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# THE LOGARITHMIC TERM OF THE SZEGÖ KERNEL FOR TWO-DIMENSIONAL GRAUERT TUBES

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**Abstract**

Every compact real-analytic Riemannian manifold has a complexification called the Grauert tube. We give an asymptotic expansion of the leading coefficient of the logarithmic term of the Szegö kernel for two-dimensional Grauert tubes.

## 1. Introduction

Every compact $C^\omega$ manifold $X$ has a complexification, that is, there exists a complex manifold $X_{\mathbb{C}}$ such that $X$ is a totally real submanifold of $X_{\mathbb{C}}$. When a Riemannian metric $g$ on $X$ is given, one can define a canonical complexification of $(X, g)$. This complexification is realized as a subset $T^r X := \{v \in TX \mid \|v\| < r\}$ of the tangent bundle $TX$ on $X$ for some $0 < r \leq \infty$, where $\|v\|$ denotes the length of a tangent vector $v$ with respect to the metric $g$. Then $X$ is identified with the zero section of $TX$, and the function $\rho(v) := 2\|v\|^2$ is strictly plurisubharmonic in $T^r X$. We call the complex manifold $T^r X$ the Grauert tube of radius $r$ over $X$. The purpose of this paper is to investigate the asymptotic behavior, as $r \to +0$, of the leading coefficient of the logarithmic term of the Szegö kernel for the two-dimensional Grauert tube.

For a relatively compact strongly pseudoconvex domain $\Omega$ with $C^\infty$ boundary $\partial\Omega$ in an $n$-dimensional complex manifold, we choose a contact form $\theta$ on $\partial\Omega$ called a pseudo-hermitian structure, and consider the volume form $\theta \wedge (d\theta)^{n-1}$ on $\partial\Omega$. It was shown in [2] and [1] that the Szegö kernel $S_\theta$ for $\Omega$ with respect to $\theta \wedge (d\theta)^{n-1}$ can be written in the form

$$S_\theta(z, \overline{z}) = \frac{\varphi_\theta(z)}{r(z)^n} + \psi_\theta(z) \log r(z), \tag{1.1}$$

where $\varphi_\theta$ and $\psi_\theta$ are functions on $\Omega$ which are smooth up to $\partial\Omega$, and $r(z)$ is a defining function for $\Omega$ with $r(z) > 0$ in $\Omega$. We note that the leading coefficient of the logarithmic term $\psi_\theta|_{\partial\Omega}$ is independent of the choice of $r(z)$ and gives a pseudo-hermitian invariant of $\theta$. More precisely speaking, it can be written as a linear combination of complete contractions, with respect to the Levi-form, of the tensor products of the Tanaka-Webster curvature, torsion and their covariant derivatives. In case $n = 2$, an explicit form of $\psi_\theta|_{\partial\Omega}$ was given in [5].

When $n = 2$, $\psi_\theta|_{\partial\Omega}$ is a constant multiple of the $Q$-curvature in real three-dimensional CR manifolds ([3]), which is a reason why we are interested in $\psi_\theta|_{\partial\Omega}$. We briefly explain the $Q$-curvature in conformal and CR geometries here. The $Q$-curvature in conformal geometry is a scalar Riemannian invariant which is conformally invariant up to an error given by a conformally invariant power of the Laplacian. In dimension two, the $Q$-curvature is equal to a half of the Gaussian curvature. On the other hand, let $M$ be a $(2n-1)$-dimensional CR manifold, and let $C$ be the bundle of complex $(n,0)$-forms on $M$. Then, for a pseudo-hermitian structure $\theta$ on $M$, the Fefferman metric $h_\theta$ is defined on the circle bundle $\widetilde{C} := (C \setminus M)/\mathbb{R}^+$ ([7]). Since the conformal $Q$-curvature of $h_\theta$ on $\widetilde{C}$ is invariant under the $S^1$-action, we can regard it as a function $Q_\theta$ on $M$. This $Q_\theta$ is called the CR $Q$-curvature on $M$. It should be noted that the integral $L_M := \int_M Q_\theta \theta \wedge (d\theta)^{n-1}$ gives a CR invariant. When $n = 2$, we have $L_M = 0$ for any CR manifold $M$. In general dimensions, it was shown in [3] that if $\theta$ is an invariant contact form, then $Q_\theta = 0$, and consequently $L_M = 0$. However, it is not known whether we can choose $\theta$ such that $Q_\theta \equiv 0$ for any CR manifold $M$. Moreover, many problems on $Q_\theta$ and $L_M$ remain unsolved. Our investigation of $\psi_\theta|_{\partial\Omega}$ for two-dimensional Grauert tubes is a trial to study the $Q$-curvature in CR geometry.

Let $(X, g)$ be an $n$-dimensional connected compact Riemannian manifold, and let $T^rX$ be the Grauert tube of radius $r$ over $X$. We define the function $\rho(v) := 2\|v\|^2$ for $v \in T^rX$, and set $\Omega_\varepsilon := \{v \in T^rX \mid \rho(v) < \varepsilon^2\}$ for $0 < \varepsilon < \sqrt{2}r$. Then $M_\varepsilon := \{\rho = \varepsilon^2\}$ is a strongly pseudoconvex CR manifold. We take a pseudo-hermitian structure $\theta^{(\varepsilon)} := \iota_\varepsilon^* \left(-\sqrt{-1}\,\partial\rho\right)$ on $M_\varepsilon$, where $\iota_\varepsilon$ is the embedding of $M_\varepsilon$ into $T^rX$. The Szegö kernel $S_{\theta^{(\varepsilon)}}$ can be written in the form (1.1). Setting $SX := \{U \in TX \mid \|U\| = 1\}$, we can identify $SX$ with $M_\varepsilon$ by the map $U \mapsto \varepsilon U/\sqrt{2}$. We regard $\psi_0^{(\varepsilon)} := \psi_{\theta^{(\varepsilon)}}|_{M_\varepsilon}$ as a function on $SX$ with the parameter $\varepsilon$ by this identification, and investigate the asymptotic behavior of $\psi_0^{(\varepsilon)}$ as $\varepsilon \to +0$.

We can now state our main result as follows:

**Theorem.** *Assume that $n = 2$. Then $\psi_0^{(\varepsilon)}$ has the following expansion as $\varepsilon \to +0$:*

$$\psi_0^{(\varepsilon)} = \sum_{l=0}^{L} F_l(U)\varepsilon^{2l} + O(\varepsilon^{2(L+1)}) \quad \text{for all} \quad L \geq 0, \tag{1.2}$$

*where $f = O(\varepsilon^m)$ is a function of $U$ and $\varepsilon$ such that $f/\varepsilon^m$ is bounded as $\varepsilon \to +0$. The coefficient $F_0$ is given by*

$$F_0 = -\frac{1}{120\pi^2}\left\{\nabla_i\nabla_j K - \frac{1}{2}(\nabla_k\nabla^k K)g_{ij}\right\}u^i u^j, \tag{1.3}$$

*where $K$ is the Gaussian curvature, $(\nabla_i\nabla_j K)u^i u^j$ is a contraction of $\nabla^2 K \otimes U \otimes U$, and $-\nabla_k\nabla^k$ is the Laplacian.*

We note that the tensor in the braces of (1.3) is the trace-free part of the Hessian of $K$. This means that the integral of $F_0$ on each fiber of $M_\varepsilon$ vanishes.

The proof of the theorem is based on a method of Kan. In [6], when $n = 2$, Kan gave expressions of the Tanaka-Webster connection by using $\rho$, and proved that the Burns-Epstein invariant $\mu^{(\varepsilon)}$ on $M_\varepsilon$ has the following expansion as $\varepsilon \to +0$:

$$(1.4)\quad \mu^{(\varepsilon)} = \frac{3}{16\pi\varepsilon^2}\int_X dV_X - \frac{1}{8\pi}\int_X K\,dV_X + \sum_{l=1}^{L} F_l^{\mu}\varepsilon^{2l} + O\big(\varepsilon^{2(L+1)}\big) \quad \text{for all} \quad L \geq 1,$$

where $dV_X$ is the volume form on $(X, g)$. She determined the coefficient of the constant term of (1.4) by calculating the invariant on the boundary of the Grauert tubes over $S^2$, without giving coefficients of terms of $\rho$ in detail. We determinine coefficients of lower order terms of $\rho$ explicitly (Proposition 4.2) by the method stated in [6], and then prove the theorem by direct calculation. We also give expansions of the Tanaka-Webster scalar curvature and torsion (Propositions 4.4 and 4.5).

This paper is organized as follows: In Section 2, we recall the definitions of three-dimensional CR manifolds, pseudo-hermitian structures and Szegö kernels, and state a result of Hirachi on the expression of $\psi_\theta|_{\partial\Omega}$. In Section 3, we recall the definition of Grauert tubes and the calculations of the Tanaka-Webster connection by Kan. In Section 4, we give coefficients of some lower order terms of $\rho$, and prove the theorem. Finally, in Section 5, we state the method, due to Kan, of determining the coefficients.

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## 2. Pseudo-hermitian structures and Szegö kernels

Let $M$ be a real three-dimensional $C^\infty$ manifold in a two-dimensional complex manifold $W$. We define a one-dimensional complex subbundle $T^{1,0}M \subset \mathbb{C}TM$ by

$$T^{1,0}M := T^{1,0}W \cap \mathbb{C}TM.$$

We call $T^{1,0}M$ a CR structure on $M$. We assume that $M$ is strongly pseudoconvex, that is, there exists a real non-vanishing one-form $\theta$ such that $\theta(V) = 0$ for any local section $V$ of $T^{1,0}M$ and that the Hermitian form on $T^{1,0}M$ defined by

$$L_\theta(V, \overline{V'}) := -\sqrt{-1}\,d\theta(V \wedge \overline{V'})$$

is positive-definite. We say that such a form $\theta$ defines a pseudo-hermitian structure.

Let $T$ be the unique vector field on $M$ such that $\theta(T) = 1$ and $T \rfloor d\theta = 0$, and let $Z_1$ be any local frame of $T^{1,0}M$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, where $Z_{\bar{1}} := \overline{Z_1}$, satisfies

$$d\theta = \sqrt{-1}\,h_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$

for some positive function $h_{1\bar{1}}$. In terms of this frame, the Tanaka-Webster connection $D$ on $\mathbb{C}TM$ is defined by the following relations for one-forms $\omega_1{}^1$ and $\tau_1$:

$$DZ_1 = \omega_1{}^1 \otimes Z_1, \quad DZ_{\bar{1}} = \omega_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \quad DT = 0,$$
$$d\theta^1 = \theta^1 \wedge \omega_1{}^1 + \theta \wedge \tau^1,$$
$$\omega_{1\bar{1}} + \omega_{\bar{1}1} = dh_{1\bar{1}},$$
$$\tau_1 \wedge \theta^1 = 0,$$

where $\omega_{\bar{1}}{}^{\bar{1}} := \overline{\omega_1{}^1}$ and we use $h_{1\bar{1}}$ and its inverse $h^{1\bar{1}}$ to raise or lower indices. We can write $\tau_1 = A_{11}\theta^1$, and we call $\tau_1$ the Tanaka-Webster torsion form. The curvature of the Tanaka-Webster connection is

$$d\omega_1{}^1 = R_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + W_1\theta^1 \wedge \theta - W_{\bar{1}}\theta^{\bar{1}} \wedge \theta.$$

The Tanaka-Webster scalar curvature is defined by $R := R_1{}^1$. If $T_{\alpha_1\cdots\alpha_k}$ are components of a tensor, then its $r$-th covariant derivative is the tensor with components denoted by $T_{\alpha_1\cdots\alpha_k,\gamma_1\cdots\gamma_r}$.

Let $\Omega$ be a relatively compact domain with $C^\infty$ boundary $\partial\Omega$ in $W$. If a volume form $\sigma$ on $\partial\Omega$ is specified, then the Szegö kernel $S$ is defined as the reproducing kernel associated with the Hardy space $H^2_\sigma(\Omega)$ consisting of holomorphic functions in $\Omega$ which have $L^2$ boundary values with respect to $\sigma$. For any complete orthonormal system $\{h_j\}$ of $H^2_\sigma(\Omega)$, $S$ is given by

$$S(z, \bar{z}) = \sum_{j=1}^{\infty} |h_j(z)|^2.$$

Assume that $\Omega$ is strongly pseudoconvex. We take a pseudo-hermitian structure $\theta$ on $\partial\Omega$, and consider the Szegö kernel $S_\theta$ for $\Omega$ with respect to the volume form $\theta \wedge d\theta$ on $\partial\Omega$. It was shown in [2] and [1] that $S_\theta$ admits an expansion

$$S_\theta(z, \bar{z}) = \frac{\varphi_\theta(z)}{r(z)^2} + \psi_\theta(z) \log r(z)$$

where $\varphi_\theta$ and $\psi_\theta$ are functions on $\Omega$ which are smooth up to $\partial\Omega$, and $r(z)$ is a defining function for $\Omega$ with $r(z) > 0$ in $\Omega$. As for $\psi_\theta|_{\partial\Omega}$, it was proved in [5] that it is given by

$$\psi_\theta|_{\partial\Omega} = \frac{1}{24\pi^2}\left(-R_{,1}{}^{1} - R_{,\bar{1}}{}^{\bar{1}} - 2\,\mathrm{Im}\,A_{11,}{}^{11}\right), \tag{2.1}$$

where $R_{,1}{}^1$, $R_{,\bar{1}}{}^{\bar{1}}$ and $A_{11,}{}^{11}$ are the contractions of the second covariant derivatives.

## 3. A pseudo-hermitian structure on the boundary of Grauert tubes

Let $(X, g)$ be an $n$-dimensional connected compact $C^\omega$ Riemannian manifold, and let $\gamma : \mathbb{R} \to X$ be a geodesic. We define a map $\psi_\gamma : \mathbb{C} \to TX$ by

$$\psi_\gamma(\xi + \sqrt{-1}\,\eta) := \eta\dot{\gamma}(\xi).$$

It was proved in [9] that, for some $0 < r \le \infty$, $T^rX := \{v \in TX \mid \|v\| < r\}$ admits the complex structure such that $\psi_\gamma$ is holomorphic for every geodesic $\gamma$ on $X$. We call the complex manifold $T^rX$ the Grauert tube of radius $r$ over $X$.

Define the function $\rho : T^rX \to \mathbb{R}$ by $\rho(v) := 2\|v\|^2$ for $v \in T^rX$. Then the following theorem holds:

**Theorem 3.1** ([4], [8]). *The function $\rho$ has the following properties*:
(1) *$\rho$ is strictly plurisubharmonic,*
(2) *$X = \rho^{-1}(0)$, where we identify the zero section of $TX$ with $X$,*
(3) *the Kähler metric $G$ obtained from the Kähler form $\sqrt{-1}\,\partial\overline{\partial}\rho/2$ is compatible with $g$, that is, $G|_X = g$,*
(4) *$(\partial\overline{\partial}\sqrt{\rho}\,)^n = 0$ in $T^rX \setminus X$, and*
(5) *if $\widetilde{\rho}$ is another function which satisfies all the conditions above, then $\widetilde{\rho} = \rho$.*

Set $\Omega_\varepsilon := \{v \in T^rX \mid \rho(v) < \varepsilon^2\}$ and $M_\varepsilon := \partial\Omega_\varepsilon$ for $0 < \varepsilon < \sqrt{2}\,r$. Then (1) of this theorem shows that $M_\varepsilon$ is a strongly pseudoconvex CR manifold.

We assume that $n = 2$. For a geodesic normal coordinate system $(x^1, x^2)$ centered at $p \in X$, let $(z^1, z^2) = (x^1 + \sqrt{-1}\,y^1, x^2 + \sqrt{-1}\,y^2)$ be the local holomorphic coordinate system. Then $(\sqrt{-1}\,y^1, \sqrt{-1}\,y^2)$ corresponds to the vector $y^i(\partial/\partial x^i)(p)$, and for $y^i(\partial/\partial x^i)(p) \in M_\varepsilon$, there exists a vector $u^i(\partial/\partial x^i)(p) \in SX := \{U \in TX \mid \|U\| = 1\}$ such that

$$y^1 = \frac{1}{\sqrt{2}}\varepsilon u^1, \qquad y^2 = \frac{1}{\sqrt{2}}\varepsilon u^2. \tag{3.1}$$

Take a pseudo-hermitian structure $\theta^{(\varepsilon)} := \iota_\varepsilon^*(-\sqrt{-1}\,\partial\rho)$ on $M_\varepsilon$, where $\iota_\varepsilon$ is the embedding of $M_\varepsilon$ into $T^rX$. We define local sections of $\mathbb{C}T(T^rX \setminus X)$ by

$$T = \frac{\sqrt{-1}}{2\rho P}\left(A\frac{\partial}{\partial z^2} - \overline{A}\frac{\partial}{\partial \overline{z^2}} - B\frac{\partial}{\partial z^1} + \overline{B}\frac{\partial}{\partial \overline{z^1}}\right),$$

$$Z_1 = \frac{1}{\sqrt{2}\,\rho^{1/2}P^{1/2}}\left(\frac{\partial\rho}{\partial z^2}\frac{\partial}{\partial z^1} - \frac{\partial\rho}{\partial z^1}\frac{\partial}{\partial z^2}\right),$$

where

$$A = \frac{\partial\rho}{\partial\overline{z^2}}\frac{\partial^2\rho}{\partial z^1\partial\overline{z^1}} - \frac{\partial\rho}{\partial\overline{z^1}}\frac{\partial^2\rho}{\partial z^1\partial\overline{z^2}}, \quad B = \frac{\partial\rho}{\partial\overline{z^2}}\frac{\partial^2\rho}{\partial z^2\partial\overline{z^1}} - \frac{\partial\rho}{\partial\overline{z^1}}\frac{\partial^2\rho}{\partial z^2\partial\overline{z^2}}, \tag{3.2}$$

$$P = \frac{\partial^2\rho}{\partial z^1\partial\overline{z^1}}\frac{\partial^2\rho}{\partial z^2\partial\overline{z^2}} - \frac{\partial^2\rho}{\partial z^1\partial\overline{z^2}}\frac{\partial^2\rho}{\partial z^2\partial\overline{z^1}},$$

and, $\rho^{1/2}$ and $P^{1/2}$ denote the positive branches of $\sqrt{\rho}$ in $T^rX \setminus X$ and $\sqrt{P}$ in $T^rX$, respectively (note that $P > 0$ in $T^rX$ since $\rho$ is strictly plurisubharmonic). For $\varepsilon > 0$, $T^{(\varepsilon)} := T|_{M_\varepsilon}$ and $Z_1^{(\varepsilon)} := Z_1|_{M_\varepsilon}$ are local sections of $\mathbb{C}TM_\varepsilon$, $T^{(\varepsilon)}$ satisfies $\theta^{(\varepsilon)}(T^{(\varepsilon)}) = 1$ and $T^{(\varepsilon)}\rfloor d\theta^{(\varepsilon)} = 0$, and $Z_1^{(\varepsilon)}$ is a local frame of $T^{1,0}M_\varepsilon$. Setting

$$\theta^1 := \frac{1}{\sqrt{2}\,\rho^{1/2}P^{1/2}}\left(Adz^1 + Bdz^2\right),$$

we see that $\{\theta^{(\varepsilon)}, \theta^{(\varepsilon)1}, \theta^{(\varepsilon)\bar{1}}\}$, where $\theta^{(\varepsilon)1} := \iota_\varepsilon^*(\theta^1)$, is the coframe dual to $\{T^{(\varepsilon)}, Z_1^{(\varepsilon)}, Z_{\bar{1}}^{(\varepsilon)}\}$ and that $d\theta^{(\varepsilon)} = \sqrt{-1}\,\theta^{(\varepsilon)1} \wedge \theta^{(\varepsilon)\bar{1}}$.

The Tanaka-Webster connection form, the torsion form and the curvature are given as follows:

$$\omega_1^{(\varepsilon)1} = \sqrt{-1}\,a^{(\varepsilon)}\theta^{(\varepsilon)} + b^{(\varepsilon)}\theta^{(\varepsilon)1} - \overline{b^{(\varepsilon)}}\theta^{(\varepsilon)\bar{1}}, \quad \tau_1^{(\varepsilon)} = A_{11}^{(\varepsilon)}\theta^{(\varepsilon)1},$$
$$d\omega_1^{(\varepsilon)1} = R_{1\bar{1}}^{(\varepsilon)}\theta^{(\varepsilon)1} \wedge \theta^{(\varepsilon)\bar{1}} + W_1^{(\varepsilon)}\theta^{(\varepsilon)1} \wedge \theta^{(\varepsilon)} - W_{\bar{1}}^{(\varepsilon)}\theta^{(\varepsilon)\bar{1}} \wedge \theta^{(\varepsilon)},$$

where $a^{(\varepsilon)}$, $b^{(\varepsilon)}$, $R_{1\bar{1}}^{(\varepsilon)}$, $A_{11}^{(\varepsilon)}$ and $W_1^{(\varepsilon)}$ are restrictions to $M_\varepsilon$ of $a$, $b$, $R_{1\bar{1}}$, $A_{11}$ and $W_1$ which are defined by

$$a = -\frac{1}{2\rho} + \frac{1}{4\rho P^2}\left(A\frac{\partial P}{\partial z^2} + \overline{A}\frac{\partial P}{\partial\overline{z^2}} - B\frac{\partial P}{\partial z^1} - \overline{B}\frac{\partial P}{\partial\overline{z^1}}\right), \tag{3.3}$$

$$b = \frac{1}{2P}Z_1P, \tag{3.4}$$

$$R_{1\bar{1}} = -a - 2|b|^2 - Z_1\overline{b} - Z_{\bar{1}}b, \tag{3.5}$$

$$A_{11} = \frac{\sqrt{-1}}{2\sqrt{2}\,\rho^{3/2}P^{3/2}}\left(\overline{A}Z_1\overline{B} - \overline{B}Z_1\overline{A}\right), \tag{3.6}$$

$$W_1 = \sqrt{-1}\,ab + \overline{b}A_{11} + \sqrt{-1}\,Z_1a - Tb.$$

## 4. Proof of the theorem

In Sections 4 and 5, $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ denote multi-indices with $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2 \geq 0$. We use the following propositions in order to prove the theorem:

**Proposition 4.1** ([6]). *Let $g = g_{ij}dx^i \otimes dx^j$. Then $\rho$ is of the following form:*

$$\rho(z^1, z^2) = 2g_{11}(x)(y^1)^2 + 4g_{12}(x)y^1y^2 + 2g_{22}(x)(y^2)^2 + \sum_{|\alpha|\geq 4,\ \text{even}} \varphi_\alpha(x)y^\alpha, \tag{4.1}$$

*where each $\varphi_\alpha$ is a real-analytic function.*

Let $K(x)$ be the Gaussian curvature of $X$, and let

$$K := K(0), \quad K_{\gamma_1 \cdots \gamma_j} := \frac{\partial^j K}{\partial x^{\gamma_1} \cdots \partial x^{\gamma_j}}(0).$$

**Proposition 4.2.** *Set* $|y|^2 = (y^1)^2 + (y^2)^2$ *and* $Y = x^2 y^1 - x^1 y^2$. *Then*

$$\rho(z^1, z^2) = 2|y|^2 + \{f_1(x) + f_2(y)\} Y^2 + O(|xy|^7), \tag{4.2}$$

*where*

$$f_1(x) = -\frac{K}{3} - \frac{1}{6}\left(K_1 x^1 + K_2 x^2\right) + \left(\frac{K^2}{45} - \frac{K_{11}}{20}\right)(x^1)^2 - \frac{K_{12}}{10} x^1 x^2 + \left(\frac{K^2}{45} - \frac{K_{22}}{20}\right)(x^2)^2,$$

$$f_2(y) = \left(\frac{K^2}{15} + \frac{K_{11}}{60}\right)(y^1)^2 + \frac{K_{12}}{30} y^1 y^2 + \left(\frac{K^2}{15} + \frac{K_{22}}{60}\right)(y^2)^2,$$

*and* $O(|xy|^m)$ *is a function of the form*

$$\sum_{|\alpha|+|\beta|=m} a_{\alpha\beta}(x, y) x^\alpha y^\beta$$

*for real-analytic functions* $a_{\alpha\beta}$.

We first give the proof of the theorem in the rest of this section, and then prove Proposition 4.2 in the next section.

We write $h \in \mathcal{F}$ if $h$ is a real-analytic function on $T^r X$ and the Taylor expansion of $h$ at $y = 0$ can be written in the form

$$\sum_{|\alpha|:\ \text{even}} F_\alpha(x) y^\alpha + \sqrt{-1} \sum_{|\alpha|:\ \text{odd}} F_\alpha(x) y^\alpha \tag{4.3}$$

for real-valued functions $F_\alpha$. In particular, we write $h \in \mathcal{F}_0$ if $h \in \mathcal{F}$ and $h$ is a real-valued function. The following properties hold for $h$, $h' \in \mathcal{F}$:

(1) $h + h'$, $hh' \in \mathcal{F}$,

(2) $\partial h/\partial z^1$, $\partial h/\partial z^2$, $\partial h/\partial \overline{z^1}$, $\partial h/\partial \overline{z^2} \in \mathcal{F}$, and

(3) $\rho T h$, $\rho^{1/2} Z_1 h \in \mathcal{F}$.

**Lemma 4.3.** *Let* $\psi_0$ *be a function on* $T^r X \setminus X$ *such that* $\psi_0|_{M_\varepsilon} = \psi_0^{(\varepsilon)} := \psi_{\theta^{(\varepsilon)}}|_{M_\varepsilon}$ *for* $0 < \varepsilon < \sqrt{2}\, r$. *Then* $\rho^3 \psi_0 \in \mathcal{F}_0$.

Proof. It follows from $\rho \in \mathcal{F}$ that $A$, $B$, $P \in \mathcal{F}$. Considering the Taylor expansion of $P^{k/2}$ at the origin for $k \in \mathbb{Z}$, we have $P^{k/2} \in \mathcal{F}$. By the definitions of $a$, $b$, $R_{1\bar{1}}$ and $A_{11}$, we see $\rho a$, $\rho^{1/2} b$, $\rho R_{1\bar{1}}$, $\sqrt{-1}\,\rho^2 A_{11} \in \mathcal{F}$. Since $R_{1\bar{1}}$ is real, we have $\rho R_{1\bar{1}} \in \mathcal{F}_0$.

We note that $R^{(\varepsilon)} = R^{(\varepsilon)}_{1\bar{1}} = R_{1\bar{1}}|_{M_\varepsilon}$. Set

(4.4) $$R_{,1}{}^{1} + R_{,\bar{1}}{}^{\bar{1}} := 2(Z_{\bar{1}}Z_1 R_{1\bar{1}} + \bar{b} Z_1 R_{1\bar{1}}) - \sqrt{-1}\, T R_{1\bar{1}}$$

and

(4.5) $$A_{11,}{}^{11} := Z_{\bar{1}}^2 A_{11} + 2\left(Z_{\bar{1}}\bar{b}\right) A_{11} + 3\bar{b} Z_{\bar{1}} A_{11} + 2\bar{b}^2 A_{11}.$$

Then we have $\rho^2(R_{,1}{}^{1} + R_{,\bar{1}}{}^{\bar{1}})$, $\sqrt{-1}\,\rho^3 A_{11,}{}^{11} \in \mathcal{F}$ and

$$R^{(\varepsilon)1}_{,1} + R^{(\varepsilon)\bar{1}}_{,\bar{1}} = (R_{,1}{}^{1} + R_{,\bar{1}}{}^{\bar{1}})\big|_{M_\varepsilon}, \qquad A^{(\varepsilon)}_{11,}{}^{11} = A_{11,}{}^{11}\big|_{M_\varepsilon}.$$

Hence, we see $\rho^3\psi_0 \in \mathcal{F}$ by (2.1). Since $\psi_0$ is real, we obtain $\rho^3\psi_0 \in \mathcal{F}_0$. □

We now calculate $R_{1\bar{1}}$ and $A_{11}$. First, we calculate $A$, $B$ and $P$ by differentiating (4.2) and using (3.2). Next, differentiating $P$, by the definitions of $a$ and $b$, we get them. Then, differentiating $\bar{b}$ with respect to $Z_1$ and using (3.5), we obtain

(4.6)
$$\begin{aligned} R_{1\bar{1}} = \frac{1}{2\rho}\Bigg[ & 1 + \frac{2}{3}\left(K + K_1x^1 + K_2x^2\right)|y|^2 \\ & + \left\{\left(-\frac{K_{11}}{15} + \frac{4}{15}K_{22} - \frac{4}{45}K^2\right)(y^1)^2 - \frac{2}{3}K_{12}y^1y^2 \right. \\ & \left. + \left(\frac{4}{15}K_{11} - \frac{K_{22}}{15} - \frac{4}{45}K^2\right)(y^2)^2\right\}|y|^2 \\ & + \left\{\frac{K_{11}}{3}(x^1)^2 + \frac{2}{3}K_{12}x^1x^2 + \left(\frac{K_{22}}{3} - \frac{K^2}{9}\right)(x^2)^2\right\}(y^1)^2 \\ & + \left\{\left(\frac{K_{11}}{3} - \frac{K^2}{9}\right)(x^1)^2 + \frac{2}{3}K_{12}x^1x^2 + \frac{K_{22}}{3}(x^2)^2\right\}(y^2)^2 \\ & + \frac{2}{9}K^2x^1x^2y^1y^2 + O(|xy|^5)\Bigg]. \end{aligned}$$

Differentiating $\overline{A}$ and $\overline{B}$ with respect to $Z_1$ and using (3.6), we get

(4.7)
$$\begin{aligned} A_{\bar{1}\bar{1}} := {} & \overline{A_{11}} \\ = {} & \frac{1}{4\rho^2}\Bigg[-\frac{4}{3}K\left(x^1y^1 + x^2y^2\right)|y|^2 + \frac{1}{3}\left(K_1y^1 + K_2y^2\right)|y|^4 \\ & - \left\{K_1(x^1)^2 + \frac{2}{3}K_2x^1x^2 + \frac{K_1}{3}(x^2)^2\right\}y^1|y|^2 \\ & - \left\{\frac{K_2}{3}(x^1)^2 + \frac{2}{3}K_1x^1x^2 + K_2(x^2)^2\right\}y^2|y|^2 \end{aligned}$$

$$
\begin{aligned}
&+\left\{\left(\frac{2}{5}K_{11}+\frac{22}{45}K^2\right)x^1+\frac{K_{12}}{5}x^2\right\}(y^1)^3|y|^2\\
&+\left\{\frac{3}{5}K_{12}x^1+\left(\frac{K_{11}}{5}+\frac{K_{22}}{5}+\frac{22}{45}K^2\right)x^2\right\}(y^1)^2y^2|y|^2\\
&+\left\{\left(\frac{K_{11}}{5}+\frac{K_{22}}{5}+\frac{22}{45}K^2\right)x^1+\frac{3}{5}K_{12}x^2\right\}y^1(y^2)^2|y|^2\\
&+\left\{\frac{K_{12}}{5}x^1+\left(\frac{2}{5}K_{22}+\frac{22}{45}K^2\right)x^2\right\}(y^2)^3|y|^2\\
+\sqrt{-1}\Bigg[&4|y|^2-\frac{4}{3}K|y|^4-\frac{2}{3}KY^2\\
&-\frac{4}{3}\left(K_1x^1+K_2x^2\right)|y|^4+\frac{2}{9}K^2|y|^2Y^2+2f_2(y)Y^2\\
&+\left\{\left(\frac{2}{15}K_{11}+\frac{14}{45}K^2\right)(y^1)^2+\frac{4}{15}K_{12}y^1y^2+\left(\frac{2}{15}K_{22}+\frac{14}{45}K^2\right)(y^2)^2\right\}|y|^4\\
&-\left\{\left(\frac{2}{3}K_{11}+\frac{2}{9}K^2\right)(x^1)^2+\frac{4}{3}K_{12}x^1x^2+\left(\frac{2}{3}K_{22}-\frac{2}{9}K^2\right)(x^2)^2\right\}(y^1)^2|y|^2\\
&-\left\{\left(\frac{2}{3}K_{11}-\frac{2}{9}K^2\right)(x^1)^2+\frac{4}{3}K_{12}x^1x^2+\left(\frac{2}{3}K_{22}+\frac{2}{9}K^2\right)(x^2)^2\right\}(y^2)^2|y|^2\\
&-\frac{8}{9}K^2x^1x^2y^1y^2|y|^2\Bigg]+O(|x|^3)+O(|xy|^7)\Bigg].
\end{aligned}
$$

By (4.4) and differentiation of (4.6), we have

$$
\left(R_{,1}{}^{1}+R_{,\bar{1}}{}^{\bar{1}}\right)\Big|_{x=0}=\frac{1}{\rho^2}\left[\left\{\frac{K_{11}}{3}(y^1)^2+\frac{2}{3}K_{12}y^1y^2+\frac{K_{22}}{3}(y^2)^2\right\}|y|^2+O(|y|^5)\right].
$$

Differentiating $\overline{b}$ and the conjugate of (4.7) with respect to $Z_{\bar{1}}$ and using (4.5), we have

$$
\begin{aligned}
A_{11,}{}^{11}\Big|_{x=0}=\frac{1}{\rho^3}\Bigg[&-\frac{1}{3}\left(K_1y^1+K_2y^2\right)|y|^4\\
&+\sqrt{-1}\Bigg\{\left(\frac{K_{11}}{15}-\frac{2}{5}K_{22}\right)(y^1)^2\\
&\qquad+\frac{14}{15}K_{12}y^1y^2-\left(\frac{2}{5}K_{11}-\frac{K_{22}}{15}\right)(y^2)^2\Bigg\}|y|^4+O(|y|^7)\Bigg].
\end{aligned}
$$

Therefore, by these two equations, (2.1) and Lemma 4.3, substituting (3.1), for $\varepsilon > 0$, we obtain

$$
\psi_0^{(\varepsilon)}\Big|_{x=0}=\frac{1}{24\pi^2}\left[\frac{1}{10}\nabla_k\nabla^kK-\frac{1}{5}\left\{K_{11}(u^1)^2+2K_{12}u^1u^2+K_{22}(u^2)^2\right\}+O(\varepsilon^2)\right].
$$

This completes the proof of the theorem.

REMARK. Since $\rho R_{1\bar{1}} \in \mathcal{F}_0$ and $\sqrt{-1}\,\rho^2 A_{\bar{1}\bar{1}} \in \mathcal{F}$ by the proof of Lemma 4.3, setting $x = 0$ and substituting (3.1) into (4.6) and (4.7), we have the following propositions:

**Proposition 4.4.** *The Tanaka-Webster scalar curvature $R^{(\varepsilon)}$ has the following expansion as $\varepsilon \to +0$:*

$$R^{(\varepsilon)} = \frac{1}{2\varepsilon^2} + \sum_{l=0}^{L} F_l^R(U)\varepsilon^{2l} + O(\varepsilon^{2(L+1)}) \quad \textit{for all} \quad L \geq 0,$$

*where*

$$F_0^R = \frac{1}{6}K, \qquad F_1^R = -\frac{1}{90}K^2 + \left\{\frac{1}{30}(\nabla_k\nabla^k K)g_{ij} - \frac{1}{24}\nabla_i\nabla_j K\right\}u^i u^j.$$

**Proposition 4.5.** *The Tanaka-Webster torsion $A_{\bar{1}}^{(\varepsilon)1}$ has the following expansion as $\varepsilon \to +0$:*

$$A_{\bar{1}}^{(\varepsilon)1} = \frac{\sqrt{-1}}{2\varepsilon^2} + \sum_{l=0}^{L} F_l^\tau(U)\varepsilon^l + O(\varepsilon^{L+1}) \quad \textit{for all} \quad L \geq 0,$$

*where*

$$F_0^\tau = -\frac{\sqrt{-1}}{12}K, \qquad F_1^\tau = \frac{1}{48\sqrt{2}}(\nabla_i K)u^i,$$
$$F_2^\tau = \frac{7\sqrt{-1}}{720}K^2 + \frac{\sqrt{-1}}{240}\left(\nabla_i\nabla_j K\right)u^i u^j.$$

## 5. Determination of lower order terms of $\rho$

We give the proof of Proposition 4.2 by using Proposition 4.1 and a method of Kan. It follows from (4) of Theorem 3.1 that

$$(\partial\bar{\partial}\sqrt{\rho})^2 = -\frac{1}{4\rho^2}\left\{\partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho - \rho(\partial\bar{\partial}\rho \wedge \partial\bar{\partial}\rho)\right\} = 0,$$

that is,

$$\partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho = \rho(\partial\bar{\partial}\rho \wedge \partial\bar{\partial}\rho).$$

This equation is equivalent to the following one:

$$(5.1) \qquad 2\rho\left\{\left(\frac{\partial^2\rho}{\partial(x^1)^2} + \frac{\partial^2\rho}{\partial(y^1)^2}\right)\left(\frac{\partial^2\rho}{\partial(x^2)^2} + \frac{\partial^2\rho}{\partial(y^2)^2}\right)\right.$$

$$
\begin{aligned}
&- \left( \frac{\partial^2 \rho}{\partial x^1 \partial x^2} + \frac{\partial^2 \rho}{\partial y^1 \partial y^2} \right)^2 - \left( \frac{\partial^2 \rho}{\partial x^1 \partial y^2} - \frac{\partial^2 \rho}{\partial x^2 \partial y^1} \right)^2 \Bigg\} \\
= & \left\{ \left( \frac{\partial \rho}{\partial x^1} \right)^2 + \left( \frac{\partial \rho}{\partial y^1} \right)^2 \right\} \left( \frac{\partial^2 \rho}{\partial (x^2)^2} + \frac{\partial^2 \rho}{\partial (y^2)^2} \right) \\
&+ \left\{ \left( \frac{\partial \rho}{\partial x^2} \right)^2 + \left( \frac{\partial \rho}{\partial y^2} \right)^2 \right\} \left( \frac{\partial^2 \rho}{\partial (x^1)^2} + \frac{\partial^2 \rho}{\partial (y^1)^2} \right) \\
&- 2 \left( \frac{\partial \rho}{\partial x^1} \frac{\partial \rho}{\partial x^2} + \frac{\partial \rho}{\partial y^1} \frac{\partial \rho}{\partial y^2} \right) \left( \frac{\partial^2 \rho}{\partial x^1 \partial x^2} + \frac{\partial^2 \rho}{\partial y^1 \partial y^2} \right) \\
&+ 2 \left( \frac{\partial \rho}{\partial x^2} \frac{\partial \rho}{\partial y^1} - \frac{\partial \rho}{\partial x^1} \frac{\partial \rho}{\partial y^2} \right) \left( \frac{\partial^2 \rho}{\partial x^1 \partial y^2} - \frac{\partial^2 \rho}{\partial x^2 \partial y^1} \right).
\end{aligned}
$$

By using Proposition 4.1, we set

$$
\begin{aligned}
\rho(z^1, z^2) &= 2g_{11}(x)(y^1)^2 + 4g_{12}(x) y^1 y^2 + 2g_{22}(x)(y^2)^2 \\
&\quad + \sum_{j=0}^{4} \varphi_j(x)(y^1)^{4-j}(y^2)^j + \sum_{|\alpha| \geq 6, \ \text{even}} \varphi_\alpha(x) y^\alpha.
\end{aligned} \tag{5.2}
$$

The Taylor expansions of $g_{ij}$ are given as follows:

$$
\begin{aligned}
g_{11}(x) &= 1 + (x^2)^2 f(x) + O(|x|^5), \\
g_{12}(x) &= -x^1 x^2 f(x) + O(|x|^5), \\
g_{22}(x) &= 1 + (x^1)^2 f(x) + O(|x|^5),
\end{aligned}
$$

where

$$
f(x) = -\frac{K}{6} - \frac{1}{12}\left(K_1 x^1 + K_2 x^2\right) + \left( \frac{K^2}{90} - \frac{K_{11}}{40} \right)(x^1)^2 - \frac{K_{12}}{20} x^1 x^2 + \left( \frac{K^2}{90} - \frac{K_{22}}{40} \right)(x^2)^2.
$$

Substituting (5.2) into (5.1) and comparing the coefficients of $(y^1)^4$ on both sides of (5.1), we have

$$
\begin{aligned}
\varphi_0(x) = & -\frac{1}{3} \frac{\partial^2 g_{11}}{\partial (x^1)^2} + \frac{1}{6(g_{11} g_{22} - g_{12}^2)} \left\{ g_{11} \left( \frac{\partial g_{11}}{\partial x^2} - 2 \frac{\partial g_{12}}{\partial x^1} \right)^2 \right. \\
& \left. + 2 g_{12} \frac{\partial g_{11}}{\partial x^1} \left( \frac{\partial g_{11}}{\partial x^2} - 2 \frac{\partial g_{12}}{\partial x^1} \right) + g_{22} \left( \frac{\partial g_{11}}{\partial x^1} \right)^2 \right\} \\
= & \left( \frac{K^2}{15} + \frac{K_{11}}{60} \right)(x^2)^2 + O(|x|^3).
\end{aligned}
$$

Arguments similar to above yield

$$\varphi_1(x) = -\left(\frac{2}{15}K^2 + \frac{K_{11}}{30}\right)x^1x^2 + \frac{K_{12}}{30}(x^2)^2 + O(|x|^3),$$
$$\varphi_2(x) = \left(\frac{K^2}{15} + \frac{K_{11}}{60}\right)(x^1)^2 - \frac{K_{12}}{15}x^1x^2 + \left(\frac{K^2}{15} + \frac{K_{22}}{60}\right)(x^2)^2 + O(|x|^3),$$
$$\varphi_3(x) = \frac{K_{12}}{30}(x^1)^2 - \left(\frac{2}{15}K^2 + \frac{K_{22}}{30}\right)x^1x^2 + O(|x|^3),$$
$$\varphi_4(x) = \left(\frac{K^2}{15} + \frac{K_{22}}{60}\right)(x^1)^2 + O(|x|^3).$$

Hence one can write

$$\rho(z^1, z^2) = 2|y|^2 + \{f_1(x) + f_2(y)\}Y^2 + \sum_{j=0}^{6} p_j(y^1)^{6-j}(y^2)^j + O(|xy|^7). \tag{5.3}$$

We substitute (5.3) into (5.1) and set $x = 0$. Comparing the coefficients of $(y^1)^{6-j}(y^2)^j$ on both sides of (5.1), we have $p_j = 0$ for $j = 0, \ldots, 6$. We therefore obtain Proposition 4.2.

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