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On the Unknotted Sphere S^2 in E^4

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The construction of a locally flat, knotted sphere introduced by Artin [1] has given rise to a series of further investigations in this direction, [2], [3]. The construction is simply thus: Let E^2 be a plane in E^3 which is in turn in E^4, and let κ be a knot in E^3 having a segment ab in common with E^2, otherwise contained wholly in the positive half E^3_+ of E^3. Call the arc $\kappa^0 = \overline{\kappa - ab}$ an *open knot* with end points a, b. Artin obtained the desired sphere S^2 by rotating the open knot κ^0 around E^2 as axis in E^4. He showed that the fundamental group of $E^4 - S^2$ is isomorphic to the knot group of κ, that is, to the fundamental group of $E^3 - \kappa$. Fox and Milnor [4] showed that if a locally flat sphere S^2 in E^4

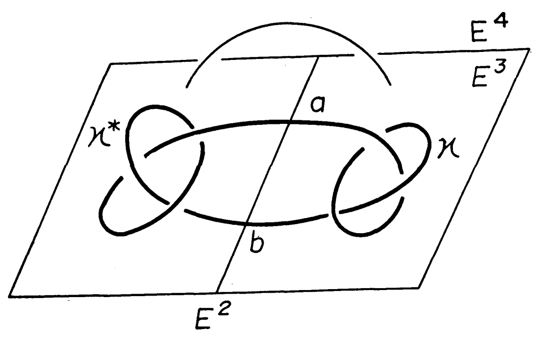

Fig. 1

is cut by an E^3, and if the intersection $S^2 \frown E^3$ is a knot, which they called a null-equivalent knot, then the Alexander polynomial of this knot must be of the form $f(x)f(x^{-1})x^n$. As it happens, the Alexander polynomial of $S^2 \frown E^3$ is $\Delta^2(x)$ for the sphere S^2 of Artin type, for then the knot in question is the product[1)] of κ, of Alexander polynomial $\Delta(x)$, with its symmetric image κ^* with respect to E^2, as will be seen in the figure.

Now the question is: what can be concluded about the knottedness of a given locally flat sphere $S^2 \subset E^4$ from the information about that of $S^2 \frown E^3$ for any hyperplane E^3 of E^4? This and other related questions

1) "sum" would be a better terminology.

are still open; in the present note we shall only show that there is a class of non-trivial knots, called doubly null-equivalent knots, of which each $\kappa \subset E^3$ admits an unknotted sphere $S^2 \subset E^4$ to pass through such that $\kappa = S^2 \cap E^3$.

A cylindrical surface in E^3 bounded by a pair of simple closed curves κ' and κ'' will be called *unknotted*, if it is isotopic to a ringed region on a plane of E^3.

Let T be a torus in E^3 with a boundary κ, which is a knot. Such a torus can be brought isotopically into the Seifert normal form [5],

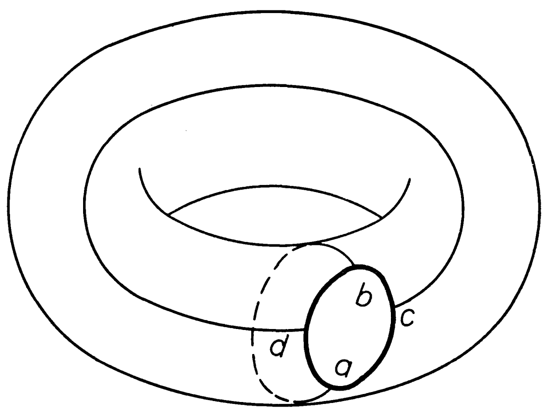

Fig. 2

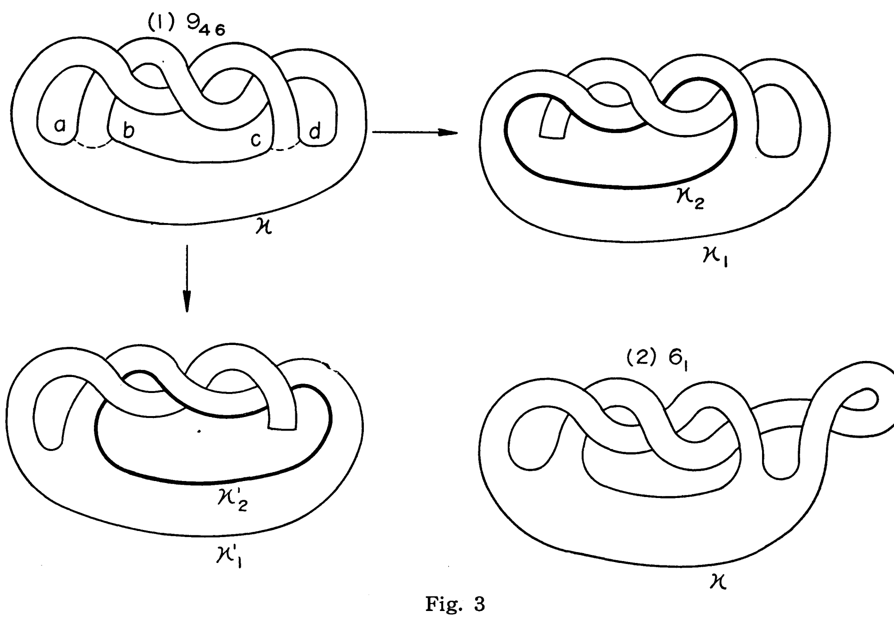

Fig. 3

cf. Fig. 3, (1) and (2). Now, if there is an arc ab joining two points a and b of κ on T such that an unknotted cylindrical surface may be obtained by cutting T along ab, then κ is a *null-equivalent knot,* [4], [6] (cf. also [7], p. 134). If there is moreover another arc joining points c and d of κ on T which is disjoint from ab and not homotopic to ab and which has the same property as above, then κ will be called a *doubly null-equivalent knot.* Call ab and cd *conjugate cross-cuts.* In Fig. 3, (1) represents the knot 9_{46} of the knot table in [8] and, by taking ab and cd as conjugate cross-cuts, it is seen to be a doubly null-equivalent knot, while (2) is the knot 6_1 with the same Alexander polynomial as that of 9_{46}, but is undecided whether or not it is doubly null-equivalent.

The theorem we are to prove is the following:

Theorem. *Let κ be a doubly null-equivalent knot in a hyperplane E^3 of E^4. Then there is a trivial sphere S^2 in E^4 whose intersection with E^3 coincides with κ.*

Proof will be divided into several steps.

1st step. First we define a continuous family of curves Γ_t, $-3 \leqq t \leqq 3$, on the standard 2-dimensional sphere Σ^2 in E^3, which is essentially a topological map of the family of general lemniscates

$$(*) \qquad ((x-1)^2+y^2)((x+1)^2+y^2) = k^2$$

for $0 \leqq k \leqq 2$ on the northern hemisphere H_+ of Σ^2 and its symmetric image on the southern hemisphere H_- (cf. Fig. 4):

Γ_3 is the image of the foci $k=0$ of (*) and consists of a pair of points α'_3 and α''_3.

Γ_t for $3>t>1$ is the image of (*) for $0<k<1$ and consists each of a pair of simple closed curves Γ'_t and Γ''_t around α'_3 and α''_3 respectively.

Γ_1 is the image of the ordinary 8-shaped lemniscate $k=1$ of (*).

Γ_t for $1>t\geqq 0$ is the image of (*) for $1<k\leqq 2$ and is a simple closed curve. Especially Γ_0 is the equator of Σ^2.

Further let $\Gamma_{-t}(3\geqq t>0)$ be the symmetric image of Γ_t with respect to the equatorial plane of Σ^2.

On the basis of Γ_t we now define a continuous family of disjoint surfaces Φ_t filling up the full sphere Δ^3 of Σ^2, as follows:

Let Φ_3 coincide with Γ_3, that is, with points α'_3 and α''_3.

Let Φ_t for $3>t>2$ consist each of a pair of disjoint hemispheres bounded by Γ'_t and Γ''_t respectively.

Let Φ_2 be a pair of hemispheres having a single point in common and bounded each by Γ'_2 and Γ''_2 respectively.

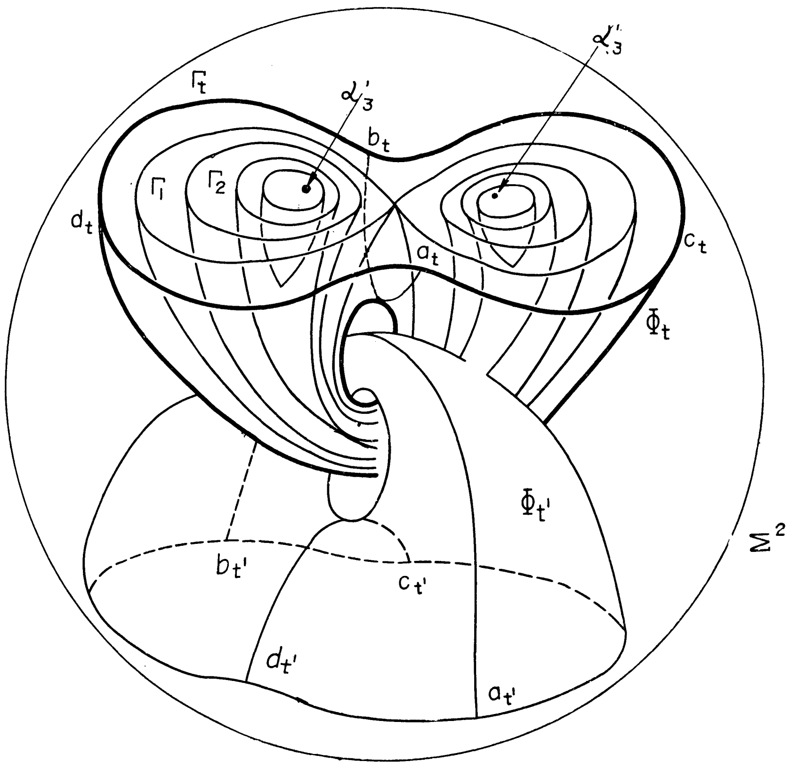

Fig. 4

Let Φ_t for $2>t>1$ be each a cylindrical surface bounded by Γ_t' and Γ_t''.

Let Φ_1 be a torus bounded by the 8-shaped curve Γ_1.

Finally let Φ_t be for $1>t\geqq 0$ a torus bounded by Γ_t.

For negative t, $0\geqq t\geqq -3$, the family of surfaces $\{\Phi_t\}$ should be as a whole homeomorphic to $\{\Phi_{-t}\}$ defined above, $\Phi_0=\{\Phi_t\}\frown\{\Phi_{-t}\}$ being mapped onto itself by this homeomorphism.

2nd step.

We now provide in the hyperplane $x_4=0$, which we denote by E_0^3, a continuous family of not necessarily disjoint surfaces T_t, $-3\leqq t\leqq 3$, of the following kind (cf. Fig. 5, where T_t are shaded):

$\kappa_0=\kappa$ is the given doubly null-equivalent knot spanned with a torus T_0, with conjugate cross-cuts a_0b_0 and c_0d_0.

For $0<t<1$, T_t is a torus bounded by a knot κ_t.

T_1 is a torus bounded by the union κ_1 of two trivial knots κ_1' and κ_1'' having in common a single point $a_1=b_1$, which is the limit of the cross-cut a_0b_0 on T_0.

For $1<t<2$, T_t is an unknotted cylindrical surface bounded by a pair of trivial knots κ'_t and κ''_t.

T_2 is the union of two disks bounded by κ'_2 and by κ''_2 respectively and having a single inner point in common.

For $2<t<3$, T_t consists of two disjoint disks bounded by knots κ'_t and κ''_t respectively.

T_3 consists of a pair of distinct points κ'_3 and κ''_3.

For $-3\leqq t<0$, T_t is homeomorphic with T_{-t}, provided that the common point of κ'_{-1} and κ''_{-1} of T_{-1} is the limit of the cross-cuts c_0d_0 of T_0.

Final step.

Now let E^3_t be the family of parallel hyperplane $x_4=t$ in E^4 for $-3\leqq t\leqq 3$.

To each t of $-3\leqq t\leqq 3$ project the surface T_t just defined in E^3_0 into E^3_t, and denote it by F_t. Then, since T_t, and hence F_t, is homeomorphic to Φ_t, the union $\bigcup_{-3\leqq t\leqq 3} F_t=D$ is clearly a full sphere in E^4, and consequently its boundary $\dot{D}=\bigcup_{-3\leqq t\leqq 3}\kappa_t$, $\kappa_t=\kappa'_t\cup\kappa''_t$, must be a trivial sphere S^2 in E^4. But $S^2\cap E^3_0$ is nothing other than the original knot $\kappa_0=\kappa$, which proves our theorem.

REMARK. By the same method of proof it can be easily shown that any product of doubly null-equivalent knots has the same property as the doubly null-equivalent knot in the theorem.

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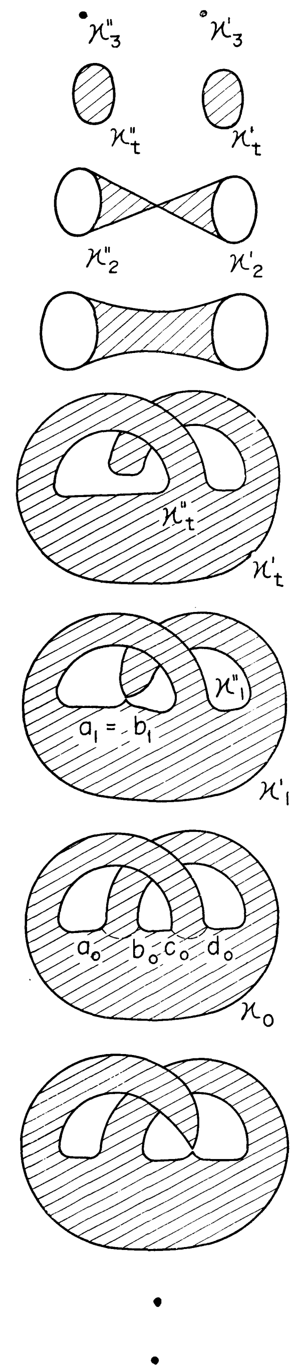

Fig. 5

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