Title	The logarithmic term of the Szegö kernel for two-dimensional Grauert tubes
Author(s)	Koizumi, Eisuke
Citation	Osaka Journal of Mathematics. 2005, 42(2), p. 339-351
Version Type	VoR
URL	https://doi.org/10.18910/8160
rights	
Note	

Koizumi, E.
Osaka J. Math.
42 (2005), 339–351

THE LOGARITHMIC TERM OF THE SZEGÖ KERNEL FOR TWO-DIMENSIONAL GRAUERT TUBES

EISUKE KOIZUMI

(Received October 20, 2003)

Abstract

Every compact real-analytic Riemannian manifold has a complexification called the Grauert tube. We give an asymptotic expansion of the leading coefficient of the logarithmic term of the Szegö kernel for two-dimensional Grauert tubes.

1. Introduction

Every compact C^ω manifold X has a complexification, that is, there exists a complex manifold $X_{\mathbb{C}}$ such that X is a totally real submanifold of $X_{\mathbb{C}}$. When a Riemannian metric g on X is given, one can define a canonical complexification of (X, g). This complexification is realized as a subset $T^r X := \{v \in TX \mid \|v\| < r\}$ of the tangent bundle TX on X for some $0 < r \leq \infty$, where $\|v\|$ denotes the length of a tangent vector v with respect to the metric g. Then X is identified with the zero section of TX, and the function $\rho(v) := 2\|v\|^2$ is strictly plurisubharmonic in $T^r X$. We call the complex manifold $T^r X$ the Grauert tube of radius r over X. The purpose of this paper is to investigate the asymptotic behavior, as $r \to +0$, of the leading coefficient of the logarithmic term of the Szegö kernel for the two-dimensional Grauert tube.

For a relatively compact strongly pseudoconvex domain Ω with C^∞ boundary $\partial\Omega$ in an n-dimensional complex manifold, we choose a contact form θ on $\partial\Omega$ called a pseudo-hermitian structure, and consider the volume form $\theta \wedge (d\theta)^{n-1}$ on $\partial\Omega$. It was shown in [2] and [1] that the Szegö kernel S_θ for Ω with respect to $\theta \wedge (d\theta)^{n-1}$ can be written in the form

$$S_\theta(z, \overline{z}) = \frac{\varphi_\theta(z)}{r(z)^n} + \psi_\theta(z) \log r(z), \tag{1.1}$$

where φ_θ and ψ_θ are functions on Ω which are smooth up to $\partial\Omega$, and $r(z)$ is a defining function for Ω with $r(z) > 0$ in Ω. We note that the leading coefficient of the logarithmic term $\psi_\theta|_{\partial\Omega}$ is independent of the choice of $r(z)$ and gives a pseudo-hermitian invariant of θ. More precisely speaking, it can be written as a linear combination of complete contractions, with respect to the Levi-form, of the tensor products of the Tanaka-Webster curvature, torsion and their covariant derivatives. In case $n = 2$, an explicit form of $\psi_\theta|_{\partial\Omega}$ was given in [5].

When $n = 2$, $\psi_\theta|_{\partial\Omega}$ is a constant multiple of the Q-curvature in real three-dimensional CR manifolds ([3]), which is a reason why we are interested in $\psi_\theta|_{\partial\Omega}$. We briefly explain the Q-curvature in conformal and CR geometries here. The Q-curvature in conformal geometry is a scalar Riemannian invariant which is conformally invariant up to an error given by a conformally invariant power of the Laplacian. In dimension two, the Q-curvature is equal to a half of the Gaussian curvature. On the other hand, let M be a $(2n-1)$-dimensional CR manifold, and let C be the bundle of complex $(n,0)$-forms on M. Then, for a pseudo-hermitian structure θ on M, the Fefferman metric h_θ is defined on the circle bundle $\widetilde{C} := (C \setminus M)/\mathbb{R}^+$ ([7]). Since the conformal Q-curvature of h_θ on $\widetilde{C}$ is invariant under the S^1-action, we can regard it as a function Q_θ on M. This Q_θ is called the CR Q-curvature on M. It should be noted that the integral $L_M := \int_M Q_\theta \theta \wedge (d\theta)^{n-1}$ gives a CR invariant. When $n = 2$, we have $L_M = 0$ for any CR manifold M. In general dimensions, it was shown in [3] that if θ is an invariant contact form, then $Q_\theta = 0$, and consequently $L_M = 0$. However, it is not known whether we can choose θ such that $Q_\theta \equiv 0$ for any CR manifold M. Moreover, many problems on Q_θ and L_M remain unsolved. Our investigation of $\psi_\theta|_{\partial\Omega}$ for two-dimensional Grauert tubes is a trial to study the Q-curvature in CR geometry.

Let (X, g) be an n-dimensional connected compact Riemannian manifold, and let T^rX be the Grauert tube of radius r over X. We define the function $\rho(v) := 2\|v\|^2$ for $v \in T^rX$, and set $\Omega_\varepsilon := \{v \in T^rX \mid \rho(v) < \varepsilon^2\}$ for $0 < \varepsilon < \sqrt{2}r$. Then $M_\varepsilon := \{\rho = \varepsilon^2\}$ is a strongly pseudoconvex CR manifold. We take a pseudo-hermitian structure $\theta^{(\varepsilon)} := \iota_\varepsilon^* \left(-\sqrt{-1}\,\partial\rho\right)$ on M_ε, where ι_ε is the embedding of M_ε into T^rX. The Szegö kernel $S_{\theta^{(\varepsilon)}}$ can be written in the form (1.1). Setting $SX := \{U \in TX \mid \|U\| = 1\}$, we can identify SX with M_ε by the map $U \mapsto \varepsilon U/\sqrt{2}$. We regard $\psi_0^{(\varepsilon)} := \psi_{\theta^{(\varepsilon)}}|_{M_\varepsilon}$ as a function on SX with the parameter ε by this identification, and investigate the asymptotic behavior of $\psi_0^{(\varepsilon)}$ as $\varepsilon \to +0$.

We can now state our main result as follows:

Theorem. *Assume that $n = 2$. Then $\psi_0^{(\varepsilon)}$ has the following expansion as $\varepsilon \to +0$:*

$$\psi_0^{(\varepsilon)} = \sum_{l=0}^{L} F_l(U)\varepsilon^{2l} + O(\varepsilon^{2(L+1)}) \quad \text{for all} \quad L \geq 0, \tag{1.2}$$

where $f = O(\varepsilon^m)$ is a function of U and ε such that f/ε^m is bounded as $\varepsilon \to +0$. The coefficient F_0 is given by

$$F_0 = -\frac{1}{120\pi^2}\left\{\nabla_i\nabla_j K - \frac{1}{2}(\nabla_k\nabla^k K)g_{ij}\right\} u^i u^j, \tag{1.3}$$

where K is the Gaussian curvature, $(\nabla_i\nabla_j K)u^i u^j$ is a contraction of $\nabla^2 K \otimes U \otimes U$, and $-\nabla_k\nabla^k$ is the Laplacian.

We note that the tensor in the braces of (1.3) is the trace-free part of the Hessian of K. This means that the integral of F_0 on each fiber of M_ε vanishes.

The proof of the theorem is based on a method of Kan. In [6], when $n = 2$, Kan gave expressions of the Tanaka-Webster connection by using ρ, and proved that the Burns-Epstein invariant $\mu^{(\varepsilon)}$ on M_ε has the following expansion as $\varepsilon \to +0$:

$$\text{(1.4)}\quad \mu^{(\varepsilon)} = \frac{3}{16\pi\varepsilon^2}\int_X dV_X - \frac{1}{8\pi}\int_X K\, dV_X + \sum_{l=1}^{L} F_l^\mu \varepsilon^{2l} + O\big(\varepsilon^{2(L+1)}\big) \quad \text{for all} \quad L \geq 1,$$

where dV_X is the volume form on (X, g). She determined the coefficient of the constant term of (1.4) by calculating the invariant on the boundary of the Grauert tubes over S^2, without giving coefficients of terms of ρ in detail. We determinine coefficients of lower order terms of ρ explicitly (Proposition 4.2) by the method stated in [6], and then prove the theorem by direct calculation. We also give expansions of the Tanaka-Webster scalar curvature and torsion (Propositions 4.4 and 4.5).

This paper is organized as follows: In Section 2, we recall the definitions of three-dimensional CR manifolds, pseudo-hermitian structures and Szegö kernels, and state a result of Hirachi on the expression of $\psi_\theta|_{\partial\Omega}$. In Section 3, we recall the definition of Grauert tubes and the calculations of the Tanaka-Webster connection by Kan. In Section 4, we give coefficients of some lower order terms of ρ, and prove the theorem. Finally, in Section 5, we state the method, due to Kan, of determining the coefficients.

ACKNOWLEDGEMENT. The author is grateful to Professor Kengo Hirachi for his helpful suggestion and advice. Also, the author would like to thank Professor Satoru Shimizu for his helpful advice and encouragement.

2. Pseudo-hermitian structures and Szegö kernels

Let M be a real three-dimensional C^∞ manifold in a two-dimensional complex manifold W. We define a one-dimensional complex subbundle $T^{1,0}M \subset \mathbb{C}TM$ by

$$T^{1,0}M := T^{1,0}W \cap \mathbb{C}TM.$$

We call $T^{1,0}M$ a CR structure on M. We assume that M is strongly pseudoconvex, that is, there exists a real non-vanishing one-form θ such that $\theta(V) = 0$ for any local section V of $T^{1,0}M$ and that the Hermitian form on $T^{1,0}M$ defined by

$$L_\theta(V, \overline{V'}) := -\sqrt{-1}\, d\theta(V \wedge \overline{V'})$$

is positive-definite. We say that such a form θ defines a pseudo-hermitian structure.

Let T be the unique vector field on M such that $\theta(T) = 1$ and $T \rfloor d\theta = 0$, and let Z_1 be any local frame of $T^{1,0}M$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, where $Z_{\bar{1}} := \overline{Z_1}$, satisfies

$$d\theta = \sqrt{-1}\, h_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$

for some positive function $h_{1\bar{1}}$. In terms of this frame, the Tanaka-Webster connection D on $\mathbb{C}TM$ is defined by the following relations for one-forms ${\omega_1}^1$ and τ_1:

$$DZ_1 = {\omega_1}^1 \otimes Z_1, \quad DZ_{\bar{1}} = {\omega_{\bar{1}}}^{\bar{1}} \otimes Z_{\bar{1}}, \quad DT = 0,$$
$$d\theta^1 = \theta^1 \wedge {\omega_1}^1 + \theta \wedge \tau^1,$$
$$\omega_{1\bar{1}} + \omega_{\bar{1}1} = dh_{1\bar{1}},$$
$$\tau_1 \wedge \theta^1 = 0,$$

where ${\omega_{\bar{1}}}^{\bar{1}} := \overline{{\omega_1}^1}$ and we use $h_{1\bar{1}}$ and its inverse $h^{1\bar{1}}$ to raise or lower indices. We can write $\tau_1 = A_{11}\theta^1$, and we call τ_1 the Tanaka-Webster torsion form. The curvature of the Tanaka-Webster connection is

$$d{\omega_1}^1 = R_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + W_1\theta^1 \wedge \theta - W_{\bar{1}}\theta^{\bar{1}} \wedge \theta.$$

The Tanaka-Webster scalar curvature is defined by $R := {R_1}^1$. If $T_{\alpha_1 \cdots \alpha_k}$ are components of a tensor, then its r-th covariant derivative is the tensor with components denoted by $T_{\alpha_1 \cdots \alpha_k, \gamma_1 \cdots \gamma_r}$.

Let Ω be a relatively compact domain with C^∞ boundary $\partial\Omega$ in W. If a volume form σ on $\partial\Omega$ is specified, then the Szegö kernel S is defined as the reproducing kernel associated with the Hardy space $H^2_\sigma(\Omega)$ consisting of holomorphic functions in Ω which have L^2 boundary values with respect to σ. For any complete orthonormal system $\{h_j\}$ of $H^2_\sigma(\Omega)$, S is given by

$$S(z, \overline{z}) = \sum_{j=1}^{\infty} |h_j(z)|^2.$$

Assume that Ω is strongly pseudoconvex. We take a pseudo-hermitian structure θ on $\partial\Omega$, and consider the Szegö kernel S_θ for Ω with respect to the volume form $\theta \wedge d\theta$ on $\partial\Omega$. It was shown in [2] and [1] that S_θ admits an expansion

$$S_\theta(z, \overline{z}) = \frac{\varphi_\theta(z)}{r(z)^2} + \psi_\theta(z) \log r(z)$$

where φ_θ and ψ_θ are functions on Ω which are smooth up to $\partial\Omega$, and $r(z)$ is a defining function for Ω with $r(z) > 0$ in Ω. As for $\psi_\theta|_{\partial\Omega}$, it was proved in [5] that it is given by

$$\psi_\theta|_{\partial\Omega} = \frac{1}{24\pi^2}\left(-{R_{,1}}^1 - {R_{,\bar{1}}}^{\bar{1}} - 2\,\mathrm{Im}\,{A_{11,}}^{11}\right), \tag{2.1}$$

where ${R_{,1}}^1$, ${R_{,\bar{1}}}^{\bar{1}}$ and ${A_{11,}}^{11}$ are the contractions of the second covariant derivatives.

3. A pseudo-hermitian structure on the boundary of Grauert tubes

Let (X, g) be an n-dimensional connected compact C^ω Riemannian manifold, and let $\gamma\colon \mathbb{R} \to X$ be a geodesic. We define a map $\psi_\gamma\colon \mathbb{C} \to TX$ by

$$\psi_\gamma(\xi + \sqrt{-1}\,\eta) := \eta\dot{\gamma}(\xi).$$

It was proved in [9] that, for some $0 < r \leq \infty$, $T^r X := \{v \in TX \mid \|v\| < r\}$ admits the complex structure such that ψ_γ is holomorphic for every geodesic γ on X. We call the complex manifold $T^r X$ the Grauert tube of radius r over X.

Define the function $\rho\colon T^r X \to \mathbb{R}$ by $\rho(v) := 2\|v\|^2$ for $v \in T^r X$. Then the following theorem holds:

Theorem 3.1 ([4], [8]). *The function ρ has the following properties*:
(1) *ρ is strictly plurisubharmonic,*
(2) *$X = \rho^{-1}(0)$, where we identify the zero section of TX with X,*
(3) *the Kähler metric G obtained from the Kähler form $\sqrt{-1}\,\partial\overline{\partial}\rho/2$ is compatible with g, that is, $G|_X = g$,*
(4) *$(\partial\overline{\partial}\sqrt{\rho}\,)^n = 0$ in $T^r X \setminus X$, and*
(5) *if $\widetilde{\rho}$ is another function which satisfies all the conditions above, then $\widetilde{\rho} = \rho$.*

Set $\Omega_\varepsilon := \{v \in T^r X \mid \rho(v) < \varepsilon^2\}$ and $M_\varepsilon := \partial\Omega_\varepsilon$ for $0 < \varepsilon < \sqrt{2}\,r$. Then (1) of this theorem shows that M_ε is a strongly pseudoconvex CR manifold.

We assume that $n = 2$. For a geodesic normal coordinate system (x^1, x^2) centered at $p \in X$, let $(z^1, z^2) = (x^1 + \sqrt{-1}\,y^1, x^2 + \sqrt{-1}\,y^2)$ be the local holomorphic coordinate system. Then $(\sqrt{-1}\,y^1, \sqrt{-1}\,y^2)$ corresponds to the vector $y^i(\partial/\partial x^i)(p)$, and for $y^i(\partial/\partial x^i)(p) \in M_\varepsilon$, there exists a vector $u^i(\partial/\partial x^i)(p) \in SX := \{U \in TX \mid \|U\| = 1\}$ such that

$$y^1 = \frac{1}{\sqrt{2}}\varepsilon u^1, \qquad y^2 = \frac{1}{\sqrt{2}}\varepsilon u^2. \tag{3.1}$$

Take a pseudo-hermitian structure $\theta^{(\varepsilon)} := \iota_\varepsilon^*(-\sqrt{-1}\,\partial\rho)$ on M_ε, where ι_ε is the embedding of M_ε into $T^r X$. We define local sections of $\mathbb{C}T(T^r X \setminus X)$ by

$$T = \frac{\sqrt{-1}}{2\rho P}\left(A\frac{\partial}{\partial z^2} - \overline{A}\frac{\partial}{\partial \overline{z^2}} - B\frac{\partial}{\partial z^1} + \overline{B}\frac{\partial}{\partial \overline{z^1}}\right),$$

$$Z_1 = \frac{1}{\sqrt{2}\,\rho^{1/2}P^{1/2}}\left(\frac{\partial\rho}{\partial z^2}\frac{\partial}{\partial z^1} - \frac{\partial\rho}{\partial z^1}\frac{\partial}{\partial z^2}\right),$$

where

$$A = \frac{\partial\rho}{\partial\overline{z^2}}\frac{\partial^2\rho}{\partial z^1\partial\overline{z^1}} - \frac{\partial\rho}{\partial\overline{z^1}}\frac{\partial^2\rho}{\partial z^1\partial\overline{z^2}}, \quad B = \frac{\partial\rho}{\partial\overline{z^2}}\frac{\partial^2\rho}{\partial z^2\partial\overline{z^1}} - \frac{\partial\rho}{\partial\overline{z^1}}\frac{\partial^2\rho}{\partial z^2\partial\overline{z^2}}, \tag{3.2}$$

$$P = \frac{\partial^2 \rho}{\partial z^1 \partial \overline{z^1}} \frac{\partial^2 \rho}{\partial z^2 \partial \overline{z^2}} - \frac{\partial^2 \rho}{\partial z^1 \partial \overline{z^2}} \frac{\partial^2 \rho}{\partial z^2 \partial \overline{z^1}},$$

and, $\rho^{1/2}$ and $P^{1/2}$ denote the positive branches of $\sqrt{\rho}$ in $T^r X \setminus X$ and $\sqrt{P}$ in $T^r X$, respectively (note that $P > 0$ in $T^r X$ since ρ is strictly plurisubharmonic). For $\varepsilon > 0$, $T^{(\varepsilon)} := T|_{M_\varepsilon}$ and $Z_1^{(\varepsilon)} := Z_1|_{M_\varepsilon}$ are local sections of $\mathbb{C}TM_\varepsilon$, $T^{(\varepsilon)}$ satisfies $\theta^{(\varepsilon)}(T^{(\varepsilon)}) = 1$ and $T^{(\varepsilon)} \rfloor d\theta^{(\varepsilon)} = 0$, and $Z_1^{(\varepsilon)}$ is a local frame of $T^{1,0}M_\varepsilon$. Setting

$$\theta^1 := \frac{1}{\sqrt{2}\,\rho^{1/2} P^{1/2}} \left(A dz^1 + B dz^2\right),$$

we see that $\{\theta^{(\varepsilon)}, \theta^{(\varepsilon)1}, \theta^{(\varepsilon)\bar{1}}\}$, where $\theta^{(\varepsilon)1} := \iota_\varepsilon^*(\theta^1)$, is the coframe dual to $\{T^{(\varepsilon)}, Z_1^{(\varepsilon)}, Z_{\bar{1}}^{(\varepsilon)}\}$ and that $d\theta^{(\varepsilon)} = \sqrt{-1}\,\theta^{(\varepsilon)1} \wedge \theta^{(\varepsilon)\bar{1}}$.

The Tanaka-Webster connection form, the torsion form and the curvature are given as follows:

$$\omega_1^{(\varepsilon)1} = \sqrt{-1}\, a^{(\varepsilon)}\theta^{(\varepsilon)} + b^{(\varepsilon)}\theta^{(\varepsilon)1} - \overline{b^{(\varepsilon)}}\theta^{(\varepsilon)\bar{1}}, \quad \tau_1^{(\varepsilon)} = A_{11}^{(\varepsilon)}\theta^{(\varepsilon)1},$$
$$d\omega_1^{(\varepsilon)1} = R_{1\bar{1}}^{(\varepsilon)}\theta^{(\varepsilon)1} \wedge \theta^{(\varepsilon)\bar{1}} + W_1^{(\varepsilon)}\theta^{(\varepsilon)1} \wedge \theta^{(\varepsilon)} - W_{\bar{1}}^{(\varepsilon)}\theta^{(\varepsilon)\bar{1}} \wedge \theta^{(\varepsilon)},$$

where $a^{(\varepsilon)}$, $b^{(\varepsilon)}$, $R_{1\bar{1}}^{(\varepsilon)}$, $A_{11}^{(\varepsilon)}$ and $W_1^{(\varepsilon)}$ are restrictions to M_ε of a, b, $R_{1\bar{1}}$, A_{11} and W_1 which are defined by

$$a = -\frac{1}{2\rho} + \frac{1}{4\rho P^2}\left(A\frac{\partial P}{\partial z^2} + \overline{A}\frac{\partial P}{\partial \overline{z^2}} - B\frac{\partial P}{\partial z^1} - \overline{B}\frac{\partial P}{\partial \overline{z^1}}\right), \tag{3.3}$$

$$b = \frac{1}{2P} Z_1 P, \tag{3.4}$$

$$R_{1\bar{1}} = -a - 2|b|^2 - Z_1\overline{b} - Z_{\bar{1}}b, \tag{3.5}$$

$$A_{11} = \frac{\sqrt{-1}}{2\sqrt{2}\,\rho^{3/2}P^{3/2}}\left(\overline{A}Z_1\overline{B} - \overline{B}Z_1\overline{A}\right), \tag{3.6}$$

$$W_1 = \sqrt{-1}\,ab + \overline{b}A_{11} + \sqrt{-1}\,Z_1 a - Tb.$$

4. Proof of the theorem

In Sections 4 and 5, $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ denote multi-indices with α_1, α_2, β_1, $\beta_2 \geq 0$. We use the following propositions in order to prove the theorem:

Proposition 4.1 ([6]). *Let $g = g_{ij}dx^i \otimes dx^j$. Then ρ is of the following form*:

$$\rho(z^1, z^2) = 2g_{11}(x)(y^1)^2 + 4g_{12}(x)y^1y^2 + 2g_{22}(x)(y^2)^2 + \sum_{|\alpha|\geq 4,\ \text{even}} \varphi_\alpha(x)y^\alpha, \tag{4.1}$$

where each φ_α is a real-analytic function.

Let $K(x)$ be the Gaussian curvature of X, and let

$$K := K(0), \quad K_{\gamma_1 \cdots \gamma_j} := \frac{\partial^j K}{\partial x^{\gamma_1} \cdots \partial x^{\gamma_j}}(0).$$

Proposition 4.2. *Set* $|y|^2 = (y^1)^2 + (y^2)^2$ *and* $Y = x^2 y^1 - x^1 y^2$. *Then*

$$\rho(z^1, z^2) = 2|y|^2 + \{f_1(x) + f_2(y)\} Y^2 + O(|xy|^7), \tag{4.2}$$

where

$$f_1(x) = -\frac{K}{3} - \frac{1}{6}\left(K_1 x^1 + K_2 x^2\right) + \left(\frac{K^2}{45} - \frac{K_{11}}{20}\right)(x^1)^2 - \frac{K_{12}}{10} x^1 x^2 + \left(\frac{K^2}{45} - \frac{K_{22}}{20}\right)(x^2)^2,$$
$$f_2(y) = \left(\frac{K^2}{15} + \frac{K_{11}}{60}\right)(y^1)^2 + \frac{K_{12}}{30} y^1 y^2 + \left(\frac{K^2}{15} + \frac{K_{22}}{60}\right)(y^2)^2,$$

and $O(|xy|^m)$ *is a function of the form*

$$\sum_{|\alpha|+|\beta|=m} a_{\alpha\beta}(x, y) x^\alpha y^\beta$$

for real-analytic functions $a_{\alpha\beta}$.

We first give the proof of the theorem in the rest of this section, and then prove Proposition 4.2 in the next section.

We write $h \in \mathcal{F}$ if h is a real-analytic function on $T^r X$ and the Taylor expansion of h at $y = 0$ can be written in the form

$$\sum_{|\alpha|:\ \text{even}} F_\alpha(x) y^\alpha + \sqrt{-1} \sum_{|\alpha|:\ \text{odd}} F_\alpha(x) y^\alpha \tag{4.3}$$

for real-valued functions F_α. In particular, we write $h \in \mathcal{F}_0$ if $h \in \mathcal{F}$ and h is a real-valued function. The following properties hold for h, $h' \in \mathcal{F}$:
(1) $h + h'$, $hh' \in \mathcal{F}$,
(2) $\partial h/\partial z^1$, $\partial h/\partial z^2$, $\partial h/\partial \overline{z^1}$, $\partial h/\partial \overline{z^2} \in \mathcal{F}$, and
(3) $\rho T h$, $\rho^{1/2} Z_1 h \in \mathcal{F}$.

Lemma 4.3. *Let* ψ_0 *be a function on* $T^r X \setminus X$ *such that* $\psi_0|_{M_\varepsilon} = \psi_0^{(\varepsilon)} := \psi_{\theta^{(\varepsilon)}}|_{M_\varepsilon}$ *for* $0 < \varepsilon < \sqrt{2}\, r$. *Then* $\rho^3 \psi_0 \in \mathcal{F}_0$.

Proof. It follows from $\rho \in \mathcal{F}$ that A, B, $P \in \mathcal{F}$. Considering the Taylor expansion of $P^{k/2}$ at the origin for $k \in \mathbb{Z}$, we have $P^{k/2} \in \mathcal{F}$. By the definitions of a, b, $R_{1\bar{1}}$ and A_{11}, we see ρa, $\rho^{1/2} b$, $\rho R_{1\bar{1}}$, $\sqrt{-1}\,\rho^2 A_{11} \in \mathcal{F}$. Since $R_{1\bar{1}}$ is real, we have $\rho R_{1\bar{1}} \in \mathcal{F}_0$.

We note that $R^{(\varepsilon)} = R^{(\varepsilon)}_{1\bar{1}} = R_{1\bar{1}}|_{M_\varepsilon}$. Set

$$R_{,1}{}^{1} + R_{,\bar{1}}{}^{\bar{1}} := 2(Z_{\bar{1}}Z_1R_{1\bar{1}} + \bar{b}Z_1R_{1\bar{1}}) - \sqrt{-1}\,TR_{1\bar{1}} \tag{4.4}$$

and

$$A_{11,}{}^{11} := Z_{\bar{1}}^2A_{11} + 2\left(Z_{\bar{1}}\bar{b}\right)A_{11} + 3\bar{b}Z_{\bar{1}}A_{11} + 2\bar{b}^2A_{11}. \tag{4.5}$$

Then we have $\rho^2(R_{,1}{}^{1} + R_{,\bar{1}}{}^{\bar{1}})$, $\sqrt{-1}\,\rho^3A_{11,}{}^{11} \in \mathcal{F}$ and

$$R^{(\varepsilon)1}_{,1} + R^{(\varepsilon)\bar{1}}_{,\bar{1}} = (R_{,1}{}^{1} + R_{,\bar{1}}{}^{\bar{1}})\big|_{M_\varepsilon}, \qquad A^{(\varepsilon)}_{11,}{}^{11} = A_{11,}{}^{11}\big|_{M_\varepsilon}.$$

Hence, we see $\rho^3\psi_0 \in \mathcal{F}$ by (2.1). Since ψ_0 is real, we obtain $\rho^3\psi_0 \in \mathcal{F}_0$. □

We now calculate $R_{1\bar{1}}$ and A_{11}. First, we calculate A, B and P by differentiating (4.2) and using (3.2). Next, differentiating P, by the definitions of a and b, we get them. Then, differentiating $\bar{b}$ with respect to Z_1 and using (3.5), we obtain

$$\begin{aligned} R_{1\bar{1}} = \frac{1}{2\rho}\Bigg[& 1 + \frac{2}{3}\left(K + K_1x^1 + K_2x^2\right)|y|^2 \\ & + \left\{\left(-\frac{K_{11}}{15} + \frac{4}{15}K_{22} - \frac{4}{45}K^2\right)(y^1)^2 - \frac{2}{3}K_{12}y^1y^2 \right. \\ & \left. + \left(\frac{4}{15}K_{11} - \frac{K_{22}}{15} - \frac{4}{45}K^2\right)(y^2)^2\right\}|y|^2 \\ & + \left\{\frac{K_{11}}{3}(x^1)^2 + \frac{2}{3}K_{12}x^1x^2 + \left(\frac{K_{22}}{3} - \frac{K^2}{9}\right)(x^2)^2\right\}(y^1)^2 \\ & + \left\{\left(\frac{K_{11}}{3} - \frac{K^2}{9}\right)(x^1)^2 + \frac{2}{3}K_{12}x^1x^2 + \frac{K_{22}}{3}(x^2)^2\right\}(y^2)^2 \\ & + \frac{2}{9}K^2x^1x^2y^1y^2 + O(|xy|^5)\Bigg]. \end{aligned} \tag{4.6}$$

Differentiating $\overline{A}$ and $\overline{B}$ with respect to Z_1 and using (3.6), we get

(4.7)

$$\begin{aligned} A_{\bar{1}\bar{1}} & := \overline{A_{11}} \\ & = \frac{1}{4\rho^2}\Bigg[-\frac{4}{3}K\left(x^1y^1 + x^2y^2\right)|y|^2 + \frac{1}{3}\left(K_1y^1 + K_2y^2\right)|y|^4 \\ & \quad - \left\{K_1(x^1)^2 + \frac{2}{3}K_2x^1x^2 + \frac{K_1}{3}(x^2)^2\right\}y^1|y|^2 \\ & \quad - \left\{\frac{K_2}{3}(x^1)^2 + \frac{2}{3}K_1x^1x^2 + K_2(x^2)^2\right\}y^2|y|^2 \end{aligned}$$

$$
\begin{aligned}
&+\left\{\left(\frac{2}{5}K_{11}+\frac{22}{45}K^2\right)x^1+\frac{K_{12}}{5}x^2\right\}(y^1)^3|y|^2\\
&+\left\{\frac{3}{5}K_{12}x^1+\left(\frac{K_{11}}{5}+\frac{K_{22}}{5}+\frac{22}{45}K^2\right)x^2\right\}(y^1)^2y^2|y|^2\\
&+\left\{\left(\frac{K_{11}}{5}+\frac{K_{22}}{5}+\frac{22}{45}K^2\right)x^1+\frac{3}{5}K_{12}x^2\right\}y^1(y^2)^2|y|^2\\
&+\left\{\frac{K_{12}}{5}x^1+\left(\frac{2}{5}K_{22}+\frac{22}{45}K^2\right)x^2\right\}(y^2)^3|y|^2\\
+\sqrt{-1}&\left[4|y|^2-\frac{4}{3}K|y|^4-\frac{2}{3}KY^2\right.\\
&-\frac{4}{3}\left(K_1x^1+K_2x^2\right)|y|^4+\frac{2}{9}K^2|y|^2Y^2+2f_2(y)Y^2\\
&+\left\{\left(\frac{2}{15}K_{11}+\frac{14}{45}K^2\right)(y^1)^2+\frac{4}{15}K_{12}y^1y^2+\left(\frac{2}{15}K_{22}+\frac{14}{45}K^2\right)(y^2)^2\right\}|y|^4\\
&-\left\{\left(\frac{2}{3}K_{11}+\frac{2}{9}K^2\right)(x^1)^2+\frac{4}{3}K_{12}x^1x^2+\left(\frac{2}{3}K_{22}-\frac{2}{9}K^2\right)(x^2)^2\right\}(y^1)^2|y|^2\\
&-\left\{\left(\frac{2}{3}K_{11}-\frac{2}{9}K^2\right)(x^1)^2+\frac{4}{3}K_{12}x^1x^2+\left(\frac{2}{3}K_{22}+\frac{2}{9}K^2\right)(x^2)^2\right\}(y^2)^2|y|^2\\
&\left.-\frac{8}{9}K^2x^1x^2y^1y^2|y|^2\right]+O(|x|^3)+O(|xy|^7)\Bigg].
\end{aligned}
$$

By (4.4) and differentiation of (4.6), we have

$$
\left(R_{,1}{}^{1}+R_{,\bar{1}}{}^{\bar{1}}\right)\Big|_{x=0}=\frac{1}{\rho^2}\left[\left\{\frac{K_{11}}{3}(y^1)^2+\frac{2}{3}K_{12}y^1y^2+\frac{K_{22}}{3}(y^2)^2\right\}|y|^2+O(|y|^5)\right].
$$

Differentiating $\overline{b}$ and the conjugate of (4.7) with respect to $Z_{\bar{1}}$ and using (4.5), we have

$$
\begin{aligned}
A_{11,}{}^{11}\Big|_{x=0}=\frac{1}{\rho^3}\Bigg[&-\frac{1}{3}\left(K_1y^1+K_2y^2\right)|y|^4\\
&+\sqrt{-1}\left\{\left(\frac{K_{11}}{15}-\frac{2}{5}K_{22}\right)(y^1)^2\right.\\
&\left.+\frac{14}{15}K_{12}y^1y^2-\left(\frac{2}{5}K_{11}-\frac{K_{22}}{15}\right)(y^2)^2\right\}|y|^4+O(|y|^7)\Bigg].
\end{aligned}
$$

Therefore, by these two equations, (2.1) and Lemma 4.3, substituting (3.1), for $\varepsilon>0$, we obtain

$$
\psi_0^{(\varepsilon)}\Big|_{x=0}=\frac{1}{24\pi^2}\left[\frac{1}{10}\nabla_k\nabla^kK-\frac{1}{5}\left\{K_{11}(u^1)^2+2K_{12}u^1u^2+K_{22}(u^2)^2\right\}+O(\varepsilon^2)\right].
$$

This completes the proof of the theorem.

REMARK. Since $\rho R_{1\bar{1}} \in \mathcal{F}_0$ and $\sqrt{-1}\,\rho^2 A_{\bar{1}\bar{1}} \in \mathcal{F}$ by the proof of Lemma 4.3, setting $x = 0$ and substituting (3.1) into (4.6) and (4.7), we have the following propositions:

Proposition 4.4. *The Tanaka-Webster scalar curvature $R^{(\varepsilon)}$ has the following expansion as $\varepsilon \to +0$:*

$$R^{(\varepsilon)} = \frac{1}{2\varepsilon^2} + \sum_{l=0}^{L} F_l^R(U)\varepsilon^{2l} + O(\varepsilon^{2(L+1)}) \quad \text{for all} \quad L \geq 0,$$

where

$$F_0^R = \frac{1}{6}K, \qquad F_1^R = -\frac{1}{90}K^2 + \left\{\frac{1}{30}(\nabla_k\nabla^k K)g_{ij} - \frac{1}{24}\nabla_i\nabla_j K\right\}u^i u^j.$$

Proposition 4.5. *The Tanaka-Webster torsion $A_{\bar{1}}^{(\varepsilon)1}$ has the following expansion as $\varepsilon \to +0$:*

$$A_{\bar{1}}^{(\varepsilon)1} = \frac{\sqrt{-1}}{2\varepsilon^2} + \sum_{l=0}^{L} F_l^\tau(U)\varepsilon^{l} + O(\varepsilon^{L+1}) \quad \text{for all} \quad L \geq 0,$$

where

$$F_0^\tau = -\frac{\sqrt{-1}}{12}K, \qquad F_1^\tau = \frac{1}{48\sqrt{2}}(\nabla_i K)u^i,$$
$$F_2^\tau = \frac{7\sqrt{-1}}{720}K^2 + \frac{\sqrt{-1}}{240}\left(\nabla_i\nabla_j K\right)u^i u^j.$$

5. Determination of lower order terms of ρ

We give the proof of Proposition 4.2 by using Proposition 4.1 and a method of Kan. It follows from (4) of Theorem 3.1 that

$$(\partial\bar{\partial}\sqrt{\rho})^2 = -\frac{1}{4\rho^2}\left\{\partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho - \rho(\partial\bar{\partial}\rho \wedge \partial\bar{\partial}\rho)\right\} = 0,$$

that is,

$$\partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho = \rho(\partial\bar{\partial}\rho \wedge \partial\bar{\partial}\rho).$$

This equation is equivalent to the following one:

$$(5.1) \qquad 2\rho\left\{\left(\frac{\partial^2\rho}{\partial(x^1)^2} + \frac{\partial^2\rho}{\partial(y^1)^2}\right)\left(\frac{\partial^2\rho}{\partial(x^2)^2} + \frac{\partial^2\rho}{\partial(y^2)^2}\right)\right.$$

$$- \left(\frac{\partial^2 \rho}{\partial x^1 \partial x^2} + \frac{\partial^2 \rho}{\partial y^1 \partial y^2} \right)^2 - \left(\frac{\partial^2 \rho}{\partial x^1 \partial y^2} - \frac{\partial^2 \rho}{\partial x^2 \partial y^1} \right)^2 \Bigg\}$$

$$= \left\{ \left(\frac{\partial \rho}{\partial x^1} \right)^2 + \left(\frac{\partial \rho}{\partial y^1} \right)^2 \right\} \left(\frac{\partial^2 \rho}{\partial (x^2)^2} + \frac{\partial^2 \rho}{\partial (y^2)^2} \right)$$

$$+ \left\{ \left(\frac{\partial \rho}{\partial x^2} \right)^2 + \left(\frac{\partial \rho}{\partial y^2} \right)^2 \right\} \left(\frac{\partial^2 \rho}{\partial (x^1)^2} + \frac{\partial^2 \rho}{\partial (y^1)^2} \right)$$

$$- 2 \left(\frac{\partial \rho}{\partial x^1} \frac{\partial \rho}{\partial x^2} + \frac{\partial \rho}{\partial y^1} \frac{\partial \rho}{\partial y^2} \right) \left(\frac{\partial^2 \rho}{\partial x^1 \partial x^2} + \frac{\partial^2 \rho}{\partial y^1 \partial y^2} \right)$$

$$+ 2 \left(\frac{\partial \rho}{\partial x^2} \frac{\partial \rho}{\partial y^1} - \frac{\partial \rho}{\partial x^1} \frac{\partial \rho}{\partial y^2} \right) \left(\frac{\partial^2 \rho}{\partial x^1 \partial y^2} - \frac{\partial^2 \rho}{\partial x^2 \partial y^1} \right).$$

By using Proposition 4.1, we set

$$\rho(z^1, z^2) = 2g_{11}(x)(y^1)^2 + 4g_{12}(x) y^1 y^2 + 2g_{22}(x)(y^2)^2 + \sum_{j=0}^{4} \varphi_j(x)(y^1)^{4-j}(y^2)^j + \sum_{|\alpha| \geq 6,\ \text{even}} \varphi_\alpha(x) y^\alpha. \tag{5.2}$$

The Taylor expansions of g_{ij} are given as follows:

$$g_{11}(x) = 1 + (x^2)^2 f(x) + O(|x|^5),$$
$$g_{12}(x) = -x^1 x^2 f(x) + O(|x|^5),$$
$$g_{22}(x) = 1 + (x^1)^2 f(x) + O(|x|^5),$$

where

$$f(x) = -\frac{K}{6} - \frac{1}{12}\left(K_1 x^1 + K_2 x^2\right) + \left(\frac{K^2}{90} - \frac{K_{11}}{40} \right)(x^1)^2 - \frac{K_{12}}{20} x^1 x^2 + \left(\frac{K^2}{90} - \frac{K_{22}}{40} \right)(x^2)^2.$$

Substituting (5.2) into (5.1) and comparing the coefficients of $(y^1)^4$ on both sides of (5.1), we have

$$\varphi_0(x) = -\frac{1}{3} \frac{\partial^2 g_{11}}{\partial (x^1)^2} + \frac{1}{6(g_{11} g_{22} - g_{12}^2)} \left\{ g_{11} \left(\frac{\partial g_{11}}{\partial x^2} - 2 \frac{\partial g_{12}}{\partial x^1} \right)^2 + 2 g_{12} \frac{\partial g_{11}}{\partial x^1} \left(\frac{\partial g_{11}}{\partial x^2} - 2 \frac{\partial g_{12}}{\partial x^1} \right) + g_{22} \left(\frac{\partial g_{11}}{\partial x^1} \right)^2 \right\}$$

$$= \left(\frac{K^2}{15} + \frac{K_{11}}{60} \right)(x^2)^2 + O(|x|^3).$$

Arguments similar to above yield

$$\varphi_1(x) = -\left(\frac{2}{15}K^2 + \frac{K_{11}}{30}\right)x^1x^2 + \frac{K_{12}}{30}(x^2)^2 + O(|x|^3),$$
$$\varphi_2(x) = \left(\frac{K^2}{15} + \frac{K_{11}}{60}\right)(x^1)^2 - \frac{K_{12}}{15}x^1x^2 + \left(\frac{K^2}{15} + \frac{K_{22}}{60}\right)(x^2)^2 + O(|x|^3),$$
$$\varphi_3(x) = \frac{K_{12}}{30}(x^1)^2 - \left(\frac{2}{15}K^2 + \frac{K_{22}}{30}\right)x^1x^2 + O(|x|^3),$$
$$\varphi_4(x) = \left(\frac{K^2}{15} + \frac{K_{22}}{60}\right)(x^1)^2 + O(|x|^3).$$

Hence one can write

$$\rho(z^1, z^2) = 2|y|^2 + \{f_1(x) + f_2(y)\}\,Y^2 + \sum_{j=0}^{6} p_j(y^1)^{6-j}(y^2)^j + O(|xy|^7). \tag{5.3}$$

We substitute (5.3) into (5.1) and set $x = 0$. Comparing the coefficients of $(y^1)^{6-j}(y^2)^j$ on both sides of (5.1), we have $p_j = 0$ for $j = 0, \dots, 6$. We therefore obtain Proposition 4.2.

References

[1] L. Boutet de Monvel and J. Sjöstrand: *Sur la singularité des noyau de Bergman et de Szegö*, Astérisque **34–35** (1976), 123–164.

[2] C. Fefferman: *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.

[3] C. Fefferman and K. Hirachi: *Ambient metric construction of Q-curvature in conformal and CR geometries*, Math. Res. Lett. **10** (2003), 819–831.

[4] V. Guillemin and M. Stenzel: *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Differ. Geom. **34** (1993), 561–570.

[5] K. Hirachi: *Scalar pseudo-hermitian invariants and the Szegö kernel on three-dimensional CR manifolds*, "Complex Geometry," Lecture Notes in Pure and Appl. Math. **143** (1993), 67–76.

[6] S.-J. Kan: *The asymptotic expansion of a CR invariant and Grauert tubes*, Math. Ann. **304** (1996), 63–92.

[7] J.M. Lee: *The Fefferman metric and pseudohermitian invariants*, Trans. Amer. Math. Soc. **296** (1986), 411–429.

[8] L. Lempert and R. Szöke: *Global solutions of the homogeneous complex Monge-Ampère equations and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. **290** (1991), 689–712.

[9] R. Szöke: *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991), 409–428.

Mathematical Institute
Tohoku University
Sendai 980-8578, Japan
e-mail: s98m13@math.tohoku.ac.jp