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<td><strong>Author(s)</strong></td>
<td>Itô, Seizô</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Mathematical Journal. 6(2) P.167-P.185</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1954-12</td>
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<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
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<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/8122">https://doi.org/10.18910/8122</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/8122</td>
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The Fundamental Solution of the Parabolic Equation in a Differentiable Manifold, II

By Seizō Itô

§ 0. Introduction (and supplements to the previous paper). Recently we have shown the existence of the fundamental solution of parabolic differential equations in a differentiable manifold (under some assumptions) in a previous paper\(^1\) which will be quoted here as [FS]. We have set no boundary condition in [FS], while we shall here show the existence of the fundamental solution of parabolic differential equations with some boundary conditions in a compact subdomain of a differentiable manifold.

We shall first add the following supplements 1°) and 2°) to [FS], as we shall quote not only the results obtained in the paper but also the procedures used in it:

1°) CORRECTIONS. Throughout the paper [FS]

for \(\exp \{M_1(t-s)^{1/2}\}\), read \(\exp \{M_1(t-s)^{1/2}\}\);

for \(\exp \{2M_1(t-s)^{1/2}\}\), read \(\exp \{2M_1(t-s)^{1/2}\}\).

In the inequality (3.4),

for \((t-s)^{-(m+1)/2}\), read \((t-s)^{-m+1}/2\).

2°) The proof of Theorem 4 in [FS, §4] is available only for the case: \(t_0 = \infty\). Instead of completing the proof, we are enough to establish a slightly ameliorated theorem as follows:

Theorem 4. i) The function \(u(t, x; s, y)\) is non-negative, and \(\int_M u(t, x; s, y)d_ay \leq \exp \{\lambda(t-s)\}\) where \(\lambda = \sup_{t, x} \lambda(t, x)\); ii) if especially \(c(t, x) \equiv 0\), then \(\int_M u(t, x; s, y)d_ay = 1\).

We see that \(|\lambda| \leq K(\infty)\) by virtue of the assumption II) in [FS, p. 76]. To prove this theorem, we consider the functions

\[ f_\varepsilon(t, x) = \int_M u(t, x; s, y)f(y)d_ay \]

and

---

\(^1\) S. Itô: The fundamental solution of the parabolic equation in a differentiable manifold, Osaka Math. J. 5 (1953) 75–92.
\begin{equation}
(0.2) \quad g^{(r_n)}_s(t, x) = f_s(t, x) \exp \left\{ -\left(\frac{t-s}{\tau-s}\right)^n \right\}
\end{equation}

where \( f(x) \) is an arbitrary function continuous on \( M \), with a compact support \( \subset M \) and satisfying \( 0 \leq f(x) \leq 1 \), and \( n \) is a natural number \( \geq 2 \) and \( s < \tau < t_0 \). Then \( g^{(r_n)}_s(t, x) \) is continuous in \([s, t_0] \times M\) and

\begin{equation}
(0.3) \quad g^{(r_n)}_s(s, x) = f(x), \quad \text{consequently} \quad 0 \leq g^{(r_n)}_s(s, x) \leq 1.
\end{equation}

By virtue of [FS, (3.10)] and the correction 1°) stated just above, we have

\begin{equation}
(0.4) \quad |f_s(t, x)| \leq M \exp \{M(t-s)\}
\end{equation}

for a suitable constant \( M > 0 \).

**Lemma A.** If \( c(t, x) \leq 0 \), then the function \( g^{(r_n)}_s(t, x) \) takes neither positive maximum nor negative minimum at any point in \([s, t_0] \times M\).

The proof may be achieved by the well known method and so will be omitted.

**Lemma B.** If \( c(t, x) \leq 0 \), then \( u(t, x; s, y) \geq 0 \) and \( \int_M u(t, x; s, y)d_x y \leq 1 \).

**Proof.** By virtue of the continuity of \( u(t, x; s, y) \) (see [FS, Theorem 1]), it is sufficient to prove that \( 0 \leq f_s(t, x) \leq 1 \) for any function \( f(x) \) satisfying the above stated conditions (see (0.1)).

Suppose that \( f_s(t, x) > 1 \) for some \( t_i > s_i \) and \( x_i \). Then, if we take \( \tau \) and \( \tau' \) such that \( t_i < \tau < \tau' < t_0 \) and sufficiently large \( n \), we have

\[ g^{(r_n)}_{t_i}(t_i, x_i) > 1 \]

and

\[ |g^{(r_n)}_{t_i}(t_i, x_i)| > |g^{(r_n)}_{t_i}(t, x)| \quad \text{for any} \quad t > \tau' \quad \text{and} \quad x \in M \]

by virtue of (0.2) and (0.4). From this fact and (0.3), it follows that \( g^{(r_n)}_s(t, x) \) takes the positive maximum at some point in \([s, t_0] \times M\); this contradicts Lemma A. Hence we have \( f_s(t, x) \leq 1 \).

Similar argument shows that, if \( f_s(t, x) < 0 \) for some \( t_i < s_i \) and \( x_i \), there exist \( \tau \) and \( n \) such that \( g^{(r_n)}_{t_i}(t, x) \) takes the negative minimum at some point in \([s, t_0] \times M\) contradictly to Lemma A. Hence we get \( f_s(t, x) \geq 0 \), q.e.d.

**Proof of Theorem 4.** Let \( u(t, x; s, y) \) be the fundamental solution of the equation \( Lf = 0 \). Then we may easily prove that the function

\[ u_s(t, x; s, y) = e^{-M(t-s)}u(t, x; s, y) \]
is the fundamental solution of the equation \((L - \lambda)f = 0\). Since \(c(t, x) - \lambda \leq 0\), we have
\[
\mathcal{u}_\lambda(t, x; s, y) \geq 0 \quad \text{and} \quad \int_M \mathcal{u}_\lambda(t, x; s, y) d_\alpha y \leq 1
\]
by Lemma B, and hence
\[
u(t, x; s, y) \geq 0 \quad \text{and} \quad \int_M \nu(t, x; s, y) d_\alpha y \leq e^{\lambda(t-s)}.
\]
Finally, if \(c(t, x) = 0\), we may apply Theorem 2 in [FS] to the function \(f(t, x) = 1\) and we get
\[
\int_M \nu(t, x; s, y) d_\alpha y = 1, \quad \text{q.e.d.}
\]

§ 1. Fundamental notions and main results. We shall say, by definition, that a function \(f(x)\) defined on a subset \(E\) of the Euclidean \(m\)-space \(\mathbb{R}^m\) satisfies the generalized Lipschitz condition in \(E\) if, for any \(x \in E\), there exist positive numbers \(N, \delta, \gamma\) (each of them may depend on \(x\)) such that
\[
|f(x) - f(y)| \leq N \sum_1^m |x^i - y^i|^{\gamma}
\]
whenever \(y \in E\) and \(|x^i - y^i| \leq \delta (i = 1, \ldots, m)\), where \((x^i)\) and \((y^i)\) denote the coordinates of \(x\) and \(y\) respectively.

A function \(f(x)\) defined on a domain \(G \subset \mathbb{R}^m\) is said to be of \(C^{\nu,\ell}\)-class if \(f(x)\) is of \(C^{\nu}\)-class in the usual sense and each partial derivative of \(k\)-th order of \(f(x)\) satisfies the generalized Lipschitz condition in \(G\). A manifold of \(C^{\nu,\ell}\)-class, a hypersurface of \(C^{\nu,\ell}\)-class, etc. should be understood analogously.

Let \(M\) be an \(m\)-dimensional manifold of \(C^{\nu,\ell}\)-class, and \(G\) be a domain in \(M\) such that the closure \(\overline{G}\) is compact and the boundary \(B = \overline{G} - G\) consists of a finite number of hypersurfaces of \(m-1\)-dimension and of \(C^{\nu,\ell}\)-class.

Under a canonical coordinate around \(x \in M\), we understand any local coordinate which maps a neighbourhood of \(x\) onto the interior of the unit sphere in \(\mathbb{R}^m\) and especially transforms \(x\) to the centre of the sphere. For each \(x \in M\) and any fixed canonical coordinate around \(x\), we denote by \(U(x)\) the neighbourhood of \(x\) of the form
\[
\{ y \in M; \sum_1^m (y^i - x^i)^2 < \varepsilon \} \quad \text{where} \quad 0 < \varepsilon \leq 1.
\]
We understand the partial derivatives of a function \(f(x)\) (defined on \(\overline{G}\)) at \(\xi \in B\) as follows: \(\partial f(\xi) / \partial x^i = \alpha_i (\xi \in B)\), \(i = 1, \ldots, m\), means that
\[
f(x) = f(\xi) + \alpha_i (x^i - \xi^i) + o(\sum_1^m |x^i - \xi^i|) \quad \text{for any} \quad x \in U(\xi) \cap \overline{G}
\]
where \(U(\xi)\) is a coordinate neighbourhood of \(\xi\).

2) Cf. Footnote 1) in [FS].
We fix \( s, t \) such that \(-\infty < s < t < \infty\) and consider the parabolic differential operator \( L \):

(1.1) \[ L = L_{ts} = A_{ts} - \frac{\partial}{\partial t}, \quad (x \in \bar{G}, \; s < t < t_0) \]

where

(1.2) \[ A_{ts} = a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x) \frac{\partial}{\partial x^i} + c(t, x) \]

and \( \| a^{ij}(t, x) \| \) is a strictly positive-definite symmetric matrix for each \( <t, x> \in (s, t_0) \times \bar{G} \); \( a^{ij}(t, x) \) and \( b^i(t, x) \) are transformed between any two local coordinates by means of (1.3) and (1.4) in [FS]. We assume that

(A.1) the functions

\[ \frac{\partial a^{ij}(t, x)}{\partial t}, \; \frac{\partial^2 a^{ij}(t, x)}{\partial x^i \partial x^j}, \; \frac{\partial b^i(t, x)}{\partial x^i}, \; (i, j, h, k, l = 1, \ldots, m) \]

and \( c(t, x) \)

satisfy the generalized Lipschitz condition in \([s_0, t_0] \times \bar{G}\).

We define the partial derivative \( \partial f(\xi)/\partial n_{\xi} \) to the outer transversal direction \( n_{\xi} \) as follows: when \( B \) is represented by \( \psi(x) = \psi(x_1, \ldots, x_m) = 0 \) with respect to a local coordinate around \( \xi \) and \( \psi(x) > 0 \) in \( G \), we set

(1.3) \[ \frac{\partial f(\xi)}{\partial n_{\xi}} = -\frac{\partial f(\xi)}{\partial x^i} \cdot \frac{\partial \psi(\xi)}{\partial x^i} \cdot a^{ij}(t, \xi); \]

this notion is independent of the special choice of the local coordinate around \( \xi \) by virtue of the transformation rule for \( a^{ij}(t, x) \) (see [FS (1.3)]). If we take a local coordinate with respect to which \( a^{ij}(t, \xi) = \delta^{ij} \) i.e. \( a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} = \text{Laplacian} \) at the point \( <t, \xi> \) (fixed), then \( \partial f(\xi)/\partial n_{\xi} \) means the partial derivative to the outer normal direction to \( B \). We consider the boundary condition:

(B_{a(t)}) \[ \alpha(t, \xi)f(\xi) + (1 - \alpha(t, \xi)) \frac{\partial f(\xi)}{\partial n_{\xi}} = 0 \quad (\xi \in B) \]

for each \( t \), where \( \alpha(t, \xi) \) is a function on \([s_0, t_0] \times B\), of \( C^1 \)-class in \( t \) and of \( C^{2-} \)-class in \( \xi \) and \( 0 \leq \alpha(t, \xi) \leq 1 \). We shall say that a function \( f(t, x) \) on \((s_0, t_0) \times \bar{G}\) satisfies the boundary condition \((B_{a(t)})\) if it satisfies \((B_{a(t)})\) for any \( t \in (s_0, t_0) \).

We define the metric tensor \( a_{ij}(x) \), as stated in [FS, p. 79], and consider the measure \( da = \sqrt{a(x)}dx^1 \cdots dx^m \) \( (a(x) = \det \| a_{ij}(x) \|) \) and de-
fine the adjoint operator $L^*$ resp. $A^*$ of $L$ resp. $A$ with respect to this measure. If $M$ is an orientable Riemannian manifold with a metric tensor $g_{ij}(x)$ a priori, then it is natural to take the measure $d_\sigma x = \sqrt{g(x)} dx^1 \cdots dx^n \quad (g(x) = \det \| g_{ij}(x) \|)$ in place of $d_\sigma x$; in this case, it is sufficient only to replace $a(x)$ by $g(x)$ throughout the course of the present paper, while $a_{ij}(x)$ should not be replaced by $g_{ij}(x)$.

We assume further that:

(A.2) the following relations hold on the set

\begin{align}
\{ \langle t, \xi \rangle ; \, \alpha(t, \xi) = 1 \} \times B : \\
\frac{\partial a^{ij}(t, \xi)}{\partial n^j} = 0 \quad (i, j = 1, \ldots, m) \quad \text{and} \\
b^i(t, \xi) = \frac{1}{\sqrt{a(\xi)}} \frac{\partial}{\partial x^i} [\sqrt{a(\xi)} a^{ij}(t, \xi)] \quad (i = 1, \ldots, m).
\end{align}

Under the above stated conditions (A.1) and (A.2), we shall consider the parabolic differential equations $Lf = 0$ and $L^*f^* = 0$ in the domain $G$ with the boundary condition $(B_f)$. By definition, a function $u(t, x; s, y), \, s_0 < s < t < t_0, \, x, y \in \bar{G}$, is called a fundamental solution of the parabolic equation $Lf = 0$ with the boundary condition $(B_u)$ if, for any $s$ and any continuous function $f(x)$ which is continuous in $G$ and satisfies the condition $(f^*_{\beta})$, the function

\begin{equation}
\begin{aligned}
f(t, x) &= \int_G u(t, x; s, y) f(y) dy \\
&= \int \int G \int_G u(t, x; s, y) f(y) dy d_\sigma x
\end{aligned}
\end{equation}

satisfies the conditions:

1. $f(t, x)$ is of $C^1$-class in $t$ and of $C^2$-class in $x$, and satisfies the equation $Lf = 0$ as well as the boundary condition $(B_u)$

2. \( \lim_{t \to s^+} f(t, x) = f(x) \) uniformly on $\bar{G}$.

A function $u^*(s, y; t, x), \, s_0 < s < t < t_0; \, x, y \in G$, is called a fundamental solution of the adjoint equation $L^*f^* = 0$ (of the equation $Lf = 0$) with the boundary condition $(B_f)$ if, for any $t$ and any continuous function

3) It is true that $\partial a^{ij}/\partial n^j$ depends on the local coordinate, but the condition (1.4) is independent of it, because, if $\| a^{ij} \|$ is changed into $\| a^{ij} \|$ by means of the coordinate transformation $(x^i) \rightarrow (x^i)$, then we get $\frac{\partial a^{ij}}{\partial n^j} = \frac{\partial x^k}{\partial x^i} \frac{\partial x^j}{\partial x^k} \frac{\partial a^{kl}}{\partial n^l}$ by virtue of $[FS, (1.3)]$.

4), 5) Cf. $[FS$, Definition 2$]$. The conditions corresponding to (1.9) and (1.9*) in $[FS]$ follow from (1.7) and (1.7*) respectively in the case where $G$ is compact.
\( f(x) \) on \( G \), the function

\[
(1.6^*) \quad f^*(s, y) = \int_G u^*(s, y; t, x)f(x)dx \quad (s \leq t)
\]

satisfies the conditions\(^6\):

\[
(1.7^*) \quad \begin{cases} 
\text{\( f^*(s, y) \) is of } C^1 \text{-class in } s \text{ and of } C^2 \text{-class in } y, \text{ and satisfies} \\
\text{the equation } L^*f^* = 0 \text{ as well as the boundary condition } (B_a) \end{cases}
\]

and

\[
(1.8^*) \quad \lim_{t \to s^+} f^*(s, y) = f(y)
\]

pointwisely in \( G \) and also strongly in \( L^1(G) \).

The purpose of the present paper is to prove the following theorems, which are literally the same as those in [FS]\(^6\) except the statements concerning the boundary condition.

**Theorem 1.** There exists a function \( u(t, x; s, y) \) of \( C^1 \)-class in \( t \) and \( s(s_0 < s < t < t_0) \) and of \( C^2 \)-class in \( x \) and \( y \) \((x, y \in G)\), with the following properties:

i) \( u(t, x; s, y) \) is a fundamental solution of the equation \( Lf = 0 \) with the boundary condition \( (B_a) \).

ii) \( u^*(s, y; t, x) = u(t, x; s, y) \) is a fundamental solution of the adjoint equation \( L^*f^* = 0 \) with the boundary condition \( (B_a) \).

iii) \( L_t u(t, x; s, y) = 0, \ L^*_t u(t, x; s, y) = 0 \) and \( u(t, x; s, y) \) satisfies the boundary condition \( (B_a) \) as a function of \( <t, x> \) and also as a function of \( <s, y> \).

iv) \( \int_G u(t, x, \tau, z)u(\tau, z; s, y)dx = u(t, x; s, y), \ s < \tau < t \).

**Theorem 2.** Let \( u(t, x; s, y) \) and \( u^*(s, y; t, x) \) be the functions stated in Theorem 1.

i) If a function \( f(t, x) \) on \((s, t_0) \times G \) satisfies (1.7) and (1.8) where \( f(x) \) is continuous in \( G \) and satisfies \( (B_a) \), then it is expressible by (1.6).

ii) If a function \( f^*(s, y) \) on \((s_0, t) \times G \) satisfies (1.7*) and (1.8*) where \( f(x) \) is a continuous function on \( G \), then it is expressible by (1.6*).

**Theorem 3.** If a function \( v(t, x; s, y) \) is continuous in the region:
\( s_0 < s < t < t_0; x, y \in G, \) and satisfies the condition i) or ii) in Theorem 1, then it is identical with \( u(t, x; s, y) \) stated in Theorem 1.

**Theorem 4.** i) \( u(t, x; s, y) \geq 0 \) and \( \int_G u(t, x; s, y)dx \leq e^{\lambda(t-s)} \) where

---

\(^6\) As for Theorem 4, see the supplement to [FS] in §0 of the present paper.
$\lambda = \sup_{t,x} c(t, x); \text{ ii) if } c(t, x) \equiv 0 \text{ in the differential operator } A_{ts}, \text{ and if } c(t, \xi) \equiv 1 \text{ in the boundary condition } (B_{a}), \text{ then } \int_{\partial U(t, x; s, y)} d_{a} y = 1.$

We shall show, in another paper\(^7\), the existence of the fundamental solution of the parabolic differential equation with a boundary condition considered in a domain whose closure is not compact.

§ 2. Preliminaries. The following lemma may be proved by means of Lebesgue’s convergence theorem, and will be useful throughout the present paper:

**Lemma 1.** Let $(X, \mu)$ be a measure space, and assume that

i) $f(t, x)$ is measurable in $x \in X$ for each $t \in (t_1, t_2)$,

ii) $f(t, x)$ is differentiable in $t$ for a.a. $x \in X$ and

iii) there exists a measurable function $\varphi(x)$ such that

$$\left| \frac{\partial f(t, x)}{\partial t} \right| \leq \varphi(x) \text{ in } (t_1, t_2) \text{ and } \int_{x} \varphi(x) d\mu(x) < \infty.$$

Then

$$\frac{d}{dt} \int_{x} f(t, x) d\mu(x) = \int_{x} \frac{\partial f(t, x)}{\partial t} d\mu(x).$$

Now let $G, B$ and $A_{ts}$ be as stated in §1 and $z$ be any fixed point in $B$. Then, for any canonical coordinate (see §1) around $z, B \cap U_{t}(z)$ is represented by means of $\psi(x^1, \ldots, x^m) = 0$ where $\psi$ is a function of $C^{1-\varepsilon}$-class. Hence, considering a suitable coordinate transformation in $U_{t}(z)$, we may show that

**Lemma 2.** There exists a canonical coordinate $(x^1)$ around $z$ such that $B \cap U_{t}(z)$ is expressible by $x^1 = 0$ and that $x^1 > 0$ in $G \cap U_{t}(z)$.

Next we shall prove that

**Lemma 3.** Let $(x^1)$ be a canonical coordinate as stated in Lemma 2, and consider the coordinate transformation: $(x^i) \rightarrow (x_{ts}^i)$, for each $t(s \leq t \leq t_0)$, defined by

$$\begin{align*}
\left[ x^1_{ts} = \varphi^t(t, x) \equiv \gamma x^1 \right. \\
\left[ x^j_{ts} = \varphi^t(t, x) \equiv \gamma \left\{ \frac{a_{ij}(t, \xi_x)}{a_{11}(t, \xi_x)} x^1 + x^j \right\}, \quad j = 2, \ldots, m, \right.
\end{align*}$$

\(^7\) See the author’s paper: Fundamental solutions of parabolic differential equations and eigenfunction expansions for elliptic differential equations, forthcoming to Nagoya Mathematical Journal.
where \( \xi = \langle 0, x^2, \ldots, x^m \rangle \in \mathcal{B} \) for \( x = \langle x^1, \ldots, x^n \rangle \in U_i(z) \) and \( \gamma \) is a suitable positive constant. Then there exists \( \delta = \delta_* > 0 \) such that

i) \( U_8(z) \subset U_1(z) \subset U_0(z) \) and \( U_{8/3}(z) \subset U_{1/3}(z) \) for any \( t \), ii) \( \mathcal{B} \) is represented by \( x_1^1 = 0 \) in \( U_1(z) \) and iii) if \( a^{ij}(t, x) \) is changed into \( a^{ij}_s(t, x) \) by means of this transformation (\( i, j = 1, \ldots, m \)), then

\[
(2.2) \quad a^{ij}_s(t, \xi) = a^{ij}_1(t, \xi) = 0 \quad \text{and} \quad a^{ij}_s(t, \xi) = a^{ij}_1(t, \xi) = 0, \quad j = 2, \ldots, m,
\]

for any \( \xi \in \mathcal{B} \cap U_1(z) \), where \( U_1(z) = \{ y \in \mathcal{M} \mid \sum_j (y^j - x^j)^2 < \varepsilon \} \) and \( \| a^{ij}_1(t, x) \| = \| a^{ij}_s(t, x) \|^{-1} \). The mapping \( \phi_s(x) = \langle \phi^1(t, x), \ldots, \phi^m(t, x) \rangle \) of \( U_8(z) \) into \( U_1(z) \) is one-to-one and of \( C^{3, \varepsilon} \)-class in \( x \) and \( a^{ij}_s(t, x) \), \( i, j = 1, \ldots, m \), are of \( C^1 \)-class in \( t \) and of \( C^{2, \varepsilon} \)-class in \( x \).

**Proof.** We notice that \( a^{ij}(t, x) \geq 0 \) in \( U_1(z) \), and consider the coordinate transformation (2.1) around \( z \). Then \( x_1^1 = 0 \) if and only if \( x^1 = 0 \), and we have for any \( \xi = \langle 0, x^2, \ldots, x^m \rangle \in \mathcal{B} \cap U_1(z) \)

\[
(2.3) \quad \begin{align*}
\left( \frac{\partial x^1}{\partial x^j} \right)_{x = \xi} &= \gamma \delta^1_j, \\
\left( \frac{\partial x^1}{\partial x^j} \right)_{x = \xi} &= \gamma \left( -\frac{a^{1j}_s(t, \xi)}{a^{11}_s(t, \xi)} \delta^1_j + \delta^1_i \right) \quad (\delta^1_j \text{: Kronecker's delta})
\end{align*}
\]

for \( 1 \leq j \leq m \) and \( 2 \leq j \leq m \). Hence the Jacobian

\[
\frac{\partial (x^1, \ldots, x^m)}{\partial (x^1, \ldots, x^m)}
\]

is bounded away from zero in \( U_1(z) \) for suitable \( \varepsilon, 0 < \varepsilon < 1 \) which may be chosen independently of \( t \) by virtue of the continuity of \( a^{ij}(t, x) \) on the compact set \([s_0, t_0] \times U_1(z)\) for any \( \varepsilon \) \( (0 < \varepsilon < 1) \), and hence the transformation (2.1) is well defined in \( U_1(z) \). Considering the continuity of \( a^{ij}(t, x) \) on \([s_0, t_0] \times U_1(z)\) again, we may determine \( \gamma > 0 \) so that \( U_8(z) \subset U_1(z) \subset U_0(z) \) and \( U_{8/3}(z) \subset U_{1/3}(z) \) for any \( t \). By virtue of the transformation rule for \( a^{ij} \) (see [FS, (1.3)]), we have, for any \( \langle t, \xi \rangle \in [s_0, t_0] \times (\mathcal{B} \cap U_1(z)) \) and for \( j \geq 2 \),

\[
a^{11}_s(t, \xi) = \left( \frac{\partial x^1}{\partial x^j} \right)_{x = \xi} \cdot \left( \frac{\partial x^1}{\partial x^i} \right)_{x = \xi} a^{ki}(t, \xi)
\]

\[
= -\gamma^2 \frac{a^{1j}_s(t, \xi)}{a^{11}_s(t, \xi)} a^{11}(t, \xi) + \gamma^2 a^{1j}(t, \xi) = 0 \quad \text{(see (2.3))},
\]

and consequently we get (2.2). The last part of Lemma 3 is also evident by means of the above arguments.
§ 3. Local construction of a quasi-parametrix. Let $G, B$ and $A_{tx}$ be as before, let $z$ be any fixed point in $B$, and let $(x^t)$ and $(x^t_x)$ ($s_0 \leq t \leq t_0$) be canonical coordinates around $z$ as stated in Lemma 3. Then we have

$$\frac{\partial f(\xi)}{\partial n_\xi} = -\frac{\partial f(\xi)}{\partial x^t_\xi} a_{tx}^t(t, \xi) + \frac{\partial f(\xi)}{\partial x^t_\xi} a_{tx}^t(t, \xi)$$

(3.1) \[ \xi \in B \]

for any function $f(x)$ of $C^1$-class, and hence the assumption (1.4) implies that

$$\frac{\partial a_{tx}^t(t, \xi)}{\partial x^t_\xi} = 0 \quad \text{on } \{ \langle t, \xi \rangle ; \alpha(t, \xi) = 1 \} ,$$

(3.2)

Now we put for $s_0 \leq s < t \leq t_0$ and $X, Y \in \mathbb{R}^m$

$$V_0(A_{t,s}; t, X; s, Y) = (t-s)^{-\frac{m}{2}} \exp \left[ -\frac{A_{j,j}(t-s)^{X^j-Y^j}(X^j-Y^j)}{4(t-s)} \right]$$

(3.3)

and define for $s_0 \leq s < t \leq t_0$ and $x, y \in U_\delta(z) \setminus \bar{G}$ ($\delta = \delta_0$ as stated in Lemma 3)

$$V(t, x; s, y) = V_0(a_{tx}^s(t, x); t, \phi_x(t, x); s, \phi_y(s, y)) \quad \text{(see Lemma 3)}$$

(3.4)

where $\phi_x(t, x; s, y) = \langle -\phi^1(s, y), \phi^2(s, y), \ldots, \phi^m(s, y) \rangle$. Further we put

$$\rho(t, x; s, y) = \frac{2(t-s)\alpha(t, \xi_{tx})}{2(t-s)\alpha(t, \xi_{tx}) + \phi^1(s, y)[1-\alpha(t, \xi_{tx}) \exp \{-|\phi^1(t, x)|^2\}]},$$

$$q(t, x; s, y) = \frac{\phi^1(s, y)[1-\alpha(t, \xi_{tx}) \exp \{-|\phi^1(t, x)|^2\}]}{2(t-s)\alpha(t, \xi_{tx}) + \phi^1(s, y)[1-\alpha(t, \xi_{tx}) \exp \{-|\phi^1(t, x)|^2\}]}$$

(3.5)

where $\xi_{tx}$ is the point \((x)\) defined by the equations:

$$\phi^j(t, \xi_{tx}) = 0, \quad \phi^j(t, \xi_{tx}) = \phi^j(t, x) \quad \text{for } j \geq 2;$$

such $\xi_{tx}$ is uniquely determined for any $x \in U_\delta(z)$ and any $t$ by virtue of Lemma 3.

Applying (3.1), (3.2), Lemma 1 and Lemma 3 to (3.3), (3.4) and (3.5), and making use of the fact that $\partial f/\partial n_\xi$ is independent of the local coordinate, we obtain
S. Ito

\[ \frac{\partial V(t, \xi)}{\partial n_{t, \xi}} = 0 \]

and

\[ \frac{\partial V(t, \xi; s, y)}{\partial n_{t, \xi}} = -a_{11}^t(t, \xi) \left. \right|_{-a_{11}^t(t, \xi)} \frac{-q^1(s, y) \cdot V(t, \xi; s, y)}{2(t-s)} \]

\[ = \frac{-q^1(s, y)}{2(t-s)} V(t, \xi; s, y) \]

for \( (t, \xi) \) such that \( \xi \in B \cap U_0(z) \) and \( \alpha(t, \xi) = 1 \), and we get also

\[ \frac{\partial p(t, \xi; s, y)}{\partial n_{t, \xi}} = \frac{\partial q(t, \xi; s, y)}{\partial n_{t, \xi}} = 0 \]

and

\[ V(t, \xi; s, y) = \tilde{V}(t, \xi; s, y) \]

for any \( \xi \in B \cap U_0(z) \). We define

\[ W_s(t, x; s, y) = p(t, x; s, y) J_s(y) \frac{V(t, x; s, y) - \tilde{V}(t, x; s, y)}{V(t, x)} + q(t, x; s, y) J_s(y) \frac{V(t, x; s, y) + \tilde{V}(t, x; s, y)}{V(t, x)} \]

where

\[ J_s(y) = \frac{\partial [\varphi^1(s, y), \ldots, \varphi^n(s, y)]}{\partial [y', \ldots, y^n]} \] (Jacobian).

Then we may prove from (3.6—9) and by simple calculation that

\[ \alpha(t, \xi) W_s(t, \xi; s, y) + (1 - \alpha(t, \xi)) \frac{\partial W_s(t, \xi; s, y)}{\partial n_x} = 0 \]

for \( \xi \in U_0(z) \cap B \),

that is, \( W_s(t, x; s, y) \) satisfies the boundary condition \((B_0)\) as a function of \( (t, x) \in [s_0, t_0] \times U_0(x) \). Since

\[ \int_{U_0(z) \cap G} V(t, x; s, y) J_s(y) dy + \int_{U_0(z) \cap G} \tilde{V}(t, x; s, y) J_s(y) dy \]

\[ \leq \int_{\mathbb{R}^n} V_0(a_{11}^t(t, x); t, \varphi_t(x); s, Y) dY = V(t, x) \]

\[ (dy = dy^1 \ldots dy^n, \ dY = dY^1 \ldots dY^n) \]

and since the denominators and numerators in the right-hand side of (3.5) are positive for any \( x, y \in U_0(z) \cap G \), we get
(3.14) \[ \int_{U_b(z) \cap G} |W_s(t, x; s, y)| dy \leq 1 \quad \text{for any} \quad x \in U_b(z) \cap \bar{G}. \]

Now we have the following

**Lemma 4.** If \( f(x) \) is continuous in \( \bar{G} \) and vanishes outside \( U_b(z) \), then

(3.15) \[ \lim_{t \to s} \int_{U_b(z) \cap G} \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} f_s(y) dy = f(x) \]

uniformly in \( U_b(z) \cap \bar{G} \).

**Proof.** By virtue of (3.3) and the uniform continuity of \( \phi^s(t, x) \) on \([s_0, t_0], U_b(z)\), we may show that

\[ \lim_{t \to s} \int_{R^n} \frac{V_0(a^s_1(t, x); t, \phi^s(t, x); s, Y)}{V_0(a^s_1(t, x))} F(Y) dY = F(\phi^s(x)) \]

uniformly in \( U_b(z) \cap \bar{G} \).

for any continuous function \( F(Y) \) with a compact support; and hence, if especially \( F(\bar{Y}) = F(Y) \) where \( \bar{Y} = \langle -Y^1, Y^2, \ldots, Y^m \rangle \) for \( Y = \langle Y^1, Y^2, \ldots, Y^m \rangle \), then

\[ \lim_{t \to s} \int_{R^n(\bar{Y} > 0)} \frac{V_0(a^s_1(t, x); t, \phi^s(t, x); s, Y) + V_0(a^s_1(t, x); t, \phi^s(t, x); s, \bar{Y})}{V_0(a^s_1(t, x))} F(Y) dY = F(\phi^s(x)) \quad \text{uniformly in} \quad U_b(z) \cap \bar{G}. \]

Putting

\[ F(Y) = F(\bar{Y}) = \begin{cases} f(\phi^s^{-1}(Y)) & \text{if} \quad \sum_t (Y^t)^2 < 1 \\ 0 & \text{if not} \end{cases} \]

in the above relation, and considering (3.4) and (3.11), we obtain (3.15).

**Lemma 5.** If \( f(x) \) is such a function as stated in Lemma 3 and if \( D \) is an open set containing \( \mathbb{B}^{(s)} = \{ \xi \in \mathbb{B}; \alpha(s, \xi) = 1 \} \), where \( s \) is any fixed real number \( s_0 < s < t_0 \), then

\[ \lim_{t \to s} \int_{U_b(z) \cap \bar{G}} W_s(t, x; s, y) f(y) dy = f(x) \quad \text{uniformly in} \quad U_b(z) \cap \bar{G} - D. \]

**Proof.** Let \( \varepsilon \) be an arbitrary positive number. Then, by virtue of Lemma 4, there exists \( \Delta > 0 \) such that

(3.16) \[ \left| \int_{U_b(z) \cap \bar{G}} \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} J_s(y) f(y) dy - f(x) \right| < \frac{\varepsilon}{4} \]

for any \( x \in U_b(z) \cap \bar{G} \) whenever \( s < t < s + \Delta. \). On the other hand, by
virtue of (3.13) and (3.14), there exists \( \eta_1 > 0 \) such that

\[
(3.17) \quad \left| \int_{U^1(0) \cap \{ \varphi^1(s, y) < \eta_1 \} \cap \mathcal{G}} V(t, x; s, y) + \bar{V}(t, x; s, y) J_s(y) f(y) dy \right| < \frac{\varepsilon}{4}
\]

and that

\[
(3.18) \quad \left| \int_{U^1(0) \cap \{ \varphi^1(s, y) < \eta_1 \} \cap \mathcal{G}} \frac{W_z(t, x; s, y) f(y) dy}{V(t, x)} \right| < \frac{\varepsilon}{4}.
\]

Since \( 1 - \alpha(t, \xi_{\varepsilon,s}) \exp \{-|\varphi^1(t, x)|^2\} \geq 0 \) for any \( t \) and any \( x \in U^1(0) \cap \overline{\mathcal{G}} - \mathcal{D} \), there exists \( \eta_2 > 0 \) such that

\[
1 - \alpha(t, \xi_{\varepsilon,s}) \exp \{-|\varphi^1(t, x)|^2\} \geq \eta_2 \quad \text{(see (3.5))}
\]

for any \( t \) and any \( x \in U^1(0) \cap \overline{\mathcal{G}} - \mathcal{D} \). Hence \( \varphi^1(s, y) \geq \eta_1 \) implies that

\[
|1 - q(t, x; s, y)| = |p(t, x; s, y)| \leq (t - s)/\eta_1 \eta_2
\]

for any \( t \geq s \) and any \( x \in U^1(0) \cap \overline{\mathcal{G}} - \mathcal{D} \), and hence it follows from (3.10) and (3.13) that there exists \( \Delta_2 > 0 \) such that

\[
(3.18) \quad \left| \int_{U^1(0) \cap \{ \varphi^1(s, y) \geq \eta_1 \} \cap \mathcal{G}} \left\{ W_z(t, x; s, y) - \frac{V(t, x; s, y) + \bar{V}(t, x; s, y) J_s(y)}{V(t, x)} f(y) dy \right\} \right| < \frac{\varepsilon}{4}
\]

for any \( x \in U^1(0) \cap \overline{\mathcal{G}} - \mathcal{D} \) whenever \( s \leq t \leq s + \Delta_2 \). Since \( f(y) = 0 \) for \( y \in \overline{\mathcal{G}} - U^1(0) \), it follows from (3.16–19) that

\[
|\int_{\mathcal{G}} W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon \quad \text{for any } x \in U^1(0) \cap \overline{\mathcal{G}} - \mathcal{D}
\]

whenever \( s \leq t \leq s + \min \{ \Delta_1, \Delta_2 \} \). Thus we obtain Lemma 5.

**Lemma 6.** Assume that \( f(x) \) is continuous in \( \overline{\mathcal{G}} \), vanishes outside \( U^1(0) \) and satisfies the boundary condition \((B_{\alpha(x)})\). Then

\[
\lim_{t \to s} \int_{\mathcal{G}} W_z(t, x; s, y) f(y) dy = f(x) \quad \text{uniformly in } U^1(0) \cap \overline{\mathcal{G}}.
\]

**Proof.** Let \( \varepsilon \) be an arbitrary positive number, and put

\[
\mathcal{D} = \{ x \in \overline{\mathcal{G}} : |f(x)| < \varepsilon/5 \} \cup \{ \bar{x} \in \overline{\mathcal{G}} : |f(x)| < \varepsilon/5 \}
\]

where \( \bar{x} = \langle -x^1, x^2, \ldots, x^n \rangle \) for \( x = \langle x^1, x^2, \ldots, x^n \rangle \). Then, by virtue of the assumption of this lemma, \( \mathcal{D} \) is an open set containing \( \mathcal{B}^\varepsilon = \{ \xi \in \mathcal{B} : \alpha(s, \xi) = 1 \} \) and hence, by Lemma 5, there exists \( \Delta > 0 \) such that
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(3.20) \[ |\mathcal{I}_G W_s(t, x; s, y) f(y)dy - f(x)| < \varepsilon \] for any \( x \in U_\delta(z) \cap \bar{G} \cap D \)
whenever \( s < t < s + \Delta \). On the other hand, by Lemma 4, there exists \( \Delta' > 0 \) such that
\[
\int_G \frac{V(t, x; s, y) + \tilde{V}(t, x; s, y)}{V(t, x)} |J_s(y)dy| < \frac{2}{5} \varepsilon
\]
for any \( x \in U_\delta(z) \cap \bar{G} \cap D \)
whenever \( s < t < s + \Delta' \). Hence, considering the non-negativity of \( V(t, x; s, y), \tilde{V}(t, x; s, y) \) and \( J_s(y) \) (see the proof of Lemma 3) and using the facts: \( 0 \leq p(t, x; s, y) \leq 1 \) and \( 0 \leq q(t, x; s, y) \leq 1 \), we obtain from (3.10) that
\[
|\mathcal{I}_G W_s(t, x; s, y) f(y)dy| < \frac{4}{5} \varepsilon
\]
for any \( x \in U_\delta(z) \cap \bar{G} \cap D \)
and accordingly
(3.21) \[ |\mathcal{I}_G W_s(t, x; s, y) f(y)dy - f(x)| < \varepsilon \] for any \( x \in U_\delta(z) \cap \bar{G} \cap D \)
whenever \( s < t < s + \Delta' \). From (3.20) and (3.21) we get
\[
|\mathcal{I}_G W_s(t, x; s, y) f(y)dy - f(x)| < \varepsilon
\]
whenever \( s < t < s + \min(\Delta, \Delta') \). Thus we obtain Lemma 6.

Next, let \( f(\tau, y) \) be a continuous function on \( (s, t_0) \times G \) which vanishes outside \( U_\delta(z) \) and satisfies the condition:
\[ \int_s^{t_0} \int_G |f(\tau, y)| dyd\tau < \infty, \]
and put
\[
f(t, x, \tau) = \mathcal{I}_G W_s(t, x; \tau, y) f(\tau, y) dy, \quad t > \tau > s,
F(t, x) = \int_s^t f(t, x, \tau) d\tau.
\]
Then we have

Lemma 7. i) \( f(t, x, \tau) \) and \( F(t, x) \) satisfy the boundary condition (B_s) in \( U_\delta(z) \cap \bar{G} \); ii) for any \( s'(t_0) > s' > s \)
\[
\lim_{\tau \to t} \int_G f(\tau, x) W_s(\tau, x; s', y)dy = f(s', y) \text{ in } G \cap U_\delta(z);
\]
iii) if \( f(\tau, y) \) satisfies the generalized Lipschitz condition in \( (s, t_0) \times G \),
then
\[
\frac{\partial F(t, x)}{\partial t} = f(t, x) + \int_s^t \int_G \frac{\partial W_s(t, x; \tau, y)}{\partial t} f(\tau, y)dyd\tau,
A_{t_0} F(t, x) = \int_s^t \int_G A_{t_0} W_s(t, x; \tau, y) f(\tau, y)dyd\tau.
\]
OUTLINE OF THE PROOF. The proposition i) may be shown by means of (3.12) and Lemma 1, and the proposition ii) may be proved similarly to [FS, Lemma 2]. The proposition iii) is proved as follows. Considering the fact that the mapping $\varphi_t(x)$ is one-to-one and of $C^{2,t}$-class for any $t$ (see Lemma 3), using the same idea as in [FS, Lemmas 1 and 3] and applying Lemma 1 (§ 1), we may show that

$$
\frac{\partial f(t, x, \tau)}{\partial t} = \int_{\mathcal{G}} \frac{\partial W_s(t, x; \tau, y)}{\partial t} f(\tau, y) dy,
$$

$$
\frac{\partial f(t, x, \tau)}{\partial x^i} = \int_{\mathcal{G}} \frac{\partial W_s(t, x; \tau, y)}{\partial x^i} f(\tau, y) dy,
$$

and

$$
\frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} = \int_{\mathcal{G}} \frac{\partial^2 W(t, x; \tau, y)}{\partial x^i \partial x^j} f(\tau, y) dy.
$$

and

$$
\lim_{t \to t' \to \tau} f(t, x, t') = f(\tau, x)
$$

and that there exist $M > 0$ and $\gamma = \gamma(t, x) > 0$ such that

$$
\frac{\partial f(t', s, \tau)}{\partial t'} \leq M(t-s)^{-(1-\frac{\gamma}{2})}
$$

whenever $s < \tau < t \leq t'$; further we have

$$
\int_s^t \left| \frac{\partial f(t, x, \tau)}{\partial x^i} \right| d\tau < \infty \quad \text{and} \quad \int_s^t \left| \frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} \right| d\tau < \infty.
$$

Hence we may prove the proposition iii) by the same manner as in [FS, Lemma 4].

**Lemma 8.** If $\omega(t, x)$ is a function of $C^1$-class in $t$ and of $C^2$-class in $x$, and vanishes outside $U_8(z)$, then there exists a constant $M_0 > 0$ such that

$$
|L_{t,s} [\omega(t, x)W_s(t, x; s, y)]| \leq M_0(t-s)^{-\frac{m+1}{2}} \exp \left\{ -\frac{M_0 \sum |x^i-y^i|^2}{4(t-s)} \right\}.
$$

This may be proved similarly to [FS, Lemma 5].

Finally we define a quasi-parametrix $W_s(t, x; s, y)$ around any inner point $z$ of $G$ as follows. We fix a canonical coordinate $(x^i)$ around $z$ satisfying $U_z(z) \subset G$ and put

$$
x^i_t = 1 \quad |\delta_z = 1
$$

$$
x^i_t \equiv \varphi^i(t, x) = x^i, i = 1, \ldots, m, \text{ for any } t
$$
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(Consequently \( q_t(x) = \langle x', \ldots, x^n \rangle \) and \( a_t^i(t, x) = a_i(t, x) \)—cf. Lemma 3). Using this local coordinate, we define \( V(t, x; s, y) \) and \( V(t, x) \) by means of (3.3) and (3.4), and put

\[
W_s(t, x; s, y) = \frac{V(t, x; s, y)}{V(t, x)} \quad (s_0 < s < t < t_0; x, y \in U_t(z)).
\]

Then we may easily prove that Lemmas 6, 8 and Lemma 7 ii), iii) hold for \( W_s(t, x; s, y) \) defined here. (See Lemmas 2, 4 and 5 in [FS].)

§ 4. Global construction of a quasi-parametrix and a fundamental solution. For each \( z \in \tilde{G} (= G + B) \), we fix canonical coordinates \((x^i)\) and \((x'_i)\) around \( z \) as stated in §2, and put

\[
U(z, \varepsilon) = \{ x \in M; \sum_i (x^i - x'^i)^2 < \varepsilon \} \quad (\varepsilon > 0).
\]

Since \( G \) is compact, there exists a finite sequence \( \{z_1, \ldots, z_N\} \subset \tilde{G} \) such that

\[
\tilde{G} \subset \bigcup_{v=1}^N U(z_v, \delta_v/3) \quad \text{where} \quad \delta_v = \delta_v(z_v) \quad \text{(see §2)},
\]

and then, since

\[
z_v \in G \implies U(z_v, \delta_v) \subset G \quad \text{(see §2)},
\]

we have

\[
B \subset \bigcup_{z_v \in B} U(z_v, \delta_v/3).
\]

Let \( \omega(\lambda) \) be a function of \( C^{\infty, \varepsilon} \)-class in \( 0 \leq \lambda \leq \infty \) such that \( \omega(\lambda) = 1 \) or 0 if \( 0 \leq \lambda \leq 1/3 \) or \( \lambda \geq 2/3 \) respectively and that \( 0 \leq \omega(\lambda) \leq 1 \) for any \( \lambda \), and put for each \( \nu \)

\[
\omega_\nu(t, x) = \begin{cases} \omega(\sum_i [x^i - (z_v)_i])^\nu & \text{for} \ x \in \tilde{G} \cap U(z_v, \delta_v) \\ 0 & \text{for} \ x \in \tilde{G} - U(z_v, \delta_v). \end{cases}
\]

Then \( \omega_\nu(t, x), \nu = 1, \ldots, N, \) are of \( C^1 \)-class in \( t \) and of \( C^{\infty, \varepsilon} \)-class in \( x \in \tilde{G}, \) and

\[
\frac{\partial \omega_\nu(t, \xi)}{\partial n_t} = 0 \quad \text{for any} \ \langle t, \xi \rangle \in [s_0, t_0] \times B;
\]

this may be proved by considering the local coordinate \((x'_i)\) around \( z_v \) for each \( t \) since the operator \( \partial / \partial n_t \) is independent of the special choice of the local coordinate.
Now let \( a_s(x) \) be the restriction of \( a(x) = \det \|a_{ij}(x)\| \) (see §1) to \( U(z_v, \delta_v) \) with the local coordinate \((x^i)\) around \( z\) stated above, and put, for \( s_0 < s < t < t_0 \),

\[
W_s(t, x; s, y) = \begin{cases} \frac{\sum_s \omega_s(t, x) \omega_s(s, y) W_s(t, x; s, y)}{\sum_s \omega_s(t, x) \sqrt{a_s(y)}} & \text{if } s_0 < s < t < t_0, \\ 0 & \text{if not.} \end{cases}
\]

We define a quasi-parametrix:

\[
Z(t, x; s, y) = \frac{\sum_s \omega_s(t, x) \omega_s(s, y) W_s(t, x; s, y)}{\sum_s \omega_s(t, x) \sqrt{a_s(y)}} \quad \text{for } s_0 < s < t < t_0, \quad x, y \in G
\]

Then \( Z(t, x; s, y) \) is of \( C^1 \)-class in \( t \) and \( s \), and of \( C^{\alpha, \ell} \)-class in \( x \) and \( y \), and it follows from (3.12), (4.2), (4.3) and (4.4) that

(4.5) \( \alpha(t, \xi) Z(t, \xi; s, y) + (1 - \alpha(t, \xi)) \frac{\partial Z(t, \xi; s, y)}{\partial n_\xi} = 0 \quad (\xi \in B), \)

that is, \( Z(t, x; s, y) \) satisfies the boundary condition \((B_\alpha)\) as a function of \( \langle t, x \rangle \). Further, by virtue of Lemmas 6, 7 and 8, we obtain the following three lemmas.

**Lemma 9.** i) If \( f(x) \) is continuous in \( G \), then

\[
\lim_{t \to s} \int_G Z(t, x; s, y) f(y) d_a y = f(x) \quad \text{in } G
\]

if especially \( f(x) \) satisfies the boundary condition \((B_{\alpha(x)})\), then the above convergence is uniform in \( G \).

ii) If \( f(t, x) \) is continuous in \([s, t_0] \times \bar{G}\), then

\[
\lim_{t \to s} \int_G f(t, x) Z(t, x; s, y) d_a y = f(s, y) \quad \text{in } G
\]

**Lemma 10.** If \( f(\tau, y) \) is continuous in \((s, t_0) \times \bar{G}\) and satisfies the condition: \( \int_s \int_G |f(\tau, y)| d_a y d\tau < \infty \), then

\[
f(t, x, \tau) = \int_G Z(t, x; \tau, y) f(\tau, y) d_a y \quad (t < \tau < s)
\]

and

\[
F(t, x) = \int_s^t f(t, x, \tau) d\tau
\]

satisfy the boundary condition \((B_\alpha)\); if further \( f(\tau, y) \) satisfies the generalized Lipschitz condition in \((s, t_0) \times \bar{G}\), then

\[
\begin{align*}
\frac{\partial F(t, x)}{\partial t} &= f(t, x) + \int_s^t \int_G \frac{\partial Z(t, x; \tau, y)}{\partial t} f(\tau, y) d_a y d\tau, \\
A_{ts} F(t, x) &= \int_s^t \int_G A_{ts} Z(t, x; \tau, y) f(\tau, y) d_a y d\tau.
\end{align*}
\]
Lemma 11. $Z(t, x; s, y)$ satisfies all inequalities stated in [FS, Lemma 8] for a suitable constant $M > 0$.

Thus we see that $Z(t, x; s, y)$ has all properties stated in [FS, §2]. Hence, starting from this quasi-parametrix $Z(t, x; s, y)$, we may construct $u(t, x; s, y)$ in the entirely same way as in [FS, §3]. We may also construct $u^*(t, x; s, y)$ in the similar manner for the adjoint equation $L^*f^* = 0$ with the same boundary condition $(B_a)$. The functions $u(t, x; s, y)$ and $u^*(t, x; s, y)$ defined here have the properties stated in [FS, §3] where the manifold $M$ should be replaced by the compact domain $G$ and the uniformity of the convergence in [FS, (3.13)] may be proved if and only if $f(x)$ is the limit of a uniformly convergent sequence of functions satisfying the the boundary condition $(B_n(x)$. Moreover $u(t, x; s, y)$ and $u^*(t, x; s, y)$ satisfy the boundary condition $(B_a)$ as functions of $<t, x>- see Lemma 10 and the procedure of the construction of $u(t, x; s, y)$ (in [FS, §3]).

§5. Proof of Theorems.

Lemma 12. If $f(x)$ and $h(x)$ are functions of $C^2$-class on $G$ satisfying the boundary condition $(B_{a(x)})$ ($t$ fixed), then

$$
\int_G f(x) \cdot A^*_e h(x) d_a x = \int_G A^*_e f(x) \cdot h(x) d_a x.
$$

PROOF. By partial integration, we obtain the Green's formula:

$$
\int_G f(x) \cdot A^*_e h(x) d_a x = \int_G A^*_e f(x) \cdot h(x) d_a x
$$

$$
- \int_N \left\{ f(\xi) \frac{\partial h(\xi)}{\partial n_t} - \frac{\partial f(\xi)}{\partial n_t} h(\xi) \right\} d_\xi
$$

$$
+ \int_N \left\{ \frac{\partial}{\partial x^2} \left[ \sqrt{a(\xi)} aH(t, \xi) \right] - \sqrt{a(\xi)} b^t(t, \xi) \right\} \frac{\partial \psi(\xi)}{\partial x^t} f(x) h(x) d_\xi
$$

where $d_\xi = d\xi^1, ..., d\xi^{m-1}$ is the hypersurface area on $B$ and $\psi(x)$ is such function that $\psi(x) = 0$ determines $B$ and that $\psi(x) > 0$ in $G$. But the right-hand side equals zero by virtue of the boundary condition $(B_{a(x)})$ and the assumption (1.5). Hence we obtain Lemma 12.

From this lemma we obtain the following (see [FS, Lemma 11])

Lemma 13. If a function $f^*(s, y)$ on $(s_0, t) \times G$ satisfies (1.7*) and $(B_a)$, then

---

8) This assumption for $f(x)$ is equivalent to the following one: $f(\xi) = 0$ on $B^{(a)} = \{ \xi \in B; a(s, \xi) = 1 \}$
\[ \int_{\sigma} f^{*}(\tau, x)u(\tau, x; s, y)d_\alpha x = f^{*}(s, y) \text{ for any } \tau \in (s, t). \]

Therefore, we may see that:

**Proof of Theorems 1, 2 and 3** may be performed in the same way as the proof of the corresponding theorems in [FS] (see [FS, pp. 89-90]). It seems not to be necessary to repeat the entirely same argument. The propositions concerning the boundary condition which are not included in [FS] may be easily proved from properties of \( u(t, x; s, y) \) and \( u^{*}(t, x; s, y) \) stated in § 4 of the present paper.

In order to prove Theorem 4, we consider, as in § 0, the functions
\[ (5.1) \quad f_s(t, x) = \int_{\sigma} u(t, x; s, y) f(y) d_\alpha y \]
and
\[ (5.2) \quad g(t, x) = g^{(\psi, n)}(t, x) = f_s(t, x) \exp \left\{ -\left( \frac{t-s}{\tau-s} \right)^n \right\} \]
where \( f(x) \) is an arbitrary continuous function on \( G \) such that \( 0 \leq f(x) \leq 1 \) and the support of \( f(x) \) is a compact set contained in the domain \( G \), and \( \tau \) and \( n \) are as stated in § 0. Then \( g^{(\psi, n)}(t, x) \) is continuous in \((s, t_0) \times \bar{G} \) and satisfies (0.3), (0. 4) and the boundary condition \( (B_{a}) \).

**Lemma 14.** If \( c(t, x) \leq 0 \), then the function \( g(t, x) \) takes neither positive maximum nor negative minimum at any point in \((s, t_0) \times \bar{G} \) (for any fixed \( \tau, n \) and \( s \)).

**Proof.** It is easily proved by the well known method that \( g(t, x) \) takes neither positive maximum nor negative minimum at any point in the open set \((s, t_0) \times G \).

Suppose that:
\[ (5.3) \quad g(t, x) \text{ takes the positive maximum at } < t_1, \xi_1 > \in (s, t_0) \times B. \]

\( f_s(t, x) \) satisfies \( Lf = 0 \) in \((s, t_0) \times \bar{G} \) as may be seen from the properties of \( u(t, x; s, y) \), where the partial derivatives at any \( \xi \in B \) should be understood as defined in § 1, and \( g(t, x) \) satisfies the boundary condition \( (B_{a}) \) as well as \( f_s(t, x) \). We adopt a canonical coordinate around \( \xi_1 \) as stated in Lemma 3. Then we obtain from (5.3), (3.1) and \( (B_{a}) \) that \( \partial g(t_1, \xi_1) / \partial x_1 \leq 0 \) and that
\[ \alpha(t_1, \xi_1)g(t_1, \xi_1) - (1 - \alpha(t_1, \xi_1))a^{11}(t_1, \xi_1) \frac{\partial g(t_1, \xi_1)}{\partial x_1} = 0. \]
Since \( g(t_1, \xi_1) > 0 \) and \( a^{11}(t_1, \xi_1) > 0 \), it follows that \( \alpha(t_1, \xi_1) \) should be
zero, consequently \( \partial g(t_1, \xi_j)/\partial x_i = 0 \), and accordingly \( \partial^2 g(t_1, \xi_j)/(\partial x_i^2)^2 \leq 0 \) by virtue of (5.3). Moreover, since \( \langle t_1, \xi \rangle \) may be considered as the maximising point of \( g(t, \xi) \) restricted to \( (s, t_0) \times B \), we have

\[ \sum_{i,j \geq 2} a_{ij}^l(t_1, \xi_j) \frac{\partial^2 g(t_1, \xi)}{\partial x_i \partial x_j} \leq 0 \text{ and } b_i^l(t_1, \xi_i) \frac{\partial g(t_1, \xi_i)}{\partial x_i} = 0 \]

where we use the following facts: \( a_{ij}^l(t_1, \xi_j) = a_{ij}^l(t_1, \xi_j) = 0 \) for \( j \geq 2 \) (see Lemma 3) and accordingly \( ||a_{ij}^l(t_1, \xi_j)||_i, j = 2, \ldots, m \) is a positive-definite symmetric matrix. Thus we get \( Ag(t_1, \xi_i) \leq 0 \), and hence

\[ 0 = \frac{\partial g(t_1, \xi_i)}{\partial t} = Ag(t_1, \xi_i) - \frac{n(t_1 - s)^{n-1}}{(t-s)^n} g(t_1, \xi_i) < 0; \]

that is a contradiction. Hence the function \( g(t, x) \) on \( (s, t_0) \times \mathcal{G} \) does not take the positive maximum at any point in \( (s, t_0) \times B \). Similarly it does not take the negative minimum at any point in \( (s, t_0) \times B \).

**Proof of Theorem 4** may be performed by means of the entirely same manner as in § 0 by making use of Lemma 14 in place of Lemma A in § 0. We omit to repeat here the argument in § 0.

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(Received August 17, 1954)