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ON THE BORSUK-ULAM THEOREM FOR $\mathbb{Z}_p^a$ ACTIONS ON $S^{2n-1}$ AND MAPS $S^{2n-1} \to \mathbb{R}^m$

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1. Introduction

In [3] we raised the following question:

Let $G$ be a finite group acting properly (as a group of homeomorphisms) on the $n$-sphere. For any (topological) $m$-manifold $M$ and any map $f$: $S^n \to M$ let $A(f) = \{x \in S^n | f(x) = f(xg), \text{all } g \in G\}$. What can be deduced about $\dim A(f)$?

In case $M = \mathbb{R}^m$, euclidean $m$-space, $A(f)$ is the set of solutions of $(|G| - 1)m + 1$ equations in $n + 1$ unknowns so one might hope to get

$$\dim A(f) \geq n - (|G| - 1)m.$$  

If $G$ is cyclic of prime order then (1.1) actually holds even for maps $f$: $S^n \to \mathbb{R}^m$ provided $M^m$ is compact (for $G = \mathbb{Z}_p$ and $m = n$ assume also that $f_*=0$: $H_n(S^n; \mathbb{Z}_p) \to H_n(M^n; \mathbb{Z}_p)$), see [3]. In this note we consider $G = \mathbb{Z}_p^a$, cyclic of odd prime power order, and we restrict attention to maps into $\mathbb{R}^m$.

Our results are expressed in two theorems:

Mod $p^a$ Borsuk-Ulam theorem: For any proper action of $\mathbb{Z}_p^a$ on $S^{2n-1}$, $p$ an odd prime, and any map $f$: $S^{2n-1} \to \mathbb{R}^m$ one has

$$\dim A(f) \geq (2n - 1) - (p^a - 1)m - [m(a - 1)p^a - (ma + 2)p^{a-1} + m + 3].$$

Mod $p^a$ Borsuk-Ulam anti-theorem: Consider the standard linear action of $\mathbb{Z}_p^a$ on $S^{2n-1}$. Assume $a > 1$ and $p^a \neq 9$. If $2n - 1 \leq (p^a - 1)m + (2p - 3)m - 1$ then there exists a map $f$: $S^{2n-1} \to \mathbb{R}^m$ with $A(f) = \Phi$.

Notice that the anti-theorem says that (1.1) fails whenever $a > 1$ and $p^a \neq 9$; the theorem gives $m(a - 1)p^a - (ma + 2)p^{a-1} + m + 3$ as an upper bound for this failure. For $a = 1$ this upper bound is 1, so for $G = \mathbb{Z}_p$ we are 1 off our previous results [3].

Remarks. 1. $\dim$ means covering dimension.
2. For $p^a = 9$ and $m > 1$ there is a result similar to the anti-theorem. We leave that to the interested reader.
3. In private correspondence with M. Nakaoka I have recently learned that (1.1) holds for $Z_p$-actions on mod $p$ homology spheres $S^n$ and maps $f: S^n \to M^m$ without the restriction of niceness of $f$ imposed by me in [3].

2. Proof of theorem

Let $\mu: S \times G \to S$ be a proper action of the cyclic group $G$ of odd prime power order $p^a=q=2k+1$ on the $(2n-1)$ sphere $S$. Denote by $\eta$ the corresponding principal $G$-bundle $S \to S/\mu$ over the orbit space $S/\mu$. For a complex $G$-module $M$ let $\eta[M]$ be the complex vector bundle $S \times_G M \to S/\mu$. The correspondence $M \mapsto \eta[M]$ gives rise to a ring homomorphism $\alpha: \mathcal{R}G \to K^0(S/\mu)$ where $\mathcal{R}G$ is the complex representation ring for $G$ while $K^0$ denotes complex $K$-theory. Denote by $L$ the standard 1-dimensional complex $G$-module, i.e. $L=C$, the field of complex numbers, and, fixing a generator $g_0$ for $G$, $g_0 \circ = \exp(2\pi i \frac%}{3} \frac{\nu}{\nu+1}$. Then $\mathcal{R}G = \mathbb{Z}[\rho]/(\rho^a-1)$ where $\rho$ is the class of $L$. Finally, put $\lambda = \eta[L]$ and for any map $f: S \to R^m$ let $\lambda_f$ be the restriction of $\lambda$ to $A(f)/\mu \subseteq S/\mu$.

Now the mod $p^a$ Borsuk-Ulam theorem is essentially contained in

**Lemma 1.** If $d \lambda_f$ has a never vanishing section then $d \geq n-1+p^a-1$.

**Proof.** Assume that $d \lambda_f$ has a never vanishing section. We first show

\begin{equation}
(2. 1) \quad \alpha(P(\rho)) = (\rho-1)^a \cdot (\rho^a-1)^m \in (\rho-1)^* \cdot \mathbb{Z}[\rho]/(\rho^a-1).
\end{equation}

Recall that the $i$th Atiyah class $a_i(\xi)$ of an $n$-dimensional complex vector bundle $\xi$ is given by $a_i(\xi) = \gamma^i(\xi-n)$ with $\gamma^i$ as in [1]. Then we have the usual Whitney duality, namely $a_i(\xi \oplus \xi_2) = \sum_{i \in \mathbb{N}} a_i(\xi) a_k(\xi_2)$, also for any line bundle $\xi$, $a_1(\xi) = \xi - 1$. Therefore it is immediate that $\alpha P(\rho) = a_{d+mk}(\Lambda)$ where $\Lambda$ is the vector bundle $d \lambda \oplus m[\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^k]$, and so (2.1) follows from

\begin{equation}
(2. 2) \quad \operatorname{Ker}(\alpha: \mathcal{R}G \to K^0(S/\mu)) \subseteq (\rho-1)^* \cdot \mathcal{R}G,
\end{equation}

(2. 3) \quad $\Lambda$ admits a never vanishing section.

To get (2.2) we compare $\mu$ with the standard linear action $\mu_o: S \times G \to S$ obtained by viewing $S$ as the unit sphere in $nL=L \oplus \cdots \oplus L$. $S^{2m-1}/\mu_o$ is a $(2n-1)$-dimensional cell complex and $\eta: S \to S/\mu$ is $(2n-1)$-universal in the sense of [5]. Hence there is a bundle map

\[
\begin{array}{ccc}
S & \xrightarrow{b} & S \\
\downarrow \eta_0 & & \downarrow \eta \\
S/\mu_0 & \xrightarrow{b} & S/\mu.
\end{array}
\]
Furthermore, it is obvious that

\[ \mathcal{R}G \xrightarrow{\alpha} K^0(S/\mu) \xrightarrow{\alpha_0} K^0(S/\mu_0) \]

commutes, so Ker \( \alpha \subseteq \text{Ker } \alpha_0 \). But \( \alpha_0 \) fits into an exact sequence (see [1])

\[ \mathcal{R}G \xrightarrow{\varphi} \mathcal{R}G \xrightarrow{\alpha_0} K^0(S/\mu_0) \]

where \( \varphi \) is multiplication by \( \lambda_-(n\rho)=(1-\rho)^n \), so Ker \( \alpha_0 \subseteq (\rho-1)^n \cdot \mathcal{R}G \).

In [3] it is shown that \( f \) gives rise to a section \( S \) of \( m(\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^k) \) which vanishes precisely on \( A(f)/\mu \) (see especially Digression 1, p. 171–2 and Step 3, p. 180–1 of [3]). \( s \) and the given section \( s_0 \) of \( d\lambda_f \) go together to prove (2.3). This completes the proof of (2.1).

Our next step is to show that (2.1) is actually equivalent to the inequality

\[ d \geq n-1+p^{a-1} - \frac{1}{2} am(p^a-p^{a-1}) \]

The equivalence is obvious if \( n<d+mk \), so assume \( n \geq d+mk \). Lift (2.1) to the polynomial ring \( \mathbb{Z}[x] \) to get the equivalent

\[ (2.1.1) \quad \exists g, h \in \mathbb{Z}[x]: P(x) = g(x)(x-1)^n + h(x)(x^q-1) \]

Now \( P(x) = (x-1)^{d+km} \Pi_{j=2}^f f_j(x)^{m_{k/j}}; (x^q-1) = (x-1) \Pi_{j=2}^f f_j(x) \) where \( f_j \) is the \( j \)th cyclotomic polynomial and \([k/j]\) is the integral part of \( k/j \). Hence, if (2.1.1) holds then \( g \) is divisible by \( \Pi_{j=2}^f f_j(x) \) and \( h \) is divisible by \( (x-1)^{d+km} \).

So, putting \( \varepsilon_j=0 \) if \( j \nmid p^a \), \( \varepsilon_j=1 \) if \( j \mid p^a \), (2.1.1) implies (and is clearly implied by)

\[ (2.1.2) \quad \exists \bar{g}, \bar{h} \in \mathbb{Z}[x]: \Pi_{j=2}^f (x-1)^{d+km} + \bar{h}(x) \cdot f_\psi(x) \]

Let \( \gamma \) be a primitive \( q^s \)th root of unity and consider the projection \( \mathbb{Z}[x] \to \mathbb{Z}[\gamma] \subseteq C \). Its kernel is the ideal generated by \( f_\psi(x) \) so (2.1.2) is equivalent to

\[ (2.1.3) \quad (\gamma-1)^{n-d-km} \Pi_{j=2}^f (\gamma^{m_{k/j}})^{n_{k/j}-1} \in \mathbb{Z}[\gamma] \]

Now \( \mathbb{Z}[\gamma] \) is precisely the algebraic integers of the field \( Q(\gamma) \) and \( (\gamma-1)Z[\gamma] \) is the unique prime ideal in \( Z[\gamma] \) lying above \( pZ \), see e.g. [6]. Let \( \mathcal{N}: Q(\gamma) \to Q \) be the norm map for the extension \( Q(\gamma)/Q \). It is then an immediate consequence of classical ideal theory for Dedekind extensions that (2.1.3) is equivalent to

\[ (2.1.4) \quad \mathcal{N}(\gamma-1)^{n-d-km} \Pi_{j=1}^f (f_\psi(\gamma))^{m_{k/j}-1} \in \mathbb{Z} \]

The norms involved here are not hard to compute, so rearranging (2.1.4) slightly it takes the desired form

\[ d \geq n-1+p^{a-1} - \frac{1}{2} am(p^a-p^{a-1}) \]
If $A(f)/\mu$ happens to be a CW complex then of course we have $(\dim A(f)/\mu < 2d) \Rightarrow (d \lambda_f$ has a never vanishing section), and the above lemma can then be translated into a condition on $\dim A(f)/\mu$. Since also $\dim A(f) = \dim A(f)/\mu$ (because $A(f) \to A(f)/\mu$ is a finite covering and $\dim$ has the monotonicity and sum-properties, see [4]) this completes the proof of the mod $p^a$ Borsuk-Ulam theorem. $A(f)/\mu$, however, need not be a CW complex so we need to know the following

**Lemma 2.** If $\lambda$ is a complex line bundle over a compact metric space $X$ of covering dimension $< 2d$ then $d\lambda$ admits a never vanishing section.

I certainly do not believe that this lemma is unknown. However, nor do I know of any reference for it, so a proof of it is given as an appendix.

### 3. Proof of the anti-theorem

Consider the standard linear action $\mu_0$ of $G = Z_p^a$ on $S^{2N-1}$, $N$ big, i.e. view $S^{2N-1}$ as the unit sphere in $NL = L \oplus \cdots \oplus L$. $S^{2N-1}/\mu_0$ is a CW-complex with $S^{2N-1}/\mu_0$ as $(2n-1)$-skeleton. Let $\xi$ be the vector bundle $S^{2N-1} \times_G IG \to S^{2N-1}/\mu_0$ where $IG$ is the augmentation ideal of the real group algebra $RG$.

We notice that the anti-theorem is a consequence of

\[(3.1) \quad m \xi \text{ admits a never vanishing section over the } [p^a-1]m + (2p-3)m-1]-\text{skeleton.}\]

Indeed, it is well known how a section $s$ of $m \xi$ over the $(2n-1)$-skeleton corresponds to an equivariant map $F: S^{2n-1} \to m(IG) = IG \oplus IG \oplus \cdots \oplus IG$. If $i: IG \to RG$ is the inclusion then equivariance of $F$ means that $(i \oplus \cdots \oplus i)F$ has the form $(i \oplus \cdots \oplus i)F(x) = (\Sigma f_1(xg^{-1})g, \cdots, \Sigma f_m(xg^{-1})g)$ for well defined continuous maps $f_i: S^{2n-1} \to R$. Put $f = (f_1, \cdots, f_m)$ and notice that $A(f) = \phi$ is equivalent to $s$ having no zeros.

If we have shown (3.1) for $m=1$ then it follows for general $m$ by noticing that $m \xi \cong \Delta^*(\xi \times \cdots \times \xi)$ for any skeletal approximation $\Delta: S^{2N-1}/\mu_0 \to S^{2N-1}/\mu_0 \times \cdots \times S^{2N-1}/\mu_0$ to the $m$-fold diagonal. Hence, assume $m=1$. In [3] we showed that the mod $p$ Euler class of $\xi$ vanishes whenever $a > 1$. If we further exclude the case $p^a = 9$ then the same proof shows that the integral Euler class vanishes. Hence $\xi$ does have a never vanishing section over the $2k$ skeleton. The obstructions to extending this section over the successive skeleta lie in $H^{2k+i}(S^{2N-1}/G; \pi_{2k-1+i}(S^{2k-1})) \cong H^{2k+i}(G; \pi_{2k-1+i}(S^{2k-1}))$. For $0 < i < 2p-3$ the homotopy group in question has vanishing $p$-primary component so the obstructions vanish and we do have our desired section over the $(2k+2p-4)$-skeleton.

**Remark.** In the above we have made strong use of the fact that $\xi$ admits
a complex structure so that $\xi$ is orientable and hence no twisting of coefficients occur.

4. Remarks on the case $G=\mathbb{Z}_{25}$, $m=1$, linear action

For $G=\mathbb{Z}_{25}$ and $m=1$ our results show that there exists a map $f: S^{29}\to R$ with $A(f)=\phi$ whereas every map $f: S^{33}\to R$ has $A(f)=\pm \phi$. In fact every map $f: S^{31}\to R$ has $A(f)=\phi$ as we now show. Suppose that $A(f_0)=\phi$ for some $f_0: S^{31}\to R$. Then

$$s_0(xG) = (x, \pi(\Sigma_g f_0(xg^{-1})g))$$

defines a cross-section $s_0$ of $\xi$ over the 31-skeleton. $(\pi: RG\to IG$ is given by $\pi(\Sigma r_g g) = \Sigma r_g (g-1))$. The obstruction to extending $s_0$ further lie in $H^{21+i}(S^{2N-1}; \mathbb{Z}_{25})$. Since the 5-primary component of $\pi_{31+i}(S^{2n})$ is zero for $0\leq i < 6$ we get a never vanishing section over the 37-skeleton. As in §3 this gives an $f: S^{37}\to R$ with $A(f)=\phi$. But that contradicts the above result for maps $S^{33}\to R$.

Unfortunately for $p^a>25$ our positive and negative results are too far apart to close the gap between them by means as trivial as the above.

Appendix. Proof of lemma 2

Let $\Delta$ be the abstract 4d–1 simplex and $|\Delta|$ its standard realization in $R^{4d}$. By the general embedding theorem for compacta (see e.g. p. 139 of [2]) $X$ can be taken as a closed subspace of $|\Delta|$. Let $K_n$ be the subcomplex of $\Delta^{(m)}$ (mth barycentric subdivision of $\Delta$) spanned by all 4d–1 simplices $\tau$ for which $|\tau| \cap X = \emptyset$. Then $K_n$ is a subcomplex of the barycentric subdivision of $K_{n-1}$ so the inclusion $i_n: |K_n| \to |K_{n-1}|$ admits a simplicial approximation $\varphi_n: K_n \to K_{n-1}$. Also $\{ |K_n| \}$ is cofinal in the (downward) directed set of all neighborhoods of $X$ in $|\Delta|$, so for any abelian group $A$ we have $\check{H}^*(X; A) = \lim_{\to} H^*(|K_n|; A)$, where as usual $\check{H}^*$ is Cech cohomology, while $H^*$ can be taken as any ordinary cohomology theory. Since line bundles are characterized by the first Chern class $c_1 \in \check{H}^1(\{ X \}; Z)$ it follows that $\lambda$ admits an extension $\lambda_N$ over $|K_N|$ for $N$ sufficiently large. Fix such an $N$ and define (inductively, for $n>N$) $\lambda_n = |\varphi_n| \ast \lambda_{n-1}$. Let $\sigma_n$ be the sphere bundle associated with $d\lambda_n$. Since $\lambda \approx \lambda_n | X$, $n\geq N$, it is clearly sufficient to show that $\sigma_n$ admits a cross-section when $n$ is sufficiently large, in other words, if we let $k$ be the maximal number such that for some $n \geq N \sigma_n$ admits a cross-section over the $k$-skeleton $|K_k^1|$ of $K_n$ then we must show $k \geq 4d-1$. Suppose $k<4d-1$. Choose $n \geq N$ such that $\sigma_n \mid |K_n^1|$ has a cross-section, $s$, say. Consider the restriction $s'$ of $s$ to the $(k-1)$-skeleton and the obstruction $c$ to extending $s'$ over the $(k+1)$-skeleton (obstruction in the sense of [5]). $c \in H^{k+1}(|K_n|; \pi)$ where $\pi = \pi_k(S^{4d-1})$, and —
by maximality of $k - c \neq 0$. Since $k$ is clearly $\geq 2d - 1$ our assumption on \( \dim X \) assures that \( H^{*+1}(X; \pi) = \lim \sup \{H^{*+1}(\{K_j\}; \pi) \} \) vanishes so there is an $m > n$ such that $c|K_m| = |\varphi|^*c = 0$; here $\varphi$ is an abbreviation for $\varphi_{n+1}\varphi_{n+2} \cdots \varphi_{m}: K_m \to K_n$. Now $\sigma_m = |\varphi|^*\sigma_n$ and $|\varphi|$ is skeleton preserving so $s$ gives rise to a cross-section $s_i$ of $\sigma_m|K_m^i|$. Moreover, if $s_i'$ is the restriction of $s_i$ to $|K_m^{i-1}|$ then the obstruction to extending $s_i'$ over $|K_m^{i+1}|$ is precisely $|\varphi|^*c$. But $|\varphi|^*c = 0$ so $s_i'$ does extend over $|K_m^{i+1}|$, thus contradicting the maximality of $k$.

**References**