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The Fundamental Solution of the Parabolic Equation in a Differentiable Manifold, II

By Seizô Itô

§0. Introduction (and supplements to the previous paper). Recently we have shown the existence of the fundamental solution of parabolic differential equations in a differentiable manifold (under some assumptions) in a previous paper¹⁾ which will be quoted here as [FS]. We have set no boundary condition in [FS], while we shall here show the existence of the fundamental solution of parabolic differential equations with some boundary conditions in a compact subdomain of a differentiable manifold.

We shall first add the following supplements 1°) and 2°) to [FS], as we shall quote not only the results obtained in the paper but also the procedures used in it:

1°) CORRECTIONS. Throughout the paper [FS]

for $\exp \{M_1(t-s)^{\frac{1}{2}}\}$, read $\exp \{M_1(t-s)\}$; for $\exp \{2M_1(t-s)^{\frac{1}{2}}\}$, read $\exp \{2M_1(t-s)\}$.

In the inequality (3.4),

for $(t-s)^{-(\frac{m}{2}+1)}$, read $(t-s)^{-\frac{m+1}{2}}$

2°) The proof of Theorem 4 in [FS, §4] is available only for the case: $t_0 = \infty$. Instead of completing the proof, we are enough to establish a slightly ameliorated theorem as follows:

Theorem 4. i) The function u(t, x; s, y) is non-negative, and $\int_{\mathcal{M}} u(t, x; s, y) d_a y \leq \exp \{\lambda(t-s)\}$ where $\lambda = \sup_{t,x} c(t, x);$ ii) if especially $c(t, x) \equiv 0$, then $\int_{\mathcal{M}} u(t, x; s, y) d_a y = 1$.

We see that $|\lambda| \leq K(<\infty)$ by virtue of the assumption II) in [FS, p. 76]. To prove this theorem, we consider the functions

(0.1)
$$f_s(t, x) = \int_{\mathcal{M}} u(t, x; s, y) f(y) d_a y$$

and

¹⁾ S. Itô: The fundamental solution of the parabolic equation in a differentiable manifold, Osaka Math. J. 5 (1953) 75-92.

S. Itô

(0.2)
$$g_{s}^{(\tau,n)}(t,x) = f_{s}(t,x) \exp\left\{-\left(\frac{t-s}{\tau-s}\right)^{n}\right\}$$

where f(x) is an arbitrary function continuous on M, with a compact support $\subset M$ and satisfying $0 \leq f(x) \leq 1$, and n is a natural number ≥ 2 and $s < \tau < t_0$. Then $g_s^{(\tau,n)}(t,x)$ is continuous in $[s, t_0) \times M$ and

$$(0.3) g_s^{(\tau,n)}(s,x) \equiv f(x), \text{ consequently } 0 \leq g_s^{(\tau,n)}(s,x) \leq 1.$$

By virtue of [FS, (3.10)] and the correction 1°) stated just above, we have

$$(0.4) |f_s(t,x)| \leq M \exp\{M(t-s)\}$$

for a suitable constant M > 0.

Lemma A. If $c(t,x) \leq 0$, then the function $g_s^{(\tau,n)}(t,x)$ takes neither positive maximum nor negative minimum at any point in $(s, t_0) \times M$.

The proof may be achieved by the well known method and so will be omitted.

Lemma B. If
$$c(t, x) \leq 0$$
, then $u(t, x; s, y) \geq 0$ and $\int_{\mathcal{M}} u(t, x; s, y) d_a y \leq 1$.

PROOF. By virtue of the continuity of u(t, x; s, y) (see [FS, Theorem 1]), it is sufficient to prove that $0 \le f_s(t, x) \le 1$ for any function f(x) satisfying the above stated conditions (see (0, 1)).

Suppose that $f_{s_1}(t_1, x_1) > 1$ for some $t_1 > s_1$ and x_1 . Then, if we take τ and τ' such that $t_1 < \tau < \tau' < t_0$ and sufficiently large n, we have

$$g_{s_1}^{(\tau,n)}(t_1,x_1) > 1$$

and

$$|g_{s_1}^{(\tau,n)}(t_1,x_1)| \ge |g_{s_1}^{(\tau,n)}(t,x)|$$
 for any $t \ge \tau'$ and $x \in M$

by virtue of (0.2) and (0.4). From this fact and (0.3), it follows that $g_{s_1}^{(\tau,n)}(t,x)$ takes the positive maximum at some point in $(s,t_0) \times M$; this contradicts Lemma A. Hence we have $f_s(t,x) \leq 1$.

Similar argument shows that, if $f_{s_1}(t_1, x_1) < 0$ for some $t_1 < s_1$ and x_1 , there exist τ and n such that $g_{s_1}^{(\tau,n)}(t, x)$ takes the negative minimum at some point in $(s, t_0) \times \mathbf{M}$ contradictly to Lemma A. Hence we get $f_s(t, x) \ge 0$, q.e.d.

PROOF OF THEOREM 4. Let u(t, x; s, y) be the fundamental solution of the equation Lf = 0. Then we may easily prove that the function

$$u_{\lambda}(t, x; s, y) = e^{-\lambda(t-s)}u(t, x; s, y)$$

is the fundamental solution of the equation $(L-\lambda)f=0$. Since $c(t, x) -\lambda \leq 0$, we have

 $u_{\lambda}(t, x; s, y) \ge 0$ and $\int_{\mathcal{M}} u_{\lambda}(t, x; s, y) d_a y \le 1$

by Lemma B, and hence

$$u(t, x; s, y) \ge 0$$
 and $\int_{\mathcal{M}} u(t, x; s, y) d_a y \le e^{\lambda(t-s)}$

Finally, if $c(t, x) \equiv 0$, we may apply Theorem 2 in [FS] to the function $f(t, x) \equiv 1$ and we get

$$\int_{M} u(t, x; s, y) d_a y = 1, \qquad \text{q.e.d.}$$

§1. Fundametal notions and main results. We shall say, by definition, that a function f(x) defined on a subset E of the Euclidean m-space \mathbb{R}^m satisfies the generalized Lipschitz condition in E if, for any $x \in E$, there exist positive numbers N, δ and γ (each of them may depend on x) such that $|f(x)-f(y)| \leq N \sum_i |x^i-y^i|^{\gamma}$ whenever $y \in E$ and $|x^i-y^i| \leq \delta(i=1,\ldots,m)$, where (x^i) and (y^i) denote the coordinates of x and y respectively².

A function f(x) defined on a domain $G \subset \mathbb{R}^m$ is said to be of $C^{k,L}$ class if f(x) is of C^k -class in the usual sense and each partial derivative of k-th order of f(x) satisfies the generalized Lipschitz condition in G. A manifold of $C^{k,L}$ -class, a hypersurface of $C^{k,L}$ -class, etc. should be understood analogously.

Let M be an *m*-dimensional manifold of $C^{4,L}$ -class, and G be a domain in M such that the closure \overline{G} is compact and the boundary $B = \overline{G} - G$ consists of a finite number of hypersurfaces of *m*-1 dimension and of $C^{4,L}$ -class.

Under a canonical coordinate around $x \in M$, we understand any local coordinate which maps a neighbourhood of x onto the interier of the unit sphere in \mathbb{R}^m and especially transforms x to the centre of the sphere. For each $x \in M$ and any fixed canonical coordinate around x, we denote by $U_{\varepsilon}(x)$ the neighbourhood of x of the form

 $\{y \in \boldsymbol{M}; \sum (y^i - x^i)^2 < \varepsilon\}$ where $0 < \varepsilon \leq 1$.

We understand the partial derivatives of a function f(x) (defined on \overline{G}) at $\xi \in \mathbf{B}$ as follows: $\partial f(\xi) / \partial x^i = \alpha_i$ ($\xi \in \mathbf{B}$), i = 1, ..., m, means that

$$f(x) = f(\xi) + \alpha_i (x^i - \xi^i) + o(\sum_i |x^i - \xi^i|) \text{ for any } x \in U(\xi) \cap \overline{G}$$

where $U(\xi)$ is a coordinate neighbourhood of ξ .

²⁾ Cf. Footnote 1) in [FS].

Ś. Itô

We fix s_0 and t_0 such that $-\infty < s_0 < t_0 < \infty$ and consider the parabolic differential operator L:

(1.1)
$$L \equiv L_{tx} = A_{tx} - \frac{\partial}{\partial t}, \quad (x \in \bar{G}, s_0 < t < t_0)$$

where

(1.2)
$$A \equiv A_{tx} = a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x) \frac{\partial}{\partial x^i} + c(t, x)$$

and $||a^{ij}(t, x)||$ is a strictly positive-definite symmetric matrix for each $\langle t, x \rangle \in (s_0, t_0) \times \overline{\mathbf{G}}; a^{ij}(t, x)$ and $b^i(t, x)$ are transformed between any two local coordinates by means of (1.3) and (1.4) in [FS]. We assume that

(A. 1) the functions

$$\frac{\partial a^{ij}(t,x)}{\partial t}, \frac{\partial^3 a^{ij}(t,x)}{\partial x^h \partial x^k \partial x^l}, \frac{\partial b^i(t,x)}{\partial x^k} \quad (i,j,h,k,l = 1, ..., m)$$

and $c(t,x)$

satisfy the generalized Lipschitz condition in $[s_0, t_0] \times \overline{G}$.

We define the partial derivative $\partial f(\xi)/\partial n_{t\xi}$ to the outer transversal direction $n_{t\xi}$ as follows: when **B** is represented by $\psi(x) \equiv \psi(x^1, \ldots, x^m) = 0$ with respect to a local coordinate around ξ and $\psi(x) > 0$ in **G**, we set

(1.3)
$$\frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t\xi}} = -\frac{\partial f(\xi)}{\partial x^i} \cdot \frac{\partial \psi(\xi)}{\partial x^j} a^{ij}(t,\xi);$$

this notion is independent of the special choice of the local coordinate around ξ by virtue of the transformation rule for $a^{ij}(t, x)$ (see [FS. (1.3)]). If we take a local coordinate with respect to which $a^{ij}(t,\xi) = \delta^{ij}$ i.e. $a^{ij}(t,\xi) \frac{\partial^2}{\partial x^i \partial x^j} = Laplacian$ at the point $\langle t,\xi \rangle$ (fixed), then $\partial f(\xi) / \partial n_{i\xi}$ means the partial derivative to the outer normal direction to **B**. We consider the boundary condition:

$$(B_{\alpha(t)}) \qquad \qquad \alpha(t,\xi)f(\xi) + \{1 - \alpha(t,\xi)\}\frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \quad (\xi \in \boldsymbol{B})$$

for each t, where $\alpha(t,\xi)$ is a function on $[s_0, t_0] \times B$, of C^1 -class in tand of $C^{2,t}$ -class in ξ and $0 \leq \alpha(t,\xi) \leq 1$. We shall say that a function f(t, x) on $(s_0, t_0) \times \overline{G}$ satisfies the boundary condition (B_{α}) if it satisfies $(B_{\alpha(t)})$ for any $t \in (s_0, t_0)$.

We define the metric tensor $a_{ij}(x)$, as stated in [FS, p. 79], and consider the measure $d_a x = \sqrt{a(x)} dx^1 \cdots dx^m (a(x) = \det || a_{ij}(x) ||)$ and de-

fine the adjoint operator L^* resp. A^* of L resp. A with respect to this measure. If M is an orientable Riemannian manifold with a metric tensor $g_{ij}(x)$ a priori, then it is natural to take the measure $d_g x = \sqrt{g(x)} dx^1 \cdots dx^m$ ($g(x) = \det ||g_{ij}(x)||$) in place of $d_a x$; in this case, it is sufficient only to replace a(x) by g(x) throughout the course of the present paper, while $a_{ij}(x)$ should not be replaced by $g_{ij}(x)$.

We assume further that:

(A. 2) the following relations hold on the set

$$\{\langle t, \xi \rangle; \alpha(t, \xi) \neq 1\} (\langle [s_0, t_0] \times \boldsymbol{B} \rangle):$$

(1.4)
$$\frac{\partial a^{ij}(t,\xi)}{\partial \boldsymbol{n}_{i\xi}} = 0^{3j} \quad (i,j=1,\ldots,m) \quad and$$

(1.5)
$$b^{i}(t,\xi) = \frac{1}{\sqrt{a(\xi)}} \cdot \frac{\partial}{\partial x^{j}} \left[\sqrt{a(\xi)} a^{ij}(t,\xi) \right] \ (i=1,\ldots,m) \,.$$

Under the above stated conditions (A. 1) and (A. 2), we shall consider the parabolic differential equations Lf=0 and $L^*f^*=0$ in the domain G with the boundary condition (B_{α}) .

By definition, a function u(t, x; s, y), $s_0 < s < t < t_0; x, y \in \overline{G}$, is called a *fundamental solution of the parabolic equation* Lf = 0 with the boundary condition (B_{α}) if, for any s and any function f(x) which is continuous in \overline{G} and satisfies the condition $(B_{\alpha(s)})$, the function

(1.6)
$$f(t, x) = \int_{G} u(t, x; s, y) f(y) d_a y \quad (t < s)$$

satisfies the conditions⁴):

(1.7) $\begin{cases} f(t, x) \text{ is of } C^1-\text{class in } t \text{ and of } C^2-\text{class in } x, \text{ and satisfies the equation } Lf=0 \text{ as well as the boundary condition } (B_{\alpha}) \end{cases}$

and

(1.8)
$$\lim_{x \to 0} f(t, x) = f(x)$$
 uniformly on \overline{G} .

A function $u^*(s, y; t, x)$, $s_0 < s < t < t_0; x, y \in \overline{G}$, is called a fundamental solution of the adjoint equation $L^*f^* = 0$ (of the equation Lf = 0) with the boundary condition (B_a) if, for any t and any continuous function

³⁾ It is true that $\partial a^{ij}/\partial n_t$ depends on the local coordinate, but the condition (1.4) is independent of it, because, if $||a^{ij}||$ is changed into $||\bar{a}^{ij}||$ by means of the coordinate transformation $(x^i) \rightarrow (\bar{x}^i)$, then we get $\frac{\partial \bar{a}^{ij}}{\partial n_t} = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} \cdot \frac{\partial a^{ki}}{\partial n_t}$ by virtue of [FS, (1.3)].

^{4), 5)} Cf. [FS, Definition 2]. The conditions corresponding to (1.9) and (1.9^{*}) in [FS] follow from (1.7) and (1.7^{*}) respectively in the case where \overline{G} is compact.

f(x) on **G**, the function

(1.6*)
$$f^*(s, y) = \int_G u^*(s, y; t, x) f(x) d_a x \quad (s < t)$$

satisfies the conditions⁵:

(1.7*) $\begin{cases} f^*(s, y) \text{ is of } C^1-\text{class in } s \text{ and of } C^2-\text{class in } y, \text{ and satisfies} \\ \text{the equation } L^*f^*=0 \text{ as well as the boundary condition } (B_{\alpha}) \end{cases}$

and

(1.8*)
$$\lim_{x \to 0} f^*(s, y) = f(y)$$

pointwisely in G and also strongly in $L^{1}(G)$.

The purpose of the present paper is to prove the following theorems, which are literally the same as those in [FS]⁶ except the statements concerning the boundary condition.

Theorem 1. There exists a function u(t, x, s, y) of C^1 -class in t and $s(s_0 < s < t < t_0)$ and of C^2 -class in x and y $(x, y \in \overline{G})$, with the following properties:

i) u(t, x; s, y) is a fundamental solution of the equation Lf = 0 with the boundary condition (B_{α}) ,

ii) $u^*(s, y; t, x) = u(t, x; s, y)$ is a fundamental solution of the adjoint equation $L^*f^* = 0$ with the boundary condition (B_{α}) ,

iii) $L_{tx}u(t, x; s, y) = 0$, $L_{sy}^*u(t, x; s, y) = 0$ and u(t, x; s, y) satisfies the boundary condition (B_{α}) as a function of $\langle t, x \rangle$ and also as a function of $\langle s, y \rangle$,

iv) $\int_G u(t, x, \tau, z) u(\tau, z; s, y) d_a z = u(t, x; s, y), s < \tau < t.$

Theorem 2. Let u(t, x; s, y) and $u^*(s, y; t, x)$ be the functions stated in Theorem 1.

i) If a function f(t, x) on $(s, t_0) \times \overline{G}$ satisfies (1.7) and (1.8) where f(x) is continuous in \overline{G} and satisfies (B_{α}) , then it is expressible by (1.6).

ii) If a function $f^*(s, y)$ on $(s_0, t) \times \overline{G}$ satisfies (1.7*) and (1.8*) where f(x) is a continuous function on \overline{G} , then it is expressible by (1.6*).

Theorem 3. If a function v(t, x; s, y) is continuous in the region: $s_0 < s < t < t_0; x, y \in \overline{G}$, and fatisfies the condition i) or ii) in Theorem 1, then it is identical with u(t, x; s, y) stated in Theorem 1.

Theorem 4. i) $u(t, x; s, y) \ge 0$ and $\int_{G} u(t, x; s, y) d_a y \le e^{\lambda(t-s)}$ where

⁶⁾ As for Theorem 4, see the supplement to [FS] in §0 of the present paper.

 $\lambda = \sup_{t,x} c(t,x)$; ii) if $c(t,x) \equiv 0$ in the differential operator A_{tx} and if $\alpha(t,\xi) \equiv 1$ in the boundary condition (B_{α}) , then $\int_{G} u(t,x;s,y) d_{\alpha} y = 1$.

We shall show, in another paper⁷, the existence of the fundamental solution of the parabolic differential equation with a boundary condition considered in a domain whose closure is not compact.

§2. Preliminaries. The following lemma may be proved by means of Lebesgue's convergence theorem, and will be useful throughout the present paper:

Lemma 1. Let (X, μ) be a measure space, and assume that

i) $f(t, \chi)$ is measurable in $\chi \in X$ for each $t \in (t_1, t_2)$,

- ii) f(t, X) is differentiable in t for a.a. $X \in X$ and
- iii) there exists a measurable function $\varphi(X)$ such that

$$\left|\frac{\partial f(t, \chi)}{\partial t}\right| \leq \varphi(\chi) \text{ in } (t_1, t_2) \text{ and } \int_{\chi} \varphi(\chi) d\mu(\chi) < \infty$$

Then

$$\frac{d}{dt}\int_{x}f(t, X)d\mu(X) = \int_{x}\frac{\partial f(t, X)}{\partial t}d\mu(X) \,.$$

Now let G, B and A_{tx} be as stated in §1 and z be any fixed point in **B**. Then, for any canonical coordinate (see §1) around $z, B \cap U_1(z)$ is represented by means of $\psi(x^1, \ldots, x^m) = 0$ where ψ is a function of $C^{4, t}$ -class. Hence, considering a suitable coordinate transformation in $U_1(z)$, we may show that

Lemma 2. There exists a canonical coordinate (x^i) around z such that $B \bigcup U_1(z)$ is expressible by $x^1 = 0$ and that $x^1 > 0$ in $G \cap U_1(z)$. Next we shall prove that

Lemma 3. Let (x^i) be a canonical coordinate as stated in Lemma 2, and consider the coordinate transformation: $(x^i) \rightarrow (x^i_t)$, for each $t(s_0 \leq t \leq t_0)$, defined by

(2.1)
$$\begin{cases} x_t^1 = \varphi^1(t, x) \equiv \gamma x^1 \\ x_t^j = \varphi^j(t, x) \equiv \gamma \left\{ -\frac{a^{1j}(t, \xi_x)}{a^{11}(t, \xi_x)} x^1 + x^j \right\}, \quad j = 2, \dots, m \end{cases}$$

⁷⁾ See the author's paper: Fundamental solutions of parabolic differential equations and eigenfunction expensions for elliptic differential equations, forthcoming to Nagoya Mathematical Journal.

where $\xi_x = \langle 0, x^2, ..., x^m \rangle (\in B)$ for $x = \langle x^1, ..., x^m \rangle \in U_1(z)$ and γ is a suitable positive constant. Then there exists $\delta = \delta_z > 0$ such that i) $U_{\delta}(z) \subset U_1^{\epsilon}(z) \subset U_1(z)$ and $U_{\delta/3}(z) \subset U_{1/3}^{\epsilon}(z)$ for any t, ii) B is represented by $x_t^1 = 0$ in $U_1^{\epsilon}(z)$ and iii) if $a^{ij}(t, x)$ is changed into $a_{\varphi}^{ij}(t, x)$ by means of this transformation (i, j = 1, ..., m), then

(2.2) $a_{\varphi}^{ij}(t,\xi) = a_{\varphi}^{j1}(t,\xi) = 0$ and $a_{1j}^{\varphi}(t,\xi) = a_{j1}^{\varphi}(t,\xi) = 0$, $j=2,\ldots,m$,

for any $\xi \in \mathbf{B} \cap U_1^t(z)$, where $U_{\varepsilon}^t(x) = \{y \in \mathbf{M} ; \sum_i (y_i^t - x_i^t)^2 < \varepsilon\}$ and $||a_{ij}^{\varphi}(t, x)|| = ||a_{\varphi}^{ij}(t, x)||^{-1}$. The mapping $\varphi_t(x) = \langle \varphi^1(t, x), \dots, \varphi^m(t, x) \rangle$ of $U_{\delta}(z)$ into $U_1^t(z)$ is one-to-one and of $C^{3,L}$ -class in x, and $a_{\varphi}^{ij}(t, x)$, $i, j = 1, \dots, m$, are of C^1 -class in t and of $C^{2,L}$ -class in x.

PROOF. We notice that $a^{11}(t, x) > 0$ in $U_1(z)$, and consider the coordinate transformation (2.1) around z. Then $x_t^1 = 0$ if and only if $x^1 = 0$, and we have for any $\xi = \langle 0, x^2, \dots, x^m \rangle \in B \cap U_1(z)$

(2.3)
$$\begin{cases} \left(\frac{\partial x_{t}^{i}}{\partial x^{k}}\right)_{x=\xi} = \gamma \delta_{k}^{1} \\ \left(\frac{\partial x_{t}^{j}}{\partial x^{k}}\right)_{x=\xi} = \gamma \left\{-\frac{a^{1j}(t,\xi)}{a^{11}(t,\xi)}\delta_{k}^{1} + \delta_{k}^{j}\right\} \quad (\delta_{k}^{j}: \text{ Kronecker's delta})\end{cases}$$

for $1 \leq j \leq m$ and $2 \leq j \leq m$. Hence the Jacobian

$$\frac{\partial(x_t^1,\ldots,x_t^m)}{\partial(x^1,\ldots,x^m)}$$

is bounded away from zero in $U_{\varepsilon_1}(z)$ for suitable ε_1 $(0 < \varepsilon_1 < 1)$ which may be chosen independently of t by virtue of the continuity of $a^{ij}(t, x)$ on the compact set $[s_0, t_0] \times \overline{U_{\varepsilon}(z)}$ for any ε $(0 < \varepsilon < 1)$, and hence the transformation (2.1) is well defined in $U_{\varepsilon_1}(z)$. Considering the continuity of $a^{ij}(t, x)$ on $[s_0, t_0] \times \overline{U_{\varepsilon_1}(z)}$ again, we may determine γ and $\delta > 0$ so that $U_{\delta}(z) < U_1^i(z) < U_1(z)$ and $U_{\delta/3}(z) < U_{1/3}^i(z)$ for any t. By virtue of the transformation rule for a^{ij} (see [FS, (1.3)]), we have, for any $< t, \xi > \in [s_0, t_0] \times (\mathbf{B} \cap U_1^i(z))$ and for $j \ge 2$,

$$\begin{aligned} a_{\varphi}^{1j}(t,\xi) &= \left(\frac{\partial x_t^1}{\partial x^k}\right)_{x=\xi} \cdot \left(\frac{\partial x_t^j}{\partial x^1}\right)_{x=\xi} a^{kl}(t,\xi) \\ &= -\gamma^2 \frac{a^{1j}(t,\xi)}{a^{11}(t,\xi)} a^{11}(t,\xi) + \gamma^2 a^{1j}(t,\xi) = 0 \quad (\text{see } (2.3)) \,, \end{aligned}$$

and consequently we get (2.2). The last part of Lemma 3 is also evident by means of the above arguments.

§3. Local construction of a quasi-parametrix. Let G, B and A_{tx} be as before, let z be any fixed point in B, and let (x^i) and (x^i_t) $(s_0 \le t \le t_0)$ be canonical coordinates around z as stated in Lemma 3. Then we have

(3.1)
$$\frac{\partial f(\xi)}{\partial \boldsymbol{n}_t} = -\frac{\partial f(\xi)}{\partial \boldsymbol{x}_t^i} a_{\varphi}^{i1}(t,\xi) = -a_{\varphi}^{11}(t,\xi) \frac{\partial f(\xi)}{\partial \boldsymbol{x}_t^1} \qquad (\xi \in \boldsymbol{B})$$

for any function f(x) of C¹-class, and hence the assumption (1.4) implies that

(3.2)
$$\frac{\partial a_{ij}^{\varphi}(t,\xi)}{\partial x_i^1} = 0 \quad \text{on } \{ \langle t,\xi \rangle; \alpha(t,\xi) \neq 1 \},$$

Now we put for $s_0 \leq s < t \leq t_0$ and $X, Y \in \mathbb{R}^m$

(3.3)
$$\begin{cases} V_0(A_{ij}; t, X; s, Y) = (t-s)^{-\frac{m}{2}} \exp\left[-\frac{A_{ij}(X^i - Y^i)(X^j - Y^j)}{4(t-s)}\right] \\ V_0(A_{ij}) = \int_{\mathbb{R}^m} \exp\left[-\frac{A_{ij}Y^iY^j}{4}\right] dY^1 \dots dY^m, \end{cases}$$

and define for $s_0 \leq s < t \leq t_0$ and $x, y \in U_{\delta}(z) \cap \bar{G}$ ($\delta = \delta_z$ as stated in Lemma 3)

(3.4)
$$\begin{cases} V(t, x; s, y) = V_0(a_{ij}^{\varphi}(t, x); t, \varphi_t(x); s, \varphi_s(y)) & \text{(see Lemma 3)} \\ \bar{V}(t, x; s, y) = V_0(a_{ij}^{\varphi}(t, x); t, \varphi_t(x); s, \bar{\varphi}_s(y)) \\ V(t, x) = V_0(a_{ij}^{\varphi}(t, x)) \end{cases}$$

where $\bar{\varphi}_s(y) = \langle -\varphi^1(s, y), \varphi^2(s, y), \dots, \varphi^m(s, y) \rangle$. Further we put

(3.5)
$$\begin{cases} p(t, x; s, y) \\ = \frac{2(t-s) \cdot \alpha(t, \xi_{tx})}{2(t-s)\alpha(t, \xi_{tx}) + \varphi^{1}(s, y)[1-\alpha(t, \xi_{tx}) \exp\{-|\varphi^{1}(t, x)|^{2}\}]} \\ q(t, x; s, y) \\ = \frac{\varphi^{1}(s, y)[1-\alpha(t, \xi_{tx}) \exp\{1-|\varphi^{1}(t, x)|^{2}\}]}{2(t-s)\alpha(t, \xi_{tx}) + \varphi^{1}(s, y)[1-\alpha(t, \xi_{tx}) \exp\{-|\varphi^{1}(t, x)|^{2}\}]} \end{cases}$$

where ξ_{tx} is the point $(\in B)$ defined by the equations:

$$\varphi^{\scriptscriptstyle 1}(t,\xi_{\scriptscriptstyle tx})=0$$
, $\varphi^{\scriptscriptstyle j}(t,\xi_{\scriptscriptstyle tx})=\varphi^{\scriptscriptstyle j}(t,x)$ for $j\geq 2$;

such ξ_{tx} is uniquely determined for any $x \in U_{\delta}(z)$ and any t by virtue of Lemma 3.

Applying (3.1), (3.2), Lemma 1 and Lemma 3 to (3.3), (3.4) and (3.5), and making use of the fact that $\partial f / \partial n_{t\xi}$ is independent of the local coordinate, we obtain

(3.6)
$$\frac{\partial V(t,\xi)}{\partial \boldsymbol{n}_{t\xi}} = 0$$

and

(3.7)
$$\frac{\partial V(t,\xi;s,y)}{\partial n_{t\xi}} = -a_{\varphi}^{11}(t,\xi) \cdot \left\{ -a_{11}^{\varphi}(t,\xi) \cdot \frac{-\varphi^{1}(s,y)}{2(t-s)} V(t,\xi;s,y) \right\}$$
$$= \frac{-\varphi^{1}(s,y)}{2(t-s)} V(t,\xi;s,y)$$

for $<\!\!t,\xi\!\!>$ such that $\xi \in {m B} igcap U_{\delta}(z)$ and $lpha(t,\xi) \! \not = \!\!1$, and we get also

(3.8)
$$\frac{\partial p(t,\xi;s,y)}{\partial \boldsymbol{n}_{t\xi}} = \frac{\partial q(t,\xi;s,y)}{\partial \boldsymbol{n}_{t\xi}} = 0$$

and

(3.9)
$$V(t, \xi; s, y) = \overline{V}(t, \xi; s, y)$$

for any $\xi \in \mathbf{B} \cap U_{\delta}(\mathbf{z})$. We define

(3.10)
$$W_{s}(t, x; s, y) = p(t, x; s, y) J_{s}(y) \frac{V(t, x; s, y) - \overline{V}(t, x; s, y)}{V(t, x)} + q(t, x; s, y) J_{s}(y) \frac{V(t, x; s, y) + \overline{V}(t, x; s, y)}{V(t, x)}$$

where

(3.11)
$$J_s(y) = \frac{\partial \left[\varphi^1(s, y), \dots, \varphi^m(s, y)\right]}{\partial \left[y^1, \dots, y^m\right]} \quad (\text{Jacobian}).$$

Then we may prove from (3.6-9) and by simple calculation that

(3.12)
$$\alpha(t,\xi)W_{z}(t,\xi;s,y) + \{1 - \alpha(t,\xi)\}\frac{\partial W_{z}(t,\xi;s,y)}{\partial n_{tx}} = 0$$

for $\xi \in U_{\delta}(z) \cap B$,

that is, $W_z(t, x; s, y)$ satisfies the boundary condition (B_a) as a function of $\langle t, x \rangle \in [s_0, t_0] \times U_{\delta}(x)$. Since

$$(3.13) \quad \int_{U_{\delta}(z)\cap G} V(t,x\,;\,s,y) J_{s}(y) dy + \int_{U_{\delta}(z)\cap G} \overline{V}(t,x\,;\,s,y) J_{s}(y) dy$$
$$\leq \int_{R^{m}} V_{0}(a^{\varphi}_{ij}(t,x)\,;\,t,\varphi_{t}(x)\,;\,s,Y) dY = V(t,x)$$
$$(dy = dy^{1} \cdots dy^{m},\ dY = dY^{1} \cdots dY^{m})$$

and since the denominators and numerators in the right-hand side of (3.5) are positive for any $x, y \in U_{\delta}(z) \cap G$, we get

$$(3.14) \quad \int_{U_{\delta}(z) \cap G} |W_{z}(t, x; s, y)| dy \leq 1 \quad \text{for any} \quad x \in U_{\delta}(z) \cap \overline{G}.$$

Now we have the following

Lemma 4. If f(x) is continuous in \overline{G} and vanishes outside $U_{\delta}(z)$, then

(3.15)
$$\lim_{t \neq s} \int_{G} \frac{V(t, x; s, y) + V(t, x; s, y)}{V(t, x)} f(y) J_{s}(y) dy = f(x)$$

uniformly in $U_{\delta}(z) \cap \overline{G}$.

PROOF. By virtue of (3.3) and the uniform continuity of $\varphi^{j}(t, x)$ on $[s_0, t_0] \times \overline{U_{\delta}(z)}$, we may show that

$$\lim_{t \neq s} \int_{R^m} \frac{V_0(a_{ij}^{\varphi}(t, x); t, \varphi_t(x); s, Y)}{V_0(a_{ij}^{\varphi}(t, x))} F(Y) dY = F(\varphi_s(x))$$

uniformly in $U_{\delta}(z) \cap \overline{G}$

for any continuous function F(Y) with a compact support; and hence, if especially $F(\bar{Y}) = F(Y)$ where $\bar{Y} = \langle -Y^1, Y^2, \dots, Y^m \rangle$ for $Y = \langle Y^1, Y^2, \dots, Y^m \rangle$, then

$$\lim_{t \neq s} \int_{R^m(Y^1 > 0)} \frac{V_0(a_{ij}^{\varphi}(t, x); t, \varphi_t(x); s, Y) + V_0(a_{ij}^{\varphi}(t, x); t, \varphi_t(x); s, \overline{Y})}{V_0(a_{ij}^{\varphi}(t, x))} F(Y) dY$$

= $F(\varphi_s(x))$ uniformly in $U_{\delta}(z) \cap \overline{G}$.

Putting

$$F(Y) = F(\overline{Y}) = \begin{cases} f(\varphi_s^{-1}(Y)) & \text{if } \sum_{i} (Y^{i})^2 < 1 \\ 0 & \text{if not} \end{cases}$$

in the above relation, and considering (3.4) and (3.11), we obtain (3.15).

Lemma 5. If f(x) is such a function as stated in Lemma 3 and if **D** is an open set containing $\mathbf{B}^{(s)} = \{\xi \in \mathbf{B}; \alpha(s, \xi) = 1\}$, where s is any fixed real number $(s_0 < s < t_0)$, then

$$\lim_{t \neq s} \int_{G} W_{z}(t, x; s, y) f(y) dy = f(x) \quad uniformly \text{ in } U_{\delta}(z) \bigwedge \overline{G} - D.$$

PROOF. Let ε be an arbitrary positive number. Then, by virtue of Lemma 4, there exists $\Delta_1 > 0$ such that

(3.16)
$$\left|\int_{U_{\delta}(z)\cap \bar{G}}\frac{V(t,x;s,y)+V(t,x;s,y)}{V(t,x)}J_{s}(y)f(y)\,dy-f(x)\right| < \frac{\varepsilon}{4}$$

for any $x \in U_{\delta}(z) \cap \overline{G}$ whenever $s < t < s + \Delta_1$. On the other hand, by

virtue of (3.13) and (3.14), there exists $\eta_1 > 0$ such that

$$(3.17) \left| \int_{U_{\delta}(z) \cap \{\varphi^{1}(s, y) < \eta_{1}\} \cap \overline{G}} \frac{V(t, x; s, y) + \overline{V}(t, x; s, y)}{V(t, x)} J_{s}(y) f(y) dy \right| < \frac{\varepsilon}{4}$$
and that

and that

(3.18)
$$\left|\int_{U_{\delta}(z) \cap \{\varphi^{1}(s,y) < \eta_{1}\} \cap \overline{G}} W_{z}(t,x;s,y)f(y)dy\right| < \frac{\varepsilon}{4}.$$

Since $1-\alpha(t,\xi_{tx}) \exp\{-|\varphi^1(t,x)|^2\} > 0$ for any t and any $x \in U_{\delta}(z) \cap \overline{G} - B^{(s)}$, there exists $\eta_2 > 0$ such that

$$1 - \alpha(t, \xi_{tx}) \exp\{-|\varphi^{1}(t, x)|^{2}\} \ge \eta_{2} \quad (\text{see } (3.5))$$

for any t and any $x \in U_{\delta}(z) \cap \overline{G} - D$. Hence $\varphi^{1}(s, y) \geq \eta_{1}$ implies that

$$|1-q(t, x; s, y)| = |p(t, x; s, y)| \leq (t-s)/\eta_1\eta_2$$

for any $t \ge s$ and any $x \in U_{\delta}(z) \cap \overline{G} - D$, and hence it follows from (3.10) and (3.13) that there exists $\Delta_2 > 0$ such that

(3.18)
$$\left| \int_{U_{\delta}(z) \cap \{\varphi^{1}(s, y) \ge \eta_{1}\} \cap \overline{G}} \left\{ W_{z}(t, x; s, y) - \frac{V(t, x; s, y) + \overline{V}(t, x; s, y)}{V(t, x)} J_{s}(y) \right\} f(y) dy \right| < \frac{\varepsilon}{4}$$

for any $x \in U_{\delta}(z) \cap G - \overline{D}$ whenever $s < t < s + \Delta_2$. Since f(y) = 0 for $y \in \overline{G} - U_{\delta}(z)$, it follows from (3.16–19) that

$$|\int_{\boldsymbol{G}} W_{\boldsymbol{z}}(t, x; s, y) f(y) dy - f(x)| \leq \varepsilon$$
 for any $x \in U_{\delta}(z) \cap \overline{\boldsymbol{G}} - \boldsymbol{D}$

whenever $s < t < s + \min\{\Delta_1, \Delta_2\}$. Thus we obtain Lemma 5.

Lemma 6. Assume that f(x) is continuous in \overline{G} , vanishes outside $U_{s}(z)$ and satisfies the boundary condition $(B_{\alpha(s)})$. Then

$$\lim_{t \neq s} \int_{\mathcal{G}} W_z(t, x; s, y) f(y) dy = f(x) \quad uniformly \ in \ U_{\delta}(z) \cap \overline{G}.$$

PROOF. Let ε be an arbitrary positive number, and put

$$\boldsymbol{D} = \{x ; x \in \bar{\boldsymbol{G}}, |f(x)| < \varepsilon/5\} \bigcup \{\bar{x} ; x \in \bar{\boldsymbol{G}}, |f(x)| < \varepsilon/5\}$$

where $\bar{x} = \langle -x^1, x^2, ..., x^m \rangle$ for $x = \langle x^1, x^2, ..., x^m \rangle$. Then, by virtue of the assumption of this lemma, **D** is an open set containing $B^{(s)} = \{\xi \in B; \alpha(s, \xi) = 1\}$ and hence, by Lemma 5, there exists $\Delta > 0$ such that

$$(3.20) \quad |\int_{\boldsymbol{G}} W_{\boldsymbol{x}}(t, \boldsymbol{x}; \boldsymbol{s}, \boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y} - f(\boldsymbol{x})| \leq \varepsilon \quad \text{for any } \boldsymbol{x} \in U_{\delta}(\boldsymbol{z}) \cap \boldsymbol{G} - \boldsymbol{D}$$

whenever $s < t < s + \Delta$. On the other hand, by Lemma 4, there exists $\Delta' > 0$ such that

$$\int_{\boldsymbol{G}} \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} |f(y)| J_{s}(y) dy < \frac{2}{5} \varepsilon$$

for any $x \in U_{\delta}(z) \cap \bar{\boldsymbol{G}} \cap \boldsymbol{D}$

whenever $s < t < s + \Delta'$. Hence, considering the non-negativity of V(t, x; s, y), $\overline{V}(t, x; s, y)$ and $J_s(y)$ (see the proof of Lemma 3) and using the facts: $0 \le p(t, x; s, y) \le 1$ and $0 \le q(t, x; s, y) \le 1$, we obtain from (3.10) that

$$\left|\int_{\boldsymbol{G}} W_{\boldsymbol{z}}(t, x; s, y) f(y) dy\right| < \frac{4}{5} \varepsilon$$
 for any $x \in U_{\delta}(z) \cap \bar{\boldsymbol{G}} \cap \boldsymbol{D}$

and accordingly

(3.21) $|\int_{G} W_{z}(t, x; s, y) f(y) dy - f(x)| \leq \varepsilon$ for any $x \in U_{\delta}(z) \cap \overline{G} \cap D$ whenever $s < t < s + \Delta'$. From (3.20) and (3.21) we get

$$|\int_{G} W_{z}(t, x; s, y) f(y) dy - f(x)| \leq \varepsilon \text{ for any } x \in U_{\delta}(z) \cap \tilde{G}$$

whenever $s < t < s + \min\{\Delta, \Delta'\}$. Thus we obtain Lemma 6.

Next, let $f(\tau, y)$ be a continuous function on $(s, t_0) \times G$ which vanishes outside $U_{\delta}(z)$ and satisfies the condition: $\int_s^t \int_G |f(\tau, y)| dy d\tau < \infty$, and put

$$f(t, x, \tau) = \int_G W_z(t, x; \tau, y) f(\tau, y) dy, \quad t > \tau > s,$$

$$F(t, x) = \int_s^t f(t, x, \tau) d\tau.$$

Then we have

Lemma 7. i) $f(t, x, \tau)$ and F(t, x) satisfy the boundary condition (B_{α}) in $U_{\delta}(z) \cap B$; ii) for any $s'(t_{0} > s' > s)$

$$\lim_{\tau \downarrow s'} \int_G f(\tau, x) W_z(\tau, x; s', y) dx = f(s', y) \text{ in } G \cap U_{\delta}(z);$$

iii) if $f(\tau, y)$ satisfies the generalized Lipschitz condition in $(s, t_0) \times \overline{G}$, then

$$\frac{\partial F(t, x)}{\partial t} = f(t, x) + \int_{s}^{t} \int_{G} \frac{\partial W_{z}(t, x; \tau, y)}{\partial t} f(\tau, y) dy d\tau ,$$
$$A_{tx}F(t, x) = \int_{s}^{t} \int_{G} A_{tx}W_{z}(t, x; \tau, y) f(\tau, y) dy d\tau .$$

OUTLINE OF THE PROOF. The proposition i) may be shown by means of (3.12) and Lemma 1, and the proposition ii) may be proved similarly to [FS, Lemma 2]. The proposition iii) is proved as follows. Considering the fact that the mapping $\varphi_t(x)$ is one-to-one and of $C^{2,t}$ -class for any t (see Lemma 3), using the same idea as in [FS, Lemmas 1 and 3] and applying Lemma 1 (§1), we may show that

$$\frac{\partial f(t, x, \tau)}{\partial t} = \int_{\boldsymbol{G}} \frac{\partial W_z(t, x; \tau, y)}{\partial t} f(\tau, y) dy,$$

$$\frac{\partial f(t, x, \tau)}{\partial x^i} = \int_{\boldsymbol{G}} \frac{\partial W_z(t, x; \tau, y)}{\partial x^i} f(\tau, y) dy,$$

$$\frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} = \int_{\boldsymbol{G}} \frac{\partial^2 W(t, x; \tau, y)}{\partial x^i \partial x^j} f(\tau, y) dy.$$

and

$$\lim_{\substack{t>t'>\tau\\t\neq\tau}} f(t, x, t') = f(\tau, x)$$

and that there exist M > 0 and $\gamma = \gamma(t, x) > 0$ such that

$$\frac{\partial f(t', s, \tau)}{\partial t'} \leq M(t-s)^{-(1-\frac{\gamma}{2})} \text{ whenever } s < \tau < t \leq t';$$

further we have

$$\int_{s}^{t} \left| \frac{\partial f(t, x, \tau)}{\partial x^{i}} \right| d\tau < \infty \text{ and } \int_{s}^{t} \left| \frac{\partial^{2} f(t, x, \tau)}{\partial x^{i} \partial x^{j}} \right| d\tau < \infty.$$

Hence we may prove the proposition iii) by the same manner as in [FS, Lemma 4].

Lemma 8. If $\omega(t, x)$ is a function of C^1 -class in t and of C^2 -class in x, and vanishes outside $U_{\delta}(z)$, then there exists a constant $M_0 > 0$ such that

$$|L_{tx}[\omega(t, x) W_{z}(t, x; s, y)]| \leq M_{0}(t-s)^{-\frac{m+1}{2}} \exp\left\{-\frac{M_{0}\sum_{i}(x^{i}-y^{i})^{2}}{4(t-s)}\right\}.$$

This may be proved similarly to [FS, Lemma 5].

Finally we define a quasi-parametrix $W_z(t, x; s, y)$ around any inner point z of G as follows. We fix a canonical coordinate (x^i) around z satisfying $U_1(z) \subset G$ and put

$$\delta_z = 1$$

 $x_t^i \equiv \varphi^i(t, x) = x^i, i = 1, ..., m$, for any t

(consequently $\varphi_t(x) = \langle x^1, \dots x^m \rangle$ and $a_{ij}^{\varphi}(t, x) = a_{ij}(t, x)$ —cf. Lemma 3). Using this local coordinate, we define V(t, x; s, y) and V(t, x) by means of (3.3) and (3.4), and put

$$W_{z}(t, x; s, y) = \frac{V(t, x; s, y)}{V(t, x)} \qquad (s_{0} < s < t < t_{0}; x, y \in U_{1}(z)).$$

Then we may easily prove that Lemmas 6, 8 and Lemma 7 ii), iii) hold for $W_z(t, x; s, y)$ defined here. (See Lemmas 2, 4 and 5 in [FS].)

§4. Gloval construction of a quasi-parametrix and a fundamental solution. For each $z \in \overline{G}(=G+B)$, we fix canonical coordinates (x^i) and (x^i_i) around z as stated in §2, and put

$$U(z, \varepsilon) = \{x \in \boldsymbol{M}; \sum_{i} (x^{i} - z^{i})^{2} < \varepsilon\} \qquad (\varepsilon > 0).$$

Since \bar{G} is compact, there exists a finite sequence $\{z_1, \ldots, z_N\} \subset \bar{G}$ such that

(4.1)
$$\overline{G} \subset \bigcup_{\nu=1}^{N} U(z_{\nu}, \delta_{\nu}/3) \text{ where } \delta_{\nu} = \delta_{z_{\nu}} \text{ (see § 2),}$$

and then, since

(4.2)
$$z_{\nu} \in G$$
 implies $U(z_{\nu}, \delta_{\nu}) \subset G$ (see §2),

we have

(4.3)
$$\boldsymbol{B} \subset \bigcup_{\boldsymbol{z}_{\nu} \in \boldsymbol{B}} U(\boldsymbol{z}_{\nu}, \, \delta_{\nu}/3) \, .$$

Let $\omega(\lambda)$ be a function of $C^{2, t}$ -class in $0 \leq \lambda < \infty$ such that $\omega(\lambda) = 1$ or 0 if $0 \leq \lambda \leq 1/3$ or $\lambda \geq 2/3$ respectively and that $0 \leq \omega(\lambda) \leq 1$ for any λ , and put for each ν

$$\omega_{\nu}(t, x) = \begin{cases} \omega(\sum_{i} [x_{t}^{i} - (z_{\nu})_{t}^{i}]^{2}) & \text{for } x \in \bar{\boldsymbol{G}} \cap U(z_{\nu}, \delta_{\nu}) \\ 0 & \text{for } x \in \bar{\boldsymbol{G}} - U(z_{\nu}, \delta_{\nu}) . \end{cases}$$

Then $\omega_{\nu}(t, x)$, $\nu = 1, ..., N$, are of C^1 -class in t and of $C^{2, t}$ -class in $x \in \overline{G}$, and

(4.4)
$$\frac{\partial \omega_{\nu}(t,\xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \quad \text{for any } \langle t,\xi \rangle \in [s_0,t_0] \times \boldsymbol{B};$$

this may be proved by considering the local coordinate (x_t^i) around z_v for each t since the operator $\partial/\partial n_t$ is independent of the special choice of the local coordinate.

Now let $a_{\nu}(x)$ be the restriction of $a(x) = \det || a_{ij}(x) ||$ (see §1) to $U(z_{\nu}, \delta_{\nu})$ with the local coordinate (x^i) around z stated above, and put, for $s_0 < s < t < t_0$,

$$W_{\nu}(t, x; s, y) = \begin{cases} W_{z_{\nu}}(t, x; s, y) \text{ (as stated in § 3) if } x, y \in U(z_{\nu}, \delta_{\nu}) \cap G \\ 0 \text{ if not.} \end{cases}$$

We define a quasi-parametrix:

$$Z(t, x; s, y) = \frac{\sum_{\nu} \omega_{\nu}(t, x) \omega_{\nu}(s, y) W_{\nu}(t, x; s, y)}{\sum_{\nu} \omega_{\nu}(t, x)^2 \sqrt{a_{\nu}(y)}} \begin{pmatrix} s_0 < s < t < t_0 \\ x, y \in \bar{G} \end{pmatrix}$$

Then Z(t, x; s, y) is of C^1 -class in t and s, and of $C^{2, t}$ -class in x and y, and it follows from (3.12), (4.2), (4.3) and (4.4) that

(4.5)
$$\alpha(t,\xi)Z(t,\xi;s,y) + \{1-\alpha(t,\xi)\} \frac{\partial Z(t,\xi;s,y)}{\partial \boldsymbol{n}_{t\xi}} = 0 \quad (\xi \in \boldsymbol{B}),$$

that is, Z(t, x; s, y) satisfies the boundary condition (B_{α}) as a function of $\langle t, x \rangle$. Further, by virtue of Lemmas 6, 7 and 8, we obtain the following three lemmas.

Lemma 9. i) If f(x) is continuous in G, then

 $\lim \int_G Z(t, x; s, y) f(y) d_a y = f(x) in G;$

if especially f(x) satisfies the boundary condition $(B_{\alpha(s)})$, then the above convergence is uniform in \overline{G} .

ii) if f(t, x) is continuous in $[s, t_0) \times G$, then

$$\lim \int_G f(t, x) Z(t, x; s, y) d_a x = f(s, y) \ in \ G.$$

Lemma 10. If $f(\tau, y)$ is continuous in $(s, t_0) \times \overline{G}$ and satisfies the condition: $\int_s^t \int_G |f(\tau, y)| d_a y d\tau < \infty$, then

$$f(t, x, \tau) = \int_G Z(t, x; \tau, y) f(\tau, y) d_a y \qquad (t < \tau < s)$$

and

$$F(t, x) = \int_s^t f(t, x, \tau) d\tau$$

satisfy the boundary condition (B_{α}) ; if further $f(\tau, y)$ satisfies the generalized Lipschitz condition in $(s, t_0) \times \overline{G}$, then

$$\begin{cases} \frac{\partial F(t, x)}{\partial t} = f(t, x) + \int_{s}^{t} \int_{G} \frac{\partial Z(t, x; \tau, y)}{\partial t} f(\tau, y) d_{a} y d\tau, \\ A_{tx}F(t, x) = \int_{s}^{t} \int_{G} A_{tx}Z(t, x; \tau, y) f(\tau, y) d_{a} y d\tau. \end{cases}$$

Lemma 11. Z(t, x; s, y) satisfies all inequalities stated in [FS, Lemma 8] for a suitable constant M > 0.

Thus we see that Z(t, x; s, y) has all properties stated in [FS, §2]. Hence, starting from this quasi-parametrix Z(t, x; s, y), we may construct u(t, x; s, y) in the entirely same way as in [FS, §3]. We may also construct $u^*(t, x; s, y)$ in the similar manner for the adjoint equation $L^* f^* = 0$ with the same boundary condition (B_{α}) . The functions u(t, x; s, y) and $u^*(t, x; s, y)$ defined here have the properties stated in [FS, §3] where the manifold M should be replaced by the compact domain \overline{G} and the uniformity of the convergence in [FS, (3.13)] may be proved if and only if f(x) is the limit of a uniformly convergent sequence of functions satisfying the the boundary condition $(B_{\alpha(s)})^{\$}$. Moreover u(t, x; s, y) and $u^*(t, x; s, y)$ satisfy the boundary condition (B_{α}) as functions of $\langle t, x \rangle$ — see Lemma 10 and the procedure of the construction of u(t, x; s, y) (in [FS, §3]).

§5. Proof of Theorems.

Lemma 12. If f(x) and h(x) are functions of C^2 -class on \overline{G} satisfying the boundary condition $(B_{\alpha(t)})$ (t: fixed), then

$$\int_G f(x) \cdot A_{tx} h(x) d_a x = \int_G A_{tx}^* f(x) \cdot h(x) d_a x.$$

PROOF. By partial integration, we obtain the Green's formula:

$$\begin{split} \int_{G} f(x) \cdot A_{tx} h(x) d_{a} x - \int_{G} A_{tx}^{*} f(x) \cdot h(x) d_{a} x \\ &= \int_{B} \left\{ f(\xi) \frac{\partial h(\xi)}{\partial \boldsymbol{n}_{t}} - \frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t}} h(\xi) \right\} \tilde{d}\xi \\ &+ \int_{B} \left\{ \frac{\partial}{\partial x^{j}} \left[\sqrt{a(\xi)} a^{ij}(t,\xi) \right] - \\ &- \sqrt{a(\xi)} b^{i}(t,\xi) \right\} \frac{\partial \psi(\xi)}{\partial x^{i}} f(x) h(x) \tilde{d}\xi \end{split}$$

where $\tilde{d\xi} = d\xi^1, \ldots, d\xi^{m-1}$ is the hypersurface area on **B** and $\psi(x)$ is such function that $\psi(x) = 0$ determines **B** and that $\psi(x) > 0$ in **G**. But the right-hand side equals zero by virtue of the boundary condition $(B_{\alpha(t)})$ and the assumption (1.5). Hence we obtain Lemma 12.

From this lemma we obtain the following (see [FS, Lemma 11])

Lemma 13. If a function $f^*(s, y)$ on $(s_0, t) \times \overline{G}$ satisfies (1.7*) and (B_{α}) , then

⁸⁾ This assumption for f(x) is equivalent to the following one: $f(\xi) = 0$ on $\mathbf{B}^{(s)} = \{\xi \in \mathbf{B} : \alpha(s, \xi) = 1\}$

 $\int_G f^*(\tau, x) u(\tau, x; s, y) d_a x = f^*(s, y) \text{ for any } \tau \in (s, t).$

Therefore, we may see that:

PROOF OF THEOREMS 1, 2 AND 3 may be performed in the same way as the proof of the corresponding theorems in [FS] (see [FS, pp. 89–90]). It seems not to be necessary to repeat the entirely same argument. The propositions concerning the boundary condition which are not included in [FS] may be easily proved from properties of u(t, x; s, y) and $u^*(t, x; s, y)$ stated in §4 of the present paper.

In order to prove Theorem 4, we consider, as in $\S 0$, the functions

(5.1)
$$f_s(t, x) = \int_G u(t, x; s, y) f(y) d_a y$$

and

(5.2)
$$g(t, x) \equiv g_s^{(\tau, n)}(t, x) = f_s(t, x) \exp\left\{-\left(\frac{t-s}{\tau-s}\right)^n\right\}$$

where f(x) is an arbitrary continuous function on G such that $0 \leq f(x) \leq 1$ and the support of f(x) is a compact set contained in the domain G, and τ and n are as stated in §0. Then $g_s^{(\tau,n)}(t,x)$ is continuous in $(s, t_0) \times \overline{G}$ and satisfies (0.3), (0.4) and the boundary condition (B_{α}) .

Lemma 14. If $c(t, x) \leq 0$, then the function g(t, x) takes neither positive maximum nor negative minimum at any point in $(s, t_0) \times \overline{G}$ (for any fixed τ , n and s).

PROOF. It is easily proved by the well known method that g(t, x) takes neither positive maximum nor negative minimum at any point in the open set $(s, t_0) \times G$.

Suppose that:

(5.3) g(t, x) takes the positive maximum at $\langle t_1, \xi_1 \rangle \in (s, t_0) \times B$.

 $f_s(t, x)$ satisfies Lf = 0 in $(s, t_0) \times \overline{G}$ as may be seen from the properties of u(t, x; s, y), where the partial derivatives at any $\xi \in \mathbf{B}$ should be understood as defined in § 1, and g(t, x) satisfies the boundary condition (B_{α}) as well as $f_s(t, x)$. We adopt a canonical coordinate around ξ_1 as stated in Lemma 3. Then we obtain from (5.3), (3.1) and (B_{α}) that $\partial g(t_1, \xi_1)/\partial x_t^1 \leq 0$ and that

$$\alpha(t_1,\xi_1)g(t_1,\xi_1) - \{1 - \alpha(t_1,\xi_1)\}a_{\varphi}^{11}(t_1,\xi_1)\frac{\partial g(t_1,\xi_1)}{\partial x_t^1} = 0.$$

Since $g(t_1, \xi_1) > 0$ and $a_{\varphi}^{11}(t_1, \xi_1) > 0$, it follows that $\alpha(t_1, \xi_1)$ should be

zero, consequently $\partial g(t_1, \xi_1)/\partial x_t^1 = 0$, and accordingly $\partial^2 g(t_1, \xi_1)/(\partial x_t^1)^2 \leq 0$ by virtue of (5.3). Moreover, since $\langle t_1, \xi_1 \rangle$ may be considered as the maximising point of $g(t, \xi)$ restricted to $(s, t_0) \times B$, we have

$$\sum_{i,j \ge 2} a_{\varphi}^{ij}(t_1,\xi_1) \frac{\partial^2 g(t_1,\xi_1)}{\partial x_t^i \partial x_t^j} \leq 0 \text{ and } b_{\varphi}^i(t_1,\xi_1) \frac{\partial g(t_1,\xi_1)}{\partial x_t^i} = 0$$

where we use the following facts: $a_{\varphi}^{ij}(t_1, \xi_1) = a_{\varphi}^{j1}(t_1, \xi_1) = 0$ for $j \ge 2$ (see Lemma 3) and accordingly $||a_{\varphi}^{ij}(t_1, \xi_1)||_{i,j=2}, ..., m$ is a positive-definite symmetric matrix. Thus we get $Ag(t_1, \xi_1) \le 0$, and hence

$$0 = \frac{\partial g(t_1, \xi_1)}{\partial t} = Ag(t_1, \xi_1) - \frac{n(t_1 - s)^{n-1}}{(\tau - s)^n}g(t_1, \xi_1) < 0;$$

that is a contradiction. Hence the function g(t, x) on $(s, t_0) \times \overline{G}$ does not take the positive maximum at any point in $(s, t_0) \times B$. Similarly it does not take the negative minimum at any point in $(s, t_0) \times B$.

PROOF OF THEOREM 4 may be performed by means of the entirely same manner as in $\S 0$ by making use of Lemma 14 in place of Lemma A in $\S 0$. We omit to repeat here the argument in $\S 0$.

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