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On the Unknotted Sphere S^2 in E^4

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The construction of a locally flat, knotted sphere introduced by Artin [1] has given rise to a series of further investigations in this direction, [2], [3]. The construction is simply thus: Let E^2 be a plane in E^3 which is in turn in E^4 , and let κ be a knot in E^3 having a segment ab in common with E^2 , otherwise contained wholly in the positive half E^3_+ of E^3 . Call the arc $\kappa^0 = \overline{\kappa - ab}$ an *open knot* with end points a, b . Artin obtained the desired sphere S^2 by rotating the open knot κ^0 around E^2 as axis in E^4 . He showed that the fundamental group of $E^4 - S^2$ is isomorphic to the knot group of κ , that is, to the fundamental group of $E^3 - \kappa$. Fox and Milnor [4] showed that if a locally flat sphere S^2 in E^4

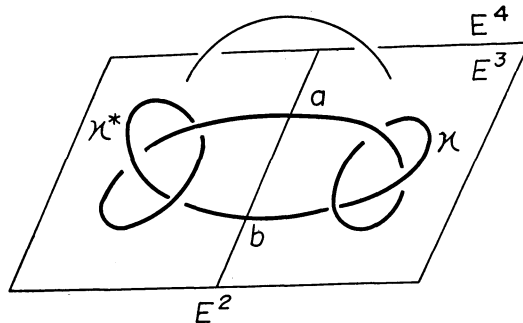


Fig. 1

is cut by an E^3 , and if the intersection $S^2 \cap E^3$ is a knot, which they called a null-equivalent knot, then the Alexander polynomial of this knot must be of the form $f(x)f(x^{-1})x^n$. As it happens, the Alexander polynomial of $S^2 \cap E^3$ is $\Delta^2(x)$ for the sphere S^2 of Artin type, for then the knot in question is the product¹⁾ of κ , of Alexander polynomial $\Delta(x)$, with its symmetric image κ^* with respect to E^2 , as will be seen in the figure.

Now the question is: what can be concluded about the knottedness of a given locally flat sphere $S^2 \subset E^4$ from the information about that of $S^2 \cap E^3$ for any hyperplane E^3 of E^4 ? This and other related questions

1) "sum" would be a better terminology.

are still open; in the present note we shall only show that there is a class of non-trivial knots, called doubly null-equivalent knots, of which each $\kappa \subset E^3$ admits an unknotted sphere $S^2 \subset E^4$ to pass through such that $\kappa = S^2 \cap E^3$.

A cylindrical surface in E^3 bounded by a pair of simple closed curves κ' and κ'' will be called *unknotted*, if it is isotopic to a ringed region on a plane of E^3 .

Let T be a torus in E^3 with a boundary κ , which is a knot. Such a torus can be brought isotopically into the Seifert normal form [5],

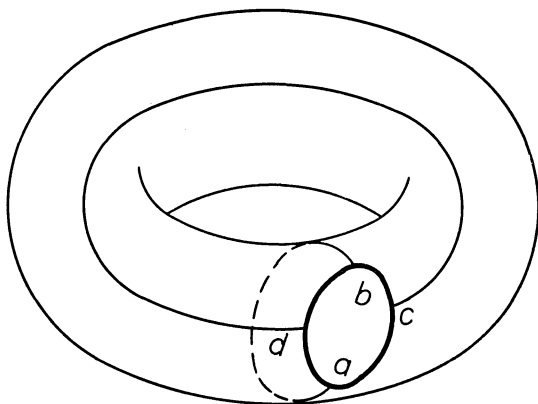


Fig. 2

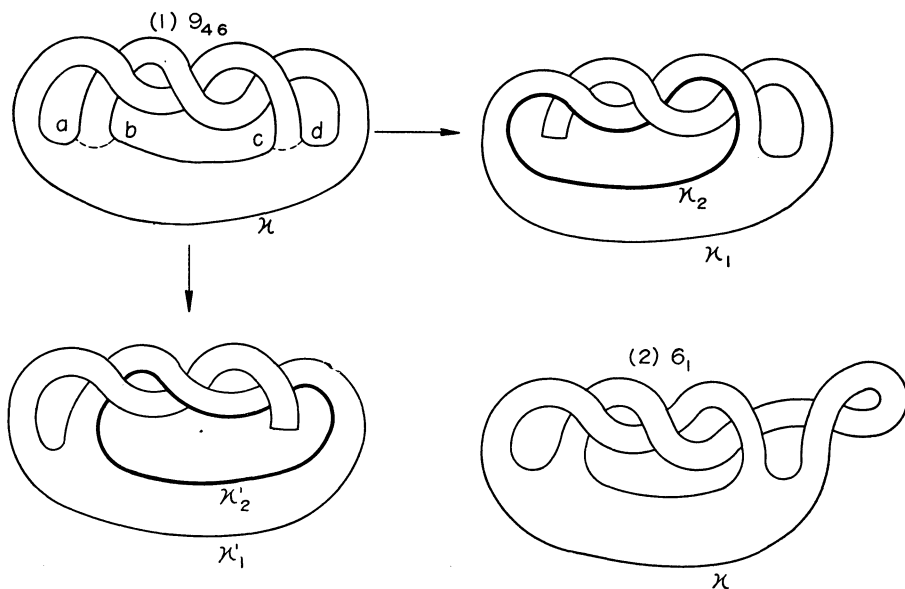


Fig. 3

cf. Fig. 3, (1) and (2). Now, if there is an arc ab joining two points a and b of κ on T such that an unknotted cylindrical surface may be obtained by cutting T along ab , then κ is a *null-equivalent knot*, [4], [6] (cf. also [7], p. 134). If there is moreover another arc joining points c and d of κ on T which is disjoint from ab and not homotopic to ab and which has the same property as above, then κ will be called a *doubly null-equivalent knot*. Call ab and cd *conjugate cross-cuts*. In Fig. 3, (1) represents the knot 9_{46} of the knot table in [8] and, by taking ab and cd as conjugate cross-cuts, it is seen to be a doubly null-equivalent knot, while (2) is the knot 6_1 with the same Alexander polynomial as that of 9_{46} , but is undecided whether or not it is doubly null-equivalent.

The theorem we are to prove is the following :

Theorem. *Let κ be a doubly null-equivalent knot in a hyperplane E^3 of E^4 . Then there is a trivial sphere S^2 in E^4 whose intersection with E^3 coincides with κ .*

Proof will be divided into several steps.

1st step. First we define a continuous family of curves Γ_t , $-3 \leq t \leq 3$, on the standard 2-dimensional sphere Σ^2 in E^3 , which is essentially a topological map of the family of general lemniscates

$$(*) \quad ((x-1)^2 + y^2)((x+1)^2 + y^2) = k^2$$

for $0 \leq k \leq 2$ on the northern hemisphere H_+ of Σ^2 and its symmetric image on the southern hemisphere H_- (cf. Fig. 4):

Γ_3 is the image of the foci $k=0$ of (*) and consists of a pair of points α'_3 and α''_3 .

Γ_t for $3 > t > 1$ is the image of (*) for $0 < k < 1$ and consists each of a pair of simple closed curves Γ'_t and Γ''_t around α'_3 and α''_3 respectively.

Γ_1 is the image of the ordinary 8-shaped lemniscate $k=1$ of (*).

Γ_t for $1 > t \geq 0$ is the image of (*) for $1 < k \leq 2$ and is a simple closed curve. Especially Γ_0 is the equator of Σ^2 .

Further let Γ_{-t} ($3 \geq t > 0$) be the symmetric image of Γ_t with respect to the equatorial plane of Σ^2 .

On the basis of Γ_t we now define a continuous family of disjoint surfaces Φ_t filling up the full sphere Δ^3 of Σ^2 , as follows:

Let Φ_3 coincide with Γ_3 , that is, with points α'_3 and α''_3 .

Let Φ_t for $3 > t > 2$ consist each of a pair of disjoint hemispheres bounded by Γ'_t and Γ''_t respectively.

Let Φ_2 be a pair of hemispheres having a single point in common and bounded each by Γ'_2 and Γ''_2 respectively.

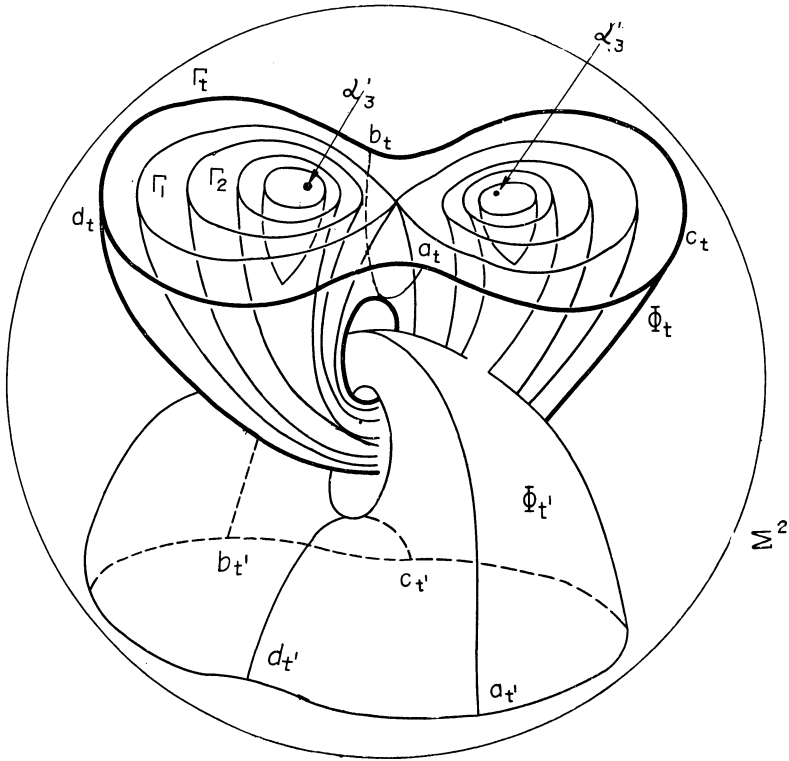


Fig. 4

Let Φ_t for $2 > t > 1$ be each a cylindrical surface bounded by Γ'_t and Γ'_t' .

Let Φ_1 be a torus bounded by the 8-shaped curve Γ_1 .

Finally let Φ_t be for $1 > t \geq 0$ a torus bounded by Γ_t .

For negative t , $0 \geq t \geq -3$, the family of surfaces $\{\Phi_t\}$ should be as a whole homeomorphic to $\{\Phi_{-t}\}$ defined above, $\Phi_0 = \{\Phi_t\} \cap \{\Phi_{-t}\}$ being mapped onto itself by this homeomorphism.

2nd step.

We now provide in the hyperplane $x_4 = 0$, which we denote by E_0^3 , a continuous family of not necessarily disjoint surfaces T_t , $-3 \leq t \leq 3$, of the following kind (cf. Fig. 5, where T_t are shaded):

$\kappa_0 = \kappa$ is the given doubly null-equivalent knot spanned with a torus T_0 , with conjugate cross-cuts $a_0 b_0$ and $c_0 d_0$.

For $0 < t < 1$, T_t is a torus bounded by a knot κ_t .

T_1 is a torus bounded by the union κ_1 of two trivial knots κ'_1 and κ''_1 having in common a single point $a_1 = b_1$, which is the limit of the cross-cut $a_0 b_0$ on T_0 .

For $1 < t < 2$, T_t is an unknotted cylindrical surface bounded by a pair of trivial knots κ'_t and κ''_t .

T_2 is the union of two disks bounded by κ'_2 and by κ''_2 respectively and having a single inner point in common.

For $2 < t < 3$, T_t consists of two disjoint disks bounded by knots κ'_t and κ''_t respectively.

T_3 consists of a pair of distinct points κ'_3 and κ''_3 .

For $-3 \leq t < 0$, T_t is homeomorphic with T_{-t} , provided that the common point of κ'_{-1} and κ''_{-1} of T_{-1} is the limit of the cross-cuts $c_0 d_0$ of T_0 .

Final step.

Now let E_t^3 be the family of parallel hyperplane $x_4 = t$ in E^4 for $-3 \leq t \leq 3$.

To each t of $-3 \leq t \leq 3$ project the surface T_t just defined in E_0^3 into E_t^3 , and denote it by F_t . Then, since T_t , and hence F_t , is homeomorphic to Φ_t , the union $\bigcup_{-3 \leq t \leq 3} F_t = D$ is clearly a full sphere in E^4 , and consequently its boundary $\dot{D} = \bigcup_{-3 \leq t \leq 3} \kappa_t$, $\kappa_t = \kappa'_t \cup \kappa''_t$, must be a trivial sphere S^2 in E^4 . But $S^2 \cap E_0^3$ is nothing other than the original knot $\kappa_0 = \kappa$, which proves our theorem.

REMARK. By the same method of proof it can be easily shown that any product of doubly null-equivalent knots has the same property as the doubly null-equivalent knot in the theorem.

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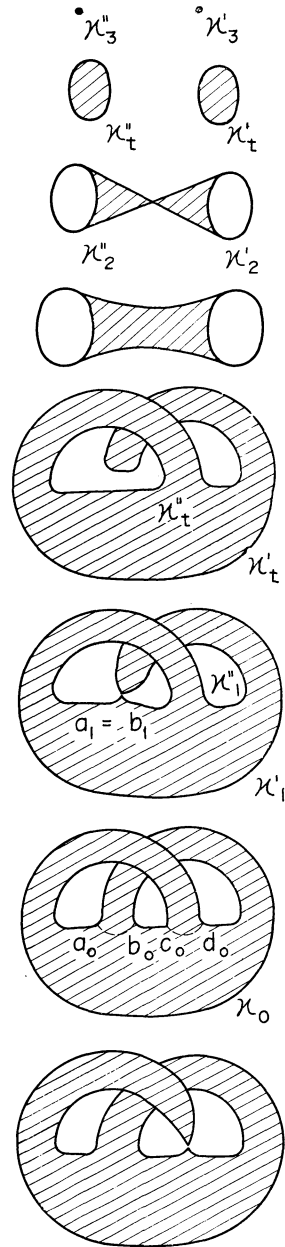


Fig. 5

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