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***On a System of Valuations of Algebraic Function Fields  
with Several Variables***

Nobuo NOBUSAWA

Let  $w$  be a discrete valuation of rank  $n$  of a rational function field  $k(x_1, x_2, \dots, x_n)$  of characteristic 0 and let  $(t_1, t_2, \dots, t_n)$  be a system of prime elements of  $w$ <sup>1)</sup>. We consider a set  $S$  of all  $w$  such that  $t_i \in k(x_i)$  ( $i = 1, 2, \dots, n$ ) and that  $t_i$  are of the first degree except one  $t_j$ . The first property to be proved in this paper is that, if  $u$  is a non zero element of  $k(x_1, x_2, \dots, x_n)$  such that  $w(u) = 0$  for all  $w$  in  $S$ , then  $u$  belongs already to the constant field  $k$ . The next one is the following: The formal power series expansions of  $k(x_1, x_2, \dots, x_n)$  by  $(t_1, t_2, \dots, t_n)$  of  $w$  in  $S$  are performed in such a way that  $k(x_1, x_2, \dots, x_n)$  can be imbedded in  $\bar{k}((t_1, t_2, \dots, t_n))^2$ . It is then our next problem to generalize the results in the case of a finite extension of  $k(x_1, x_2, \dots, x_n)$ . When  $K$  is a finite extension of  $k(x_1, x_2, \dots, x_n)$ , we denote by  $\bar{S}$  the set of all the valuations of  $K$  that are the extensions of  $w$  in  $S$ . The first property is proved to be also true in this case<sup>3)</sup>, although the second one is possible in the unramified case under some conditions. Finally it will be shown that divisors and principal divisors with respect to  $\bar{S}$  will be defined in the similar sense of Lamprecht [1].

**1. Rational function fields.**

Throughout this paper we consider only discrete valuations  $w$  of rank  $n$  of an algebraic function field  $K$  of characteristic 0 with  $n$  variables;

$$w(u) = (\alpha_1, \alpha_2, \dots, \alpha_n) \text{ for non zero elements } u \text{ of } K,$$

where  $\alpha_i$  are rational integers. We assume naturally that, for the elements  $a$  of  $k$ ,  $w(a) = (0, 0, \dots, 0) = 0$ . The order in the value group will be always defined in such a way that  $(\alpha_1, \alpha_2, \dots, \alpha_n) > (\beta_1, \beta_2, \dots, \beta_n)$  if  $\alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}$  and  $\alpha_i > \beta_i$ . A system of prime elements of  $w$

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1) For the definition, see the first section.  
2) For the definition, see the first section. Cf. [2] p. 78.  
3) For a different approach, see [1].

is a system  $(t_1, t_2, \dots, t_n)$  consisting of such elements  $t_i$  that  $w(t_1) = (1, 0, \dots, 0)$ ,  $w(t_2) = (0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $w(t_n) = (0, \dots, 0, 1)$ .  $k[[t_1, t_2, \dots, t_n]]$  is an integral domain consisting of formal power series  $\sum_{i_1, i_2, \dots, i_n \geq 0} a_{i_1, i_2, \dots, i_n} t_1^{i_1} t_2^{i_2} \dots t_n^{i_n}$  where  $a_{i_1, i_2, \dots, i_n} \in k$ , and  $k((t_1, t_2, \dots, t_n))$  is the quotient field of  $k[[t_1, t_2, \dots, t_n]]$ .

In this section we are concerned with a valuation of a rational function field  $k(x_1, x_2, \dots, x_n)$  with  $n$  variables  $x_i$ . Our consideration is also restricted to such a valuation  $w$  of  $k(x_1, x_2, \dots, x_n)$  that, for a system  $(t_1, t_2, \dots, t_n)$  of prime elements of  $w$ ,

- 1)  $t_i \in k(x_i)$  ( $i = 1, 2, \dots, n$ )
- 2)  $t_i = x_i - a_i$  or  $x_i^{-1}$  except one  $t_j$ .

We denote the set of all those valuations  $w$  of  $k(x_1, x_2, \dots, x_n)$  by  $S$ .

**Theorem 1.** *Let  $t_i$  be prime elements of  $k(x_i)$  at some places for  $i = 1, 2, \dots, n$ . If  $t_i$  are of the first degree except one  $t_j$ , that is, satisfy 2), then there exists a unique valuation  $w$  of  $k(x_1, x_2, \dots, x_n)$  such that  $(t_1, t_2, \dots, t_n)$  is a system of prime elements of  $w$ . In this case we can imbed  $k(x_1, x_2, \dots, x_n)$  in  $\bar{k}((t_1, t_2, \dots, t_n))^{(4)}$  by the formal power series expansions by  $(t_1, t_2, \dots, t_n)$ .*

*Proof.* By the valuation theory of algebraic function fields with one variable, we can first imbed  $k(x_j)$  in  $\bar{k}((t_j))$  with respect to  $k(t_j)$  in a unique way within automorphisms of  $\bar{k}/k$ . Since  $\bar{k}((t_j))$  and  $k(t_1, \dots, t_n)$  ( $=k(x_1, \dots, x_{j-1}, t_j, x_{j+1}, \dots, x_n)$ ) are algebraically independent over  $k(t_j)$ ,  $k(x_1, x_2, \dots, x_n)$  can be imbedded isomorphically in  $\bar{k}((t_j))k(t_1, \dots, t_n) \subseteq \bar{k}((t_1, t_2, \dots, t_n))$  with respect to  $k(t_1, t_2, \dots, t_n)$ . It is then clear that there exists a unique valuation of  $\bar{k}((t_1, t_2, \dots, t_n))$  such that  $(t_1, t_2, \dots, t_n)$  is a system of prime elements of it.

We denote the valuation of Theorem 1 by  $w_{(t_1, t_2, \dots, t_n)}$ .

**Theorem 2.** *Let  $f(x_1, x_2, \dots, x_n)$  and  $g(x_1, x_2, \dots, x_n)$  be two polynomials in  $k[x_1, x_2, \dots, x_n]$ . If  $\frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)} \notin k$ , then there exists a valuation  $w$  in  $S$  such that  $w\left(\frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)}\right) \neq 0$ .*

*Proof.* This theorem is a direct consequence of next Theorem 3.

**Theorem 3.** *Under the same assumption as in Theorem 2, there exist*

4) We can always imbed  $k(x_1, x_2, \dots, x_n)$  in  $\bar{k}\{t_1\}\{t_2\}\dots\{t_n\}$  with a system  $(t_1, t_2, \dots, t_n)$  of prime elements of any valuation. But  $\bar{k}((t_1, t_2, \dots, t_n)) \subseteq \bar{k}\{t_1\}\{t_2\}\dots\{t_n\}$ . Cf. [3].

valuations  $w$  and  $w'$  in  $S$  such that  $w\left(\frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)}\right) > 0$  and  $w'\left(\frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)}\right) < 0$ .

Proof. We shall prove Theorem 3 by induction. It is clear for  $n = 1$ . First assume that  $f(0, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n)$ , that is,  $f(x_1, x_2, \dots, x_n) \in k[x_2, x_3, \dots, x_n]$ . If moreover  $g(0, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n)$ , then by induction hypothesis there exist such valuations  $w_0$  and  $w'_0$  of  $k(x_2, x_3, \dots, x_n)$  that  $w_0\left(\frac{f(0, x_2, \dots, x_n)}{g(0, x_2, \dots, x_n)}\right) > 0$  and  $w'_0\left(\frac{f(0, x_2, \dots, x_n)}{g(0, x_2, \dots, x_n)}\right) < 0$ . If  $w_0 = w_{(t_2, t_3, \dots, t_n)}$  and  $w'_0 = w_{(t'_2, t'_3, \dots, t'_n)}$ , then  $w = w_{(x_1, t_2, \dots, t_n)}$  and  $w' = w_{(x_1, t'_2, \dots, t'_n)}$  are required valuations. Hence we may assume that  $f(0, x_2, \dots, x_n) \neq f(x_1, x_2, \dots, x_n)$  or  $g(0, x_2, \dots, x_n) \neq g(x_1, x_2, \dots, x_n)$ . Since  $\frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)} \notin k$ , there exist  $a_i$  ( $i = 2, 3, \dots, n$ ) in  $k$  such that  $g(x_1, a_2, \dots, a_n) \neq 0$  and  $\frac{f(x_1, a_2, \dots, a_n)}{g(x_1, a_2, \dots, a_n)} \notin k$ . If  $t_1$  and  $t'_1$  are prime elements of valuations  $w_1$  and  $w'_1$  of  $k(x_1)$  such that  $w_1\left(\frac{f(x_1, a_2, \dots, a_n)}{g(x_1, a_2, \dots, a_n)}\right) > 0$  and  $w'_1\left(\frac{f(x_1, a_2, \dots, a_n)}{g(x_1, a_2, \dots, a_n)}\right) < 0$ , then  $w = w_{(t_1, x_2-a_2, \dots, x_n-a_n)}$  and  $w' = w_{(t'_1, x_2-a_2, \dots, x_n-a_n)}$  are required valuations<sup>5)</sup>.

REMARK.  $S$  is not a minimal system which satisfies Theorem 2. For example, let  $n = 2$  and let  $S'$  be a subset of  $S$  consisting of valuations  $w_{(t_1, x_2-a)}$  and  $w_{(x_1, t_2)}$  where  $a$  are all elements of  $k$  and  $t_1$  and  $t_2$  are all the prime elements of  $k(x_1)$  and  $k(x_2)$ . Then it is seen from the proof of Theorem 3 that Theorem 3 holds with respect  $S'$  instead of  $S$ . We shall give another example in which Theorem 2 does not hold. Let  $n = 2$  and let  $S''$  be a subset of  $S'$  consisting of all the valuations  $w_{(t_1, x_2)}$  and  $w_{(x_1, t_2)}$ . Put  $u = \frac{1+x_1x_2}{1+2x_1x_2}$ . Then  $w_{(t_1, x_2)}(1+x_1x_2) = w_{(t_1, x_2)}(1+2x_1x_2) = 0$ , that is,  $w_{(t_1, x_2)}(u) = 0$ . If  $t_2 \neq x_2^{-1}$ ,  $w_{(x_1, t_2)}(1+x_1x_2) = w_{(x_1, t_2)}(1+2x_1x_2) = 0$ , that is,  $w_{(x_1, t_2)}(u) = 0$ . Lastly  $w_{(x_1, x_2^{-1})}\left(\frac{1+x_1x_2}{1+2x_1x_2}\right) = (1, -1) - (1, -1) = 0$ . Hence, for all  $w$  in  $S''$ ,  $w(u) = 0$  in spite of the fact  $u \notin k$ .

**2. Finite extensions of rational function fields.**

Let  $K$  be a finite extension of a rational function field  $k(x_1, x_2, \dots, x_n)$

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5) Note that  $f(x_1, a_2, \dots, a_n)$  is the term which contains only elements of  $k$  and  $x_1$  in the expansion of  $f(x_1, x_2, \dots, x_n)$  by  $((x_2 - a_2), (x_3 - a_3), \dots, (x_n - a_n))$ .

where  $k = K \cap \bar{k}$ . The set of all the valuations of  $K$  that are the extensions of the valuations in  $S$  will be denoted by  $\bar{S}$ .

**Theorem 4.** *If  $s$  is a non constant element of  $K$ , then there exist  $\bar{w}$  and  $\bar{w}'$  in  $\bar{S}$  such that  $\bar{w}(s) > 0$  and  $\bar{w}'(s) < 0$ .*

Proof. Let an irreducible equation of  $s$  over  $k(x_1, x_2, \dots, x_n)$  be

$$F(z) = z^m + u_1 z^{m-1} + \dots + u_m = 0,$$

where  $u_i \in k(x_1, x_2, \dots, x_n)$  ( $i = 1, 2, \dots, n$ ) and  $u_i \notin k$  for some  $i$ . By Theorem 3 there exists  $w'$  in  $S$  such that  $w'(u_i) < 0$ . We shall first prove that, for one of the extensions  $\bar{w}'$  of  $w'$ ,  $\bar{w}'(s) < 0$ . We split  $F(z)$  into irreducible factors in  $(\bar{k}\{t_1\}\{t_2\} \dots \{t_n\})(z)^{(6)}$ . Here  $(t_1, t_2, \dots, t_n)$  is a system of prime elements of  $w'$ .

$$(1) \quad F(z) = (z^{m_1} + u'_1 z^{m_1-1} + \dots + u'_{m_1})(z^{m_2} + u''_1 z^{m_2-1} + \dots + u''_{m_2}) \dots (z^{m_r} + u^{(r)}_1 z^{m_r-1} + \dots + u^{(r)}_{m_r}),$$

where  $u_i^{(j)} \in \bar{k}\{t_1\}\{t_2\} \dots \{t_n\}$ . If  $w'(u_{m_i}^{(i)}) \geq 0$  for all  $u_{m_i}^{(i)}$ , then  $w'(u_i^{(j)}) \geq 0$  for all  $u_i^{(j)}$ , and hence  $w'(u_i) \geq 0$  for all  $u_i$ . This is a contradiction. Thus  $w'(u_{m_i}^{(i)}) < 0$  for some  $i$ , in other words, for one of the extensions  $\bar{w}'$  of  $w'$ ,  $\bar{w}'(s) < 0$ <sup>(8)</sup>. Next we shall prove the existence of  $\bar{w}$  in Theorem 4. If  $u_m \notin k$ , then by Theorem 3 there exist  $w$  of  $k(x_1, x_2, \dots, x_n)$  such that  $w(u_m) > 0$ . Then  $w(u_{m_i}^{(i)}) > 0$  for some  $u_{m_i}^{(i)}$  in (1). One of the extensions of  $w$  has then a required property. If  $u_m \in k$ , put

$$z' = \frac{1}{z} \text{ and we have instead of } F(z) = 0$$

$$F'(z') = z'^m + \frac{u_{m-1}}{u_m} z'^{m-1} + \dots + \frac{1}{u_m} = 0.$$

We can conclude from the previous discussion that, for  $s' = \frac{1}{s}$ , there exists one of the extensions  $\bar{w}'$  of  $w'$  having the property  $\bar{w}'(s') < 0$ , that is,  $\bar{w}'(s) > 0$ . The proof is completed.

Let  $K$  be a simple extension of  $k(x_1, x_2, \dots, x_n)$  with a primitive element  $s$  and let an irreducible equation of  $s$  be

$$F(z) = z^m + r_1(x_1, x_2, \dots, x_n) z^{m-1} + \dots + r_m(x_1, x_2, \dots, x_n) = 0,$$

6) Cf. footnote 4).

7) Note that Hensel's lemma holds in  $\bar{k}\{t_1\}\{t_2\} \dots \{t_n\}$ . Cf. [3].

8)  $s$  is naturally algebraic over  $\bar{k}\{t_1\}\{t_2\} \dots \{t_n\}$  and the above discussion implies that, for one of the extensions  $\bar{w}'$  of  $w'$  in  $(\bar{k}\{t_1\}\{t_2\} \dots \{t_n\})(s)$ ,  $\bar{w}'(s) < 0$ . Then, for the restriction  $\bar{w}'$  of  $\bar{w}'$  to  $k(x_1, x_2, \dots, x_n)(s)$ ,  $\bar{w}'(s) < 0$  and  $\bar{w}'$  is an extension of  $w'$ .

where  $r_i(x_1, x_2, \dots, x_n) \in k(x_1, x_2, \dots, x_n)$ . We denote by  $D$  the resultant of  $F(z)$ . Let  $w = w_{[t_1, t_2, \dots, t_n]} \in S$ .

**Theorem 5.** *Under the following assumptions :*

- i)  $r_i(x_1, x_2, \dots, x_n) \in \bar{k}[[t_1, t_2, \dots, t_n]]$
- ii)  $w(D) = 0$
- iii)  $w(r_m(x_1, x_2, \dots, x_n)) = 0$ ,

$w$  is unramified in  $K/k(x_1, x_2, \dots, x_n)$  and we have  $K \subseteq \bar{k}((t_1, t_2, \dots, t_n))$  by the formal power series expansions of  $K$  by  $(t_1, t_2, \dots, t_n)$ .

Proof. By i) we may put  $r_i(x_1, x_2, \dots, x_n) = r_i(t_1, t_2, \dots, t_n) \in \bar{k}[[t_1, t_2, \dots, t_n]]$ , and

$$F(z) = z^m + r_1(t_1, t_2, \dots, t_n) z^{m-1} + \dots + r_m(t_1, t_2, \dots, t_n) = 0.$$

It is sufficient to prove that  $n$  distinct roots of  $F(z) = 0$  are found in  $\bar{k}[[t_1, t_2, \dots, t_n]]$  and are all units in  $\bar{k}[[t_1, t_2, \dots, t_n]]$ . We shall prove this by induction. First assume  $n = 1$ . The roots of  $F(z) = 0$  will be found in the following way. We denote by  $F_{t_1=0}(z)$  a polynomial obtained from  $F(z)$  by putting  $t_1 = 0$ . ii) implies  $D_{t_1=0} \neq 0$  and hence  $F_{t_1=0}(z) = 0$  has  $n$  distinct roots  $a_0$  in  $\bar{k}$ . iii) implies that none of  $a_0$  is zero. Then we put  $z = a_0 + a_1 t_1$  in  $F(z)$  and we shall define  $a_1$  such that the term of  $t_1$  may disappear. The coefficient of  $t_1$  in  $F(a_0 + a_1 t_1)$  is

$$ma_0^{m-1} a_1 + r_1(0)(m-1) a_0^{m-2} a_1 + \dots + r_{m-1}(0) a_1 + a,$$

where  $a$  is defined only by  $F(z)$  and  $a_0$ . However we have

$$F'_{t_1=0}(a_0) = ma_0^{m-1} + r_1(0)(m-1) a_0^{m-2} + \dots + r_{m-1}(0) \neq 0,$$

since  $D_{t_1=0} \neq 0$  and  $a_0$  is roots of  $F_{t_1=0}(z) = 0$ . Therefore  $a_1 = -a(F'_{t_1=0}(a_0))^{-1}$ . Next we put  $z = a_0 + a_1 t_1 + a_2 t_1^2$  and shall define  $a_2$  in  $\bar{k}$  such that the term of  $t_1^2$  in  $F(z)$  will disappear. This is done by the same method as above. This process is continued to obtain  $z = a_0 + a_1 t_1 + a_2 t_1^2 + \dots$  which are  $n$  distinct roots of  $F(z) = 0$  in  $\bar{k}[[t_1]]$  and are all units in  $\bar{k}[[t_1]]$  as desired. Now let us return to the general case. It is easily proved that the assumptions i), ii) and iii) are satisfied in  $F_{t_n=0}(z) = 0$ . By the induction hypothesis, all the roots  $f_0(t_1, t_2, \dots, t_{n-1})$  are units in  $\bar{k}[[t_1, t_2, \dots, t_{n-1}]]$ . Put  $z = f_0 + f_1 t_n$  where  $f_1 \in \bar{k}[[t_1, t_2, \dots, t_{n-1}]]$ . We shall define  $f_1$  in such a way that the term of  $t_n$  in  $F(z)$  will disappear. The coefficient of  $t_n$  in  $F(z)$  is  $mf_0^{m-1} f_1 + r_1(t_1, \dots, t_{n-1}, 0)$

$(m-1)f_0^{m-2}f_1 + \dots + r_{m-1}(t_1, \dots, t_{n-1}, 0)f_1 + f$  where  $f$  is an element of  $\bar{k}[[t_1, t_2, \dots, t_n]]$  defined only by  $F(z)$  and  $f_0$ . Since

$$F'_{t_n=0}(f_0) = mf_0^{m-1} + r_1(t_1, \dots, t_{n-1}, 0)(m-1)f_0^{m-2} + \dots + r_{m-1}(t_1, \dots, t_{n-1}, 0)$$

are units in  $\bar{k}[[t_1, t_2, \dots, t_{n-1}]]$ , we put  $f_1 = -f(F'_{t_n=0}(f_0))^{-1}$ . In the same manner we can define  $f_2, f_3, \dots$  such that  $z = f_0 + f_1t_n + f_2t_n^2 + \dots$  are  $n$  distinct roots of  $F(z) = 0$  in  $\bar{k}[[t_1, t_2, \dots, t_n]]$  and are all units. Thus the proof is completed.

**3. Divisors and principal divisors.**

We shall give in this section a definition of divisors with respect  $\bar{S}$  in the similar way as in [1]. For the sake of simplicity we assume  $n = 2$ . We can easily generalize the result for  $n \neq 2$ .

**Lemma.** *Let  $u \in k(x_1, x_2)$  and  $w_{[t_1, t_2]}(u) = (\alpha_1, \alpha_2)$  for  $w_{[t_1, t_2]}$  in  $S$ . Then  $\alpha_2 \neq 0$  for at most a finite number of  $t_2$ , and  $\alpha_1 \neq 0$  for at most a finite number of  $t_1$  when  $t_2$  is fixed<sup>9)</sup>.*

Proof.  $t_2$  can be considered to be also prime elements at some places  $\mathfrak{p}_{t_2}$  in  $k(x_1, x_2)/k(x_1)$ , and, for a finite number of  $\mathfrak{p}_{t_2}$ ,  $w_{[t_2]}(u) \neq 0$ , that is, the first part of Lemma holds<sup>10)</sup>. Next we imbed  $k(x_1, x_2)$  in  $\bar{k}((x_1, t_2))$  and get  $u = \frac{f_1(x_1, t_2)}{f_2(x_1, t_2)}$  where  $f_i(x_1, t_2) \in \bar{k}[[x_1, t_2]]$ . Then  $w_{[t_1, t_2]}(u) = w_{[t_1, t_2]}(f_1(x_1, t_2)) - w_{[t_1, t_2]}(f_2(x_1, t_2)) = (\alpha'_1, \alpha'_2) - (\alpha''_1, \alpha''_2)$ <sup>11)</sup>. But it is easy to see that  $\alpha'_1 \neq 0$  and  $\alpha''_1 \neq 0$  for a finite number of  $t_1$ .

Let  $K$  be a finite extension of  $k(x_1, x_2)$  and denote by  $w_{[t_1, t_2]}^{(K)}$  the extensions of  $w_{[t_1, t_2]}$  in  $K$ .

**Theorem 6.** *Let  $w_{[t_1, t_2]}^{(K)}(s) = (\alpha_1, \alpha_2)$  for an element  $s$  of  $K$ . Then  $\alpha_2 \neq 0$  for at most a finite number of  $t_2$ , and  $\alpha_1 \neq 0$  for at most a finite number of  $t_1$  when  $t_2$  is fixed.*

9) In this section we assume naturally that prime elements of the same place are the same.

10) We can show that, if  $w_{[t_1, t_2]}(u) = (\alpha_1, \alpha_2)$ , then  $w_{[t_2]}(u) = \alpha_2$ .

11) More exactly, if  $t_2 \neq x_2^{-1}$ ,  $u = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$  with  $f_i(x_1, x_2) \in k[x_1, x_2]$  and  $f_i(x_1, x_2) = f_i(x_1, t_2) \in \bar{k}[x_1][[t_2]]$ , and, if  $t_2 = x_2^{-1}$ ,  $u = \frac{f_1(x_1, x_2^{-1})}{f_2(x_1, x_2^{-1})}$  with  $f_i(x_1, x_2^{-1}) \in k[x_1, x_2^{-1}]$  and  $f_i(x_1, x_2^{-1}) = f_i(x_1, t_2) \in k[x_1, t_2]$ . If we denote by  $f_i^0(x_1, t_2)$  the lowest term of  $f_i(x_1, t_2)$  with respect to  $t_2$ , then  $f_i^0(x_1, 1) \in \bar{k}[x_1]$  and  $w_{[t_1]}(f_i^0(x_1, 1)) = \alpha_i$ . Here we must notice that  $f_i(x_1, 1) \in k[x_1]$  when  $t_1$  is not of the first degree.

Proof. Let the equation of  $s$  be

$$s^m + u_1 s^{m-1} + \dots + u_m = 0,$$

where  $u_j \in k(x_1, x_2)$ . Let  $w_{[t_1, t_2]}(u_j) = (\alpha_1^{(j)}, \alpha_2^{(j)})$ . It is clear from the previous lemma that  $\alpha_2^{(j)} = 0$  ( $j = 1, 2, \dots, m$ ) for almost all  $t_2$ . If  $\alpha_2 \neq 0$ , then  $w_{[t_1, t_2]}(u_i s^{m-i})$  ( $i = 0, 1, \dots, m$ ) are all different, which is a contradiction. Hence  $\alpha_2 = 0$  for almost all  $t_2$ . When  $t_2$  is fixed,  $w_{[t_1, t_2]}(u_j) = (0, \alpha_2^{(j)})$  ( $j = 1, 2, \dots, m$ ) for almost all  $t_1$  by the previous lemma. Then we can show by the similar way that  $w_{[t_1, t_2]}(s) = (0, \alpha_2)$  for almost all  $t_1$ .

Now we give a definition of divisors with respect  $\bar{S}$ . We mean by a *divisor* a mapping of  $\bar{S}$  into the value group of the valuations:

$$\mathfrak{A} = \{w_{[t_1, t_2]}^{(i)} \rightarrow (\alpha_1, \alpha_2)_{t_1, t_2; i}\},$$

where two next conditions are satisfied:

- 1)  $\alpha_2 \neq 0$  for at most a finite number of  $t_2$ ,
- 2)  $\alpha_1 \neq 0$  for at most a finite number of  $t_1$  when  $t_2$  is fixed.

Theorem 5 implies that, if  $s$  is a non zero element of  $K$ ,

$$(s) = \{w_{[t_1, t_2]}^{(i)} \rightarrow w_{[t_1, t_2]}^{(i)}(s)\}$$

is a divisor. We call this divisor a *principal divisor*. We now define addition of two divisors  $\mathfrak{A}$  and  $\mathfrak{B}$ :

$$\begin{aligned} \mathfrak{A} &= \{w_{[t_1, t_2]}^{(i)} \rightarrow (\alpha_1, \alpha_2)_{t_1, t_2; i}\} \\ \mathfrak{B} &= \{w_{[t_1, t_2]}^{(i)} \rightarrow (\beta_1, \beta_2)_{t_1, t_2; i}\} \end{aligned}$$

such as

$$\mathfrak{A} + \mathfrak{B} = \{w_{[t_1, t_2]}^{(i)} \rightarrow (\alpha_1 + \beta_1, \alpha_2 + \beta_2)_{t_1, t_2; i}\}.$$

Then clearly

$$(s_1 s_2) = (s_1) + (s_2).$$

Thus all divisors form an additive group  $D$  and all principal divisors form a subgroup  $H$  of  $D$ . And

$$s \rightarrow (s)$$

gives a homomorphic mapping of the multiplicative group  $K^*$  with the additive group  $H$ . The kernel of this homomorphism is  $k^*$  by Theorem 2.

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