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CHARACTERIZATIONS OF CONDITIONAL EXPECTATIONS FOR \( L_1(X) \)-VALUED FUNCTIONS

RYOHEI MIYADERA

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**Introduction.** The conditional expectation of a Banach-valued function is defined by means of Bochner integral, see L. Schwartz [11]. The purpose of this paper is to study sufficient conditions for a linear operator on the space of \( L_1 \)-valued integrable functions on a probability space \((\Omega, \mathcal{A}, \mu)\) to be a conditional expectation operator (in the sense of Schwartz [11]), where \( L_1 \) means the space of integrable real-valued functions over a measure space \((X, S, \lambda)\). For the case of real-valued functions, such a problem has been studied by several authors, such as T. Ando [1], R.R. Bahadur [2], R.G. Douglas [4], S.C. Moy [8], M.P. Olson [9], J. Pfanzagl [10], M.M. Rao [11], and Z. Šidák [14].

For the case of strictly convex space-valued functions D. Landers and L. Rogge [7] proved that every constant-preserving contractive projection becomes conditional expectation operators. They also show that these conditions do not characterize the conditional expectation operator for the case of \( L_1 \)-valued functions.

In Section 2 we shall reduce the problem of characterization of conditional expectations of \( L_1 \)-valued functions to the problem of operators of scalar valued integrable functions on a product space. In Section 3 we deal with the case of a measure space with ergodic transformations. Then every constant-preserving contractive projection becomes a conditional expectation operator under the additional condition that it commutes with these transformations. Then we deal with the case that \( X \) is a locally compact Hausdorff topological group and \( \lambda \) is the left Haar measure on the \( \sigma \)-ring \( S \) generated by the class of compact sets. In Section 4 we suppose that \( X = \mathbb{R}/\mathbb{Z} \), where \( \mathbb{Z} \) is the class of integers, and \( S \) is the class of Borel sets and \( \lambda \) is the Haar measure. Then properties of translation-invariant \( \sigma \)-subalgebra \( S' \) of \( S \) is considered, and we will use this result to consider the case in Section 2.

1. **Definitions and useful lemmas.** Let \( E \) be a Banach space over the reals with the norm \( \| \cdot \|_E \) and \((\Omega, \mathcal{A}, \mu)\) a probability space. Let \( L_1(\Omega, \mathcal{A}, \mu, E) \) denote the space of all \( E \)-valued Bochner integrable functions on \((\Omega,
\( \mathcal{A}, \mu \) associated with the norm defined by
\[
\|f\|_L = \int \|f(\omega)\|_E \, d\mu(\omega).
\]

For the definitions and properties of Bochner integral, see Hille and Phillips [6].

**Definition 1.** For a \( \sigma \)-subalgebra \( \mathcal{B} \) of \( \mathcal{A} \), a function \( g \) is called the conditional expectation of \( f \) given \( \mathcal{B} \) if \( g \) is weakly measurable with respect to \( \mathcal{B} \), and \( \int_B g \, d\mu = \int_B f \, d\mu \) for each \( B \in \mathcal{B} \), where the integral is Bochner integral. We denote by \( f^\mathcal{B} \) the conditional expectation of \( f \) given \( \mathcal{B} \).

We shall denote by \( R \) the space of real numbers. For each \( \varphi \in L_1(\Omega, \mathcal{A}, \mu, R) \) and \( a \in E \) we define \((\varphi \cdot a)(\omega) = \varphi(\omega) \cdot a \) for each \( \omega \in \Omega \). Then \( \|\varphi \cdot a\|_L = \|a\|_E \int |\varphi| \, d\mu \).

**Lemma 1.1.** For each \( f \in L_1(\Omega, \mathcal{A}, \mu, E) \) the conditional expectation \( f^\mathcal{B} \) of \( f \) given \( \mathcal{B} \) exists uniquely up to almost everywhere and satisfies \( \int |f(\omega)|_E \, d\mu(\omega) = \int |f(\omega)|_E \, d\mu(\omega) \).

For proof see Schwartz [12].

By the definition of conditional expectation, \((\varphi \cdot a)^\mathcal{B} = \varphi^\mathcal{B} \cdot a \) for each \( \varphi \in L_1(\Omega, \mathcal{A}, \mu, R) \) and \( a \in E \).

**Definition 2.** Let \( P \) be a linear operator of \( L_1(\Omega, \mathcal{A}, \mu, E) \) into itself. \( P \) is said to be contractive if \( \|P\| = \sup\{\|P(f)\|_L : f \in L_1(\Omega, \mathcal{A}, \mu, E) \text{ and } \|f\|_L = 1\} \leq 1 \), \( P \) is constant-preserving if \( P(1_\Omega \cdot a) = 1_\Omega \cdot a \) for each \( a \in E \) and \( P \) is called a projection if \( P \circ P = P \).

In particular a contractive operator is bounded, and hence continuous.

**Lemma 1.2.** The conditional expectation operator \((\cdot)^\mathcal{B}\) is a constant-preserving contractive projection for each \( \sigma \)-subalgebra \( \mathcal{B} \) of \( \mathcal{A} \).

This is a direct consequence of Definition 1 and Lemma 1.1.

**Lemma 1.3** (Douglas). If \( P \) is a constant-preserving contractive projection of \( L_1(\Omega, \mathcal{A}, \mu, R) \) into itself, then there exists a \( \sigma \)-subalgebra \( \mathcal{C} \) of \( \mathcal{A} \) such that \( P(f) \) is the conditional expectation of \( f \) given \( \mathcal{C} \) for each \( f \in L_1(\Omega, \mathcal{A}, \mu, R) \); i.e., \( P(f) = f^\mathcal{C} \) for each \( f \in L_1(\Omega, \mathcal{A}, \mu, R) \).

For proof see Douglas [4].
Obviously, the above lemma holds for every finite measure space \((\Omega, \mathcal{A}, \mu)\).

**Lemma 1.4.** If \(Q\) is a constant-preserving contractive projection of \(L_1(\Omega, \mathcal{A}, \mu, E)\) into itself, then for each \(\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)\) with \(0 \leq \varphi \leq 1\) and \(a \in E\) there exists a \(\mu\)-null set \(N\) such that

\[
\|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E = \|a - Q(\varphi \cdot a)(\omega)\|_E \quad \text{for each} \quad \omega \in \Omega - N.
\]

**Proof.** Since \(Q\) is constant-preserving and contractive and \(0 \leq \varphi \leq 1\),

\[
\|1_\Omega \cdot a\|_L - \|\varphi \cdot a\|_L = \|a\|_E - \|a\|_E \int \varphi \, d\mu
\]

\[
= \|a\|_E \int 1_\Omega - \varphi \, d\mu = \int \|a\|_E 1_\Omega - \varphi \, d\mu = \|1_\Omega \cdot a - \varphi \cdot a\|_L
\]

\[
\geq \|Q(1_\Omega \cdot a - \varphi \cdot a)\|_L = \|1_\Omega \cdot a - Q(\varphi \cdot a)\|_L
\]

\[
= \|1_\Omega \cdot a\|_L - \|Q(\varphi \cdot a)\|_L \geq \|1_\Omega \cdot a\|_L - \|\varphi \cdot a\|_L.
\]

Therefore it holds that

\[
\|1_\Omega \cdot a\|_L - \|Q(\varphi \cdot a)\|_L = \|1_\Omega \cdot a - Q(\varphi \cdot a)\|_L.
\]

Hence we have

\[
\int \{\|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E\} \, d\mu(\omega) = \int \|1_\Omega(\omega) \cdot a - Q(\varphi \cdot a)(\omega)\|_E \, d\mu(\omega).
\]

From the evident inequality

\[
\|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E \leq \|a - Q(\varphi \cdot a)(\omega)\|_E \quad \text{for each} \quad \omega \in \Omega, \quad \text{we have} \quad \|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E \leq \|a - Q(\varphi \cdot a)(\omega)\|_E \quad \text{for each} \quad \omega \in \Omega - N, \quad \text{where} \quad N \text{is a} \ \mu\text{-null set.}
\]

**Proposition 1.1.** Let \(Q\) be a constant-preserving contractive projection of \(L_1(\Omega, \mathcal{A}, \mu, E)\) into itself. If, for each \(\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)\) and for each nonzero element \(a\) of \(E\), there exists \(\varphi' \in L_1(\Omega, \mathcal{A}, \mu, R)\) such that \(Q(\varphi \cdot a) = \varphi' \cdot a\), then there exists a \(\sigma\)-subalgebra \(C\) of \(\mathcal{A}\) such that \(Q(f)\) is the conditional expectation of \(f\) given \(C\) for each \(f \in L_1(\Omega, \mathcal{A}, \mu, E)\).

**Proof.** \(\varphi'\) does not depend on the choice of the element \(a\) of \(E\). (See the Proof of the theorem of Landers [7].) Therefore we can define an operator \(Q'\) of \(L_1(\Omega, \mathcal{A}, \mu, R)\) into itself by \(Q'(\varphi \cdot a) = Q(\varphi \cdot a)\) for each \(a \in E\) and \(\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)\). Clearly \(Q'\) is a constant-preserving contractive projection of \(L_1(\Omega, \mathcal{A}, \mu, R)\) into itself. Therefore by Lemma 1.3 there exists a \(\sigma\)-subalgebra \(C\) such that \(Q \circ C = Q'(\varphi)\). Therefore we have \(Q(\varphi \cdot a) = \varphi \circ C \cdot a = (\varphi \cdot a) \circ C\). And hence \(Q\) is the conditional expectation operator given \(C\) by the proof of [12, Theorem 1.6.4]

In the rest of this paper we restrict ourselves to the case that \(E = L_1(X, S, \lambda, R)\), where \(X\) is a measure space and \(S\) is a \(\sigma\)-ring and \(\lambda\) is a measure on \(S\).
Lemma 1.5. Suppose that $Q$ is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If $K \subseteq S$ with $\lambda(K) < \infty$, then, for every $\varphi \in L_1(\Omega, \mathcal{A}, \mu, E)$ with $0 \leq \varphi \leq 1$ and $A \in \mathcal{A}$, we have

$$1 \geq \int_A Q(\varphi \cdot 1_K)(\omega, x) \, d\mu(\omega) \, d\lambda(x) \geq 0 \quad \text{and} \quad \int_A Q(\varphi \cdot 1_K)(\omega, x) \, d\mu(\omega) = 0, \lambda\text{-a.e.} x \text{ on } K^C.$$

Proof. By Lemma 1.4 there exists a $\mu$-null set $N$ such that $||1_K||_E - ||Q(\varphi \cdot 1_K)(\omega)||_E = ||1_K - Q(\varphi \cdot 1_K)(\omega)||_E$ for each $\omega \in \Omega - N$, since $1_K \in E$, where $1_K$ is the indicator function of $K$. Hence

$$\int 1_K \, d\lambda(x) - \int |Q(\varphi \cdot 1_K)(\omega, x)| \, d\lambda(x) = \int |1_K - Q(\varphi \cdot 1_K)(\omega, x)| \, d\lambda(x).$$

From the evident inequality

$$1_K(x) - |Q(\varphi \cdot 1_K)(\omega, x)| \leq |1_K(x) - Q(\varphi \cdot 1_K)(\omega, x)|,$$

we have for each $\omega \in \Omega - N$

$$1_K(x) - |Q(\varphi \cdot 1_K)(\omega, x)| = |1_K(x) - Q(\varphi \cdot 1_K)(\omega, x)|, \lambda\text{-a.e.} x.$$

Therefore, for each $\omega \in \Omega - N$, $0 \leq Q(\varphi \cdot 1_K)(\omega, x) \leq 1$, $\lambda\text{-a.e.}$, and $Q(\varphi \cdot 1_K)(\omega, x) = 0$, $\lambda\text{-a.e.}$ on $K^C$. Hence

$$1 \geq \int_A Q(\varphi \cdot 1_K)(\omega, x) \, d\mu(\omega) \, d\lambda(x) \geq 0 \quad \text{and} \quad \int_A Q(\varphi \cdot 1_K)(\omega, x) \, d\mu(\omega) = 0, \lambda\text{-a.e.} x \text{ on } K^C.$$

2. The case of a general measure space. Let $(X, S, \lambda)$ be a measure space. For convenience we denote $L_1(\Omega \times X, \mathcal{A} \times S, \mu \times \lambda, R)$ by $L_1(\Omega \times X)$ and $L_1(\Omega, \mathcal{A}, \mu, L_1(X, S, \lambda, R))$ by $L_1(\Omega, L_1(X))$.

Lemma 2.1. There exists a norm isomorphism of $L_1(\Omega, L_1(X))$ onto $L_1(\Omega \times X)$.

For proof see Treves [15, p. 464, Exercise 46.5]

Let $Q$ be a mapping of $L_1(\Omega, L_1(X))$ into itself. Let $i$ be the isomorphism of $L_1(\Omega, L_1(X))$ onto $L_1(\Omega \times X)$. Then $Q' = i \circ Q \circ i^{-1}$ is a mapping of $L_1(\Omega \times X)$ into itself.

Lemma 2.2. $Q$ is a contractive projection iff $Q'$ is a contractive projection.

This lemma is a direct consequence of the definition of $i$.

For $K \subseteq S$ such that $0 < \lambda(K) < \infty$ we denote $L_1(K, S \cap K, \lambda|S \cap K)$ by
Let $L_1(K)$ and $L_1(\Omega \times K, \mathcal{A} \times (S \cap K), \mu \times (\lambda/S \cap K))$ by $L_1(\Omega \times K)$. We may regard $L_1(\Omega \times K)$ as a subspace of $L_1(\Omega \times X)$ by a canonical way.

**Lemma 2.3.** If $Q$ is a constant-preserving contractive projection, then $Q'(L_1(\Omega \times K)) \subset L_1(\Omega \times K)$.

Proof. If $f \in L_1(\Omega, L_1(X))$ and $f$ is an $L_1(K)$-valued function, then by Lemma 1.5, $Q(f)$ is an $L_1(K)$-valued function. By Lemma 2.1 there exists a norm isomorphism of $L_1(\Omega, L_1(K))$ onto $L_1(\Omega \times K)$, therefore $Q'(L_1(\Omega \times K)) \subset L_1(\Omega \times K)$.

**Lemma 2.4.** Let $Q$ be a bounded transformation of $L_1(\Omega, L_1(X))$ into itself. Then $Q$ is the conditional expectation operator given $\mathcal{B}$ if and only if $Q'L_1(\Omega \times K)$ is the conditional expectation operator of $L_1(\Omega \times K)$ into itself given $\mathcal{B} \times (S \cap K)$.

Proof. Suppose that $Q$ is a conditional expectation operator given $\mathcal{B}$. Then for every $M \in \mathcal{B}$ and $N \in S \cap K$, we have $Q(1_M \cdot 1_N) = (1_M)_{\mathcal{B}} \cdot 1_N$. It follows that $Q'(L_1(\Omega \times K)) \subset L_1(\Omega \times K)$. For any $M \in \mathcal{B}, N \in S \cap K$ and $f \in L_1(\Omega \times K)$, we have

$$\int_{M \times N} Q'(f) \, d\mu \times d\lambda = \int_N \left( \int_M Q(f) \, d\mu \right) d\lambda = \int_M f \, d\mu \, d\lambda$$

Thus $Q'L_1(\Omega \times K)$ is the conditional expectation operator given $\mathcal{B} \times (S \cap K)$. Conversely, suppose that $Q'L_1(\Omega \times K)$ is the conditional expectation operator given $\mathcal{B} \times (S \cap K)$ for each $K \in S$ with $0 < \lambda(K) < \infty$. Let $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and $K \in S$ with $0 < \lambda(K) < \infty$. Then, for any $M \in \mathcal{A}$ and $N \in S$, we have

$$\int_N \left( \int_M Q(\varphi \cdot 1_K) \, d\mu \right) d\lambda = \int_{M \times N} Q'(\varphi \cdot 1_K) \, d\mu \times d\lambda = \int_M \varphi_{\mathcal{B} \cdot 1_K} \, d\mu \, d\lambda$$

It follows that $Q(\varphi \cdot 1_K) = \varphi_{\mathcal{B} \cdot 1_K}$. By linearity and continuity $Q(\varphi \cdot a) = \varphi_{\mathcal{B} \cdot a}$ for all $a \in L_1(X)$. By the proof [12, Theorem 1.6.4], $Q$ is the conditional expectation operator given $\mathcal{B}$.

Let $Q$ be a constant-preserving contractive projection on $L_1(\Omega, L_1(X))$. Then by Lemmas 1.3 and 2.3, for any $K \in S$ with finite measure, there is a $\sigma$-subalgebra $F_K$ of $\mathcal{A} \times (S \cap K)$ such that $Q'L_1(\Omega \times K)$ is the conditional expectation operator given $F_K$. Moreover, by Lemma 2.4, $Q$ is a conditional expectation operator on $L_1(\Omega, L_1(X))$ if and only if there is a $\sigma$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $F_K = \mathcal{B} \times (S \cap K)$ for all $K$.

### 3. The case of a measure space with ergodic transformations.

Let
(X, S, λ) be a measure space, S a σ-algebra, S(λ)={K; K∈S and λ(K)<∞} and Sf(λ)={K⊂X; K∩E∈S for each E∈S(λ)}.
For each K∈Sf(λ) let λ(K)=sup {λ(K∩E); E∈S(λ)}.

**Definition 3.** A measure space (X, S, λ) is localizable if each nonempty collection C⊂S(λ) has sup C∈S, in the sense that for each K∈C, λ(K−sup C)=0 and if H∈S and λ(K−H)=0 for each K∈C, then λ(sup C−H)=0.

**Definition 4.** A measure space (X, S, λ) is locally localizable if each nonempty collection C⊂S(λ) has sup C∈Sf(λ), in the sense that for each K∈C, λ(K−sup C)=0 and if H∈Sf(λ) and λ(K−H)=0 for each K∈C, then λ(sup C−H)=0.

**Definition 5.** A measure space (X, S, λ) has the finite subset property if for each K∈S, λ(K)>0, there is K′∈S with K′⊂K and 0<λ(K′)<∞.

**Lemma 3.1.** If (X, S, λ) is a locally localizable measure space with the finite subset property, then (X, Sf(λ), λ) is a localizable space which satisfies the finite subset property and λ/S=λ.

For proof see Ghosh, Morimoto and Yamada [5].

**Definition 6.** A class \{f(x, K); K∈S(λ)\} of S-measurable functions on (X, S, λ) is called a cross-section

\[
\text{if } f(x, K) = 0 \text{ on } K \subseteq \text{ and } \quad 1_{K_1 \cap K_2}(x) \cdot f(x, K_1) = 1_{K_1 \cap K_2}(x) \cdot f(x, K_2) \quad \text{(a.e.x) for each } K_1, K_2 \in S(\lambda). 
\]

**Lemma 3.2.** Suppose that a measure space (X, S, λ) is localizable. Then for each cross-section \{f(x, K); K∈S(λ)\} there exists a S-measurable function f such that \( f(\cdot, K) = f(x, K) \) (λ-a.e.x) for each K∈S(λ).

For proof see Zaanen [16]

In the rest of this section we assume that (X, S, λ) is a localizable space with the finite subset property.

**Lemma 3.3.** Let Q be a constant-preserving contractive projection of \( L_1(\Omega, \mathcal{A}, \mu, E) \) into itself, where \( E=L_1(\Omega, \mathcal{A}, \mu, R), 0\leq \varphi \leq 1, \) and \( A \in \mathcal{A} \) there exists a λ-a.e. unique S-measurable function b such that

\[
0 \leq b(x) \leq 1 \quad (\lambda\text{-a.e.x}) \quad \text{and} \quad b \cdot 1_K = 1_K \cdot \int_A Q(\varphi \cdot 1_K) \, d\mu \quad (\lambda\text{-a.e.x}) \quad \text{for each } K \in S(\lambda).
\]
Proof. By Lemma 1.5, \( \int_A Q(\varphi \cdot 1_K) \, d\mu = 0 \) on \( K \subseteq S(\lambda) \). For each \( K_1, K_2 \subseteq S(\lambda) \),
\[
1_{K_1 \cap K_2} \int_A Q(\varphi \cdot 1_{K_1}) \, d\mu = 1_{K_1 \cap K_2} \int_A Q(\varphi \cdot 1_{K_1 \cap K_2}) \, d\mu, 
\]
since \( \int_A Q(\varphi \cdot 1_{K_1 \cap K_2}) \, d\mu = 0 \) on \( K_1 \cap K_2 \). Similarly
\[
1_{K_1 \cap K_2} \int_A Q(\varphi \cdot 1_{K_2}) \, d\mu = 1_{K_1 \cap K_2} \int_A Q(\varphi \cdot 1_{K_1 \cap K_2}) \, d\mu. 
\]

Therefore \( \{ \int_A Q(\varphi \cdot 1_K) \, d\mu : K \subseteq S(\lambda) \} \) is a cross section, and hence by Lemma 3.2 there exists a \( S \)-measurable function \( b \) such that \( b \cdot 1_K = 1_K \cdot \int_A Q(\varphi \cdot 1_K) \, d\mu \) (\( \lambda \)-a.e.x.) for each \( K \subseteq S(\lambda) \). What remains is to prove the uniqueness of \( b \).

Suppose that there exists a \( S \)-measurable function \( b' \) such that \( b \cdot 1_K = b' \cdot 1_K \) (\( \lambda \)-a.e.x.) and \( \lambda(\{x : b(x) \neq b'(x)\}) = 0 \). By the finite subset property of \((X, S, \lambda)\) there exists \( \mathcal{E} \subseteq S(\lambda) \), \( \mathcal{E} \cap \{x : b(x) \neq b'(x)\} \), which leads to a contradiction, since \( b \cdot 1_E = b' \cdot 1_E \) (a.e.x.). We have proved that \( b(x) = b'(x) \) (\( \lambda \)-a.e.x.). Similarly by Lemma 1.5 and the finite subset property of \((X, S, \lambda)\) we have \( 0 \leq b(x) \leq 1 \) (\( \lambda \)-a.e.x.).

**Definition 7.** Let \( T \) be a one to one transformation of \((X, S, \lambda)\) onto itself, then \( T \) is called a bounded measurable transformation if \( T \) is a measurable transformation and there exists a positive number \( k \) such that \( \lambda(T^{-1}(A)) \leq k \cdot \lambda(A) \) for each \( A \subseteq S \).

**Definition 8.** Let \( \{T : T \in \mathcal{T}\} \) be a class of bounded measurable transformations of \( X \) onto \( X \) such that \( T^{-1}(S(\lambda)) = S(\lambda) \) for each \( T \in \mathcal{T} \). \((X, S, \lambda, T : T \in \mathcal{T}) \) is called ergodic if \( \lambda(T^{-1}(A)) = 0 \) for each \( T \in \mathcal{T} \) implies \( \lambda(A) = 0 \) or \( \lambda(A^c) = 0 \).

**Lemma 3.4.** If \((X, S, \lambda, T : T \in \mathcal{T})\) is an ergodic space, then for each bounded measurable function \( f \) on \( X \), \( f(x) = f(T(x)) \) a.e.x for each \( T \in \mathcal{T} \) implies that \( f(x) = \text{const.} \lambda \)-a.e.x.

**Proof.** Let \( f \) be a bounded measurable function on \( X \) and \( f(x) = f(T(x)) \), \( \lambda \)-a.e.x for each \( T \in \mathcal{T} \). For each real number \( d \) let \( E_d = f^{-1}((d, \infty)) \). Then \( \lambda(E_d \Delta T^{-1}(E_d)) \leq \lambda(f(x) \neq f(T(x))) = 0 \). By the definition of ergodicity \( \lambda(E_d) = 0 \) or \( \lambda(E_d^c) = 0 \), \( f \) is bounded, and hence there exists a real number \( M \) such that \( |f(x)| \leq M \), a.e.x. If \( d > M \) then, \( \lambda(E_d) = 0 \). If \( d < -M \), then \( \lambda(E_d^c) = 0 \).

Let \( c = \inf \{d : \lambda(E_d) = 0\} \). Then \( f = c \), \( \lambda \)-a.e.x.

Let \((X, S, \lambda, T : T \in \mathcal{T})\) be an ergodic measure space and \( E = L_1(X, S, \lambda, T : T \in \mathcal{T}) \). For each real valued measurable function \( a \) on \( X \) and \( T \in \mathcal{T} \), we write \( T \cdot a(x) = a(T(x)) \). Then \( T \) can be seen as a bounded linear operator of \( L_1(X, S, \lambda, \mathcal{R}) \) into itself.

**Definition 9.** Let \( Q \) be a transformation of \( L_1(\Omega, \mathcal{A}, \mu, E) \) into itself,
then \( Q \) is called covariant under \( \mathcal{D} \) if \( Q(\varphi \cdot (T \cdot a)) = T \cdot Q(\varphi \cdot a) \) for each \( \varphi \in L_1(\Omega, \mathcal{A}, \mu, R) \) and \( a \in E \) and \( T \in \mathcal{D} \).

**Theorem 1.** Let \( Q \) be a constant-preserving contractive projection which is invariant under \( \mathcal{D} \). Then \( Q = (\cdot)B \) for some \( \sigma \)-subalgebra \( B \) of \( \mathcal{A} \).

Proof. Let \( \varphi \in L_1(\Omega, \mathcal{A}, \mu, R), 0 \leq \varphi \leq 1 \) and \( A \in \mathcal{A} \) and \( T \in \mathcal{D} \). By Proposition 1.1 it is sufficient to prove that there exists \( \varphi' \in L_1(\Omega, \mathcal{A}, \mu, R) \) such that \( Q(\varphi \cdot 1_K) = \varphi' \cdot 1_K \) for each \( K \in S(\lambda) \). By Lemma 3.3 there exists a \( \mathcal{S} \)-measurable function \( b \) such that \( 0 \leq b(x) \leq 1 \) (a.e.x) and \( b \cdot 1_K = \int_A Q(\varphi \cdot 1_K) d\mu \) (\( \lambda \)-a.e.x) for each \( K \in S(\lambda) \).

\[
(T \cdot b) \cdot 1_{T^{-1}(K)} = T(b \cdot 1_K) = T \int_A Q(\varphi \cdot 1_K) d\mu
\]
\[
= \int_A T \cdot Q(\varphi \cdot 1_K) d\mu = \int_A Q(\varphi \cdot (T \cdot 1_K)) d\mu
\]
\[
= \int_A Q(\varphi \cdot 1_{T^{-1}(K)}) d\mu. \quad \text{Since } T^{-1}(S(\lambda)) = S(\lambda) \text{ we have}
\]
\[
(T \cdot b) \cdot 1_K = \int_A Q(\varphi \cdot 1_K) d\mu = b \cdot 1_K \quad \text{for each } K \in S(\lambda).
\]

By the uniqueness of \( b \) \( T \cdot b = b(\lambda \text{-a.e.x}) \). By Lemma 3.4 there exists a positive number \( k(A) \) such that \( b(x) = 1_K \cdot k(A) \) (\( \lambda \)-a.e.x). Hence \( b \cdot 1_K = 1_K \cdot k(A) \).

Let \( \{A_n, n=1, 2, \ldots\} \) be a sequence of elements of \( \mathcal{A} \) and \( A_n \cap A_m = \phi(n \neq m) \).

\[
1_K \cdot k \bigcup_{n=1}^{\infty} A_n = \int_{\bigcup_{n=1}^{\infty} A_n} Q(\varphi \cdot 1_K) d\mu = \sum_{n=1}^{\infty} \int_{A_n} Q(\varphi \cdot 1_K) d\mu
\]
\[
= \sum_{n=1}^{\infty} 1_K \cdot K(A_n) = 1_K \cdot (\sum_{n=1}^{\infty} k(A_n)).
\]

Therefore \( k \bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} k(A_n) \), this shows that \( k(\cdot) \) is a measure on \( \mathcal{A} \). \( k \) is absolutely continuous with respect to \( \mu \), since \( 1_K \cdot k(A) = \int_A Q(\varphi \cdot 1_K) d\mu \). By the Radon-Nykodym theorem there is \( \varphi' \in L_1(\Omega, \mathcal{A}, \mu, R) \) such that

\[
\int_A \varphi' \cdot 1_K d\mu = 1_K \cdot \int_A \varphi' d\mu = 1_K \cdot k(A) = \int_A Q(\varphi \cdot 1_K) d\mu.
\]

Therefore \( Q(\varphi \cdot 1_K) = \varphi' \cdot 1_K \).

**Remark.** If \( (X, S, \lambda) \) is \( \sigma \)-finite measure space, then Theorem 1 can be proved without the condition that \( T^{-1}(S(\lambda)) = S(\lambda) \).

Let \( G \) be a locally compact group and \( \lambda \) a left Haar measure on the \( \sigma \)-algebra \( S \) generated by open sets (cf. Berberian [3, Exercise 79.6, p. 263]). Then \( (G, S, \lambda) \) is a locally localizable measure space with the finite subset property.
CONDITIONAL EXPECTATIONS FOR $L_1(X)$-VALUED FUNCTIONS

(cf. Segal [13]). Let $G$ be the set of all translations on $G$. Then it is easy to see that $(G, S, \mathcal{F})$ is an ergodic measure space. Thus we obtain the following.

**Corollary 1.** A constant-preserving contractive projection on $L_1(\Omega, L_1(G))$ which is covariant under all translations is a conditional expectation operator given some $\sigma$-subalgebra of $\mathcal{F}$.

4. Properties of translation-invariant $\sigma$-algebras on $R/Z$ and a characterization of conditional expectation for $L_1(R/Z)$-valued function.

Let $X = R/Z$, where $Z$ is the class of integers. Let $\lambda$ be the Haar measure and $S$ the $\lambda$-completion of the class of Borel sets on $X$. Let $\mathcal{N}$ be the $\sigma$-ring of $\lambda$-null sets and $\alpha$ an irrational number.

We define a mapping $T_\alpha$ of $X$ onto $X$ by $T_\alpha(x) = x + \alpha \pmod{1}$. A $\sigma$-subalgebra $S'$ of $S$ is said to be $T_\alpha$-invariant if $T_\alpha(K) \subseteq S'$ for each $K \subseteq S'$. For $n=1, 2, \ldots$.

Let $S_n = \{K \subseteq S, K = K + 1/n (\lambda-a.e.)\}$.

**Lemma 4.1.** Let $U$ and $V$ be $\sigma$-subalgebras of $S$ containing $\mathcal{N}$. Then $U = V$ iff

$$(e^{2\pi i k})^U = (e^{2\pi i k})^V \quad \lambda-a.e. \quad \text{for any} \quad k \in Z.$$  

Proof. For each complex integrable function $f$ and a positive number $\varepsilon > 0$ there exist complex numbers $c_1, c_2, \ldots, c_n$ such that $\|f - \sum_{j=1}^{n} c_j e^{2\pi i j}\|_{L_1(X)}$. Since conditional expectation operator is linear continuous, we have this lemma.

**Lemma 4.2.** Let $S'$ be a $\sigma$-subalgebra of $S$ containing $\mathcal{N}$. Then $S' = S_\alpha$ iff $(e^{2\pi i k'})^S = \begin{cases} 0 & (k \equiv 0 \pmod{n}) \\ e^{2\pi i k} & (k \equiv 0 \pmod{n}) \end{cases} \quad \lambda-a.e. \quad \text{for any} \quad k \in Z.$

Proof. If $k \equiv 0 \pmod{n}$, then $\int_K e^{2\pi i k} \, dx = 0$ for each $K \subseteq S'$. This lemma is a direct consequence of this fact and Lemma 4.1.

**Lemma 4.3.** Let $S'$ be a $T_\alpha$-invariant $\sigma$-subalgebra of $S$ containing $\mathcal{N}$. Then

$$(e^{2\pi i k})(T_\alpha(x))^{S'} = e^{2\pi i k} \cdot (e^{2\pi i k})^{S'}(x) \quad \lambda-a.e. \quad \text{for any} \quad k \in Z.$$

Proof. Let $f(x) = (e^{2\pi i k})^{S'}(x)$. Since $\lambda$ and $S'$ are $T_\alpha$-invariant, for any $K \subseteq S'$

$$\int_K f(T_\alpha(x)) \, d\lambda(x) = \int_{T_\alpha(K)} f(x) \, d\lambda(x) = \int_{T_\alpha(K)} e^{2\pi i k} \, d\lambda(x)$$
\[ e^{2\pi ik} \frac{d\lambda}{dx} = e^{2\pi ik} \int e^{2\pi ik} \, d\lambda(x) \]
\[ = e^{2\pi ik} \int f(x) \, d\lambda(x) . \]

Therefore \( f(T_a(x)) = e^{2\pi ik} f(x) \).

Lemma 4.4. Let \( f \in L_2(X, S, \lambda, R) \) such that \( f(T_a(x)) = e^{2\pi ik} f(x) \) a.e. Then \( f(x) = C e^{2\pi ik} \) a.e. where \( C \) is a constant.

Proof. \( \{e^{2\pi ij}, j = 1, 2, \ldots\} \) is a complete orthogonal system in \( L_2(X, S, \lambda, R) \). Let \( f(x) = \sum_{j=1}^{\infty} c_j e^{2\pi ij} \). Since \( f(T_a(x)) = e^{2\pi ik} f(x) \) a.e., it holds that \( c_j e^{2\pi ij} = c_j e^{2\pi ik} \) for any positive integer \( j \). Therefore \( c_j = 0 \) except for \( j = k \).

Theorem 2. Let \( S' \) be a \( \sigma \)-subalgebra of \( S \) containing \( \mathfrak{N} \). Then \( S' \) is \( T_a \)-invariant if \( S' = \mathfrak{N} \) or \( S' = S_n \) for some positive integer \( n \).

Proof. Suppose that \( S' \) is \( T_a \)-invariant. By Lemma 5.3 and Lemma 4.4 there exists a complex number \( C_k \) such that \( (e^{2\pi ik})^{s'} = c_k e^{2\pi ik} \) a.e. for each positive integer \( k \). If \( S' = \mathfrak{N} \), then there exists a positive integer \( k \) such that \( (e^{2\pi ik})^{s'} \neq 0 \) (a.e.). Let \( n = \min \{k: k \text{ is a positive integer and } (e^{2\pi ik})^{s'} \neq 0 \text{ (a.e.)}\} \). Then \( e^{2\pi in} \) is \( S' \)-measurable and \( c_n = 1 \). Since \( S' \) is \( T_a \)-invariant and \( e^{2\pi in} \) is \( S_n \)-measurable, \( S_n \subseteq S' \). Therefore for each \( k \) such that \( k = 0 \) (mod \( n \)) \( C_k = 0 \). For any positive integer \( k \) there exist positives integers \( h \) and \( j \) such that \( k = h + j \) (0 \leq j < n). Since \( e^{2\pi ih} \) is \( S_n \)-measurable, it is \( S' \)-measurable. Hence \( (e^{2\pi ik})^{s'} = (e^{2\pi ih})^{s'} (e^{2\pi ij})^{s'} = 0 \) a.e. By Lemma 4.2 \( S' = S_n \). Conversely if \( S' = \mathfrak{N} \) or \( S' = S_n \) for some positive integer \( n \), then \( S' \) is \( T_a \)-invariant.

Definition 11. Let \( \psi(x) = x - [x] \). Then \( \psi \) is a mapping of \( R \) onto \( R/Z \). A subset \( K \) of \( R/Z \) is said to be an interval if \( K = \psi([a, b]) \) for some real numbers \( a, b \in R \).

Definition 12. For \( K \in S \) define \( k(K) = \max \{\lambda(H): H \text{ is an interval and } H \subseteq K\} \).

Definition 13. For each \( a \in L_1(X, S, \lambda, R) \) and \( x_0 \in X \), let \( (T_{x_0}a)(x) = a(x_0 \cdot x) \). Let \( P \) be a transformation of \( L_1(\Omega \times X, \mathcal{A} \times S, \mu \times \lambda, R) \) into itself. \( P \) is said to be translation invariant if \( T_{x_0}P(\varphi \cdot a) = P(\varphi \cdot T_{x_0}a) \) for each \( \varphi \in L_1(\Omega, \mathcal{A}, \mu, R), a \in L_1(X, S, \lambda, R) \) and \( x \in X \).

Theorem 3. Let \( P \) be a translation invariant constant-preserving contractive projection of \( L_1(\Omega \times X, \mathcal{A} \times S, \mu \times \lambda, R) \) into itself. If there exists \( K \in S \)
such that \( k(K) > 1/2, \lambda(K) < 1 \) and \( P(1_{\Omega \times K}) = 1_{\Omega \times K} \), then there exists a \( \sigma \)-subalgebra \( B \) of \( A \) such that \( P(f) = f_{B \times S} \) for each \( f \in L_1(\Omega \times X, A \times S, \mu \times \lambda, R) \).

Proof. By Lemma 1.3 there exists a \( \sigma \)-subalgebra \( C \) of \( A \times S \) such that \( P(f) = f_C \) for each \( f \in L_1(\Omega \times X, A \times S, \mu \times \lambda, R) \). Let \( i \) be the isomorphism of \( L_1(\Omega, A, \mu, L_1(X, S, \lambda, R)) \) onto \( L_1(\Omega \times X, A \times S, \mu \times \lambda, R) \) and \( Q = i^{-1} \circ P \circ i \), then \( Q \) is a translation invariant contractive projection of \( L_1(\Omega, A, \mu, L_1(X, S, \lambda, R)) \) into itself. Write \( S' = \{ K : \Omega \times K \in C \} \). Since \( P \) is translation invariant, \( S \) is a \( T_\sigma \)-invariant \( \sigma \)-subalgebra of \( S \). Therefore by Theorem 2 \( S' = \mathcal{H} \) or \( S' = S_n \) for some positive integer \( n \). Since \( k(K) > 1/2, \) \( S' = S \). This implies that for each \( K \in S \) \( P(1_{\Omega \times K}) = 1_{\Omega \times K} \). Therefore \( Q(1_{\Omega} \cdot 1_K) = 1_{\Omega} \cdot 1_K \). By the arbitrariness of \( K \) we have \( Q(1_{\Omega} \cdot a) = 1_{\Omega} \cdot a \) for each \( a \in L_1(X, S, \lambda, R) \), and hence \( Q \) is a constant-preserving contractive projection. Therefore by Corollary 1 there exists \( B \) such that \( Q(f) = f_{B \times S} \) for each \( f \in L_1(\Omega, A, \mu, L_1(X, S, \lambda, R)) \). By Lemma 2.4 \( P(f) = f_{B \times S} \) for each \( f \in L_1(\Omega \times X, A \times S, \mu \times \lambda, R) \).

Remark. In Theorem 3 for the transformation \( P \) of \( L_1(\Omega \times X, A \times S, \mu \times \lambda, R) \) into itself constant-preserving means \( P(1_{\Omega \times X}) = 1_{\Omega \times X} \).

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References


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