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Author(s)	Nobusawa, Nobuo
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Nobusawa, Nobuo
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On a System of Valuations of Algebraic Function Fields with Several Variables

Nobuo NOBUSAWA

Let w be a discrete valuation of rank n of a rational function field $k(x_1, x_2, \cdots, x_n)$ of characteristic 0 and let $(t_1, t_2, \cdots, t_n)$ be a system of prime elements of w[1]. We consider a set S of all w such that $t_i \in k(x_i)$ $(i = 1, 2, \cdots, n)$ and that t_i are of the first degree except one t_j. The first property to be proved in this paper is that, if u is a non zero element of $k(x_1, x_2, \cdots, x_n)$ such that $w(u) = 0$ for all w in S, then u belongs already to the constant field k. The next one is the following: The formal power series expansions of $k(x_1, x_2, \cdots, x_n)$ by $(t_1, t_2, \cdots, t_n)$ of w in S are performed in such a way that $k(x_1, x_2, \cdots, x_n)$ can be imbedded in $\bar{k}((t_1, t_2, \cdots, t_n))$[2]. It is then our next problem to generalize the results in the case of a finite extension of $k(x_1, x_2, \cdots, x_n)$. When K is a finite extension of $k(x_1, x_2, \cdots, x_n)$, we denote by $\bar{S}$ the set of all the valuations of K that are the extensions of w in S. The first property is proved to be also true in this case[3], although the second one is possible in the unramified case under some conditions. Finally it will be shown that divisors and principal divisors with respect to $\bar{S}$ will be defined in the similar sense of Lamprecht [1].

1. Rational function fields.

Throughout this paper we consider only discrete valuations w of rank n of an algebraic function field K of characteristic 0 with n variables;

$$w(u) = (\alpha_1, \alpha_2, \cdots, \alpha_n) \text{ for non zero elements } u \text{ of } K,$$

where α_i are rational integers. We assume naturally that, for the elements a of k, $w(a) = (0, 0, \cdots, 0) = 0$. The order in the value group will be always defined in such a way that $(\alpha_1, \alpha_2, \cdots, \alpha_n) > (\beta_1, \beta_2, \cdots, \beta_n)$ if $\alpha_n = \beta_n, \cdots, \alpha_{i+1} = \beta_{i+1}$ and $\alpha_i > \beta_i$. *A system of prime elements of w*

1) For the definition, see the first section.

2) For the definition, see the first section. Cf. [2] p. 78.

3) For a different approach, see [1].

is a system $(t_1, t_2, \cdots, t_n)$ consisting of such elements t_i that $w(t_1) = (1, 0, \cdots, 0)$, $w(t_2) = (0, 1, 0, \cdots, 0)$, $\cdots$, $w(t_n) = (0, \cdots, 0, 1)$. $k[[t_1, t_2, \cdots, t_n]]$ is an integral domain consisting of formal power series $\sum_{i_1, i_2, \cdots i_n \geqq 0} a_{i_1, i_2, \dots, i_n} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ where $a_{i_1, i_2, \dots, i_n} \in k$, and $k((t_1, t_2, \cdots, t_n))$ is the quotient field of $k[[t_1, t_2, \cdots, t_n]]$.

In this section we are conserned with a valuation of a rational function field $k(x_1, x_2, \cdots, x_n)$ with n variables x_i. Our consideration is also restricted to such a valuation w of $k(x_1, x_2, \cdots, x_n)$ that, for a system $(t_1, t_2, \cdots, t_n)$ of prime elements of w,

1) $t_i \in k(x_i)$ $(i = 1, 2, \cdots, n)$
2) $t_i = x_i - a_i$ or x_i^{-1} except one t_j.

We denote the set of all those valuations w of $k(x_1, x_2, \cdots, x_n)$ by S.

Theorem 1. *Let t_i be prime elements of $k(x_i)$ at some places for $i = 1, 2, \cdots, n$. If t_i are of the first degree except one t_j, that is, satisfy 2), then there exists a unique valuation w of $k(x_1, x_2, \cdots, x_n)$ such that $(t_1, t_2, \cdots, t_n)$ is a system of prime elements of w. In this case we can imbed $k(x_1, x_2, \cdots, x_n)$ in $\bar{k}((t_1, t_2, \cdots, t_n))$[4] by the formal power series expasions by $(t_1, t_2, \cdots, t_n)$.*

Proof. By the valuation theory of algebraic function fields with one variable, we can first imbed $k(x_j)$ in $\bar{k}((t_j))$ with respect to $k(t_j)$ in a unique way within automorphisms of $\bar{k}/k$. Since $\bar{k}((t_j))$ and $k(t_1, \cdots, t_n)$ $(= k(x_1, \cdots, x_{j-1}, t_j, x_{j+1}, \cdots, x_n))$ are algebraically independent over $k(t_j)$, $k(x_1, x_2, \cdots, x_n)$ can be imbedded isomorphically in $\bar{k}((t_j))\, k(t_1, \cdots, t_n) \subseteq \bar{k}((t_1, t_2, \cdots, t_n))$ with respect to $k(t_1, t_2, \cdots, t_n)$. It is then clear that there exists a unique valuation of $\bar{k}((t_1, t_2, \cdots, t_n))$ such that $(t_1, t_2, \cdots, t_n)$ is a system of prime elements of it.

We denote the valuation of Theorem 1 by $w_{(t_1, t_2, \dots, t_n)}$.

Theorem 2. *Let $f(x_1, x_2, \cdots, x_n)$ and $g(x_1, x_2, \cdots, x_n)$ be two polynomials in $k[x_1, x_2, \cdots, x_n]$. If $\frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_n)} \notin k$, then there exists a valuation w in S such that $w\left(\frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_n)}\right) \neq 0$.*

Proof. This theorem is a direct consequence of next Theorem 3.

Theorem 3. *Under the same assumption as in Theorem 2, there exist*

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4) We can always imbed $k(x_1, x_2, \cdots, x_n)$ in $\bar{k}\{t_1\}\{t_2\} \cdots \{t_n\}$ with a system $(t_1, t_2, \cdots, t_n)$ of prime elements of any valuation. But $\bar{k}((t_1, t_2, \cdots, t_n)) \subsetneqq \bar{k}\{t_1\}\{t_2\} \cdots \{t_n\}$. Cf. [3].

valuations w *and* w' *in* S *such that* $w\left(\frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_n)}\right)>0$ *and* $w'\left(\frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_n)}\right)<0$.

Proof. We shall prove Theorem 3 by induction. It is clear for $n=1$. First assume that $f(0, x_2, \cdots, x_n)=f(x_1, x_2, \cdots, x_n)$, that is, $f(x_1, x_2, \cdots, x_n)\in k[x_2, x_3, \cdots, x_n]$. If moreover $g(0, x_2, \cdots, x_n)=g(x_1, x_2, \cdots, x_n)$, then by induction hypothesis there exist such valuations w_0 and w_0' of $k(x_2, x_3, \cdots, x_n)$ that $w_0\left(\frac{f(0, x_2, \cdots, x_n)}{g(0, x_2, \cdots, x_n)}\right)>0$ and $w_0'\left(\frac{f(0, x_2, \cdots, x_n)}{g(0, x_2, \cdots, x_n)}\right)<0$. If $w_0=w_{[t_2, t_3, \cdots, t_n]}$ and $w_0'=w_{[t_2', t_3', \cdots, t_n']}$, then $w=w_{[x_1, t_2, \cdots, t_n]}$ and $w'=w_{[x_1, t_2', \cdots, t_n']}$ are required valuations. Hence we may assume that $f(0, x_2, \cdots, x_n)\neq f(x_1, x_2, \cdots, x_n)$ or $g(0, x_2, \cdots, x_n)\neq g(x_1, x_2, \cdots, x_n)$. Since $\frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_n)}\notin k$, there exist a_i $(i=2, 3, \cdots, n)$ in k such that $g(x_1, a_2, \cdots, a_n)\neq 0$ and $\frac{f(x_1, a_2, \cdots, a_n)}{g(x_1, a_2, \cdots, a_n)}\notin k$. If t_1 and t_1' are prime elements of valuations w_1 and w_1' of $k(x_1)$ such that $w_1\left(\frac{f(x_1, a_2, \cdots, a_n)}{g(x_1, a_2, \cdots, a_n)}\right)>0$ and $w_1'\left(\frac{f(x_1, a_2, \cdots, a_n)}{g(x_1, a_2, \cdots, a_n)}\right)<0$, then $w=w_{[t_1, x_2-a_2, \cdots, x_n-a_n]}$ and $w'=w_{[t_1', x_2-a_2, \cdots, x_n-a_n]}$ are required valuations[5].

Remark. S is not a minimal system which satisfies Theorem 2. For example, let $n=2$ and let S' be a subset of S consisting of valuations $w_{[t_1, x_2-a]}$ and $w_{[x_1, t_2]}$ where a are all elements of k and t_1 and t_2 are all the prime elements of $k(x_1)$ and $k(x_2)$. Then it is seen from the proof of Theorem 3 that Theorem 3 holds with respect S' instead of S. We shall give another example in which Theorem 2 does not hold. Let $n=2$ and let S'' be a subset of S' consisting of all the valuations $w_{[t_1, x_2]}$ and $w_{[x_1, t_2]}$. Put $u=\frac{1+x_1x_2}{1+2x_1x_2}$. Then $w_{[t_1, x_2]}(1+x_1x_2)=w_{[t_1, x_2]}(1+2x_1x_2)=0$, that is, $w_{[t_1, x_2]}(u)=0$. If $t_2\neq x_2^{-1}$, $w_{[x_1, t_2]}(1+x_1x_2)=w_{[x_1, t_2]}(1+2x_1x_2)=0$, that is, $w_{[x_1, t_2]}(u)=0$. Lastly $w_{[x_1, x_2^{-1}]}\left(\frac{1+x_1x_2}{1+2x_1x_2}\right)=(1, -1)-(1, -1)=0$. Hence, for all w in S'', $w(u)=0$ in spite of the fact $u\notin k$.

2. Finite extensions of rational function fields.

Let K be a finite extension of a rational function field $k(x_1, x_2, \cdots, x_n)$

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5) Note that $f(x_1, a_2, \cdots, a_n)$ is the term which contains only elements of k and x_1 in the expansion of $f(x_1, x_2, \cdots, x_n)$ by $((x_2-a_2), (x_3-a_3), \cdots, (x_n-a_n))$.

where $k = K \cap \bar{k}$. The set of all the valuations of K that are the extensions of the valuations in S will be denoted by $\bar{S}$.

Theorem 4. *If s is a non constant element of K, then there exist $\bar{w}$ and $\bar{w}'$ in $\bar{S}$ such that $\bar{w}(s) > 0$ and $\bar{w}'(s) < 0$.*

Proof. Let an irreducible equation of s over $k(x_1, x_2, \cdots, x_n)$ be

$$F(z) = z^m + u_1 z^{m-1} + \cdots + u_m = 0,$$

where $u_i \in k(x_1, x_2, \cdots, x_n)$ $(i = 1, 2, \cdots, n)$ and $u_i \notin k$ for some i. By Theorem 3 there exists w' in S such that $w'(u_i) < 0$. We shall first prove that, for one of the extensions $\bar{w}'$ of w', $\bar{w}'(s) < 0$. We split $F(z)$ into irreducible factors in $(\bar{k}\{t_1\}\{t_2\} \cdots \{t_n\})(z)$[6]. Here $(t_1, t_2, \cdots, t_n)$ is a system of prime elements of w'.

$$(1) \quad F(z) = (z^{m_1} + u_1' z^{m_1-1} + \cdots + u'_{m_1})(z^{m_2} + u_1'' z^{m_2-1} + \cdots + u''_{m_2}) \cdots (z^{m_r} + u_1^{(r)} z^{m_r-1} + \cdots + u^{(r)}_{m_r}),$$

where $u_i^{(j)} \in \bar{k}\{t_1\}\{t_2\} \cdots \{t_n\}$. If $w'(u^{(i)}_{m_i}) \geqq 0$ for all $u^{(i)}_{m_i}$, then $w'(u_i^{(j)}) \geqq 0$ for all $u_i^{(j)}$[7], and hence $w'(u_i) \geqq 0$ for all u_i. This is a contradiction. Thus $w'(u^{(i)}_{m_i}) < 0$ for some i, in other words, for one of the extensions $\bar{w}'$ of w', $\bar{w}'(s) < 0$[8]. Next we shall prove the existence of $\bar{w}$ in Theorem 4. If $u_m \notin k$, then by Theorem 3 there exist w of $k(x_1, x_2, \cdots, x_n)$ such that $w(u_m) > 0$. Then $w(u^{(i)}_{m_i}) > 0$ for some $u^{(i)}_{m_i}$ in (1). One of the extensions of w has then a required property. If $u_m \in k$, put $z' = \frac{1}{z}$ and we have instead of $F(z) = 0$

$$F'(z') = z'^m + \frac{u_{m-1}}{u_m} z'^{m-1} + \cdots + \frac{1}{u_m} = 0.$$

We can conclude from the previous discussion that, for $s' = \frac{1}{s}$, there exists one of the extensions $\bar{w}'$ of w' having the property $\bar{w}'(s') < 0$, that is, $\bar{w}'(s) > 0$. The proof is completed.

Let K be a simple extension of $k(x_1, x_2, \cdots x_n)$ with a primitive element s and let an irreducible equation of s be

$$F(z) = z^m + r_1(x_1, x_2, \cdots, x_n) z^{m-1} + \cdots + r_m(x_1, x_2, \cdots, x_n) = 0,$$

6) Cf. footnote 4).

7) Note that Hensel's lemma holds in $\bar{k}\{t_1\}\{t_2\} \cdots \{t_n\}$. Cf. [3].

8) s is naturally algebraic over $\bar{k}\{t_1\}\{t_2\} \cdots \{t_n\}$ and the above discussion implies that, for one of the extensions $\bar{\bar{w}}'$ of w' in $(\bar{k}\{t_1\}\{t_2\} \cdots \{t_n\})(s)$, $\bar{\bar{w}}'(s) < 0$. Then, for the restriction $\bar{w}'$ of $\bar{\bar{w}}'$ to $k(x_1, x_2, \cdots, x_n)(s)$, $\bar{w}'(s) < 0$ and $\bar{w}'$ is an extension of w'.

where $r_i(x_1, x_2, \cdots, x_n) \in k(x_1, x_2, \cdots, x_n)$. We denote by D the resultant of $F(z)$. Let $w = w_{[t_1, t_2, \cdots t_n]} \in S$.

Theorem 5. *Under the following assumptions:*

i) $r_i(x_1, x_2, \cdots, x_n) \in \bar{k}[[t_1, t_2, \cdots, t_n]]$

ii) $w(D) = 0$

iii) $w(r_m(x_1, x_2, \cdots, x_n)) = 0$,

w is unramified in $K/k(x_1, x_2, \cdots, x_n)$ and we have $K \subseteq \bar{k}((t_1, t_2, \cdots, t_n))$ by the formal power series expansions of K by $(t_1, t_2, \cdots, t_n)$.

Proof. By i) we may put $r_i(x_1, x_2, \cdots, x_n) = r(t_1, t_2, \cdots, t_n) \in \bar{k}[[t_1, t_2, \cdots, t_n]]$, and

$$F(z) = z^m + r_1(t_1, t_2, \cdots, t_n)\, z^{m-1} + \cdots + r_m(t_1, t_2, \cdots, t_n) = 0\,.$$

It is sufficient to prove that n distinct roots of $F(z) = 0$ are found in $\bar{k}[[t_1, t_2, \cdots, t_n]]$ and are all units in $\bar{k}[[t_1, t_2, \cdots, t_n]]$. We shall prove this by induction. First assume $n=1$. The roots of $F(z)=0$ will be found in the following way. We denote by $F_{t_1=0}(z)$ a polynomial obtained from $F(z)$ by putting $t_1=0$. ii) implies $D_{t_1=0} \neq 0$ and hence $F_{t_1=0}(z)=0$ has n distinct roots a_0 in $\bar{k}$. iii) implies that none of a_0 is zero. Then we put $z=a_0+a_1t_1$ in $F(z)$ and we shall define a_1 such that the term of t_1 may disappear. The coefficient of t_1 in $F(a_0+a_1t_1)$ is

$$ma_0^{m-1}a_1 + r_1(0)(m-1)\, a_0^{m-2}a_1 + \cdots + r_{m-1}(0)\, \dot{a}_1 + a\,,$$

where a is defined only by $F(z)$ and a_0. However we have

$$F'_{t_1=0}(a_0) = ma_0^{m-1} + r_1(0)(m-1)\, a_0^{m-2} + \cdots + r_{m-1}(0) \neq 0\,,$$

since $D_{t_1=0} \neq 0$ and a_0 is roots of $F_{t_1=0}(z) = 0$. Therefore $a_1 = -a(F'_{t_1=0}(a_0))^{-1}$. Next we put $z = a_0 + a_1t_1 + a_2t_1^2$ and shall define a_2 in $\bar{k}$ such that the term of t_1^2 in $F(z)$ will disappear. This is done by the same method as above. This process is continued to obtain $z = a_0 + a_1t_1 + a_2t_1^2 + \cdots$ which are n distinct roots of $F(z) = 0$ in $\bar{k}[[t_1]]$ and are all units in $\bar{k}[[t_1]]$ as desired. Now let us return to the general case. It is easily proved that the assumptions i), ii) and iii) are satisfied in $F_{t_n=0}(z) = 0$. By the induction hypothesis, all the roots $f_0(t_1, t_2, \cdots, t_{n-1})$ are units in $\bar{k}[[t_1, t_2, \cdots, t_{n-1}]]$. Put $z = f_0 + f_1t_n$ where $f_1 \in \bar{k}[[t_1, t_2, \cdots, t_{n-1}]]$. We shall define f_1 in such a way that the term of t_n in $F(z)$ will disappear. The coefficient of t_n in $F(z)$ is $mf_0^{m-1}f_1 + r_1(t_1, \cdots, t_{n-1}, 0)$

$(m-1)f_0^{m-2}f_1+\cdots+r_{m-1}(t_1,\cdots,t_{n-1},0)f_1+f$ where f is an element of $\bar{k}[[t_1, t_2,\cdots,t_n]]$ defined only by $F(z)$ and f_0. Since

$$F'_{t_n=0}(f_0)=mf_0^{m-1}+r_1(t_1,\cdots,t_{n-1},0)(m-1)f_0^{m-2}+\cdots$$
$$+r_{m-1}(t_1,\cdots,t_{n-1},0)$$

are units in $\bar{k}[[t_1, t_2,\cdots,t_{n-1}]]$, we put $f_1=-f(F'_{t_n=0}(f_0))^{-1}$. In the same manner we can define $f_2, f_3,\cdots$ such that $z=f_0+f_1t_n+f_2t_n^2+\cdots$ are n distinct roots of $F(z)=0$ in $\bar{k}[[t_1, t_2,\cdots,t_n]]$ and are all units. Thus the proof is completed.

3. Divisors and principal divisors.

We shall give in this section a definition of divisors with respect $\bar{S}$ in the similar way as in [1]. For the sake of simplicity we assume $n=2$. We can easily generalize the result for $n\neq 2$.

Lemma. *Let $u\in k(x_1, x_2)$ and $w_{[t_1,t_2]}(u)=(\alpha_1, \alpha_2)$* for $w_{[t_1,t_2]}$ *in* S. *Then $\alpha_2\neq 0$ for at most a finite number of t_2, and $\alpha_1\neq 0$ for at most a finite number of t_1 when t_2 is fixed*[9].

Proof. t_2 can be considered to be also prime elements at some places $\mathfrak{p}_{t_2}$ in $k(x_1, x_2)/k(x_1)$, and, for a finite number of $\mathfrak{p}_{t_2}$, $w_{[t_2]}(u)\neq 0$, that is, the first part of Lemma holds[10]. Next we imbed $k(x_1, x_2)$ in $\bar{k}((x_1, t_2))$ and get $u=\dfrac{f_1(x_1, t_2)}{f_2(x_1, t_2)}$ where $f_i(x_1, t_2)\in\bar{k}[[x_1, t_2]]$. Then $w_{[t_1,t_2]}(u)=w_{[t_1,t_2]}(f_1(x_1, t_2))-w_{[t_1,t_2]}(f_2(x_1, t_2))=(\alpha_1', \alpha_2')-(\alpha_1'', \alpha_2'')$[11]. But it is easy to see that $\alpha_1'\neq 0$ and $\alpha_1''\neq 0$ for a finite number of t_1.

Let K be a finite extension of $k(x_1, x_2)$ and denote by $w^{(i)}_{[t_1,t_2]}$ the extensions of $w_{[t_1,t_2]}$ in K.

Theorem 6. *Let $w^{(i)}_{[t_1,t_2]}(s)=(\alpha_1, \alpha_2)$ for an element s of K. Then $\alpha_2\neq 0$ for at most a finite number of t_2, and $\alpha_1\neq 0$ for at most a finite number of t_1 when t_2 is fixed.*

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9) In this section we assume naturally that prime elements of the same place are the same.

10) We can show that, if $w_{[t_1,t_2]}(u)=(\alpha_1, \alpha_2)$, then $w_{[t_2]}(u)=\alpha_2$.

11) More exactly, if $t_2\neq x_2^{-1}$, $u=\dfrac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$ with $f_i(x_1, x_2)\in k[x_1, x_2]$ and $f_i(x_1, x_2)=f_i(x_1, t_2)\in\bar{k}[x_1][[t_2]]$, and, if $t_2=x_2^{-1}$, $u=\dfrac{f_1(x_1, x_2^{-1})}{f_2(x_1, x_2^{-1})}$ with $f_i(x_1, x_2^{-1})\in k[x_1, x_2^{-1}]$ and $f_i(x_1, x_2^{-1})=f_i(x_1, t_2)\in k[x_1, t_2]$. If we denote by $f_i^0(x_1, t_2)$ the lowest term of $f_i(x_1, t_2)$ with respect to t_2, then $f_i^0(x_1, 1)\in\bar{k}[x_1]$ and $w_{[t_1]}(f_i^0(x_1, 1))=\alpha_1$. Here we must notice that $f_i(x_1, 1)\in k[x_1]$ when t_1 is not of the first degree.

Proof. Let the equation of s be

$$s^m + u_1 s^{m-1} + \cdots + u_m = 0\,,$$

where $u_j \in k(x_1, x_2)$. Let $w_{[t_1, t_2]}(u_j) = (\alpha_1^{(j)}, \alpha_2^{(j)})$. It is clear from the previous lemma that $\alpha_2^{(j)} = 0$ $(j = 1, 2, \cdots, m)$ for almost all t_2. If $\alpha_2 \neq 0$, then $w_{[t_1, t_2]}^{(i)}(u_i s^{m-i})$ $(i = 0, 1, \cdots, m)$ are all different, which is a contradiction. Hence $\alpha_2 = 0$ for almsot all t_2. When t_2 is fixed, $w_{[t_1, t_2]}(u_j) = (0, \alpha_2^{(j)})$ $(j = 1, 2, \cdots, m)$ for almost all t_1 by the previous lemma. Then we can show by the similar way that $w_{[t_1, t_2]}^{(i)}(s) = (0, \alpha_2)$ for almost all t_1.

Now we give a definition of divisors with respect $\bar{S}$. We mean by *a divisor* a mapping of $\bar{S}$ into the value group of the valuations:

$$\mathfrak{A} = \{w_{[t_1, t_2]}^{(i)} \rightarrow (\alpha_1, \alpha_2)_{t_1, t_2; i}\}\,,$$

where two next conditions are satisfied:

1) $\alpha_2 \neq 0$ for at most a finite number of t_2,

2) $\alpha_1 \neq 0$ for at most a finite number of t_1 when t_2 is fixed.

Theorem 5 implies that, if s is a non zero element of K,

$$(s) = \{w_{[t_1, t_2]}^{(i)} \rightarrow w_{[t_1, t_2]}^{(i)}(s)\}$$

is a divisor. We call this divisor *a principal divisor*. We now define addition of two divisors $\mathfrak{A}$ and $\mathfrak{B}$:

$$\mathfrak{A} = \{w_{[t_1, t_2]}^{(i)} \rightarrow (\alpha_1, \alpha_2)_{t_1, t_2; i}\}$$
$$\mathfrak{B} = \{w_{[t_1, t_2]}^{(i)} \rightarrow (\beta_1, \beta_2)_{t_1, t_2; i}\}$$

such as

$$\mathfrak{A} + \mathfrak{B} = \{w_{[t_1, t_2]}^{(i)} \rightarrow (\alpha_1 + \beta_1, \alpha_2 + \beta_2)_{t_1, t_2; i}\}\,.$$

Then clearly

$$(s_1 s_2) = (s_1) + (s_2)\,.$$

Thus all divisors form an additive group D and all principal divisors form a subgroup H of D. And

$$s \rightarrow (s)$$

gives a homomorphic mapping of the multiplicative group K^* with the additive group H. The kernel of this homomorphism is k^* by Theorem 2.

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