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On the Unknotted Sphere $S^2$ in $E^4$

By Hidetaka Terasaka and Fujitsugu Hosokawa

The construction of a locally flat, knotted sphere introduced by Artin [1] has given rise to a series of further investigations in this direction, [2], [3]. The construction is simply thus: Let $E^2$ be a plane in $E^3$ which is in turn in $E^4$, and let $\kappa$ be a knot in $E^3$ having a segment $ab$ in common with $E^2$, otherwise contained wholly in the positive half $E^3_+$ of $E^3$. Call the arc $\kappa^0 = \kappa - ab$ an open knot with end points $a, b$. Artin obtained the desired sphere $S^2$ by rotating the open knot $\kappa^0$ around $E^2$ as axis in $E^4$. He showed that the fundamental group of $E^4 - S^2$ is isomorphic to the knot group of $\kappa$, that is, to the fundamental group of $E^3 - \kappa$. Fox and Milnor [4] showed that if a locally flat sphere $S^2$ in $E^4$ is cut by an $E^3$, and if the intersection $S^2 \cap E^3$ is a knot, which they called a null-equivalent knot, then the Alexander polynomial of this knot must be of the form $f(x)f(x^{-1})x^n$. As it happens, the Alexander polynomial of $S^2 \cap E^3$ is $\Delta(x)$ for the sphere $S^2$ of Artin type, for then the knot in question is the product of $\kappa$, of Alexander polynomial $\Delta(x)$, with its symmetric image $\kappa^*$ with respect to $E^2$, as will be seen in the figure.

Now the question is: what can be concluded about the knottedness of a given locally flat sphere $S' \subseteq E^4$ from the information about that of $S' \cap E^3$ for any hyperplane $E^3$ of $E^4$? This and other related questions

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1) "sum" would be a better terminology.
are still open; in the present note we shall only show that there is a class of non-trivial knots, called doubly null-equivalent knots, of which each \( \kappa \subset \mathbb{E}^3 \) admits an unknotted sphere \( S^2 \subset \mathbb{E}^4 \) to pass through such that \( \kappa = S^2 \cap \mathbb{E}^4 \).

A cylindrical surface in \( \mathbb{E}^3 \) bounded by a pair of simple closed curves \( \kappa' \) and \( \kappa'' \) will be called unknotted, if it is isotopic to a ringed region on a plane of \( \mathbb{E}^3 \).

Let \( T \) be a torus in \( \mathbb{E}^3 \) with a boundary \( \kappa \), which is a knot. Such a torus can be brought isotopically into the Seifert normal form [5],

\[ \text{Fig. 2} \]

\[ \text{Fig. 3} \]
On the Unknotted Sphere $S^3$ in $E^4$

Now, if there is an arc $ab$ joining two points $a$ and $b$ of $\kappa$ on $T$ such that an unknotted cylindrical surface may be obtained by cutting $T$ along $ab$, then $\kappa$ is a null-equivalent knot, [4], [6] (cf. also [7], p. 134). If there is moreover another arc joining points $c$ and $d$ of $\kappa$ on $T$ which is disjoint from $ab$ and not homotopic to $ab$ and which has the same property as above, then $\kappa$ will be called a doubly null-equivalent knot. Call $ab$ and $cd$ conjugate cross-cuts. In Fig. 3, (1) represents the knot $9_{46}$ of the knot table in [8] and, by taking $ab$ and $cd$ as conjugate cross-cuts, it is seen to be a doubly null-equivalent knot, while (2) is the knot $6_1$ with the same Alexander polynomial as that of $9_{46}$, but is undecided whether or not it is doubly null-equivalent.

The theorem we are to prove is the following:

**Theorem.** Let $\kappa$ be a doubly null-equivalent knot in a hyperplane $E^3$ of $E^4$. Then there is a trivial sphere $S^2$ in $E^4$ whose intersection with $E^3$ coincides with $\kappa$.

**Proof** will be divided into several steps.

1st step. First we define a continuous family of curves $\Gamma_t$, $-3 \leq t \leq 3$, on the standard 2-dimensional sphere $\Sigma^2$ in $E^3$, which is essentially a topological map of the family of general lemniscates

\[(*) \quad ((x-1)^2+y^2)((x+1)^2+y^2) = k^2\]

for $0 \leq k \leq 2$ on the northern hemisphere $H_+$ of $\Sigma^2$ and its symmetric image on the southern hemisphere $H_-$ (cf. Fig. 4):

- $\Gamma_3$ is the image of the foci $k=0$ of (*) and consists of a pair of points $\alpha'_3$ and $\alpha''_3$.
- $\Gamma_t$ for $3 > t > 1$ is the image of (*) for $0 < k < 1$ and consists of a pair of simple closed curves $\Gamma'_t$ and $\Gamma''_t$ around $\alpha'_3$ and $\alpha''_3$ respectively.
- $\Gamma_1$ is the image of the ordinary 8-shaped lemniscate $k=1$ of (*).
- $\Gamma_t$ for $1 > t > 0$ is the image of (*) for $1 < k < 2$ and is a simple closed curve. Especially $\Gamma_0$ is the equator of $\Sigma^2$.

Further let $\Gamma_{-t}(3 \geq t > 0)$ be the symmetric image of $\Gamma_t$ with respect to the equatorial plane of $\Sigma^2$.

On the basis of $\Gamma_t$ we now define a continuous family of disjoint surfaces $\Phi_t$ filling up the full sphere $\Delta^3$ of $\Sigma^2$, as follows:

- Let $\Phi_t$ coincide with $\Gamma_t$, that is, with points $\alpha'_3$ and $\alpha''_3$.
- Let $\Phi_t$ for $3 > t > 2$ consist each of a pair of disjoint hemispheres bounded by $\Gamma'_t$ and $\Gamma''_t$ respectively.
- Let $\Phi_t$ be a pair of hemispheres having a single point in common and bounded each by $\Gamma'_2$ and $\Gamma''_2$ respectively.
Let $\Phi$, for $2 > t > 1$ be each a cylindrical surface bounded by $\Gamma'$ and $\Gamma''$.

Let $\Phi_i$ be a torus bounded by the 8-shaped curve $\Gamma_i$.

Finally let $\Phi_t$ be for $1 > t > 0$ a torus bounded by $\Gamma_t$.

For negative $t$, $0 \geq t \geq -3$, the family of surfaces $\{\Phi_t\}$ should be as a whole homeomorphic to $\{\Phi_i\}$ defined above, $\Phi_0 = \{\Phi_i\} \cap \{\Phi_{-i}\}$ being mapped onto itself by this homeomorphism.

2nd step.
We now provide in the hyperplane $x_4 = 0$, which we denote by $E_0^3$, a continuous family of not necessarily disjoint surfaces $T_t$; $-3 \leq t \leq 3$, of the following kind (cf. Fig. 5, where $T_t$ are shaded):

$\kappa_0 = \kappa$ is the given doubly null-equivalent knot spanned with a torus $T_o$, with conjugate cross-cuts $a_o b_o$ and $c_o d_o$.

For $0 < t < 1$, $T_t$ is a torus bounded by a knot $\kappa_t$.

$T_{1}$ is a torus bounded by the union $\kappa$ of two trivial knots $\kappa_1$ and $\kappa'_1$ having in common a single point $a_1 = b_1$, which is the limit of the cross-cut $a_o b_o$ on $T_o$. 
For $1 < t < 2$, $T_t$ is an unknotted cylindrical surface bounded by a pair of trivial knots $\kappa_t'$ and $\kappa_t''$.

$T_2$ is the union of two disks bounded by $\kappa_2'$ and by $\kappa_2''$ respectively and having a single inner point in common.

For $2 < t < 3$, $T_t$ consists of two disjoint disks bounded by knots $\kappa_t'$ and $\kappa_t''$ respectively.

$T_3$ consists of a pair of distinct points $\kappa_3'$ and $\kappa_3''$.

For $-3 \leq t < 0$, $T_t$ is homeomorphic with $T_{-t}$, provided that the common point of $\kappa_{-1}'$ and $\kappa_{-1}''$ of $T_{-1}$ is the limit of the cross-cuts $c_0d_0$ of $T_0$.

Final step.

Now let $E_t^4$ be the family of parallel hyperplane $x_i = t$ in $E^4$ for $-3 \leq t \leq 3$.

To each $t$ of $-3 \leq t \leq 3$ project the surface $T_t$ just defined in $E_0^3$ into $E_t^3$, and denote it by $F_t$. Then, since $T_t$, and hence $F_t$, is homeomorphic to $\Phi_t$, the union $\bigcup_{-3 \leq t \leq 3} F_t = D$ is clearly a full sphere in $E^4$, and consequently its boundary $\partial D = \bigcup_{-3 \leq t \leq 3} \kappa_t$, $\kappa_t = \kappa_t' \cup \kappa_t''$, must be a trivial sphere $S^2$ in $E^4$. But $S^2 \cap E_0^3$ is nothing other than the original knot $\kappa_0 = \kappa$, which proves our theorem.

REMARK. By the same method of proof it can be easily shown that any product of doubly null-equivalent knots has the same property as the doubly null-equivalent knot in the theorem.

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References