}^\varphi(t, \xi) = 0\,, \quad j = 2, \dots, m, \tag{2.2}$$

for any $\xi \in \boldsymbol{B} \cap U_1^t(z)$, *where* $U_\varepsilon^t(x) = \{y \in \boldsymbol{M}\,;\, \sum_i (y_t^i - x_t^i)^2 < \varepsilon\}$ *and* $\| a_{ij}^\varphi(t, x) \| = \| a_\varphi^{ij}(t, x) \|^{-1}$. *The mapping* $\varphi_t(x) = \langle \varphi^1(t, x), \dots, \varphi^m(t, x) \rangle$ *of* $U_\delta(z)$ *into* $U_1^t(z)$*is one-to-one and of* $C^{3,L}$*–class in* x, *and* $a_\varphi^{ij}(t, x)$, $i, j = 1, \dots, m$, *are of* C^1*–class in* t *and of* $C^{2,L}$*–class in* x.

Proof. We notice that $a^{11}(t, x) > 0$ in $U_1(z)$, and consider the coordinate transformation (2.1) around z. Then $x_t^1 = 0$ if and only if $x^1 = 0$, and we have for any $\xi = \langle 0, x^2, \dots, x^m \rangle \in \boldsymbol{B} \cap U_1(z)$

$$\begin{cases} \left(\dfrac{\partial x_t^1}{\partial x^k}\right)_{x=\xi} = \gamma \delta_k^1 \\ \left(\dfrac{\partial x_t^j}{\partial x^k}\right)_{x=\xi} = \gamma \left\{ -\dfrac{a^{1j}(t, \xi)}{a^{11}(t, \xi)} \delta_k^1 + \delta_k^j \right\} \quad (\delta_k^j : \text{Kronecker's delta}) \end{cases} \tag{2.3}$$

for $1 \leqq j \leqq m$ and $2 \leqq j \leqq m$. Hence the Jacobian

$$\frac{\partial(x_t^1, \dots, x_t^m)}{\partial(x^1, \dots, x^m)}$$

is bounded away from zero in $U_{\varepsilon_1}(z)$ for suitable ε_1 $(0 < \varepsilon_1 < 1)$ which may be chosen independently of t by virtue of the continuity of $a^{ij}(t, x)$ on the compact set $[s_0, t_0] \times \overline{U_\varepsilon(z)}$ for any ε $(0 < \varepsilon < 1)$, and hence the transformation (2.1) is well defined in $U_{\varepsilon_1}(z)$. Considering the continuity of $a^{ij}(t, x)$ on $[s_0, t_0] \times \overline{U_{\varepsilon_1}(z)}$ again, we may determine γ and $\delta > 0$ so that $U_\delta(z) \subset U_1^t(z) \subset U_1(z)$ and $U_{\delta/3}(z) \subset U_{1/3}^t(z)$ for any t. By virtue of the transformation rule for a^{ij} (see [FS, (1.3)]), we have, for any $\langle t, \xi \rangle \in [s_0, t_0] \times (\boldsymbol{B} \cap U_1^t(z))$ and for $j \geqq 2$,

$$\begin{aligned} a_\varphi^{1j}(t, \xi) &= \left(\frac{\partial x_t^1}{\partial x^k}\right)_{x=\xi} \cdot \left(\frac{\partial x_t^j}{\partial x^l}\right)_{x=\xi} a^{kl}(t, \xi) \\ &= -\gamma^2 \frac{a^{1j}(t, \xi)}{a^{11}(t, \xi)} a^{11}(t, \xi) + \gamma^2 a^{1j}(t, \xi) = 0 \quad (\text{see } (2.3))\,, \end{aligned}$$

and consequently we get (2.2). The last part of Lemma 3 is also evident by means of the above arguments.

§ 3. **Local construction of a quasi-parametrix.** Let $\boldsymbol{G}$, $\boldsymbol{B}$ and A_{tx} be as before, let z be any fixed point in $\boldsymbol{B}$, and let (x^i) and (x_t^i) $(s_0 \leqq t \leqq t_0)$ be canonical coordinates around z as stated in Lemma 3. Then we have

$$(3.1) \qquad \frac{\partial f(\xi)}{\partial \boldsymbol{n}_t} = -\frac{\partial f(\xi)}{\partial x_t^i} a_\varphi^{i1}(t,\xi) = -a_\varphi^{11}(t,\xi)\frac{\partial f(\xi)}{\partial x_t^1} \qquad (\xi \in \boldsymbol{B})$$

for any function $f(x)$ of C^1-class, and hence the assumption (1.4) implies that

$$(3.2) \qquad \frac{\partial a_{ij}^\varphi(t,\xi)}{\partial x_t^1} = 0 \quad \text{on } \{<t,\xi>;\ \alpha(t,\xi) \neq 1\}\,,$$

Now we put for $s_0 \leqq s < t \leqq t_0$ and $X, Y \in R^m$

$$(3.3) \qquad \begin{cases} V_0(A_{ij};\, t, X;\, s, Y) = (t-s)^{-\frac{m}{2}} \exp\left[-\dfrac{A_{ij}(X^i-Y^i)(X^j-Y^j)}{4(t-s)}\right] \\ V_0(A_{ij}) = \displaystyle\int_{R^m} \exp\left[-\frac{A_{ij}Y^iY^j}{4}\right] dY^1 \dots dY^m\,, \end{cases}$$

and define for $s_0 \leqq s < t \leqq t_0$ and $x, y \in U_\delta(z) \frown \bar{\boldsymbol{G}}$ ($\delta = \delta_z$ as stated in Lemma 3)

$$(3.4) \qquad \begin{cases} V(t,x;\,s,y) = V_0(a_{ij}^\varphi(t,x);\, t, \varphi_t(x);\, s, \varphi_s(y)) \quad \text{(see Lemma 3)} \\ \bar{V}(t,x;\,s,y) = V_0(a_{ij}^\varphi(t,x);\, t, \varphi_t(x);\, s, \bar{\varphi}_s(y)) \\ V(t,x) = V_0(a_{ij}^\varphi(t,x)) \end{cases}$$

where $\bar{\varphi}_s(y) = <-\varphi^1(s,y), \varphi^2(s,y), \dots, \varphi^m(s,y)>$. Further we put

$$(3.5) \qquad \begin{cases} p(t,x;\,s,y) \\ \quad = \dfrac{2(t-s)\cdot\alpha(t,\xi_{tx})}{2(t-s)\alpha(t,\xi_{tx}) + \varphi^1(s,y)[1-\alpha(t,\xi_{tx})\exp\{-|\varphi^1(t,x)|^2\}]} \\ q(t,x;\,s,y) \\ \quad = \dfrac{\varphi^1(s,y)[1-\alpha(t,\xi_{tx})\exp\{1-|\varphi^1(t,x)|^2\}]}{2(t-s)\alpha(t,\xi_{tx}) + \varphi^1(s,y)[1-\alpha(t,\xi_{tx})\exp\{-|\varphi^1(t,x)|^2\}]} \end{cases}$$

where ξ_{tx} is the point $(\in \boldsymbol{B})$ defined by the equations:

$$\varphi^1(t,\xi_{tx}) = 0\,,\ \varphi^j(t,\xi_{tx}) = \varphi^j(t,x) \quad \text{for } j \geqq 2\,;$$

such ξ_{tx} is uniquely determined for any $x \in U_\delta(z)$ and any t by virtue of Lemma 3.

Applying (3.1), (3.2), Lemma 1 and Lemma 3 to (3.3), (3.4) and (3.5), and making use of the fact that $\partial f/\partial \boldsymbol{n}_{t\xi}$ is independent of the local coordinate, we obtain

$$(3.6)\qquad \frac{\partial V(t,\xi)}{\partial \boldsymbol{n}_{t\xi}}=0$$

and

$$(3.7)\qquad \frac{\partial V(t,\xi\,;\,s,y)}{\partial \boldsymbol{n}_{t\xi}}=-a_{\varphi}^{11}(t,\xi)\cdot\left\{-a_{11}^{\varphi}(t,\xi)\cdot\frac{-\varphi^{1}(s,y)}{2(t-s)}V(t,\xi\,;\,s,y)\right\}$$
$$=\frac{-\varphi^{1}(s,y)}{2(t-s)}V(t,\xi\,;\,s,y)$$

for $<t,\xi>$ such that $\xi\in \boldsymbol{B}\frown U_{\delta}(z)$ and $\alpha(t,\xi)\neq 1$, and we get also

$$(3.8)\qquad \frac{\partial p(t,\xi\,;\,s,y)}{\partial \boldsymbol{n}_{t\xi}}=\frac{\partial q(t,\xi\,;\,s,y)}{\partial \boldsymbol{n}_{t\xi}}=0$$

and

$$(3.9)\qquad V(t,\xi\,;\,s,y)=\bar{V}(t,\xi\,;\,s,y)$$

for any $\xi\in \boldsymbol{B}\frown U_{\delta}(z)$. We define

$$(3.10)\qquad W_{z}(t,x\,;\,s,y)=p(t,x;\,s,y)J_{s}(y)\frac{V(t,x\,;\,s,y)-\bar{V}(t,x\,;\,s,y)}{V(t,x)}$$
$$+q(t,x\,;\,s,y)J_{s}(y)\frac{V(t,x\,;\,s,y)+\bar{V}(t,x\,;\,s,y)}{V(t,x)}$$

where

$$(3.11)\qquad J_{s}(y)=\frac{\partial[\varphi^{1}(s,y),\dots,\varphi^{m}(s,y)]}{\partial[y^{1},\dots,y^{m}]}\quad\text{(Jacobian).}$$

Then we may prove from (3.6—9) and by simple calculation that

$$(3.12)\qquad \alpha(t,\xi)W_{z}(t,\xi\,;\,s,y)+\{1-\alpha(t,\xi)\}\frac{\partial W_{z}(t,\xi\,;\,s,y)}{\partial \boldsymbol{n}_{tx}}=0$$
$$\text{for}\quad \xi\in U_{\delta}(z)\frown\boldsymbol{B},$$

that is, $W_{z}(t,x\,;\,s,y)$ satisfies the boundary condition (B_{α}) as a function of $<t,x>\in[s_{0},t_{0}]\times U_{\delta}(x)$. Since

$$(3.13)\qquad \int_{U_{\delta}(z)\cap \boldsymbol{G}}V(t,x\,;\,s,y)J_{s}(y)dy+\int_{U_{\delta}(z)\cap\boldsymbol{G}}\bar{V}(t,x\,;\,s,y)J_{s}(y)dy$$
$$\leqq\int_{R^{m}}V_{0}(a_{ij}^{\varphi}(t,x)\,;\,t,\varphi_{t}(x)\,;\,s,\,Y)dY=V(t,x)$$
$$(dy=dy^{1}\cdots dy^{m},\ dY=dY^{1}\cdots dY^{m})$$

and since the denominators and numerators in the right-hand side of (3.5) are positive for any $x,y\in U_{\delta}(z)\frown\boldsymbol{G}$, we get

$$(3.14)\qquad \int_{U_\delta(z)\cap G} |W_z(t,x\,;\,s,y)|\,dy \leqq 1 \quad \text{for any} \quad x\in U_\delta(z)\wedge\bar{G}.$$

Now we have the following

Lemma 4. *If $f(x)$ is continuous in $\bar{G}$ and vanishes outside $U_\delta(z)$, then*

$$(3.15)\qquad \lim_{t\downarrow s}\int_G \frac{V(t,x\,;\,s,y)+\bar{V}(t,x\,;\,s,y)}{V(t,x)} f(y)J_s(y)dy = f(x)$$

uniformly in $U_\delta(z)\wedge\bar{G}$.

PROOF. By virtue of (3. 3) and the uniform continuity of $\varphi^j(t,x)$ on $[s_0,t_0]\times\overline{U_\delta(z)}$, we may show that

$$\lim_{t\downarrow s}\int_{R^m}\frac{V_0(a^\varphi_{ij}(t,x)\,;\,t,\varphi_t(x)\,;\,s,Y)}{V_0(a^\varphi_{ij}(t,x))}F(Y)dY = F(\varphi_s(x)) \quad \text{uniformly in } U_\delta(z)\wedge\bar{G}$$

for any continuous function $F(Y)$ with a compact support; and hence, if especially $F(\bar{Y})=F(Y)$ where $\bar{Y}=\langle -Y^1, Y^2,\dots,Y^m\rangle$ for $Y=\langle Y^1,Y^2,\dots,Y^m\rangle$, then

$$\lim_{t\downarrow s}\int_{R^m(Y^1>0)}\frac{V_0(a^\varphi_{ij}(t,x)\,;\,t,\varphi_t(x)\,;\,s,Y)+V_0(a^\varphi_{ij}(t,x)\,;\,t,\varphi_t(x)\,;\,s,\bar{Y})}{V_0(a^\varphi_{ij}(t,x))}F(Y)dY$$
$$= F(\varphi_s(x)) \quad \text{uniformly in} \quad U_\delta(z)\wedge\bar{G}.$$

Putting

$$F(Y)=F(\bar{Y})=\begin{cases} f(\varphi_s^{-1}(Y)) & \text{if } \sum_i (Y^j)^2<1 \\ 0 & \text{if not}\end{cases}$$

in the above relation, and considering (3. 4) and (3. 11), we obtain (3. 15).

Lemma 5. *If $f(x)$ is such a function as stated in Lemma 3 and if $\boldsymbol{D}$ is an open set containing $\boldsymbol{B}^{(s)}=\{\xi\in\boldsymbol{B}\,;\,\alpha(s,\xi)=1\}$, where s is any fixed real number $(s_0<s<t_0)$, then*

$$\lim_{t\downarrow s}\int_G W_z(t,x\,;\,s,y)f(y)dy = f(x) \quad \textit{uniformly} \text{ in } \quad U_\delta(z)\wedge\bar{G}-\boldsymbol{D}.$$

PROOF. Let ε be an arbitrary positive number. Then, by virtue of Lemma 4, there exists $\Delta_1>0$ such that

$$(3.16)\qquad \left|\int_{U_\delta(z)\cap\bar{G}}\frac{V(t,x\,;\,s,y)+\bar{V}(t,x\,;\,s,y)}{V(t,x)}J_s(y)f(y)dy-f(x)\right|<\frac{\varepsilon}{4}$$

for any $x\in U_\delta(z)\wedge\bar{G}$ whenever $s<t<s+\Delta_1$. On the other hand, by

virtue of (3.13) and (3.14), there exists $\eta_1>0$ such that

$$(3.17)\quad \left|\int_{U_\delta(z)\cap\{\varphi^1(s,y)<\eta_1\}\cap\bar{G}} \frac{V(t,x,;s,y)+\bar{V}(t,x;s,y)}{V(t,x)} J_s(y)f(y)dy\right| < \frac{\varepsilon}{4}$$

and that

$$(3.18)\quad \left|\int_{U_\delta(z)\cap\{\varphi^1(s,y)<\eta_1\}\cap\bar{G}} W_z(t,x;s,y)f(y)dy\right| < \frac{\varepsilon}{4}.$$

Since $1-\alpha(t,\xi_{tx})\exp\{-|\varphi^1(t,x)|^2\}>0$ for any t and any $x\in U_\delta(z)\cap\bar{G}-B^{(s)}$, there exists $\eta_2>0$ such that

$$1-\alpha(t,\xi_{tx})\exp\{-|\varphi^1(t,x)|^2\}\geqq\eta_2 \qquad \text{(see (3.5))}$$

for any t and any $x\in U_\delta(z)\cap\bar{G}-D$. Hence $\varphi^1(s,y)\geqq\eta_1$ implies that

$$|1-q(t,x;s,y)|=|p(t,x;s,y)|\leqq(t-s)/\eta_1\eta_2$$

for any $t\geqq s$ and any $x\in U_\delta(z)\cap\bar{G}-D$, and hence it follows from (3.10) and (3.13) that there exists $\Delta_2>0$ such that

$$(3.18)\quad \left|\int_{U_\delta(z)\cap\{\varphi^1(s,y)\geqq\eta_1\}\cap\bar{G}}\left\{W_z(t,x;s,y)-\right.\right.$$
$$\left.\left.-\frac{V(t,x;s,y)+\bar{V}(t,x;s,y)}{V(t,x)}J_s(y)\right\}f(y)dy\right|<\frac{\varepsilon}{4}$$

for any $x\in U_\delta(z)\cap G-\bar{D}$ whenever $s<t<s+\Delta_2$. Since $f(y)=0$ for $y\in\bar{G}-U_\delta(z)$, it follows from (3.16—19) that

$$\left|\int_G W_z(t,x;s,y)f(y)dy-f(x)\right|<\varepsilon \quad \text{for any } x\in U_\delta(z)\cap\bar{G}-D$$

whenever $s<t<s+\min\{\Delta_1,\Delta_2\}$. Thus we obtain Lemma 5.

Lemma 6. *Assume that $f(x)$ is continuous in $\bar{G}$, vanishes outside $U_\delta(z)$ and satisfies the boundary condition $(B_{\alpha(s)})$. Then*

$$\lim_{t\downarrow s}\int_G W_z(t,x;s,y)f(y)dy=f(x) \quad \textit{uniformly in } U_\delta(z)\cap\bar{G}.$$

Proof. Let ε be an arbitrary positive number, and put

$$D=\{x;x\in\bar{G},|f(x)|<\varepsilon/5\}\cup\{\bar{x};x\in\bar{G},|f(x)|<\varepsilon/5\}$$

where $\bar{x}=\langle -x^1,x^2,\dots,x^m\rangle$ for $x=\langle x^1,x^2,\dots,x^m\rangle$. Then, by virtue of the assumption of this lemma, D is an open set containing $B^{(s)}=\{\xi\in B;\ \alpha(s,\xi)=1\}$ and hence, by Lemma 5, there exists $\Delta>0$ such that

(3.20) $\quad |\int_G W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon$ for any $x \in U_\delta(z) \cap \bar{G} - D$

whenever $s < t < s + \Delta$. On the other hand, by Lemma 4, there exists $\Delta' > 0$ such that

$$\int_G \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} | f(y) | J_s(y) dy < \frac{2}{5} \varepsilon$$
$$\text{for any } x \in U_\delta(z) \cap \bar{G} \cap D$$

whenever $s < t < s + \Delta'$. Hence, considering the non-negativity of $V(t, x; s, y)$, $\bar{V}(t, x; s, y)$ and $J_s(y)$ (see the proof of Lemma 3) and using the facts: $0 \leqq p(t, x; s, y) \leqq 1$ and $0 \leqq q(t, x; s, y) \leqq 1$, we obtain from (3.10) that

$$\left| \int_G W_z(t, x; s, y) f(y) dy \right| < \frac{4}{5} \varepsilon \quad \text{for any } x \in U_\delta(z) \cap \bar{G} \cap D$$

and accordingly

(3.21) $\quad |\int_G W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon$ for any $x \in U_\delta(z) \cap \bar{G} \cap D$

whenever $s < t < s + \Delta'$. From (3.20) and (3.21) we get

$$|\int_G W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon \quad \text{for any } x \in U_\delta(z) \cap \bar{G}$$

whenever $s < t < s + \min\{\Delta, \Delta'\}$. Thus we obtain Lemma 6.

Next, let $f(\tau, y)$ be a continuous function on $(s, t_0) \times G$ which vanishes outside $U_\delta(z)$ and satisfies the condition: $\int_s^t \int_G | f(\tau, y) | dy d\tau < \infty$, and put

$$f(t, x, \tau) = \int_G W_z(t, x; \tau, y) f(\tau, y)\, dy, \quad t > \tau > s,$$
$$F(t, x) = \int_s^t f(t, x, \tau)\, d\tau .$$

Then we have

Lemma 7. i) *$f(t, x, \tau)$ and $F(t, x)$ satisfy the boundary condition (B_α) in $U_\delta(z) \cap B$;* ii) *for any s' $(t_0 > s' > s)$*

$$\lim_{\tau \downarrow s'} \int_G f(\tau, x) W_z(\tau, x; s', y) dx = f(s', y) \text{ in } G \cap U_\delta(z);$$

iii) *if $f(\tau, y)$ satisfies the generalized Lipschitz condition in $(s, t_0) \times \bar{G}$, then*

$$\frac{\partial F(t, x)}{\partial t} = f(t, x) + \int_s^t \int_G \frac{\partial W_z(t, x; \tau, y)}{\partial t} f(\tau, y) dy d\tau ,$$
$$A_{tx} F(t, x) = \int_s^t \int_G A_{tx} W_z(t, x; \tau, y) f(\tau, y) dy d\tau .$$

Outline of the Proof. The proposition i) may be shown by means of (3. 12) and Lemma 1, and the proposition ii) may be proved similarly to [FS, Lemma 2]. The proposition iii) is proved as follows. Considering the fact that the mapping $\varphi_t(x)$ is one-to-one and of $C^{2,L}$-class for any t (see Lemma 3), using the same idea as in [FS, Lemmas 1 and 3] and applying Lemma 1 (§1), we may show that

$$\frac{\partial f(t, x, \tau)}{\partial t} = \int_G \frac{\partial W_z(t, x ; \tau, y)}{\partial t} f(\tau. y) dy ,$$

$$\frac{\partial f(t, x, \tau)}{\partial x^i} = \int_G \frac{\partial W_z(t, x ; \tau, y)}{\partial x^i} f(\tau, y) dy ,$$

$$\frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} = \int_G \frac{\partial^2 W(t, x ; \tau, y)}{\partial x^i \partial x^j} f(\tau, y) dy .$$

and

$$\lim_{\substack{t > t' > \tau \\ t \to \tau}} f(t, x, t') = f(\tau, x)$$

and that there exist $M > 0$ and $\gamma = \gamma(t, x) > 0$ such that

$$\frac{\partial f(t', s, \tau)}{\partial t'} \leqq M(t-s)^{-(1-\frac{\gamma}{2})} \text{ whenever } s < \tau < t \leqq t' ;$$

further we have

$$\int_s^t \left| \frac{\partial f(t, x, \tau)}{\partial x^i} \right| d\tau < \infty \text{ and } \int_s^t \left| \frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} \right| d\tau < \infty .$$

Hence we may prove the proposition iii) by the same manner as in [FS, Lemma 4].

Lemma 8. *If $\omega(t, x)$ is a function of C^1-class in t and of C^2-class in x, and vanishes outside $U_8(z)$, then there exists a constant $M_0 > 0$ such that*

$$| L_{tx}[\omega(t, x) W_z(t, x ; s, y)] | \leqq M_0 (t-s)^{-\frac{m+1}{2}} \exp \left\{ -\frac{M_0 \sum_i (x^i - y^i)^2}{4(t-s)} \right\} .$$

This may be proved similarly to [FS, Lemma 5].

Finally we define a quasi-parametrix $W_z(t, x ; s, y)$ around any inner point z of G as follows. We fix a canonical coordinate (x^i) around z satisfying $U_1(z) \subset G$ and put

$$\begin{cases} \delta_z = 1 \\ x_t^i \equiv \varphi^i(t, x) = x^i, \; i = 1, \dots, m, \text{ for any } t \end{cases}$$

(consequently $\varphi_t(x)=\langle x^1,\dots x^m\rangle$ and $a^{\varphi}_{ij}(t,x)=a_{ij}(t,x)$—cf. Lemma 3). Using this local coordinate, we define $V(t,x;s,y)$ and $V(t,x)$ by means of (3.3) and (3.4), and put

$$W_z(t,x;s,y)=\frac{V(t,x;s,y)}{V(t,x)}\qquad (s_0<s<t<t_0;\ x,y\in U_1(z)).$$

Then we may easily prove that Lemmas 6, 8 and Lemma 7 ii), iii) hold for $W_z(t,x;s,y)$ defined here. (See Lemmas 2, 4 and 5 in [FS].)

§4. **Gloval construction of a quasi-parametrix and a fundamental solution.** For each $z\in\bar{G}(=G+B)$, we fix canonical coordinates (x^i) and (x^i_t) around z as stated in §2, and put

$$U(z,\varepsilon)=\{x\in M;\ \sum_i(x^i-z^i)^2<\varepsilon\}\qquad(\varepsilon>0).$$

Since $\bar{G}$ is compact, there exists a finite sequence $\{z_1,\dots,z_N\}\subset\bar{G}$ such that

$$\text{(4.1)}\qquad \bar{G}\subset\bigcup_{\nu=1}^{N}U(z_\nu,\delta_\nu/3)\ \text{ where }\ \delta_\nu=\delta_{z_\nu}\ \text{ (see §2)},$$

and then, since

$$\text{(4.2)}\qquad z_\nu\in G\ \text{ implies }\ U(z_\nu,\delta_\nu)\subset G\ \text{ (see §2)},$$

we have

$$\text{(4.3)}\qquad B\subset\bigcup_{z_\nu\in B}U(z_\nu,\delta_\nu/3).$$

Let $\omega(\lambda)$ be a function of $C^{2,L}$-class in $0\leqq\lambda<\infty$ such that $\omega(\lambda)=1$ or 0 if $0\leqq\lambda\leqq1/3$ or $\lambda\geqq2/3$ respectively and that $0\leqq\omega(\lambda)\leqq1$ for any λ, and put for each ν

$$\omega_\nu(t,x)=\begin{cases}\omega(\sum_i[x^i_t-(z_\nu)^i_t]^2) & \text{for } x\in\bar{G}\cap U(z_\nu,\delta_\nu)\\ 0 & \text{for } x\in\bar{G}-U(z_\nu,\delta_\nu).\end{cases}$$

Then $\omega_\nu(t,x)$, $\nu=1,\dots,N$, are of C^1-class in t and of $C^{2,L}$-class in $x\in\bar{G}$, and

$$\text{(4.4)}\qquad \frac{\partial\omega_\nu(t,\xi)}{\partial \boldsymbol{n}_{t\xi}}=0\qquad\text{for any }\langle t,\xi\rangle\in[s_0,t_0]\times B;$$

this may be proved by considering the local coordinate (x^i_t) around z_ν for each t since the operator $\partial/\partial\boldsymbol{n}_t$ is independent of the special choice of the local coordinate.

Now let $a_\nu(x)$ be the restriction of $a(x)=\det\|a_{ij}(x)\|$ (see §1) to $U(z_\nu, \delta_\nu)$ with the local coordinate (x^i) around z stated above, and put, for $s_0 < s < t < t_0$,

$$W_\nu(t, x; s, y)=\begin{cases} W_{z_\nu}(t, x; s, y) \text{ (as stated in §3) if } x, y \in U(z_\nu, \delta_\nu) \cap \bar{\boldsymbol{G}} \\ 0 \quad \text{if not.}\end{cases}$$

We define a quasi-parametrix:

$$Z(t, x; s, y)=\frac{\sum_\nu \omega_\nu(t, x)\omega_\nu(s, y) W_\nu(t, x; s, y)}{\sum_\nu \omega_\nu(t, x)^2 \sqrt{a_\nu(y)}} \quad \begin{pmatrix} s_0 < s < t < t_0 \\ x, y \in \bar{\boldsymbol{G}} \end{pmatrix}$$

Then $Z(t, x; s, y)$ is of C^1-class in t and s, and of $C^{2,\Gamma}$-class in x and y, and it follows from (3.12), (4.2), (4.3) and (4.4) that

$$\alpha(t, \xi)Z(t, \xi; s, y)+\{1-\alpha(t, \xi)\}\frac{\partial Z(t, \xi; s, y)}{\partial \boldsymbol{n}_{t\xi}}=0 \quad (\xi \in \boldsymbol{B}), \tag{4.5}$$

that is, $Z(t, x; s, y)$ satisfies the boundary condition (B_α) as a function of $< t, x >$. Further, by virtue of Lemmas 6, 7 and 8, we obtain the following three lemmas.

Lemma 9. i) *If $f(x)$ is continuous in $\bar{\boldsymbol{G}}$, then*

$$\lim_{t \downarrow s} \int_G Z(t, x; s, y) f(y) d_a y = f(x) \quad \textit{in } \boldsymbol{G};$$

if especially $f(x)$ satisfies the boundary condition $(B_{\alpha(s)})$, then the above convergence is uniform in $\bar{\boldsymbol{G}}$.

ii) *if $f(t, x)$ is continuous in $[s, t_0) \times \bar{\boldsymbol{G}}$, then*

$$\lim \int_G f(t, x) Z(t, x; s, y) d_a x = f(s, y) \quad \textit{in } \boldsymbol{G}.$$

Lemma 10. *If $f(\tau, y)$ is continuous in $(s, t_0) \times \bar{\boldsymbol{G}}$ and satisfies the condition: $\int_s^t \int_G |f(\tau, y)| d_a y d\tau < \infty$, then*

$$f(t, x, \tau)=\int_G Z(t, x; \tau, y) f(\tau, y) d_a y \qquad (t < \tau < s)$$

and

$$F(t, x)=\int_s^t f(t, x, \tau) d\tau$$

satisfy the boundary condition (B_α); if further $f(\tau, y)$ satisfies the generalized Lipschitz condition in $(s, t_0) \times \bar{\boldsymbol{G}}$, then

$$\begin{cases} \dfrac{\partial F(t, x)}{\partial t}=f(t, x)+\displaystyle\int_s^t \int_G \frac{\partial Z(t, x; \tau, y)}{\partial t} f(\tau, y) d_a y d\tau, \\ A_{tx} F(t, x)=\displaystyle\int_s^t \int_G A_{tx} Z(t, x; \tau, y) f(\tau, y) d_a y d\tau. \end{cases}$$

Lemma 11. *$Z(t, x; s, y)$ satisfies all inequalities stated in* [FS, Lemma 8] *for a suitable constant* $M > 0$.

Thus we see that $Z(t, x; s, y)$ has all properties stated in [FS, §2]. Hence, starting from this quasi-parametrix $Z(t, x; s, y)$, we may construct $u(t, x; s, y)$ in the entirely same way as in [FS, §3]. We may also construct $u^*(t, x; s, y)$ in the similar manner for the adjoint equation $L^* f^* = 0$ with the same boundary condition (B_α). The functions $u(t, x; s, y)$ and $u^*(t, x; s, y)$ defined here have the properties stated in [FS, §3] where the manifold $\boldsymbol{M}$ should be replaced by the compact domain $\bar{\boldsymbol{G}}$ and the uniformity of the convergence in [FS, (3.13)] may be proved if and only if $f(x)$ is the limit of a uniformly convergent sequence of functions satisfying the the boundary condition $(B_{\alpha(s)})$[8]. Moreover $u(t, x; s, y)$ and $u^*(t, x; s, y)$ satisfy the boundary condition (B_α) as functions of $< t, x >$ — see Lemma 10 and the procedure of the construction of $u(t, x; s, y)$ (in [FS, §3]).

§5. Proof of Theorems.

Lemma 12. *If $f(x)$ and $h(x)$ are functions of C^2-class on $\bar{\boldsymbol{G}}$ satisfying the boundary condition $(\boldsymbol{B}_{\alpha(t)})$ (t: fixed), then*

$$\int_G f(x) \cdot A_{tx} h(x) d_a x = \int_G A^*_{tx} f(x) \cdot h(x) d_a x .$$

PROOF. By partial integration, we obtain the Green's formula:

$$\begin{aligned}\int_G f(x) \cdot A_{tx} h(x) d_a x &- \int_G A^*_{tx} f(x) \cdot h(x) d_a x \\ &= \int_{\boldsymbol{B}} \left\{ f(\xi) \frac{\partial h(\xi)}{\partial \boldsymbol{n}_t} - \frac{\partial f(\xi)}{\partial \boldsymbol{n}_t} h(\xi) \right\} \tilde{d}\xi \\ &+ \int_{\boldsymbol{B}} \left\{ \frac{\partial}{\partial x^j} [\sqrt{a(\xi)} a^{ij}(t, \xi)] - \right. \\ &\left. - \sqrt{a(\xi)} b^i(t, \xi) \right\} \frac{\partial \psi(\xi)}{\partial x^i} f(x) h(x) \tilde{d}\xi\end{aligned}$$

where $\tilde{d}\xi = d\xi^1, \dots, d\xi^{m-1}$ is the hypersurface area on $\boldsymbol{B}$ and $\psi(x)$ is such function that $\psi(x) = 0$ determines $\boldsymbol{B}$ and that $\psi(x) > 0$ in $\boldsymbol{G}$. But the right-hand side equals zero by virtue of the boundary condition $(\boldsymbol{B}_{\alpha(t)})$ and the assumption (1.5). Hence we obtain Lemma 12.

From this lemma we obtain the following (see [FS, Lemma 11])

Lemma 13. *If a function $f^*(s, y)$ on $(s_0, t) \times \bar{\boldsymbol{G}}$ satisfies* (1.7*) *and* (B_α), *then*

8) This assumption for $f(x)$ is equivalent to the following one: $f(\xi) = 0$ on $\boldsymbol{B}^{(s)} = \{\xi \in \boldsymbol{B};\ \alpha(s, \xi) = 1\}$

$$\int_G f^*(\tau, x)u(\tau, x; s, y)d_a x = f^*(s, y) \text{ for any } \tau \in (s, t).$$

Therefore, we may see that:

PROOF OF THEOREMS 1, 2 AND 3 *may be performed in the same way as the proof of the corresponding theorems in* [FS] (see [FS, pp. 89–90]). It seems not to be necessary to repeat the entirely same argument. *The propositions concerning the boundary condition which are not included in* [FS] *may be easily proved from properties of* $u(t, x; s, y)$ *and* $u^*(t, x; s, y)$ *stated in* § 4 *of the present paper.*

In order to prove Theorem 4, we consider, as in § 0, the functions

$$f_s(t, x) = \int_G u(t, x; s, y) f(y) d_a y \tag{5.1}$$

and

$$g(t, x) \equiv g_s^{(\tau, n)}(t, x) = f_s(t, x) \exp\left\{-\left(\frac{t-s}{\tau-s}\right)^n\right\} \tag{5.2}$$

where $f(x)$ is an arbitrary continuous function on $\boldsymbol{G}$ such that $0 \leq f(x) \leq 1$ and the support of $f(x)$ is a compact set contained in the domain $\boldsymbol{G}$, and τ and n are as stated in § 0. Then $g_s^{(\tau, n)}(t, x)$ is continuous in $(s, t_0) \times \bar{\boldsymbol{G}}$ and satisfies (0. 3), (0. 4) and the boundary condition (B_α).

Lemma 14. *If* $c(t, x) \leq 0$, *then the function* $g(t, x)$ *takes neither positive maximum nor negative minimum at any point in* $(s, t_0) \times \bar{\boldsymbol{G}}$ (*for any fixed* τ, n *and* s).

PROOF. It is easily proved by the well known method that $g(t, x)$ takes neither positive maximum nor negative minimum at any point in the open set $(s, t_0) \times \boldsymbol{G}$.

Suppose that:

(5. 3) $g(t, x)$ *takes the positive maximum at* $\langle t_1, \xi_1 \rangle \in (s, t_0) \times \boldsymbol{B}$.

$f_s(t, x)$ satisfies $Lf = 0$ in $(s, t_0) \times \bar{\boldsymbol{G}}$ as may be seen from the properties of $u(t, x; s, y)$, where the partial derivatives at any $\xi \in \boldsymbol{B}$ should be understood as defined in § 1, and $g(t, x)$ satisfies the boundary condition (B_α) as well as $f_s(t, x)$. We adopt a canonical coordinate around ξ_1 as stated in Lemma 3. Then we obtain from (5. 3), (3. 1) and (B_α) that $\partial g(t_1, \xi_1)/\partial x_t^1 \leq 0$ and that

$$\alpha(t_1, \xi_1) g(t_1, \xi_1) - \{1 - \alpha(t_1, \xi_1)\} a_\varphi^{11}(t_1, \xi_1) \frac{\partial g(t_1, \xi_1)}{\partial x_t^1} = 0.$$

Since $g(t_1, \xi_1) > 0$ and $a_\varphi^{11}(t_1, \xi_1) > 0$, it follows that $\alpha(t_1, \xi_1)$ should be zero, consequently $\partial g(t_1, \xi_1)/\partial x_t^1 = 0$, and accordingly $\partial^2 g(t_1, \xi_1)/(\partial x_t^1)^2 \leqq 0$ by virtue of (5.3). Moreover, since $< t_1, \xi_1 >$ may be considered as the maximising point of $g(t, \xi)$ restricted to $(s, t_0) \times \boldsymbol{B}$, we have

$$\sum_{i,j\geqq 2} a_\varphi^{ij}(t_1, \xi_1) \frac{\partial^2 g(t_1, \xi_1)}{\partial x_t^i \partial x_t^j} \leqq 0 \text{ and } b_\varphi^i(t_1, \xi_1) \frac{\partial g(t_1, \xi_1)}{\partial x_t^i} = 0$$

where we use the following facts: $a_\varphi^{1j}(t_1, \xi_1) = a_\varphi^{j1}(t_1, \xi_1) = 0$ for $j \geqq 2$ (see Lemma 3) and accordingly $\|a_\varphi^{ij}(t_1, \xi_1)\|_{i,j=2,\dots,m}$ is a positive-definite symmetric matrix. Thus we get $Ag(t_1, \xi_1) \leqq 0$, and hence

$$0 = \frac{\partial g(t_1, \xi_1)}{\partial t} = Ag(t_1, \xi_1) - \frac{n(t_1 - s)^{n-1}}{(\tau - s)^n} g(t_1, \xi_1) < 0;$$

that is a contradiction. Hence the function $g(t, x)$ on $(s, t_0) \times \bar{\boldsymbol{G}}$ does not take the positive maximum at any point in $(s, t_0) \times \boldsymbol{B}$. Similarly it does not take the negative minimum at any point in $(s, t_0) \times B$.

PROOF OF THEOREM 4 may be performed by means of the entirely same manner as in § 0 by making use of Lemma 14 in place of Lemma A in § 0. We omit to repeat here the argument in § 0.

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