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## ON INDECOMPOSABLE MODULES AND BLOCKS

Dedicated to Professor HIROSI NAGAO for his 60th birthday

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(Received November 30, 1984)

### Introduction

Let  $G$  be a finite group and  $F$  a field of prime characteristic  $p$ . Let  $M$  be an irreducible  $FG$ -module belonging to a block  $B$  of  $FG$  with defect group  $D$ . Then the following fact is well-known. Namely if  $M$  has height 0 in  $B$ , then  $D$  is a vertex of  $M$  and the dimension of  $D$ -source of  $M$  is prime to  $p$  (provided that  $F$  is sufficiently large). The main objective of this paper is to study an indecomposable module  $M$  which satisfies the conclusion in the above statement. In particular it will turn out that  $M_H$  has a component with the same property for  $H \leq G$  under certain circumstances (see Theorem 2.1). We shall apply our results to give new proofs to some of important theorems concerning blocks.

The notation is almost standard: We fix a complete discrete valuation ring  $R$  of characteristic 0 with  $F$  as its residue class field. We assume that the quotient field of  $R$  is a splitting one for every subgroup of  $G$ . We let  $\theta$  denote  $R$  or  $F$ . By an  $\theta G$ -module  $M$ , we understand a right  $\theta G$ -module which is finitely generated free over  $\theta$ . If  $M$  is indecomposable, we denote its vertex by  $vx(M)$ . For another module  $N$ ,  $N|M$  indicates that  $N$  is isomorphic to a direct summand of  $M$  and we say " $N$  is a component of  $M$ " if  $N$  is indecomposable. If  $n$  is an integer and  $p^m$  is the highest  $p$ -power dividing  $n$ , then we write  $m = \nu(n)$ . Finally for a block  $B$  of  $G$ , we denote by  $\delta(B)$  a defect group of  $B$ .

### 1. Sources with $\theta$ -rank prime to $p$

For later convenience, we put down the following well-known fact without proof.

**Lemma 1.1.** *Let  $M$  be an indecomposable  $\theta G$ -module with vertex  $Q$ . Let  $V$  be an indecomposable  $\theta Q$ -module. Then  $V$  is a  $Q$ -source of  $M$  if and only if  $V|M_Q$  and  $Q$  is a vertex of  $V$ .*

Let  $M$  be an indecomposable  $\theta G$ -module. We consider the following condition;

$$(*) \quad p \nmid \text{rank}_\theta V \quad \text{for a source } V \text{ of } M.$$

**Theorem 1.2.** *Let  $H$  be a subgroup of  $G$ . Let  $M$  be an indecomposable  $\theta G$ -module with vertex  $Q$  which satisfies (\*). Let  $P$  be a maximal member of  $\{Q^x \cap H \mid x \in G\}$ . Then there exists a component  $N$  of  $M_H$  such that  $P$  is a vertex of  $N$  and  $N$  satisfies (\*).*

Proof. We set  $P=Q^a \cap H$  ( $a \in G$ ) and let  $V$  be a  $Q^a$ -source of  $M$ . Then there exists a component  $W$  of  $V_P$  with  $p \nmid \text{rank}_\theta W$ . Then  $P$  is a vertex of  $W$  by Green's theorem. We may assume that  $V \mid M_{Q^a}$  and hence  $W \mid M_P$ . Let  $N$  be a component of  $M_H$  such that  $W \mid N_P$ . Then  $P \subseteq_H vx(N)$ . On the other hand,  $N \mid M_H$  means that  $vx(N) \subseteq_H Q^x \cap H$  for some  $x \in G$ . Therefore we have  $vx(N) =_H P$  by the choice of  $P$ . Moreover  $W$  is a  $P$ -source of  $N$  by Lemma 1.1. This completes the proof.

We mention a couple of remarks concerning the condition (\*).

REMARK 1.3. Let  $M$  be an indecomposable  $FG$ -module with cyclic vertex. Then  $M$  satisfies (\*).

For the proof of this fact, it is sufficient to show the following lemma, which may be, much or less, well-known.

**Lemma 1.4.** *Let  $Q = \langle x \rangle$  be a cyclic group of order  $p^s$ . Let  $M$  be an arbitrary indecomposable  $FQ$ -module. Then  $M$  satisfies (\*).*

Proof. (Watanabe) We denote by  $Q_i$  the subgroup of  $Q$  with order  $p^i$  ( $0 \leq i \leq s$ ). For each  $i$ ,  $FQ_i$  has exactly  $p^i$  indecomposable modules  $V_{ij}$  with  $\dim_F V_{ij} = j$  ( $1 \leq j \leq p^i$ ). Recall that each  $M_{ij} = (V_{ij})^Q$  is indecomposable by Green's theorem. Moreover if  $(j, p) = 1$ , then  $\nu(\dim_F M_{ij}) = \nu(|Q : Q_i|)$ . This implies that  $Q_i$  is a vertex of  $M_{ij}$  and so  $V_{ij}$  is a  $Q_i$ -source of it. Now we see that the set  $\bigcup_{i=0}^s \{M_{ij} \mid (j, p) = 1, 1 \leq j \leq p^i\}$  must be a full set of non-isomorphic indecomposable  $FQ$ -modules, since  $p^s = \sum_{i=0}^s \varphi(p^i)$  ( $\varphi$  denotes the Euler totient function). This completes the proof of Lemma 1.4.

REMARK 1.5 (Knörr [5], Theorem 4.5). Assume that  $F$  is algebraically closed. Let  $M$  be an indecomposable  $\theta G$ -module. Then if  $\nu(\text{rank}_\theta M) = \nu(|G : vx(M)|)$ ,  $M$  satisfies (\*).

As an application of Theorem 1.2, we show the following;

**Corollary 1.6.** *Let  $H$  be a normal subgroup of  $G$ . Let  $M$  be an irreducible*

*FG-module and  $N$  an irreducible constituent of  $M_H$ . Then if  $\nu(\dim_F M) = \nu(|G: vx(M)|)$ , we have  $\nu(\dim_F N) = \nu(|H: vx(N)|)$ .*

*Proof.* For the proof of this result, we may assume that  $F$  is algebraically closed. By Theorem 1.2 and Remark 1.5, there exists an irreducible constituent  $\hat{N}$  of  $M_H$  such that  $\hat{N}$  satisfies (\*). However, since a source of  $N$  and that of  $\hat{N}$  are  $G$ -conjugate to each other, we have that  $\nu(\dim_F N) = \nu(|H: vx(N)|)$  by Theorem 4.5 in [5].

As one of typical modules which satisfy (\*), let us take what is called a Scott module. For any subgroup  $X$  of  $G$ , we denote by  $I_X$  the trivial  $\theta X$ -module (an  $\theta X$ -module of rank 1 on which  $X$  acts trivially). For a  $p$ -subgroup  $Q$  of  $G$ ,  $(I_Q)^G$  has exactly one component  $S$  which contains  $I_G$  as a submodule, and then  $Q$  is a vertex of  $S$  (see Burry [2]). Following Burry, we call  $S$  the Scott  $G$ -module with vertex  $Q$ . The following theorem was suggested by Okuyama.

**Theorem 1.7.** *Let  $H$  be a subgroup of  $G$  and  $S$  the Scott  $G$ -module with vertex  $Q$ . Let  $P$  be a maximal member of  $\{Q^x \cap H \mid x \in G\}$ . Then there exists a component  $U$  of  $S_H$  which is the Scott  $H$ -module with vertex  $P$ .*

*Proof.* We prove by the induction on  $|Q|/|P|$ . If  $|Q| = |P|$ , our assertion follows immediately from Theorem 2 in [2]. So we assume that  $|Q| > |P|$ . We set  $H_1 = N_G(P)$  and let  $P_1$  be a maximal member of  $\Omega = \{Q^x \cap H_1 \mid Q^x \cap H_1 \not\supseteq P, x \in G\}$ . It is clear that  $\Omega$  is not empty. Thus by the induction hypothesis, there exists a component  $U_1$  of  $S_{H_1}$  which is the Scott  $H_1$ -module with vertex  $P_1$ . We set  $T = N_H(P)$ , then there exists a component  $\hat{U}$  of  $(U_1)_T$  which contains  $I_T$  as a submodule. However, since  $(P_1)^y \cap T = P$  for all  $y \in H_1$ ,  $\{(I_{P_1})^{H_1}\}_T$  is a direct sum of copies of  $(I_P)^T$  by Mackey decomposition theorem. Thus  $\hat{U}$  must be the Scott  $T$ -module with vertex  $P$ . Let  $U$  be a component of  $S_H$  such that  $\hat{U} \mid U_T$ . Then since  $P$  is a vertex of  $U$ ,  $U$  corresponds to  $\hat{U}$  in the Green correspondence with respect to  $(H, P, T)$ . Thus by Theorem 1 in [2],  $U$  is the Scott  $H$ -module with vertex  $P$ .

**2. Some applications to block theory**

Let  $H$  be a subgroup of  $G$  and  $b$  a block of  $H$ . Following Brauer, we call  $b$   $G$ -admissible provided  $C_G(\delta(b)) \subseteq H$ . Note that this does not depend on the particular choice of  $\delta(b)$  and  $b^G$  is defined. The following theorem was suggested by Okuyama.

**Theorem 2.1.** *Let  $b$  be a  $G$ -admissible block of  $H$ . If  $M$  is an indecomposable  $\theta G$ -module in  $B = b^G$  which has  $\delta(B)$  as a vertex and satisfies (\*), then there exists a component  $N$  of  $M_H$  which belongs to  $b$  and has  $\delta(b)$  as a vertex and*

satisfies (\*).

Proof. We prove by the induction on  $|\delta(B)|/|\delta(b)|$ . If  $|\delta(B)|=|\delta(b)|$ , our assertion follows immediately from Corollary 9 in [6] and Lemma 1.1. So we assume that  $|\delta(B)|>|\delta(b)|$ . Let  $\hat{b}$  be a root of  $b$  in  $T=\delta(b)C_G(\delta(b))$ . We set  $H_1=N_G(\delta(b))$  and  $b_1=\hat{b}^{H_1}$ . Then  $|\delta(b_1)|>|\delta(b)|$  by Brauer's first main theorem and the assumption. Thus by the induction hypothesis, there exists a component  $N_1$  of  $M_{H_1}$  in  $b_1$  such that  $\delta(b_1)$  is a vertex of  $N_1$  and  $N_1$  satisfies (\*). Since  $H_1 \triangleright T$ ,  $b_1$  covers  $\hat{b}$ . Thus by Theorem 1.2, we can show that there exists a component  $\hat{N}$  of  $(N_1)_T$  such that  $\hat{N}$  belongs to  $\hat{b}$  and  $v_x(\hat{N})=_{H_1} \delta(b_1) \cap T$ . However, since  $v_x(\hat{N}) \subseteq \delta(b) \subseteq \delta(b_1)$ , we have that  $v_x(\hat{N})=\delta(b)$  from the above. Let  $N$  be a component of  $M_H$  such that  $\hat{N}|N_T$ . Then  $N \in b$  by (3.7a) in [3]. Since  $N \in b$  and  $\hat{N}|N_T$ ,  $\delta(b)$  is a vertex of  $N$  and  $N$  satisfies (\*) by Lemma 1.1. Thus the proof is complete.

The above theorems allow us to give alternative proofs to some of important results concerning blocks.

**Corollary 2.2** (Brauer's third main theorem). *Let  $b$  be a  $G$ -admissible block of a subgroup of  $G$ . If  $b^G$  is principal, then  $b$  is principal.*

Proof. This is immediate from the above theorem by taking  $M=I_G$ , the trivial  $\theta G$ -module.

For the proofs of the following corollaries, we may assume that  $F$  is algebraically closed.

**Corollary 2.3** (Alperin and Burry [1]). *Let  $Q$  be a  $p$ -subgroup of  $G$  and  $H$  a subgroup of  $G$  such that  $H \supseteq QC_G(Q)$ . Let  $B$  be a block of  $G$ . If  $P$  is a maximal member of  $\{\delta(B)^x \cap H \mid x \in G, \delta(B)^x \cap H \supseteq Q\}$ , then there exists a block  $b$  of  $H$  such that  $b^G=B$  and  $P$  is a defect group of  $b$ .*

Proof. Let  $M$  be an irreducible  $FG$ -module in  $B$  of height 0. Then  $\nu(\dim_F M)=\nu(|G: v_x(M)|)$  and  $\delta(B)$  is a vertex of  $M$ . By Theorem 1.2 and Remark 1.5, there exists a component  $N$  of  $M_H$  such that  $P$  is a vertex of  $N$ . Let  $b$  be a block of  $H$  which contains  $N$ . Since  $C_G(P) \subseteq H$ ,  $b^G$  is defined and equals to  $B$  by (3.7a) in [3]. Furthermore by the maximality of  $P$ , we see easily that  $P$  is a defect group of  $b$ .

**Corollary 2.4** (Knörr [4]). *Let  $H$  be a normal subgroup of  $G$ . Let  $B$  be a block of  $G$  and  $b$  a block of  $H$ . If  $B$  covers  $b$ , then  $\delta(b)=_c \delta(B) \cap H$ .*

Proof. Let  $M$  be an irreducible  $FG$ -module in  $B$  of height 0. Then by Theorem 1.2 and Remark 1.5, we can show that there exists a component  $N$  of  $M_H$  such that  $N$  belongs to  $b$  and  $v_x(N)=_c \delta(B) \cap H$ . So we have  $\delta(b) \supseteq$

${}_c\delta(B) \cap H$ . On the other hand, for an irreducible  $FH$ -module  $N$  in  $b$  with  $\delta(b)$  as a vertex, there exists an irreducible  $FG$ -module  $M$  in  $B$  such that  $N \mid M_H$  (see Proposition 4.1 in [4]). Thus we have  $\delta(b) \subseteq {}_c\delta(B) \cap H$ . Combining with the above,  $\delta(b) = {}_c\delta(B) \cap H$  as asserted.

**Corollary 2.5.** *Let  $H$  be a normal subgroup of  $G$ . Let  $B$  be a block of  $G$  and  $\varphi$  an irreducible Brauer character of  $G$  in  $B$ . If  $\varphi$  has height 0, then any irreducible constituent of  $\varphi_H$  has height 0 in the block of  $H$  to which it belongs.*

Proof. This is immediate from Corollary 1.6 and Corollary 2.4.

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