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Author(s)	Terasaka, Hidetaka; Hosokawa, Fujitsugu
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***On the Unknotted Sphere  $S^2$  in  $E^4$***

By Hidetaka TERASAKA and Fujitsugu HOSOKAWA

The construction of a locally flat, knotted sphere introduced by Artin [1] has given rise to a series of further investigations in this direction, [2], [3]. The construction is simply thus: Let  $E^2$  be a plane in  $E^3$  which is in turn in  $E^4$ , and let  $\kappa$  be a knot in  $E^3$  having a segment  $ab$  in common with  $E^2$ , otherwise contained wholly in the positive half  $E^3_+$  of  $E^3$ . Call the arc  $\kappa^0 = \overline{\kappa - ab}$  an *open knot* with end points  $a, b$ . Artin obtained the desired sphere  $S^2$  by rotating the open knot  $\kappa^0$  around  $E^2$  as axis in  $E^4$ . He showed that the fundamental group of  $E^4 - S^2$  is isomorphic to the knot group of  $\kappa$ , that is, to the fundamental group of  $E^3 - \kappa$ . Fox and Milnor [4] showed that if a locally flat sphere  $S^2$  in  $E^4$

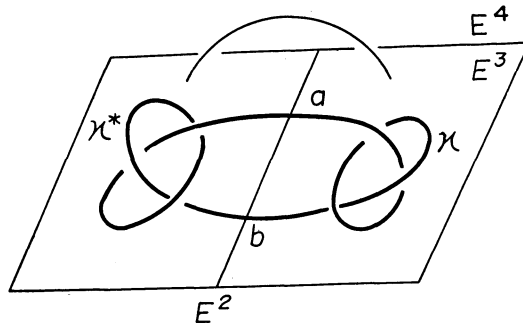


Fig. 1

is cut by an  $E^3$ , and if the intersection  $S^2 \cap E^3$  is a knot, which they called a null-equivalent knot, then the Alexander polynomial of this knot must be of the form  $f(x)f(x^{-1})x^n$ . As it happens, the Alexander polynomial of  $S^2 \cap E^3$  is  $\Delta^2(x)$  for the sphere  $S^2$  of Artin type, for then the knot in question is the product<sup>1)</sup> of  $\kappa$ , of Alexander polynomial  $\Delta(x)$ , with its symmetric image  $\kappa^*$  with respect to  $E^2$ , as will be seen in the figure.

Now the question is: what can be concluded about the knottedness of a given locally flat sphere  $S^2 \subset E^4$  from the information about that of  $S^2 \cap E^3$  for any hyperplane  $E^3$  of  $E^4$ ? This and other related questions

1) "sum" would be a better terminology.

are still open; in the present note we shall only show that there is a class of non-trivial knots, called doubly null-equivalent knots, of which each  $\kappa \subset E^3$  admits an unknotted sphere  $S^2 \subset E^4$  to pass through such that  $\kappa = S^2 \cap E^3$ .

A cylindrical surface in  $E^3$  bounded by a pair of simple closed curves  $\kappa'$  and  $\kappa''$  will be called *unknotted*, if it is isotopic to a ringed region on a plane of  $E^3$ .

Let  $T$  be a torus in  $E^3$  with a boundary  $\kappa$ , which is a knot. Such a torus can be brought isotopically into the Seifert normal form [5],

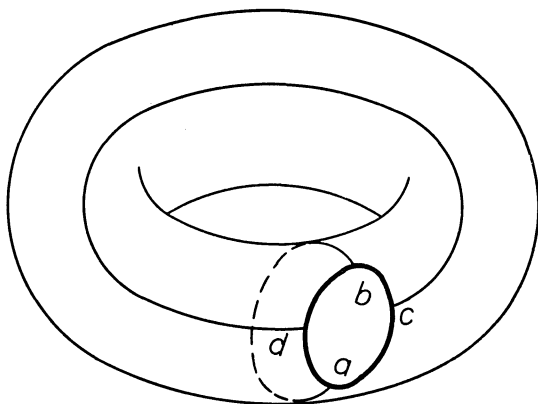


Fig. 2

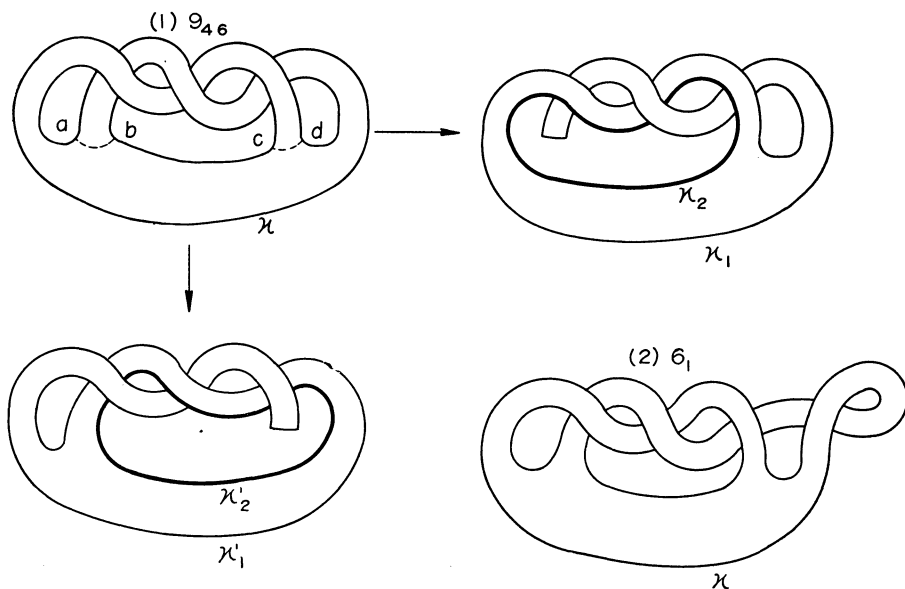


Fig. 3

cf. Fig. 3, (1) and (2). Now, if there is an arc  $ab$  joining two points  $a$  and  $b$  of  $\kappa$  on  $T$  such that an unknotted cylindrical surface may be obtained by cutting  $T$  along  $ab$ , then  $\kappa$  is a *null-equivalent knot*, [4], [6] (cf. also [7], p. 134). If there is moreover another arc joining points  $c$  and  $d$  of  $\kappa$  on  $T$  which is disjoint from  $ab$  and not homotopic to  $ab$  and which has the same property as above, then  $\kappa$  will be called a *doubly null-equivalent knot*. Call  $ab$  and  $cd$  *conjugate cross-cuts*. In Fig. 3, (1) represents the knot  $9_{46}$  of the knot table in [8] and, by taking  $ab$  and  $cd$  as conjugate cross-cuts, it is seen to be a doubly null-equivalent knot, while (2) is the knot  $6_1$  with the same Alexander polynomial as that of  $9_{46}$ , but is undecided whether or not it is doubly null-equivalent.

The theorem we are to prove is the following :

**Theorem.** *Let  $\kappa$  be a doubly null-equivalent knot in a hyperplane  $E^3$  of  $E^4$ . Then there is a trivial sphere  $S^2$  in  $E^4$  whose intersection with  $E^3$  coincides with  $\kappa$ .*

*Proof* will be divided into several steps.

1st step. First we define a continuous family of curves  $\Gamma_t$ ,  $-3 \leq t \leq 3$ , on the standard 2-dimensional sphere  $\Sigma^2$  in  $E^3$ , which is essentially a topological map of the family of general lemniscates

$$(*) \quad ((x-1)^2 + y^2)((x+1)^2 + y^2) = k^2$$

for  $0 \leq k \leq 2$  on the northern hemisphere  $H_+$  of  $\Sigma^2$  and its symmetric image on the southern hemisphere  $H_-$  (cf. Fig. 4):

$\Gamma_3$  is the image of the foci  $k=0$  of (\*) and consists of a pair of points  $\alpha'_3$  and  $\alpha''_3$ .

$\Gamma_t$  for  $3 > t > 1$  is the image of (\*) for  $0 < k < 1$  and consists each of a pair of simple closed curves  $\Gamma'_t$  and  $\Gamma''_t$  around  $\alpha'_3$  and  $\alpha''_3$  respectively.

$\Gamma_1$  is the image of the ordinary 8-shaped lemniscate  $k=1$  of (\*).

$\Gamma_t$  for  $1 > t \geq 0$  is the image of (\*) for  $1 < k \leq 2$  and is a simple closed curve. Especially  $\Gamma_0$  is the equator of  $\Sigma^2$ .

Further let  $\Gamma_{-t}$  ( $3 \geq t > 0$ ) be the symmetric image of  $\Gamma_t$  with respect to the equatorial plane of  $\Sigma^2$ .

On the basis of  $\Gamma_t$  we now define a continuous family of disjoint surfaces  $\Phi_t$  filling up the full sphere  $\Delta^3$  of  $\Sigma^2$ , as follows:

Let  $\Phi_3$  coincide with  $\Gamma_3$ , that is, with points  $\alpha'_3$  and  $\alpha''_3$ .

Let  $\Phi_t$  for  $3 > t > 2$  consist each of a pair of disjoint hemispheres bounded by  $\Gamma'_t$  and  $\Gamma''_t$  respectively.

Let  $\Phi_2$  be a pair of hemispheres having a single point in common and bounded each by  $\Gamma'_2$  and  $\Gamma''_2$  respectively.

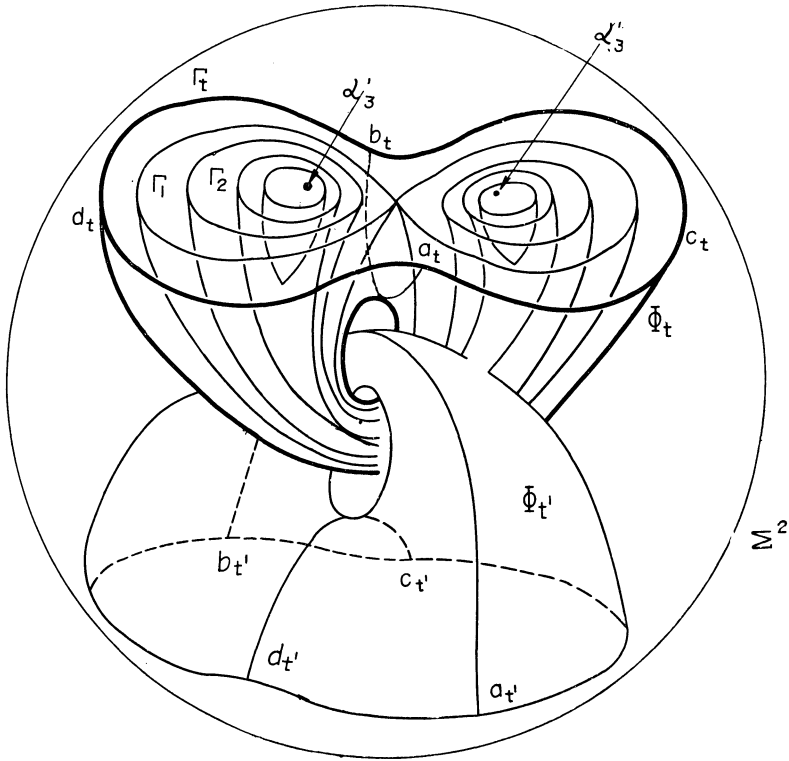


Fig. 4

Let  $\Phi_t$  for  $2 > t > 1$  be each a cylindrical surface bounded by  $\Gamma'_t$  and  $\Gamma'_t'$ .

Let  $\Phi_1$  be a torus bounded by the 8-shaped curve  $\Gamma_1$ .

Finally let  $\Phi_t$  be for  $1 > t \geq 0$  a torus bounded by  $\Gamma_t$ .

For negative  $t$ ,  $0 \geq t \geq -3$ , the family of surfaces  $\{\Phi_t\}$  should be as a whole homeomorphic to  $\{\Phi_{-t}\}$  defined above,  $\Phi_0 = \{\Phi_t\} \cap \{\Phi_{-t}\}$  being mapped onto itself by this homeomorphism.

2nd step.

We now provide in the hyperplane  $x_4 = 0$ , which we denote by  $E_0^3$ , a continuous family of not necessarily disjoint surfaces  $T_t$ ,  $-3 \leq t \leq 3$ , of the following kind (cf. Fig. 5, where  $T_t$  are shaded):

$\kappa_0 = \kappa$  is the given doubly null-equivalent knot spanned with a torus  $T_0$ , with conjugate cross-cuts  $a_0 b_0$  and  $c_0 d_0$ .

For  $0 < t < 1$ ,  $T_t$  is a torus bounded by a knot  $\kappa_t$ .

$T_1$  is a torus bounded by the union  $\kappa_1$  of two trivial knots  $\kappa'_1$  and  $\kappa''_1$  having in common a single point  $a_1 = b_1$ , which is the limit of the cross-cut  $a_0 b_0$  on  $T_0$ .

For  $1 < t < 2$ ,  $T_t$  is an unknotted cylindrical surface bounded by a pair of trivial knots  $\kappa'_t$  and  $\kappa''_t$ .

$T_2$  is the union of two disks bounded by  $\kappa'_2$  and by  $\kappa''_2$  respectively and having a single inner point in common.

For  $2 < t < 3$ ,  $T_t$  consists of two disjoint disks bounded by knots  $\kappa'_t$  and  $\kappa''_t$  respectively.

$T_3$  consists of a pair of distinct points  $\kappa'_3$  and  $\kappa''_3$ .

For  $-3 \leq t < 0$ ,  $T_t$  is homeomorphic with  $T_{-t}$ , provided that the common point of  $\kappa'_{-1}$  and  $\kappa''_{-1}$  of  $T_{-1}$  is the limit of the cross-cuts  $c_0 d_0$  of  $T_0$ .

Final step.

Now let  $E_t^3$  be the family of parallel hyperplane  $x_4 = t$  in  $E^4$  for  $-3 \leq t \leq 3$ .

To each  $t$  of  $-3 \leq t \leq 3$  project the surface  $T_t$  just defined in  $E_0^3$  into  $E_t^3$ , and denote it by  $F_t$ . Then, since  $T_t$ , and hence  $F_t$ , is homeomorphic to  $\Phi_t$ , the union  $\bigcup_{-3 \leq t \leq 3} F_t = D$  is clearly a full sphere in  $E^4$ , and consequently its boundary  $\dot{D} = \bigcup_{-3 \leq t \leq 3} \kappa_t$ ,  $\kappa_t = \kappa'_t \cup \kappa''_t$ , must be a trivial sphere  $S^2$  in  $E^4$ . But  $S^2 \cap E_0^3$  is nothing other than the original knot  $\kappa_0 = \kappa$ , which proves our theorem.

REMARK. By the same method of proof it can be easily shown that any product of doubly null-equivalent knots has the same property as the doubly null-equivalent knot in the theorem.

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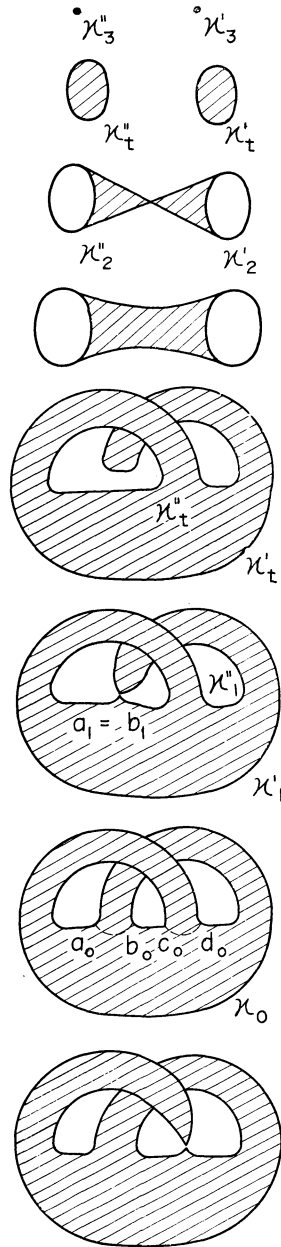


Fig. 5

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