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UNSTABLE *jO*-GROUPS AND STABLY LINEAR HOMOTOPY REPRESENTATIONS FOR *p*-GROUPS

Dedicated to Professor Fuichi Uchida on his sixtieth birthday

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0. Introduction

Since the study of homotopy representations by tom Dieck and Petrie [6], several authors studied homotopy representations in their own setting. (See for example [4], [6], [8], [9], [10], [11].)

In this paper we will study the unstable jO-groups and stably linear homotopy representations for p-groups.

Let G be a finite group. A G-homotopy representation X is a G-CW complex such that X^H is homotopy equivalent to a $(\dim X^H)$-dimensional sphere for every subgroup H of G. When X^H is empty, we set $\dim X^H = -1$ and we regard the empty set as a (-1)-dimensional sphere. We call a G-homotopy representation X linear if X is G-homotopy equivalent to a linear G-sphere $S(V)$. We also call a G-homotopy representation X stably linear if there exist representations V and W such that $X * S(W)$ is G-homotopy equivalent to $S(V)$. Let $S(G)$ be the set of subgroups of G. The dimension function $\mathrm{Dim}X : S(G) \to \mathbb{Z}$ is defined by $\mathrm{Dim}X(H) = \dim X^H + 1$. A function $\underline{n} : S(G) \to \mathbb{Z}$ is called linear if there exists a representation V such that $\underline{n} = \mathrm{Dim}S(V)$.

In the theory of homotopy representations, the Picard group $Pic(G)$, the unstable Picard group $Pic(G, \underline{n})$ and the jO-groups are relevant to the classification of G-homotopy types and the cancellation law for homotopy representations. In [6], tom Dieck and Petrie introduced the homotopy representation group $V^\infty(G)$, which is the Grothendieck group of the G-homotopy types of G-homotopy representations, and tom Dieck [2] also introduced the JO-group $JO(G)$, which is the Grothendieck group of the G-homotopy types of linear G-spheres. The calculation of these groups gives many results on classification of G-homotopy representations or linear G-spheres. In [6, Theorem 6.5], tom Dieck and Petrie proved that the torsion subgroup of $V^\infty(G)$ is isomorphic to the Picard group $Pic(G)$, which is algebraically defined and computable

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for some groups. Thus we can study the G-homotopical problem such as the classification of G-homotopy types of G-homotopy representations by the algebraic method.

E. Laitinen [8] introduced the unstable Picard group $Pic(G,\underline{n})$ and studied unstable G-homotopy types of G-homotopy representations and the cancellation law problem for G-homotopy representations. He essentially proved that the cancellation law holds for G-homotopy representations if the natural map $\alpha(\underline{n}) : Pic(G;\underline{n}) \to Pic(G)$ is injective (cf. [8, Theorem 5]) and furthermore one can see that the injectivity of $\alpha(\underline{n})$ is equivalent to the cancellation law for homotopy representations by [11, Theorem 2.6 and Proposition 3.3].

In [10] and [11], we introduced the unstable jO-group $jO(G;\underline{n})$ as a subgroup of $Pic(G;\underline{n})$, where $\underline{n}$ is a linear dimension function, and we also defined the unstable LH-group by

$$LH^{\infty}(G;\underline{n}) = Pic(G;\underline{n})/jO(G;\underline{n}),$$

and the LH-group by

$$LH^{\infty}(G) = Pic(G)/jO(G).$$

These groups play important roles in the (stable) linearity of homotopy representations. For example we can see the following result by [11, Theorem 5.2 and Corollary 5.3].

Proposition 0.1. *The following statements are equivalent.*

(1) *The natural homomorphism $i(\underline{n}) : LH^{\infty}(G;\underline{n}) \to LH^{\infty}(G)$ is injective for every linear dimension function $\underline{n}$.*

(2) *Every stably linear G-homotopy representation with linear dimension function is linear.*

In [2], tom Dieck computed the jO-group $jO(G)$ for a p-group, which is isomorphic to the torsion subgroup of $JO(G)$.

Our first result is the analogous computation of the unstable jO-group $jO(G;\underline{n})$ for a p-group G in terms of representation theory (see Theorem 4.1).

Secondly we will show the following theorem.

Theorem A. *Let G be a p-group and $\underline{n}$ a linear dimension function.*

(1) *If p is an odd prime, then the natural homomorphism $i(\underline{n}) : LH^{\infty}(G;\underline{n}) \to LH^{\infty}(G)$ is injective.*

(2) *If $p = 2$ and the natural homomorphism $\alpha(\underline{n}) : Pic(G,\underline{n}) \to Pic(G)$ is injective, then the natural homomorphism $i(\underline{n}) : LH^{\infty}(G;\underline{n}) \to LH^{\infty}(G)$ is injective.*

As mentioned in [11, p.289, Proof of Corollary B], every homotopy representation of a p-group has a linear dimension function. Thus we obtain the following by Proposition 0.1 and Theorem A.

Corollary B. *Let G be a p-group.*

(1) *If p is an odd prime, then every stably linear G-homotopy representation is linear.*

(2) *If $p = 2$ and the natural homomorphism $\alpha(\underline{n}) : Pic(G, \underline{n}) \to Pic(G)$ is injective for every dimension function $\underline{n}$, then every stably linear G-homotopy representation is linear.*

REMARK 1. Stably linear homotopy representations are not linear in general. For example if G is a cyclic group C_{pq} of order pq, where p, q are distinct odd prime, there exist a C_{pq}-homotopy representation which is stably linear but not linear. (See [11, Section 5] for the detail.)

E. Laitinen [8] showed that the cancellation law holds (and hence $\alpha(\underline{n})$ is injective) if G is a nilpotent group whose 2-Sylow subgroup is abelian. In 2-group case, unfortunately his result covers only abelian 2-groups. We note the following remark, whose proof is given in Section 5. By this remark, one can find many finite 2-groups for which $\alpha(\underline{n})$ is injective.

REMARK 2. Let G be a 2-group such that for every normal subgroup H, (1) G/H is abelian, dihedral or semi-dihedral, or (2) the center $Z(G/H)$ of G/H is not of order 2, and let $\underline{n}$ be a dimension function. Then the natural homomorphism $\alpha(\underline{n})$ is injective.

This paper is organized as follows. In Section 1, we will recall several definitions and notations for the convenience of the readers. In Sections 2 and 3, we will discuss splittings of the (unstable) Picard groups, the (unstable) jO-groups and the (unstable) LH-groups by using the splitting theorem in [9]. In Section 4, we will compute the unstable jO-groups for p-groups by using the splittings of jO-groups. In Section 5, we shall prove Theorem A. The key result for the proof is Theorem 5.8 in Section 5, which states that the injectivity of $\alpha(\underline{n}) : Pic(G; \underline{n}) \to Pic(G)$ implies the injectivity of $i(\underline{n}) : LH^{\infty}(G; \underline{n}) \to LH(G)$ for a p-group. Theorem 5.8 is proved by using some results of representation theory and the splittings of the Picard groups and the LH-groups in Sections 3 and 4.

1. Preliminaries

In this section we briefly recall the (unstable) Picard group, the (unstable) jO-group and the (unstable) LH-group. (For the detail, see [2], [5], [6], [8], [10], [11].)

Let $\phi(G)$ denote the set of conjugacy classes of subgroups of G and $C(G)$ the ring of integer-valued functions on $\phi(G)$. Let $A(G)$ denote the Burnside ring of G. We regard $A(G)$ as a subring of $C(G)$ by the usual way ([6]). We put $\overline{C}(G) = C(G)/|G|C(G)$ and $\overline{A}(G) = A(G)/|G|C(G)$. Then the Picard group is defined by

$$Pic(G) = \frac{\overline{C}(G)^*}{C(G)^*\overline{A}(G)^*},$$

where $*$ indicates the unit group.

For the definition of the unstable Picard group, we recall the unstability conditions. Let $\underline{n}$ be the dimension function of a G-homotopy representation. We say that a function $d \in C(G)$ satisfies the unstability conditions for $\underline{n}$ if the following conditions (1)–(3) hold.

(1) $d(H) = 1$ when $\underline{n}(H) = 0$;
(2) $d(H) = -1, 0, 1$ when $\underline{n}(H) = 1$;
(3) $d(H) = d(\overline{H})$ for $(H) \in \phi(G)$, where $\overline{H}$ is defined as the maximum subgroup including H such that $\underline{n}(H) = \underline{n}(\overline{H})$.

We put

$$\begin{aligned} \overline{C}(G;\underline{n})^* &= \{d \in \overline{C}(G)^* \mid d \text{ satisfies the unstability conditions.}\}, \\ \overline{A}(G;\underline{n})^* &= \{d \in \overline{A}(G)^* \mid d \text{ satisfies the unstability conditions.}\}, \\ C(G;\underline{n})^* &= \{d \in C(G)^* \mid d \text{ satisfies the unstability conditions.}\}. \end{aligned}$$

The unstable Picard group is defined by

$$Pic(G;\underline{n}) = \frac{\overline{C}(G;\underline{n})^*}{C(G;\underline{n})^*\overline{A}(G;\underline{n})^*}.$$

Let X, Y be G-homotopy representations with dimension function $\underline{n}$. In [8], the G-homotopy invariants $D(X,Y) \in Pic(G)$ and $D_{\underline{n}}(X,Y) \in Pic(G;\underline{n})$ are defined by the class of the degree function $d(f)$, where $f : Y \to X$ is a G-map such that $\deg f^H$ is prime to $|G|$ for every subgroup H.

In the case where $X = S(V)$, $Y = S(W)$, we denote $D(S(V), S(W))$ and $D_{\underline{n}}(S(V), S(W))$ by $D(V,W)$ and $D_{\underline{n}}(V,W)$ respectively.

We define the jO-group $jO(G)$, and the unstable jO-group $jO(G;\underline{n})$ as follows.

$$\begin{aligned} jO(G) &= \{D(V,W) \mid \mathrm{Dim}S(V) = \mathrm{Dim}S(W)\}, \\ jO(G;\underline{n}) &= \{D_{\underline{n}}(V,W) \mid \mathrm{Dim}S(V) = \mathrm{Dim}S(W) = \underline{n}\}. \end{aligned}$$

When $\underline{n}$ is not linear, we set $jO(G,\underline{n}) = 0$.

REMARK. The above definition of $jO(G)$ is different from the original one in [2] or [3], but it is known [6] that our $jO(G)$ is isomorphic to the original one.

The (unstable) LH-groups are defined as follows.

$$LH^{\infty}(G) = \frac{Pic(G)}{jO(G)} \qquad LH^{\infty}(G;\underline{n}) = \frac{Pic(G;\underline{n})}{jO(G;\underline{n})}.$$

Finally we recall essential isotropy subgroups (for $\underline{n}$). We call a subgroup H an essential isotropy subgroup if $\underline{n}(H) > 0$ and $H = \overline{H}$. We denote by $Iso(\underline{n})$ the set of essential isotropy subgroups.

2. The splitting theorem for the Picard groups

For the convenience, we recall the splittings of the Picard groups. The general theory of the splittings is discussed in [9].

We suppose that $\underline{n}$ is linear. We denote $Pic(G)$, $jO(G)$, $Pic(G;\underline{n})$ or $jO(G;\underline{n})$ by $B(G)$.

Let H be any normal subgroup of G and x any element of $B(G)$. We define a new element x^H of $B(G)$ as follows. Suppose that x is represented by a function $d \in \overline{C}(G)^*$. Then we set $x^H = [d^H] \in B(G)$, where d^H is defined by $d^H(L) = d(HL)$.

Lemma 2.1. *The above x^H is well-defined.*

Proof. Clearly d^H is in $\overline{C}(G)^*$. Hence $x^H \in Pic(G)$ when $B(G) = Pic(G)$. If d satisfies the unstability conditions, then d^H also satisfies the unstability conditions. In fact, if $\underline{n}(L) = 0$, then $\underline{n}(HL) = 0$. Hence $d^H(L) = d(HL) = 1$. If $\underline{n}(L) = 1$, then $\underline{n}(HL) = 0$ or 1. Hence $d^H(L) = d(HL) = \pm 1$. Since $Iso(\underline{n})$ is closed under intersection, it follows from [8] or [11, Lemma 2.2] that $HL \le H\overline{L} \le \overline{HL}$. Hence $d^H(L) = d(HL) = d(H\overline{L}) = d^H(\overline{L})$. Thus x^H is in $Pic(G;\underline{n})$ when $B(G) = Pic(G;\underline{n})$.

We next consider the case where $B(G) = jO(G)$ (resp. $jO(G;\underline{n})$). Assume that $x = D(V,W)$ (resp. $D_{\underline{n}}(V,W)$) and that $d(f) \in \overline{C}(G)^*$ represents x, where $f : S(W) \to S(V)$ is a G-map such that $\deg f^H$ is prime to $|G|$ for every H. Then it is easily checked that $d(f)^H = d(f^H)$. Hence $x^H \in B(G)$ when $B(G) = jO(G)$. Furthermore we put:

$$g = f^H * id : S(W^H) * S(W - W^H) \to S(V^H) * S(W - W^H).$$

Then $d(g)$ satisfies the unstability conditions for $\underline{n}$ and clearly $d(f^H) = d(g)$. Hence $x^H \in B(G)$ when $B(G) = jO(G;\underline{n})$. □

We define x_H for $H \trianglelefteq G$ by

$$x_H = \prod_{H \le K \trianglelefteq G} (x^K)^{\mu(H,K)},$$

where $\mu(-,-)$ is the Möbius function on the normal subgroup lattice. (For the Möbius function, see [1].) By the Möbius inversion, one can see that

$$x^K = \prod_{K \leq H \trianglelefteq G} x_H$$

The following is clear.

Lemma 2.2. *Let x, y be in $B(G)$ and H, K normal subgroups of G.*
(1) $(xy)^H = x^H y^H$ *and* $1^H = 1$.
(2) $(xy)_H = x_H y_H$ *and* $1_H = 1$.
(3) $(x^H)^K = x^{HK} = (x^K)^H$.

For every normal subgroup H, we put:

$$B(G)_H = \{x \in B(G) \mid x^H = x, x^K = 1 \text{ for any } K \text{ such that } H < K \trianglelefteq G\}.$$

By Lemma 2.2, one can see that $B(G)_H$ is a subgroup of $B(G)$.

Let $\mathcal{N}(G)$ denote the set of normal subgroups of G. We can define a homomorphism

$$\Phi_{B(G)} : B(G) \longrightarrow \prod_{H \in \mathcal{N}(G)} B(G)_H$$

by

$$\Phi_{B(G)}(x) = (x_H)_{H \in \mathcal{N}(G)}.$$

We then obtain the following splitting theorem.

Theorem 2.3 (cf. [9, Theorem 2.2]). *The above homomorphism $\Phi_{B(G)}$ is an isomorphism.*

Proof. By the Möbius inversion, one can see that a homomorphism

$$(x_H)_{H \in \mathcal{N}(G)} \longmapsto \prod_{H \in \mathcal{N}(G)} x_H$$

is the inverse of $\Phi_{B(G)}$. □

Finally we note that $B(G)_H$ is naturally isomorphic to $B(G/H)_{\{1\}}$. (For the detail, see [9].)

3. On $Pic(G;\underline{n})_H$, $jO(G)_H$ and $jO(G;\underline{n})_H$

In this section we investigate $B(G)_H$ for $B(G) = Pic(G;\underline{n})$, $jO(G)$ and $jO(G;\underline{n})$.

We assume that $\underline{n}$ is a linear dimension function. Let $\mathcal{I}(\underline{n})$ ($\subset Iso(\underline{n})$) denote the set of normal essential isotropy subgroups for $\underline{n}$.

Proposition 3.1. *For any $H \in \mathcal{N}(G) \setminus \mathcal{I}(\underline{n})$, $Pic(G;\underline{n})_H = 1$.*

Proof. We note $HK \leq \overline{H}K \leq \overline{HK}$. Indeed it is clear that $HK \leq \overline{H}K$. Since $Iso(\underline{n})$ is closed under intersection, it follows from [11, Lemma 2.2] that $\overline{H} \leq \overline{HK}$ and $K \leq \overline{HK}$. Hence $\overline{H}K \leq \overline{HK}$. By definition, $d^H(K) = d(HK)$ for any $[d] \in Pic(G;\underline{n})_H$. Since d satisfies the unstability conditions, it follows that $d(HK) = d(\overline{H}K) = d^{\overline{H}}(K)$. Hence $d^H = d^{\overline{H}}$. Since $H < \overline{H}$, it follows that $[d]^{\overline{H}} = 1$. Therefore $[d] = [d]^H = [d]^{\overline{H}} = 1$ and $Pic(G;\underline{n})_H = 1$. □

Let $\mathcal{K}(G)$ denote the set of the kernels of irreducible representations of G. We define $\mathcal{K}(\underline{n})$ as follows. Suppose that $\underline{n} = \mathrm{Dim}S(U)$. Then U decomposes into

$$U = U_1 \oplus \cdots \oplus U_r,$$

where U_i's are irreducible subrepresentations of U. We set

$$\mathcal{K}(\underline{n}) = \{\mathrm{Ker}\ U_1, \ldots, \mathrm{Ker}\ U_r\}.$$

By [8], this definition depends only on $\underline{n}$. We note the following relation.

$$\begin{array}{ccc} \mathcal{K}(G) & \subset & \mathcal{N}(G) \\ \cup & & \cup \\ \mathcal{K}(\underline{n}) & \subset & \mathcal{I}(\underline{n}) \end{array}$$

Proposition 3.2.

(1) *For any $H \in \mathcal{N}(G) \setminus \mathcal{K}(G)$, $jO(G)_H = 1$.*

(2) *For any $H \in \mathcal{N}(G) \setminus \mathcal{K}(\underline{n})$, $jO(G;\underline{n}) = 1$.*

Proof. The homomorphism $jO(G) \to jO(G)_H$ (resp. $jO(G;\underline{n}) \to jO(G;\underline{n})_H$): $x \mapsto x_H$ is surjective. Hence it suffices to show that $x_H = 1$ for any $x \in jO(G)$ (resp. $jO(G;\underline{n})$). Suppose that $x = D(V,W)$ (resp. $D_{\underline{n}}(V,W)$) and $\mathcal{K}(\underline{n}) = \{H_1, \ldots, H_r\}$, where $\underline{n} = \mathrm{Dim}S(V) = \mathrm{Dim}S(W)$. Suppose that

$$\begin{aligned} V &= V_1 \oplus \cdots \oplus V_s \\ W &= W_1 \oplus \cdots \oplus W_s, \end{aligned}$$

are irreducible decompositions of V and W. (Note that the numbers of irreducible components of V and W are equal.) We denote by $V(H_i)$ (resp. $W(H_i)$) the direct sum of V_j (resp. W_j) with kernel H_i. Then V, W are described as

$$\begin{aligned} V &= V(H_1) \oplus \cdots \oplus V(H_r) \\ W &= W(H_1) \oplus \cdots \oplus W(H_r). \end{aligned}$$

By [5, p.213], one can see that $\mathrm{Dim} S(V(H_i)) = \mathrm{Dim} S(W(H_i))$ for any i. Take G-maps $f_i : S(W(H_i)) \to S(V(H_i))$ such that $d(f_i)(H)$ is prime to $|G|$ for every H. Put

$$f = f_1 * \cdots * f_r : S(W(H_1)) * \cdots * S(W(H_r)) \longrightarrow S(V(H_1)) * \cdots * S(V(H_r)),$$

where $*$ means join. Note that $S(W(H_1)) * \cdots * S(W(H_r))$ (resp. $S(V(H_1)) * \cdots * S(V(H_r))$) is G-homeomorphic to $S(W)$ (resp. $S(V)$). Hence we have

$$x = [d(f)] = \prod_i [d(f_i)].$$

Furthermore, for $H \in \mathcal{N}(G)$, it follows that

$$x^H = \prod_{\substack{i \\ H \leq H_i}} [d(f_i)].$$

Indeed, since $S(W(H_i))^H = S(W(H_i))$ when $H \leq H_i$, and $S(W(H_i))^H$ is empty when $H \not\leq H_i$, we have

$$f^H = *_{i, H \leq H_i} f_i : *_{i, H \leq H_i} S(W(H_i)) \to *_{i, H \leq H_i} S(V(H_i)).$$

This leads the above equation. By the Möbius inversion, we obtain

$$x_H = \begin{cases} [d(f_i)], & \text{if } H = H_i \\ 1, & \text{if } H \in \mathcal{N}(G) \setminus \mathcal{K}(\underline{n}). \end{cases}$$

Hence we can see that $jO(G; \underline{n})_H = 1$ for $H \in \mathcal{N}(G) \setminus \mathcal{K}(\underline{n})$, and $jO(G)_H = 1$ for $H \in \mathcal{N}(G) \setminus \mathcal{K}(G)$. □

Thus we can see the following.

Theorem 3.3. *Via the isomorphisms* $\Phi_{B(G)}$ *for* $B(G) = Pic(G)$, $Pic(G; \underline{n})$, $jO(G)$ *and* $jO(G; \underline{n})$, *the commutative diagram:*

$$\begin{array}{ccc} Pic(G; \underline{n}) & \longrightarrow & Pic(G) \\ \cup & & \cup \\ jO(G; \underline{n}) & \longrightarrow & jO(G) \end{array}$$

is isomorphic to

$$\begin{array}{ccc}
\prod_{H\in\mathcal{I}(\underline{n})} Pic(G;\underline{n})_H & \longrightarrow & \prod_{H\in\mathcal{N}(G)} Pic(G)_H \\
\cup & & \cup \\
\prod_{H\in\mathcal{K}(\underline{n})} jO(G;\underline{n})_H & \longrightarrow & \prod_{H\in\mathcal{K}(G)} jO(G)_H.
\end{array}$$

For $H \in \mathcal{N}(G)$, we put

$$LH^\infty(G)_H = \frac{Pic(G)_H}{jO(G)_H} \quad \text{and} \quad LH^\infty(G;\underline{n})_H = \frac{Pic(G;\underline{n})_H}{jO(G;\underline{n})_H}.$$

We obtain the following.

Corollary 3.4. *The diagram*

$$\begin{array}{ccc}
LH^\infty(G;\underline{n}) & \xrightarrow{i(\underline{n})} & LH^\infty(G) \\
\cong\downarrow & & \cong\downarrow \\
\prod_{H\in\mathcal{I}(\underline{n})} LH^\infty(G;\underline{n})_H & \xrightarrow{\Pi i(\underline{n})_H} & \prod_{H\in\mathcal{N}(G)} LH^\infty(G)_H
\end{array}$$

consisting of natural homomorphisms is commutative.

4. Computation of the unstable jO-group for a p-group

We first recall the computation of $jO(G)$ and $jO(G)_H$ for a p-group G from [2] and [3].

Let m be a multiple of $2|G|$ and ξ_m a primitive m-th root of unity. Let Γ denote the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$. As is well-known, Γ is isomorphic to $\mathbb{Z}/m^*$ via the correspondence:

$$\mathbb{Z}/m^* \ni k \longmapsto (\psi^k : \xi_m \mapsto \xi_m^k) \in \Gamma.$$

We often identify Γ with $\mathbb{Z}/m^*$. Let $\mathrm{Irr}(G)$ denote the set of irreducible (real) representations of G and Let $\mathrm{Irr}(G)_H$ denote the set of irreducible representations of G with kernel H. Then Γ acts on $\mathrm{Irr}(G)$ and $\mathrm{Irr}(G)_H$ via the Galois conjugation. (See [3, pp.230–231].) Let $X(G)$ [resp. $X(G)_H$] denote the orbit set $\mathrm{Irr}(G)/\Gamma$ [resp. $\mathrm{Irr}(G)_H/\Gamma$] (i.e., the set of the Galois conjugate classes). We put

$$iO(G) = \prod_{A\in X(G)} \Gamma/\Gamma_A$$

$$iO(G)_H = \prod_{A \in X(G)_H} \Gamma/\Gamma_A,$$

where Γ_A is the isotropy subgroup at $V \in A$. By [2], for any finite group G, there are surjective homomorphisms

$$tO(G) : iO(G) \longrightarrow jO(G)$$

and

$$tO(G)_H : iO(G)_H \longrightarrow jO(G)_H$$

defined by

$$(k_A)_{A \in X(G)} \longmapsto \prod_{A \in X(G)} D(V_A, \psi^{k_A} V_A) = D(\oplus_{A \in X(G)} V_A, \oplus_{A \in X(G)} \psi^{k_A} V_A)$$

and

$$(k_A)_{A \in X(G)_H} \longmapsto \prod_{A \in X(G)_H} D(V_A, \psi^{k_A} V_A) = D(\oplus_{A \in X(G)_H} V_A, \oplus_{A \in X(G)_H} \psi^{k_A} V_A),$$

where $V_A \in A$. In particular if G is a p-group, then $tO(G)$ and $tO(G)_H$ are isomorphisms [2, Theorem 2].

We compute $jO(G;\underline{n})$ and $jO(G;\underline{n})_H$.

As is well-known, $\underline{n}$ is uniquely described as $a_1\underline{n_1} + \cdots + a_s\underline{n_s}$, where $\underline{n_i}$ is the dimension function of a sphere of some irreducible representation and a_i's are positive integers, and $\underline{n_i}$'s are linearly independent. (See [5].) Let $\mathrm{Irr}(\underline{n})$ denote the set of irreducible representations V of G such that $\mathrm{Dim}S(V) = \underline{n_i}$ for some i. Let $\mathrm{Irr}(\underline{n})_H$ denote the set of irreducible representations V of G with kernel H such that $\mathrm{Dim}S(V) = \underline{n_i}$ for some i. Since V and $\psi^k V$ have the same kernel, Γ acts on $\mathrm{Irr}(\underline{n})$ and $\mathrm{Irr}(\underline{n})_H$. We put $X(\underline{n}) = \mathrm{Irr}(\underline{n})/\Gamma$ and $X(\underline{n})_H = \mathrm{Irr}(\underline{n})_H/\Gamma$. We also put

$$iO(G;\underline{n}) = \prod_{A \in X(\underline{n})} \Gamma/\Gamma_A,$$

$$iO(G;\underline{n})_H = \prod_{A \in X(\underline{n})_H} \Gamma/\Gamma_A.$$

Let U_i $(i = 1, \ldots, s)$ be irreducible representations such that $\underline{n_i} = \mathrm{Dim}S(U_i)$ and $X(\underline{n}) = \{[U_1], \ldots, [U_s]\}$. We define a homomorphism

$$tO(\underline{n}) : iO(G;\underline{n}) \longrightarrow jO(G;\underline{n}),$$

by

$$(k_i)_i \longmapsto \prod_{i=1}^{s} \alpha_{\underline{n_i},\underline{n}} D_{\underline{n_i}}(U_i, \psi^{k_i} U_i),$$

where $\alpha_{\underline{n_i},\underline{n}} : jO(G;\underline{n_i}) \to jO(G;\underline{n})$ is the homomorphism defined by the natural way.

We also define a homomorphism $tO(G;\underline{n})_H : iO(G;\underline{n})_H \to jO(G;\underline{n})_H$ by a similar way.

Theorem 4.1.

(1) *The homomorphisms $tO(\underline{n})$ and $tO(\underline{n})_H$ are surjective.*

(2) *If G is a p-group, then $tO(\underline{n})$ and $tO(\underline{n})_H$ are isomorphisms.*

Proof. (1) By [10, p.599], for any element x of $jO(G;\underline{n})$, there exist $k_{i,j}$'s such that

$$x = D_{\underline{n}}(\oplus_{i=1}^{s} \oplus_{j=1}^{a_i} U_i, \oplus_{i=1}^{s} \oplus_{j=1}^{a_i} \psi^{k_{i,j}} U_i).$$

Furthermore $D_{\underline{n_i}}(U_i, \psi^{k_{i,j}} U_i)$ is represented by the function ${k_{i,j}}^{[(1/2)\mathrm{Dim}S(U_i)]}$, which is the degree function of some G-map $f_{i,j} : S(\psi^{k_{i,j}} U_i) \to S(U_i)$, where [ ] denotes the Gauss symbol. Then one can see that

$$D_{\underline{n_i}}(U_i, \psi^{k_{i,t}} U_i) D_{\underline{n_i}}(U_i, \psi^{k_{i,u}} U_i) = D_{\underline{n_i}}(U_i, \psi^{k_{i,t} k_{i,u}} U_i).$$

Hence x is equal to

$$\begin{aligned} & D_{\underline{n}}(\oplus_{i=1}^{s} U_i \oplus \hat{U}, \oplus_{i=1}^{s} \psi^{k_i} U_i \oplus \hat{U}) \\ = & \prod_{i=1}^{s} \alpha_{\underline{n_i},\underline{n}} D_{\underline{n_i}}(U_i, \psi^{k_i} U_i), \end{aligned}$$

where $\hat{U} = \oplus_{i=1}^{s} \oplus_{j=1}^{a_i - 1} U_i$ and $k_i = \prod_{j=1}^{a_i} k_{i,j}$.

Hence $tO(\underline{n})$ is surjective. Similarly one can see that $tO(\underline{n})_H$ is surjective.

(2) The diagram

$$\begin{array}{ccc} iO(G) & \xrightarrow{tO(G)} & jO(G) \\ \uparrow & & \uparrow \\ iO(G;\underline{n}) & \xrightarrow{tO(\underline{n})} & jO(G;\underline{n}). \end{array}$$

consisting of natural homomorphisms is commutative. If G is a p-group, then $tO(G)$ is an isomorphism and $iO(G;\underline{n}) \to iO(G)$ is injective. Hence it follows that $tO(\underline{n})$

is injective and by (1), $tO(\underline{n})$ is an isomorphism. By the same argument, one can see that $tO(\underline{n})_H$ is an isomorphism. □

5. Proof of Theorem A

In order to prove Theorem A, we recall some well-known results from finite group theory and representation theory.

The following results are well-known. (see for example [7].)

Lemma 5.1.

(1) *A non-trivial p-group has the non-trivial center.*

(2) *Every non-trivial normal subgroup of a p-group intersects the center non-trivially.*

Lemma 5.2. *If a finite group G has a faithful irreducible representation, then the center $Z(G)$ is cyclic. In particular $Z(G/H)$ is cyclic for every $H \in \mathcal{K}(G)$.*

Lemma 5.3. *Let G be a p-group. If G does not have a normal subgroup isomorphic to $C_p \times C_p$, then G is cyclic when p is an odd prime, and G is cyclic, dihedral, quaternionic or semi-dihedral when $p = 2$.*

We also notice the following.

Lemma 5.4. *For every $H \in \mathcal{I}(\underline{n})$, there exist $K_1, \ldots, K_r \in \mathcal{K}(\underline{n})$ such that $H = \bigcap_i K_i$.*

Proof. One can show this by a similar argument in [10, Lemma 4.1]. □

Proposition 5.5. *Let G be a p-group. Then $\mathcal{K}(\underline{n}) = \mathcal{K}(G) \cap \mathcal{I}(\underline{n})$.*

Proof. $\mathcal{K}(\underline{n}) \subset \mathcal{K}(G) \cap \mathcal{I}(\underline{n})$ is clear. Let H be in $\mathcal{K}(G) \cap \mathcal{I}(\underline{n})$. Then take $K_1, \ldots, K_r \in \mathcal{K}(\underline{n})$ such that $H = \bigcap_i K_i$. If $H = G$, then $K_i = G$ for any i and hence $H \in \mathcal{K}(\underline{n})$. We assume that $H \neq G$. Suppose that $H \notin \mathcal{K}(\underline{n})$. Since every K_i/H is a non-trivial normal subgroup of G/H, K_i/H intersects $Z(G/H)$ non-trivially. Since $Z(G/H)$ is a cyclic p-group, $Z(G/H) \cap \bigcap_i K_i/H$ is a nontrivial subgroup. On the other hand, $\bigcap_i K_i/H$ is trivial since $H = \bigcap_i K_i$. This is a contradiction. Therefore $H \in \mathcal{K}(\underline{n})$. □

Proposition 5.6. *Let G be a nilpotent group. Suppose that G has a normal subgroup A isomorphic to $C_p \times C_p$ (p:prime) and an irreducible faithful representation V. Let $A_0, \ldots, A_p$ be all subgroups of order p in A. Then*

(1) *One of the A_i's, say A_0, is normal in G and the other A_i's $(i = 1, \ldots, p)$ have*

the same normalizer $N(= N_G(A_i))$ of index p in G.

(2) *The restriction* $\mathrm{Res}_N V$ *splits into irreducible representations* $V_1 \oplus \cdots \oplus V_p$ *such that* $\mathrm{Res}_A V_i$ *has the kernel* A_i. *If* W *is an irreducible representation which is not isomorphic to* V, *then* $\mathrm{Res}_N W$ *does not contain a summand which is isomorphic to one of* V_i*'s.*

(3) $V = \mathrm{Ind}_N^G V_i$ $(i = 1, \ldots, p)$.

(4) *The irreducible summands* V_i*'s as in* (2) *are not Galois conjugate.*

(5) *If faithful irreducible representations* V *and* W *are not Galois conjugate, then* V_i *and* W_j *are not Galois conjugate for every* i *and* j.

Proof. For (1)–(3), see [5, p.214].

(4): Since the kernels of V_i's are distinct, V_i's are not Galois conjugate.

(5): If V_i and W_j are Galois conjugate for some i, j, then V and W are Galois conjugate by (3). □

In order to prove the theorem, we show the following lemma. Suppose that $\underline{n}$ is a linear dimension function.

Lemma 5.7. *Let* G *be a dihedral* 2*-group, a quaternion* 2*-group or a semi-dihedral* 2*-group. Then* $i(\underline{n}) : LH^\infty(G;\underline{n}) \to LH^\infty(G)$ *is injective.*

Proof. By Corollary 3.4 it suffices to show that $i(\underline{n})_H : LH^\infty(G;\underline{n})_H \to LH^\infty(G)_H$ is injective for every $H \in \mathcal{I}(\underline{n})$. It is not hard to see $LH^\infty(G;\underline{n})_H$ and $LH^\infty(G)_H$ are naturally isomorphic to $LH^\infty(G/H;\underline{n}_H)_{\{1\}}$ and $LH^\infty(G/H)_{\{1\}}$ respectively. Here $\underline{n}_H = \sum_{K\in\mathcal{N}(G)} \mu(H,K)\underline{n}^K$. Since every quotient group of G is 1, $C_2 \times C_2$, a dihedral, a quaternionic or a semi-dihedral 2-group, it suffices to show that

$$i(\underline{n})_{\{1\}} : LH^\infty(G;\underline{n})_{\{1\}} \to LH^\infty(G)_{\{1\}}$$

is injective for 1, $C_2 \times C_2$, a dihedral, a quaternionic and a semi-dihedral 2-group. For 1 and $C_2 \times C_2$, since the (unstable) LH-group vanishes, $i(\underline{n})_{\{1\}}$ is injective. We next consider in case of the dihedral 2-group

$$G = D_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, bab^{-1} = a^{-1}\rangle.$$

In this case, it can be seen that $Pic(G;\underline{n})_{\{1\}} = Pic(G)_{\{1\}}$. In fact, by definition,

$$Pic(G;\underline{n})_{\{1\}} = \{x \in Pic(G;\underline{n}) \mid x^K = 1 \text{ for any } K \text{ such that } 1 < K \trianglelefteq G\}.$$

Hence every element x of $Pic(G;\underline{n})_{\{1\}}$ is represented by a function d such that $d(L) = 1$ for $(L) \neq (1)$, $(\langle b\rangle)$, $(\langle ab\rangle)$ and such that d satisfies the unstability conditions. Since $\underline{n}$ is the dimension function of a faithful representation by $1 \in \mathcal{I}(\underline{n})$, it is seen that $\langle a\rangle$, $\langle ab\rangle$ are also in $\mathcal{I}(\underline{n})$ by computing the dimension function. Thus the unstability conditions gives no restriction, and hence $Pic(G;\underline{n})_{\{1\}}$ coincides with $Pic(G)_{\{1\}}$. This leads the injectivity of $\alpha(\underline{n})_{\{1\}}$. Similarly $jO(G;\underline{n})_{\{1\}}$ coincides with $jO(G)_{\{1\}}$. Hence $i(\underline{n})_{\{1\}}$ is injective.

In the cases of a quaternion 2-group and a semi-dihedral 2-group, by the same argument, one can also see that $i(\underline{n})_{\{1\}}$ is injective. The detail is omitted. □

Theorem 5.8. *Let G be a p-group. If $\alpha(\underline{n}) : Pic(G,\underline{n}) \to Pic(G)$ is injective, then $i(\underline{n}) : LH^\infty(G;\underline{n}) \to LH^\infty(G)$ is injective.*

Proof. Note that

$$\alpha(\underline{n})_{\{1\}} : Pic(G/H;\underline{n}_H)_{\{1\}} \to Pic(G/H)_{\{1\}}$$

is injective by the injectivity of $\alpha(\underline{n})$. As mentioned in the proof of Lemma 5.7, it suffices to show that $i(\underline{n})_{\{1\}} : LH^\infty(G;\underline{n})_{\{1\}} \to LH^\infty(G)_{\{1\}}$ is injective when $1 \in \mathcal{I}(\underline{n})$. If $1 \notin \mathcal{K}(\underline{n})$, then, by Proposition 5.5, it follows that $1 \notin \mathcal{K}(G)$. Hence $LH^\infty(G;\underline{n})_{\{1\}} = Pic(G;\underline{n})_{\{1\}}$ and $LH^\infty(G)_{\{1\}} = Pic(G)_{\{1\}}$ by Propositions 3.1 and 3.2, and hence $i(\underline{n})_{\{1\}} = \alpha(\underline{n})_{\{1\}}$. Thus it follows that $i(\underline{n})_{\{1\}}$ is injective.

We next consider the case where $1 \in \mathcal{K}(\underline{n})$. In this case we show the theorem by induction on the order of G. We showed in [11, Theorem 5.5] that $i(\underline{n})$ is injective for an abelian p-group. By this fact and Lemma 5.7, we may assume that G is not cyclic, dihedral, quaternionic or semi-dihedral. Let x be any element of $Pic(G;\underline{n})_{\{1\}}$ such that $\alpha(\underline{n})_{\{1\}}(x) \in jO(G)_{\{1\}}$. We show that x is in $jO(G;\underline{n})_{\{1\}}$. By Lemma 5.3, G has a normal subgroup A isomorphic to $C_p \times C_p$. Let N be the normal subgroup of G as in Proposition 5.6. Suppose $X(\underline{n})_{\{1\}} = \{[U_1],\dots,[U_r]\}$ and $X(G)_{\{1\}} = \{[U_1],\dots,[U_r],[U_{r+1}],\dots,[U_s]\}$. Note that $jO(G)_{\{1\}} \cong \oplus_{i=1}^s \Gamma/\Gamma_{[U_i]}$. (See section 4.) We decompose $\mathrm{Res}_N U_i$ into $\oplus_{j=1}^p U_{i,j}$ as in Proposition 5.6. Let $a_1,\dots,a_r$ denote the positive integers such that $\underline{n} = a_1\mathrm{Dim}S(U_1) + \cdots + a_r\mathrm{Dim}S(U_r)$ and let $k_1,\dots,k_r$ denote the elements such that $k_i \in \Gamma/\Gamma_{[U_i]}$, $i = 1,\dots,r$ and $\alpha(\underline{n})_{\{1\}}(x) = D(\oplus_{i=1}^s U_i, \oplus_{i=1}^s \psi^{k_i}U_i)$. Furthermore

$$\mathrm{Res}_N(\alpha(\underline{n})_{\{1\}}(x)) = D(\oplus_{i=1}^s \oplus_{j=1}^p U_{i,j}, \oplus_{i=1}^s \oplus_{j=1}^p \psi^{k_i}U_{i,j}),$$

which corresponds to $\oplus_{i=1}^s(k_i,\dots,k_i) \in \oplus_{i=1}^s \oplus_{j=1}^p \Gamma/\Gamma_{[U_{i,j}]} \subset jO(N)$. By the hypothesis of induction, $\mathrm{Res}_N x$ is in $jO(N,\mathrm{Res}_N\underline{n})$, and $\mathrm{Res}_N x$ goes to $\mathrm{Res}_N(\alpha(\underline{n})_{\{1\}}(x))$ via $\alpha(\mathrm{Res}_N\underline{n})$. Since $[U_{i,j}] \notin X(\mathrm{Res}_N\underline{n})$ for $r+1 \le i \le s$, $1 \le j \le p$, we see that $k_i = 1 \in \Gamma/\Gamma_{[U_{i,j}]}$ for $r+1 \le i \le s$, $1 \le j \le p$ and hence $D(U_{i,j},\psi^{k_i}U_{i,j}) = 1$ for $r+1 \le i \le s$, $1 \le j \le p$. Therefore $S(\psi^{k_i}U_{i,j})$ and $S(U_{i,j})$ is (stably) N-homotopy equivalent. (See [8].) Since $U_i = \mathrm{Ind}_N^G U_{i,j}$ and $\psi^{k_i}U_i = \mathrm{Ind}_N^G \psi^{k_i}U_{i,j}$ by Proposition 5.6, U_i and $\psi^{k_i}U_i$ are (stably) G-homotopy equi-valent and hence $D(U_i, \psi^{k_i} U_i) = 1$ for $r+1 \leq i \leq s$. Thus we see that

$$\alpha(\underline{n})_{\{1\}}(x) = D(\oplus_{i=1}^{r} U_i, \oplus_{i=1}^{r} \psi^{k_i} U_i).$$

Put $y = tO(\underline{n})_{\{1\}}((k_i)_i) \in jO(G;\underline{n})_{\{1\}}$. Then one can easily see that $\alpha(\underline{n})_{\{1\}}(y) = \alpha(\underline{n})_{\{1\}}(x)$. Since $\alpha(\underline{n})_{\{1\}}$ is injective, it follows that $x = y \in jO(G;\underline{n})_{\{1\}}$. Thus the proof is complete. □

Proof of Theorem A. (1): By a result of Laitinen [8, Theorem 6], the cancellation low holds for G-homotopy representations if G is a group of odd prime power order. This implies that $\alpha(\underline{n})$ is injective for a group of odd prime power order. Hence (1) holds.

(2): It directly follows from Theorem 5.8. □

Finally we give the proof of Remark 2 in Introduction.

Proof of Remark 2. It suffices to show that $\alpha(\underline{n})_{\{1\}}$ is injective when $H = 1 \in \mathcal{I}(\underline{n})$. In the case (1), we have already shown the injectivity of $\alpha(\underline{n})_{\{1\}}$. In the case (2), we set

$$\mathcal{M} = \{K \mid \text{If } L \leq K \text{ and } L \text{ is normal in } G, \text{ then } L = 1\}.$$

Since $Z(G)$ is not C_2 and also not 1, it follows that $Z(WK)$ is not C_2 for every $K \in \mathcal{M}$. Indeed since $K \cap Z(G) = 1$ for $K \in \mathcal{M}$ and $Z(G)K/K \leq Z(NK/K)$, it follows that $Z(WK)$ is not C_2, and in particular WK is neither dihedral nor semi-dihedral.

Take $x \in \operatorname{Ker} \alpha(\underline{n})_{\{1\}}$. Then x is represented by a function d such that $d(K) = 1$ for every subgroup $K \notin \mathcal{M}$. This function d is realized by the degree function $d(f)$ of some G-map $f : Y \to X$ ([11, Theorem 2.6]), where X, Y are homotopy representations with the dimension function $\underline{n}$. Then by the the same argument as in the proof of [8, Theorem 6], one can modify f to a G-homotopy equivalence by using the fact that WK is neither dihedral nor semi-dihedral for every $K \in \mathcal{M}$ (See [8, p.245, Remark]). Thus it is seen that $x = 1$ in $Pic(G;\underline{n})$ and hence $\alpha(\underline{n})_{\{1\}}$ is injective. □

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