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Osaka University
ON THE BORSUK-ULAM THEOREM FOR $\mathbb{Z}_{p^a}$ ACTIONS ON $S^{2n-1}$ AND MAPS $S^{2n-1} \to \mathbb{R}^m$

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1. Introduction

In [3] we raised the following question:
Let $G$ be a finite group acting properly (as a group of homeomorphisms) on the $n$-sphere. For any (topological) $m$-manifold $M$ and any map $f: S^n \to M$ let $A(f)=\{x \in S^n \mid f(x) = f(xg), \text{ all } g \in G\}$. What can be deduced about $\dim A(f)$?

In case $M=\mathbb{R}^m$, euclidean $m$-space, $A(f)$ is the set of solutions of $(|G|-1)m+1$ equations in $n+1$ unknowns so one might hope to get

$$\dim A(f) \geq n - (|G|-1)m.$$  

(1.1)

If $G$ is cyclic of prime order then (1.1) actually holds even for maps $f: S^n \to M^m$ provided $M^m$ is compact (for $G=\mathbb{Z}_p$ and $m=n$ assume also that $f_\ast=0: H_n(S^n; \mathbb{Z}_p) \to H_n(M^n; \mathbb{Z}_p)$), see [3]. In this note we consider $G=\mathbb{Z}_{p^a}$, cyclic of odd prime power order, and we restrict attention to maps into $\mathbb{R}^m$.

Our results are expressed in two theorems:

**Mod $p^a$ Borsuk-Ulam theorem:** For any proper action of $\mathbb{Z}_{p^a}$ on $S^{2m-1}$, $p$ an odd prime, and any map $f: S^{2m-1} \to \mathbb{R}^m$ one has

$$\dim A(f) \geq (2n-1) - (p^a - 1)m - [m(a-1)p^a - (ma + 2)p^{a-1} + m + 3].$$

**Mod $p^a$ Borsuk-Ulam anti-theorem:** Consider the standard linear action of $\mathbb{Z}_{p^a}$ on $S^{2m-1}$. Assume $a>1$ and $p^a \neq 9$. If $2n-1 \leq (p^a-1)m + (2p-3)m - 1$ then there exists a map $f: S^{2m-1} \to \mathbb{R}^m$ with $A(f) = \emptyset$.

Notice that the anti-theorem says that (1.1) fails whenever $a>1$ and $p^a \neq 9$; the theorem gives $m(a-1)p^a - (ma + 2)p^{a-1} + m + 3$ as an upper bound for this failure. For $a=1$ this upper bound is 1, so for $G=\mathbb{Z}_p$ we are 1 off our previous results [3].

**Remarks.** 1. $\dim$ means covering dimension.
2. For $p^a=9$ and $m>1$ there is a result similar to the anti-theorem. We leave that to the interested reader.
3. In private correspondence with M. Nakaoka I have recently learned that (1.1) holds for $Z_p$-actions on mod $p$ homology spheres $S^n$ and maps $f: S^n \to M^m$ without the restriction of niceness of $f$ imposed by me in [3].

2. Proof of theorem

Let $\mu: S \times G \to S$ be a proper action of the cyclic group $G$ of odd prime power order $p^a = q = 2k+1$ on the $(2n-1)$ sphere $S$. Denote by $\eta$ the corresponding principal $G$-bundle $S \to S/\mu$ over the orbit space $S/\mu$. For a complex $G$-module $M$ let $\eta[M]$ be the complex vector bundle $S \times_G M \to S/\mu$. The correspondence $M \to \eta[M]$ gives rise to a ring homomorphism $\alpha: R^G \to K^q(S/\mu)$ where $R^G$ is the complex representation ring for $G$ while $K^q$ denotes complex $K$-theory. Denote by $L$ the standard 1-dimensional complex $G$-module, i.e. $L = \mathbb{C}$, the field of complex numbers, and, fixing a generator $g_0$ for $G$, $g_0 = \exp(2\pi i \sqrt{-1}) c$. Then $R^G = Z[p]/(p^a - 1)$ where $p$ is the class of $L$. Finally, put $\lambda = \eta[L]$ and for any map $f: S \to R^m$ let $\lambda_f$ be the restriction of $\lambda$ to $A(f)/\mu \subseteq S/\mu$.

Now the mod $p^a$ Borsuk-Ulam theorem is essentially contained in

**Lemma 1.** If $d\lambda_f$ has a never vanishing section then $d \geq n-1 + p^a - 1$.

Proof. Assume that $d\lambda_f$ has a never vanishing section. We first show

\[ (2.1) \quad P(p) = (\alpha - 1)^a \{(\alpha - 1)(\alpha^2 - 1) \cdots (\alpha^k - 1)\}}. \]

Recall that the $i$th Atiyah class $a_i(\xi)$ of an $n$-dimensional complex vector bundle $\xi$ is given by $a_i(\xi) = \gamma^i(\xi - n)$ with $\gamma^i$ as in [1]. Then we have the usual Whitney duality, namely $a_i(\xi) \otimes a_i(\xi) = \sum_{i \neq j} a_j(\xi) \otimes a_j(\xi)$, also for any line bundle $\xi$, $a_i(\xi) = \xi - 1$. Therefore it is immediate that $\alpha P(p) = a_{d+mk}(\Lambda)$ where $\Lambda$ is the vector bundle $d\lambda \oplus m[\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^k]$, and so (2.1) follows from

\[ (2.2) \quad \text{Ker}(\alpha: R^G \to K^q(S/\mu) \subseteq (\alpha - 1)^a \cdot R^G) \]

(2.3) $\Lambda$ admits a never vanishing section.

To get (2.2) we compare $\mu$ with the standard linear action $\mu_0: S \times G \to S$ obtained by viewing $S$ as the unit sphere in $nL = L \oplus \cdots \oplus L$. $S^{2m-1}/\mu_0$ is a $(2n-1)$-dimensional cell complex and $\eta: S \to S/\mu$ is $(2n-1)$-universal in the sense of [5]. Hence there is a bundle map

\[
\begin{array}{ccc}
S & \xrightarrow{b} & S \\
\downarrow{\eta_0} & & \downarrow{\eta} \\
S/\mu_0 & \xrightarrow{b} & S/\mu.
\end{array}
\]
Furthermore, it is obvious that

$$
\begin{array}{c}
\mathcal{R}G \xrightarrow{\alpha} K^0(S/\mu) \\
\downarrow \alpha_0 \downarrow b^*
\end{array}
$$

commutes, so $\text{Ker } \alpha \subseteq \text{Ker } \alpha_0$. But $\alpha_0$ fits into an exact sequence (see [1])

$$
\mathcal{R}G \xrightarrow{\varphi} \mathcal{R}G \xrightarrow{\alpha_0} K^0(S/\mu_0)
$$

where $\varphi$ is multiplication by $\lambda,-(n\rho)=1-\rho)^n$, so $\text{Ker } \alpha_0 \subseteq (\rho-1)^n \cdot \mathcal{R}G$.

In [3] it is shown that $f$ gives rise to a section $S$ of $m(\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^k)$ which vanishes precisely on $A(f)/\mu$ (see especially Digression 1, p. 171–2 and Step 3, p. 180–1 of [3]). $s$ and the given section $s_0$ of $d\lambda, f$ go together to prove (2.3). This completes the proof of (2.1).

Our next step is to show that (2.1) is actually equivalent to the inequality

$$
d \geq n-1+p^{a-1}-\frac{1}{2} am(p^a-p^{a-1}).$$

The equivalence is obvious if $n<d+mk$, so assume $n \geq d+mk$. Lift (2.1) to the polynomial ring $Z[x]$ to get the equivalent

$$
(2.1.1) \quad \exists g, h \in Z[x]: P(x) = g(x)(x-1)^n + h(x)(x^q-1).
$$

Now $P(x) = (x-1)^{d+mk} \cdot \Pi_{j=1}^j f_j(x)^{m[k/i]} \cdot (x^q-1) = (x-1)^{d+km-1} \cdot \Pi_{j=1}^j f_j(x)$ where $f_j$ is the $j^{th}$ cyclotomic polynomial and $[k/j]$ is the integral part of $kJ$. Hence, if (2.1.1) holds then $g$ is divisible by $\Pi_{j=1}^j f_j(x)$ and $h$ is divisible by $(x-1)^{d+km-1}$. So, putting $\varepsilon_j = 0$ if $j \nmid p^n, \varepsilon_j = 1$ if $j \mid p^n$, (2.1.1) implies (and is clearly implied by)

$$
(2.1.2) \quad \exists \bar{g}, \bar{h} \in Z[x]: \Pi_{j=2}^j f_j(x)^{m[k/j]} \cdot (x-1)^{d-\bar{h}} = \bar{g}(x) \cdot (x-1)^{n-\bar{h}} + \bar{h}(x) \cdot f_\varphi(x).
$$

Let $\gamma$ be a primitive $q^{th}$ root of unity and consider the projection $Z[x] \to Z[\gamma] \subseteq C$. Its kernel is the ideal generated by $f_\varphi(x)$ so (2.1.2) is equivalent to

$$
(2.1.3) \quad (\gamma-1)^{d-\bar{h}} \cdot \Pi_{j=2}^j (\gamma)^{m[k/j]} \cdot (x-1)^{n-\bar{h}} \in Z[\gamma].
$$

Now $Z[\gamma]$ is precisely the algebraic integers of the field $Q(\gamma)$ and $(\gamma-1)Z[\gamma]$ is the unique prime ideal in $Z[\gamma]$ lying above $pZ$, see e.q. [6]. Let $\mathcal{N}: Q(\gamma) \to Q$ be the norm map for the extension $Q(\gamma)/Q$. It is then an immediate consequence of classical ideal theory for Dedekind extensions that (2.1.3) is equivalent to

$$
(2.1.4) \quad \mathcal{N}(\gamma-1)^{d-\bar{h}} \cdot \Pi_{j=2}^j (\mathcal{N}(f_j(\gamma)))^{m[k/j]} \in Z.
$$

The norms involved here are not hard to compute, so rearranging (2.1.4) slightly it takes the desired form $d \geq n-1+p^{a-1}-\frac{1}{2} am(p^a-p^{a-1})$. 


If $A(f)/\mu$ happens to be a CW complex then of course we have $(\dim A(f)/\mu < 2d) \Rightarrow (d \chi_f$ has a never vanishing section), and the above lemma can then be translated into a condition on $\dim A(f)/\mu$. Since also $\dim A(f) = \dim A(f)/\mu$ (because $A(f) \to A(f)/\mu$ is a finite covering and $\dim$ has the monotonicity and sum-properties, see [4]) this completes the proof of the mod $p^a$ Borsuk-Ulam theorem. $A(f)/\mu$, however, need not be a CW complex so we need to know the following

**Lemma 2.** If $\lambda$ is a complex line bundle over a compact metric space $X$ of covering dimension $< 2d$ then $d\lambda$ admits a never vanishing section.

I certainly do not believe that this lemma is unknown. However, nor do I know of any reference for it, so a proof of it is given as an appendix.

### 3. Proof of the anti-theorem

Consider the standard linear action $\mu_0$ of $G = \mathbb{Z}_{pa}$ on $S^{2N-1}$, $N$ big, i.e. view $S^{2N-1}$ as the unit sphere in $NL = L \oplus \cdots \oplus L$. $S^{2N-1}/\mu_0$ is a CW-complex with $S^{2N-1}/\mu_0$ as $(2n-1)$-skeleton. Let $\xi$ be the vector bundle $S^{2N-1} \times_G IG \to S^{2N-1}/\mu_0$ where $IG$ is the augmentation ideal of the real group algebra $RG$.

We notice that the anti-theorem is a consequence of

$$m\xi \text{ admits a never vanishing section over the } [(p^a-1)m+(2p-3)m-1]-\text{skeleton}. \tag{3.1}$$

Indeed, it is well known how a section $s$ of $m\xi$ over the $(2n-1)$-skeleton corresponds to an equivariant map $F: S^{2n-1} \to m(IG) = IG \oplus IG \oplus \cdots \oplus IG$. If $i: IG \to RG$ is the inclusion then equivariance of $F$ means that $(i \oplus \cdots \oplus i)F$ has the form $(i \oplus \cdots \oplus i)F(x) = (\Sigma_{g} f_1(xg^{-1})g, \cdots, \Sigma_{g} f_m(xg^{-1})g)$ for well defined continuous maps $f_i: S^{2n-1} \to R$. Put $f = (f_1, \cdots, f_m)$ and notice that $A(f) = \phi$ is equivalent to $s$ having no zeros.

If we have shown (3.1) for $m=1$ then it follows for general $m$ by noticing that $m\xi \cong \Delta^*(\xi \times \cdots \times \xi)$ for any skeletal approximation $\Delta: S^{2N-1}/\mu_0 \to S^{2N-1}/\mu_0 \times \cdots \times S^{2N-1}/\mu_0$ to the $m$-fold diagonal. Hence, assume $m=1$. In [3] we showed that the mod $p$ Euler class of $\xi$ vanishes whenever $a>1$. If we further exclude the case $p^2=9$ then the same proof shows that the integral Euler class vanishes. Hence $\xi$ does have a never vanishing section over the $2k$ skeleton. The obstructions to extending this section over the successive skeleta lie in $H^{2k+i}(S^{2N-1}/G; \pi_{2k-1+i}(S^{2k-1})) \cong H^{2k+i}(G; \pi_{2k-1+i}(S^{2k-1}))$. For $0 < i < 2p-3$ the homotopy group in question has vanishing $p$-primary component so the obstructions vanish and we do have our desired section over the $(2k+2p-4)$-skeleton.

**Remark.** In the above we have made strong use of the fact that $\xi$ admits
4. Remarks on the case $G = \mathbb{Z}_{25}$, $m = 1$, linear action

For $G = \mathbb{Z}_{25}$ and $m = 1$ our results show that there exists a map $f : S^{29} \to R$ with $A(f) = \phi$ whereas every map $f : S^{33} \to R$ has $A(f) \neq \phi$. In fact every map $f : S^{31} \to R$ has $A(f) \neq \phi$ as we now show. Suppose that $A(f_0) = \phi$ for some $f_0 : S^{31} \to R$. Then

$s_0(xG) = (x, \pi(\sum_g f_0(xg^{-1})g))G$

defines a cross-section $s_0$ of $\xi$ over the 31-skeleton. ($\pi : RG \to IG$ is given by $\pi(\Sigma r_g g) = \Sigma r_g g - 1$). The obstruction to extending $s_0$ further lie in $H^{23+i}(S^{2n-1})/\mathbb{Z}_{25}; \pi_{31+i}(S^{2n})$. Since the 5-primary component of $\pi_{31+i}(S^{2n})$ is zero for $0 \leq i < 6$ we get a never vanishing section over the 37-skeleton. As in §3 this gives an $f : S^{37} \to R$ with $A(f) = \phi$. But that contradicts the above result for maps $S^{33} \to R$.

Unfortunately for $p^a > 25$ our positive and negative results are too far apart to close the gap between them by means as trivial as the above.

Appendix. Proof of lemma 2

Let $\Delta$ be the abstract $4d$-1 simplex and $|\Delta|$ its standard realization in $R^{4d}$. By the general embedding theorem for compacta (see e.g. p. 139 of [2]) $X$ can be taken as a closed subspace of $|\Delta|$. Let $K_n$ be the subcomplex of $\Delta^{(n)}$ ($= n^{th}$ barycentric subdivision of $\Delta$) spanned by all $4d - 1$ simplices $\tau$ for which $|\tau| \cap X = \varnothing$. Then $K_n$ is a subcomplex of the barycentric subdivision of $K_{n-1}$ so the inclusion $i_n : |K_n| \to |K_{n-1}|$ admits a simplicial approximation $\varphi_n : K_n \to K_{n-1}$. Also $\{ |K_n| \}$ is cofinal in the (downward) directed set of all neighborhoods of $X$ in $|\Delta|$, so for any abelian group $A$ we have $H^*(X; A)$ $\approx \lim_{\to n} H^*(|K_n| ; A)$, where as usual $H^*$ is Cech cohomology, while $H^*$ can be taken as any ordinary cohomology theory. Since line bundles are characterized by the first Chern class $c_1 \in H^2(- ; \mathbb{Z})$ it follows that $\lambda$ admits an extension $\lambda_n$ over $|K_n|$ for $N$ sufficiently large. Fix such an $N$ and define (inductively, for $n \geq N$) $\lambda_n = |\varphi_n| * \lambda_{n-1}$. Let $\sigma_n$ be the sphere bundle associated with $d\lambda_n$. Since $\lambda \approx \lambda_n \cap X$, $n \geq N$, it is clearly sufficient to show that $\sigma_n$ admits a cross-section when $n$ is sufficiently large, in other words, if we let $k$ be the maximal number such that for some $n \geq N \sigma_n$ admits a cross-section over the $k$-skeleton $|K_k|$ of $K_n$ then we must show $k \geq 4d - 1$. Suppose $k < 4d - 1$. Choose $n \geq N$ such that $\sigma_n || K_n^k$ has a cross-section, $s$, say. Consider the restriction $s'$ of $s$ to the $(k-1)$-skeleton and the obstruction $c$ to extending $s'$ over the $(k+1)$-skeleton (obstruction in the sense of [5]). $c \in H^{k+1}(|K_n| ; \pi)$ where $\pi = \pi_k(S^{4d-1})$, and —
by maximality of $k - c = 0$. Since $k$ is clearly $\geq 2d - 1$ our assumption on $\dim X$ assures that $H^{k+1}(X; \pi) = \lim \limits_{\rightarrow j} H^{k+1}(|K_j|; \pi)$ vanishes so there is an $m > n$ such that $c|K_m| = |\varphi|^* c = 0$; here $\varphi$ is an abbreviation for $\varphi_n, \varphi_{n+2}, \cdots$

$\varphi_m: K_m \rightarrow K_n$. Now $\sigma_m = |\varphi|^* \sigma_n$ and $|\varphi|$ is skeleton preserving so $s$ gives rise to a cross-section $s_i$ of $\sigma_m|K_m|$. Moreover, if $s'_i$ is the restriction of $s_i$ to $|K_m^{k+1}|$ then the obstruction to extending $s'_i$ over $|K_m^{k+1}|$ is precisely $|\varphi|^* c$. But $|\varphi|^* c = 0$ so $s_i'$ does extend over $|K_m^{k+1}|$, thus contradicting the maximality of $k$.

References