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ON INDECOMPOSABLE MODULES AND BLOCKS

Dedicated to Professor Hirosi Nagao for his 60th birthday

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Introduction

Let G be a finite group and F a field of prime characteristic p. Let M be an irreducible FG-module belonging to a block B of FG with defect group D. Then the following fact is well-known. Namely if M has height 0 in B, then D is a vertex of M and the dimension of D-source of M is prime to p (provided that F is sufficiently large). The main objective of this paper is to study an indecomposable module M which satisfies the conclusion in the above statement. In particular it will turn out that M_H has a component with the same property for $H \leq G$ under certain circumstances (see Theorem 2.1). We shall apply our results to give new proofs to some of important theorems concerning blocks.

The notation is almost standard: We fix a complete discrete valuation ring R of characteristic 0 with F as its residue class field. We assume that the quotient field of R is a splitting one for every subgroup of G. We let θ denote R or F. By an θG-module M, we understand a right θG-module which is finitely generated free over θ. If M is indecomposable, we denote its vertex by $vx(M)$. For another module N, $N|M$ indicates that N is isomorphic to a direct summand of M and we say "N is a component of M" if N is indecomposable. If n is an integer and p^m is the highest p-power dividing n, then we write $m=\nu(n)$. Finally for a block B of G, we denote by $\delta(B)$ a defect group of B.

1. Sources with θ-rank prime to p

For later convenience, we put down the following well-known fact without proof.

Lemma 1.1. *Let M be an indecomposable θG-module with vertex Q. Let V be an indecomposable θQ-module. Then V is a Q-source of M if and only if $V|M_Q$ and Q is a vertex of V.*

Let M be an indecomposable θG-module. We consider the following condition;

(*) $\qquad p \not| \operatorname{rank}_\theta V \quad$ *for a source* V *of* M.

Theorem 1.2. *Let H be a subgroup of G. Let M be an indecomposable θG-module with vertex Q which satisfies* (*). *Let P be a maximal member of $\{Q^x \cap H \mid x \in G\}$. Then there exists a component N of M_H such that P is a vertex of N and N satisfies* (*).

Proof. We set $P=Q^a \cap H$ $(a \in G)$ and let V be a Q^a-source of M. Then there exists a component W of V_P with $p \not| \operatorname{rank}_\theta W$. Then P is a vertex of W by Green's theorem. We may assume that $V \mid M_{Q^a}$ and hence $W \mid M_P$. Let N be a component of M_H such that $W \mid N_P$. Then $P \subseteq_H vx(N)$. On the other hand, $N \mid M_H$ means that $vx(N) \subseteq_H Q^x \cap H$ for some $x \in G$. Therefore we have $vx(N) =_H P$ by the choice of P. Moreover W is a P-source of N by Lemma 1.1. This completes the proof.

We mention a couple of remarks concerning the condition (*).

REMARK 1.3. Let M be an indecomposable FG-module with cyclic vertex. Then M satisfies (*).

For the proof of this fact, it is sufficient to show the following lemma, which may be, much or less, well-known.

Lemma 1.4. *Let $Q=\langle x \rangle$ be a cyclic group of order p^s. Let M be an arbitrary indecomposable FQ-module. Then M satisfies* (*).

Proof. (Watanabe) We denote by Q_i the subgroup of Q with order p^i $(0 \leq i \leq s)$. For each i, FQ_i has exactly p^i indecomposable modules V_{ij} with $\dim_F V_{ij}=j$ $(1 \leq j \leq p^i)$. Recall that each $M_{ij}=(V_{ij})^Q$ is indecomposable by Green's theorem. Moreover if $(j, p)=1$, then $\nu(\dim_F M_{ij})=\nu(|Q: Q_i|)$. This implies that Q_i is a vertex of M_{ij} and so V_{ij} is a Q_i-source of it. Now we see that the set $\bigcup_{i=0}^{s} \{M_{ij} \mid (j, p)=1,\ 1 \leq j \leq p^i\}$ must be a full set of non-isomorphic indecomposable FQ-modules, since $p^s=\sum_{i=0}^{s} \varphi(p^i)$ (φ denotes the Euler totient function). This completes the proof of Lemma 1.4.

REMARK 1.5 (Knörr [5], Theorem 4.5). Assume that F is algebraically closed. Let M be an indecomposable θG-module. Then if $\nu(\operatorname{rank}_\theta M)=\nu(|G: vx(M)|)$, M satisfies (*).

As an application of Theorem 1.2, we show the following;

Corollary 1.6. *Let H be a normal subgroup of G. Let M be an irreducible*

FG-module and N an irreducible constituent of M_H. Then if $\nu(\dim_F M) = \nu(|G\colon vx(M)|)$, we have $\nu(\dim_F N) = \nu(|H\colon vx(N)|)$.

Proof. For the proof of this result, we may assume that F is algebraically closed. By Theorem 1.2 and Remark 1.5, there exists an irreducible constituent $\hat{N}$ of M_H such that $\hat{N}$ satisfies (*). However, since a source of N and that of $\hat{N}$ are G-conjugate to each other, we have that $\nu(\dim_F N) = \nu(|H\colon vx(N)|)$ by Theorem 4.5 in [5].

As one of typical modules which satisfy (*), let us take what is called a Scott module. For any subgroup X of G, we denote by I_X the trivial θX-module (an θX-module of rank 1 on which X acts trivially). For a p-subgroup Q of G, $(I_Q)^G$ has exactly one component S which contains I_G as a submodule, and then Q is a vertex of S (see Burry [2]). Following Burry, we call S the Scott G-module with vertex Q. The following theorem was suggested by Okuyama.

Theorem 1.7. *Let H be a subgroup of G and S the Scott G-module with vertex Q. Let P be a maximal member of $\{Q^x \cap H \mid x \in G\}$. Then there exists a component U of S_H which is the Scott H-module with vertex P.*

Proof. We prove by the induction on $|Q|/|P|$. If $|Q| = |P|$, our assertion follows immediately from Theorem 2 in [2]. So we assume that $|Q| > |P|$. We set $H_1 = N_G(P)$ and let P_1 be a maximal member of $\Omega = \{Q^x \cap H_1 \mid Q^x \cap H_1 \supsetneqq P,\ x \in G\}$. It is clear that Ω is not empty. Thus by the induction hypothesis, there exists a component U_1 of S_{H_1} which is the Scott H_1-module with vertex P_1. We set $T = N_H(P)$, then there exists a component $\hat{U}$ of $(U_1)_T$ which contains I_T as a submodule. However, since $(P_1)^y \cap T = P$ for all $y \in H_1$, $\{(I_{P_1})^{H_1}\}_T$ is a direct sum of copies of $(I_P)^T$ by Mackey decomposition theorem. Thus $\hat{U}$ must be the Scott T-module with vertex P. Let U be a component of S_H such that $\hat{U} \mid U_T$. Then since P is a vertex of U, U corresponds to $\hat{U}$ in the Green correspondence with respect to (H, P, T). Thus by Theorem 1 in [2], U is the Scott H-module with vertex P.

2. Some applications to block theory

Let H be a subgroup of G and b a block of H. Following Brauer, we call b G-admissible provided $C_G(\delta(b)) \subseteq H$. Note that this does not depend on the particular choice of $\delta(b)$ and b^G is defined. The following theorem was suggested by Okuyama.

Theorem 2.1. *Let b be a G-admissible block of H. If M is an indecomposable θG-module in $B = b^G$ which has $\delta(B)$ as a vertex and satisfies (*), then there exists a component N of M_H which belongs to b and has $\delta(b)$ as a vertex and* *satisfies* (*).

Proof. We prove by the induction on $|\delta(B)|/|\delta(b)|$. If $|\delta(B)|=|\delta(b)|$, our assertion follows immediately from Corollary 9 in [6] and Lemma 1.1. So we assume that $|\delta(B)|>|\delta(b)|$. Let $\hat{b}$ be a root of b in $T=\delta(b)C_G(\delta(b))$. We set $H_1=N_G(\delta(b))$ and $b_1=\hat{b}^{H_1}$. Then $|\delta(b_1)|>|\delta(b)|$ by Brauer's first main theorem and the assumption. Thus by the induction hypothesis, there exists a component N_1 of M_{H_1} in b_1 such that $\delta(b_1)$ is a vertex of N_1 and N_1 satisfies (*). Since $H_1 \triangleright T$, b_1 covers $\hat{b}$. Thus by Theorem 1.2, we can show that there exists a component $\hat{N}$ of $(N_1)_T$ such that $\hat{N}$ belongs to $\hat{b}$ and $vx(\hat{N})={}_{H_1}\delta(b_1)\cap T$. However, since $vx(\hat{N})\subseteq\delta(b)\subseteq\delta(b_1)$, we have that $vx(\hat{N})=\delta(b)$ from the above. Let N be a component of M_H such that $\hat{N}|N_T$. Then $N\in b$ by (3.7a) in [3]. Since $N\in b$ and $\hat{N}|N_T$, $\delta(b)$ is a vertex of N and N satisfies (*) by Lemma 1.1. Thus the proof is complete.

The above theorems allow us to give alternative proofs to some of important results concerning blocks.

Corollary 2.2 (Brauer's third main theorem). *Let b be a G-admissible block of a subgroup of G. If b^G is principal, then b is principal.*

Proof. This is immediate from the above theorem by taking $M=I_G$, the trivial θG-module.

For the proofs of the following corollaries, we may assume that F is algebraically closed.

Corollary 2.3 (Alperin and Burry [1]). *Let Q be a p-subgroup of G and H a subgroup of G such that $H\supseteq QC_G(Q)$. Let B be a block of G. If P is a maximal member of $\{\delta(B)^x\cap H \mid x\in G,\ \delta(B)^x\cap H\supseteq Q\}$, then there exists a block b of H such that $b^G=B$ and P is a defect group of b.*

Proof. Let M be an irreducible FG-module in B of height 0. Then $\nu(\dim_F M)=\nu(|G\colon vx(M)|)$ and $\delta(B)$ is a vertex of M. By Theorem 1.2 and Remark 1.5, there exists a component N of M_H such that P is a vertex of N. Let b be a block of H which contains N. Since $C_G(P)\subseteq H$, b^G is defined and equals to B by (3.7a) in [3]. Furthermore by the maximality of P, we see easily that P is a defect group of b.

Corollary 2.4 (Knörr [4]). *Let H be a normal subgroup of G. Let B be a block of G and b a block of H. If B covers b, then $\delta(b)={}_G\delta(B)\cap H$.*

Proof. Let M be an irreducible FG-module in B of height 0. Then by Theorem 1.2 and Remark 1.5, we can show that there exists a component N of M_H such that N belongs to b and $vx(N)={}_G\delta(B)\cap H$. So we have $\delta(b)\supseteq$

${}_G\delta(B)\cap H$. On the other hand, for an irreducible FH-module N in b with $\delta(b)$ as a vertex, there exists an irreducible FG-module M in B such that $N\,|\,M_H$ (see Proposition 4.1 in [4]). Thus we have $\delta(b)\subseteq_G\delta(B)\cap H$. Combining with the above, $\delta(b)=_G\delta(B)\cap H$ as asserted.

Corollary 2.5. *Let H be a normal subgroup of G. Let B be a block of G and φ an irreducible Brauer character of G in B. If φ has height 0, then any irreducible constituent of φ_H has height 0 in the block of H to which it belongs.*

Proof. This is immediate from Corollary 1.6 and Corollary 2.4.

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