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INDEX OF THE EXPONENTIAL MAP ON A COMPLEX SIMPLE LIE GROUP

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0. Introduction

Let \mathfrak{G} be a connected Lie group with Lie algebra G . Following Goto [2], for $g \in \mathfrak{G}$, we define the index (of the exponential map) $\text{ind}(g)$ to be the smallest positive integer q such that $g^q \in \exp G$, if it exists, otherwise, $\text{ind}(g) = \infty$. The index $\text{ind}(\mathfrak{G})$ of \mathfrak{G} is defined to be the least common multiple of all $\text{ind}(g)$ ($g \in \mathfrak{G}$).

Given a complex simple Lie algebra G with a Cartan subalgebra H , let $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$ be the highest root of G with respect to H expressed in terms of a simple root system $\{\alpha_1, \dots, \alpha_l\}$. Consider the center-free Lie group with Lie algebra G , which can be identified with the adjoint group of (all inner automorphisms of) G . In Lai [4], we proved the following theorem:

Theorem. $\{\text{ind}(g); g \in \text{Ad}(G)\} = \{1, m_1, \dots, m_l\} = \{d; d \text{ is a factor of some } m_j\}$.

The main purpose of this paper is to generalize the above result to an arbitrary (always assumed to be connected) complex simple Lie group \mathfrak{G} .

Theorem. *Let \mathfrak{G} be a complex simple Lie group with Lie algebra G . We can find certain positive integers p_0, \dots, p_l (depending on the center $Z(\mathfrak{G})$ of \mathfrak{G} , to be defined in the next section) such that*

$$\{\text{ind}(g); g \in \mathfrak{G}\} = \{d; d \text{ is a factor of some } p_j m_j (0 \leq j \leq l) \text{ with } m_0 = 1\}.$$

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1. Notation and definition of p_j 's

Let G be a complex semisimple Lie algebra with a (fixed) Cartan subalgebra H . Let Δ be the root system of G with respect to H , $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a fundamental root system of Δ , and $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$ be the highest root.

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Let B be the Killing form on G . Then for each $\alpha \in \Delta$, we can find $h_\alpha \in H$ with $B(h, h_\alpha) = \alpha(h)$ for all $h \in H$, and $e_\alpha \in G$ such that

$$G = H + \sum_{\alpha \in \Delta} \mathbb{C}e_\alpha$$

$$\begin{aligned} [h, e_\alpha] &= \alpha(h)e_\alpha, & [e_\alpha, e_\beta] &= N_{\alpha, \beta}e_{\alpha+\beta} & \text{if } \alpha+\beta \neq 0 \text{ is in } \Delta, \\ [e_\alpha, e_{-\alpha}] &= -h_\alpha, & [e_\alpha, e_\beta] &= 0 & \text{if } 0 \neq \alpha+\beta \notin \Delta. \end{aligned}$$

Let $H_0 \subset H$ be the real vector space spanned by $h_\alpha (\alpha \in \Delta)$, then $\beta|_{H_0}$ is real for any $\beta \in \Delta$. Since $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is linearly independent, we can choose $h_1, \dots, h_l \in H_0$ such that $\alpha_i(h_j) = \delta_{ij}$, $1 \leq i, j \leq l$. The lattice $\Omega = \mathbb{Z}2\pi i h_1 + \dots + \mathbb{Z}2\pi i h_l \subset iH_0$ ($i = \sqrt{-1}$) is the kernel of $\exp|_H: H \rightarrow \text{Ad}(G)$. On the other hand, let \mathfrak{G} be the simply connected Lie group with Lie algebra G , denoting $2h_\alpha/B(h_\alpha, h_\alpha)$ by h_α^* , the lattice Ω^* generated by $\{2\pi i h_\alpha^*; \alpha \in \Delta\}$ becomes the kernel of $\exp|_H: H \rightarrow \mathfrak{G}$. Ω^* is of finite index in Ω . For simplicity, we identify Δ with a subset of iH_0 by the map $\alpha \mapsto h_\alpha/2\pi i$, and introduce an inner product in iH_0 by $(h, h') = -B(h, h')/(2\pi)^2$. Then $(\alpha, h) = \alpha(h)/2\pi i$ for $\alpha \in \Delta, h \in iH_0$.

If \mathfrak{G} is a connected Lie group with G as its Lie algebra. Let Ω' be the kernel of $\exp|_H: H \rightarrow \mathfrak{G}$, then $\Omega^* \subset \Omega' \subset \Omega$, so that Ω' is an additive subgroup of finite index in Ω . For each h_j , let p_j be the smallest positive integer such that $2\pi i p_j h_j \in \Omega' (j=1, \dots, l)$. Denote by p_0 the least common multiple of $\{p_1, \dots, p_l\}$, and $m_0 = 1$.

REMARK. p_0 is the smallest positive integer such that $g^{p_0} = 1$ for any element g in the center $Z(\mathfrak{G})$ (which is equal to $\exp(\Omega)$). In case G is simple, computation shows that $p_0 = p_j$ for some $j=1, \dots, l$. (For this, see, e.g. Goto-Grosshans [3] Chapter 5.)

Let $\text{Ad}(\Delta)$ denote the Weyl group of Δ . Any element S of $\text{Ad}(\Delta)$, regarded as a linear transformation on iH_0 , can be extended to an inner automorphism of the Lie algebra G . Let $T(\Omega^*)$ be the group of translations of the euclidean space iH_0 induced by elements in Ω^* . Then, if G is simple, the group $\text{Ad}(\Delta) \cdot T(\Omega^*)$ acts transitively on the set of all cells, see Goto-Grosshans [3] Chapter 5. We summarize as follows:

Proposition. *Let G be a complex simple Lie algebra and C_0 the fundamental cell: $C_0 = \{h \in iH_0; (\alpha_1, h) > 0, \dots, (\alpha_l, h) > 0 \text{ and } (-\alpha_0, h) < 1\}$. Let \bar{C}_0 denote the closure of C_0 . Then for any h in iH_0 , we can find $U \in \text{Ad}(\Delta) \cdot T(\Omega^*) = \text{Afd}(\Delta)$ such that $h \in U\bar{C}_0$.*

In the following, we assume \mathfrak{G} is a connected simple complex Lie group.

2. Upper bound for $\text{ind}(g)$

Theorem. *For any $g \in \mathfrak{G}$, $\text{ind}(g)$ is a factor of $p_j m_j$ for some $j=0, \dots, l$.*

Any element g in \mathfrak{G} has a decomposition $g = g_0 \cdot \exp N$ into semisimple part g_0 and unipotent part $\exp N$ such that $g_0 \cdot \exp N = \exp N \cdot g_0$. Let $G(1, \text{Ad } g_0)$ denote the 1-eigenspace of $\text{Ad } g_0$ in G . Then $G(1, \text{Ad } g_0)$ is a subalgebra of G and $N \in G(1, \text{Ad } g_0)$.

By Gantmacher [1], g_0 is conjugate to some element in $\exp H$. Hence, to prove our theorem, it suffices to consider elements g whose semisimple part lies in $\exp H$, i.e., $g = \exp h_0 \cdot \exp N$, $h_0 \in H$ and $N \in G(1, \text{Ad } \exp h_0)$. Let $\Delta(h_0) = \{\alpha \in \Delta; \text{Ad } \exp h_0 \cdot e_\alpha = e_\alpha\} = \{\alpha \in \Delta; \alpha(h_0) \in 2\pi i\mathbb{Z}\}$. Then $G(1, \text{Ad } \exp h_0) = H + \sum_{\alpha \in \Delta(h_0)} \mathbb{C}e_\alpha$, and $\Delta(h_0)$ is a subsystem of Δ , we can choose a simple root system $\Pi(h_0) = \{\beta_1, \dots, \beta_r\}$ for $\Delta(h_0)$.

Lemma 1. *To find an upper bound for $\text{ind}(g)$ ($g \in \mathfrak{G}$), it suffices to consider elements with semisimple part $\exp h_0$, where $h_0 \in iH_0$ and $\Pi(h_0)$ has cardinality $l = \text{rank of } G$.*

Proof. Assume that $h_0 = x_1 h_1 + \dots + x_l h_l$ for some complex numbers x_i . For each $j = 1, \dots, r$, since $(\text{Ad } \exp h_0 - 1) \cdot e_{\beta_j} = 0$, we have $\beta_j(h_0) = 2\pi i k_j$ for some $k_j \in \mathbb{Z}$. If k_j are all zero, then $[h_0, N] = 0$ for any $N \in G(1, \text{Ad } \exp h_0)$, so that $\exp h_0 \cdot \exp N = \exp(h_0 + N)$, and $\text{ind}(\exp h_0 \cdot \exp N) = 1$. So we assume that some $k_j \neq 0$, after this.

Since $\exp h_0 = \exp(h_0 + \Omega')$, if we can find a positive integer d and integers n_1, \dots, n_l such that for $h = dh_0 + \sum_{j=1}^l 2\pi i n_j p_j h_j$, $[h, dN] = 0$, then $\text{ind}(\exp h_0 \cdot \exp N)$ divides d . For this, it suffices to choose d and n_j with $\alpha(h) = 0$ for all $\alpha \in \Delta(h_0)$, or equivalently, for all $\alpha \in \Pi(h_0)$. Therefore, the problem reduces to finding d so that $\beta_i(\sum_{j=1}^l n_j p_j h_j) = -dk_i$ has integral solutions n_1, \dots, n_l .

Choose $\beta_{r+1}, \dots, \beta_l \in \Delta$ so that $\{\beta_1, \dots, \beta_l\}$ is a maximal linearly independent subset of Δ . We write $\beta_i = \sum_{j=1}^l q_{ij} \alpha_j$ where $q_{ij} \in \mathbb{Z}$. Consider the following system of linear equations:

$$\begin{aligned} q_{i1} p_1 n_1 + \dots + q_{il} p_l n_l &= -k_i & i &= 1, \dots, r; \\ q_{i1} p_1 n_1 + \dots + q_{il} p_l n_l &= 0 & i &= r+1, \dots, l. \end{aligned}$$

Since $(q_{ij} p_j)$ is a nonsingular integral matrix with determinant $p_1 \dots p_l \cdot \det(q_{ij})$ (which is not zero by the choice of β_j 's and the fact that p_j are positive), and k_i are integers, this has a rational solution, say, r_1, \dots, r_l .

Let $h_0' = \sum_{j=1}^l 2\pi i r_j p_j h_j \in iH_0$, then $\beta_1, \dots, \beta_l \in \Delta(h_0')$. Suppose we can find a positive integer d' and integers n_1', \dots, n_l' such that $\beta(d'h_0' + \sum_{j=1}^l 2\pi i n_j' p_j h_j) = 0$ for all $\beta \in \Delta(h_0')$, then $(n_1, \dots, n_l) = (n_1', \dots, n_l')$ is the solution for the following system of linear equations:

$$\begin{aligned} \sum_{j=1}^l q_{ij} p_j n_j &= -d' k_i & i &= 1, \dots, r; \\ \sum_{j=1}^l q_{ij} p_j n_j &= 0 & i &= r+1, \dots, l. \end{aligned}$$

Thus we can find $n_j \in \mathbb{Z}$ such that $\beta_i(\sum_{j=1}^l 2\pi i n_j p_j h_j) = -2\pi i d' k_i$ ($i=1, \dots, r$). Hence for $h = d'h_0 + \sum_{j=1}^l 2\pi i n_j p_j h_j$, we have $\beta_i(h) = 0$ ($i=1, \dots, r$) and so $\beta(h) = 0$ for all $\beta \in \Delta(h_0)$.

We have proved that $\text{ind}(\exp h_0 \cdot \exp N)$ is a factor of $\text{ind}(\exp h'_0 \cdot \exp N)$. Therefore, we may replace h_0 by h'_0 which satisfies Lemma 1. ||

Let S be in the Weyl group $Ad(\Delta)$. Then S can be extended to an inner automorphism σ of the Lie algebra G , which can be extended to an inner automorphism of the Lie group \mathfrak{G} . Clearly $\text{ind}(g) = \text{ind}(\sigma g)$. Therefore, to find an upper bound for $\text{ind}(g)$ ($g \in \mathfrak{G}$), we may replace g (whose semisimple part is $\exp h_0$) by an element whose semisimple part is $\exp Sh_0$ ($S \in Ad(\Delta)$).

On the other hand, $\exp h_0 = \exp(h_0 + \Omega^*)$ (because $\Omega^* \subset \Omega'$), so we may replace h_0 by $T(\Omega^*)h_0$. We get the following lemma by applying the proposition we stated at the end of section 1.

Lemma 2. *Let $-\alpha_0 = m_1\alpha_1 + \dots + m_l\alpha_l$ be the highest root. To find an upper bound for $\text{ind}(g)$ ($g \in \mathfrak{G}$), it suffices to consider elements whose semisimple part have the form $\exp h$, $h \in iH_0$ with $0 \leq (\alpha_1, h), \dots, 0 \leq (\alpha_l, h)$ and $(-\alpha_0, h) \leq 1$.*

Let $\tilde{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ be the extended simple root system. The following two lemmas, proved in [4], being properties of simple Lie algebras, can be applied in the present case too. For a proof, please see [4] or Goto-Grosshans [3] Chapter 8.

Lemma 3. *Let $h \in \bar{C}_0$ be an element satisfying Lemma 1, then $\Pi' = \tilde{\Pi} \cap \Delta(h)$ is a simple root system for $\Delta(h)$ with respect to a suitable ordering.*

Since $\Pi(h) = \tilde{\Pi} \cap \Delta(h)$ has cardinality l . If $\Pi(h) = \Pi$, then $\Delta(h) = \Delta$ and $h \in \Omega$, in this case, $\text{ind}(\exp h \cdot \exp N)$ is a factor of p_0 ($= p_0 m_0$) because $p_0 h \in \Omega'$.

Lemma 4. *If $\Pi(h) \neq \Pi$ has cardinality l , then $h = 2\pi i h_j / m_j$ for some $j = 1, \dots, l$ such that $m_j > 1$.*

In the case $m_j = 1$, we have $\Pi(2\pi i h_j / m_j) = \Pi$.

Conclusion. Let \mathfrak{G} be a connected complex simple Lie group. To find an upper bound for $\{\text{ind}(g); g \in \mathfrak{G}\}$, it suffices to consider elements $g \in \mathfrak{G}$ whose semisimple part has the form $\exp 2\pi i h_j / m_j$ for some $j = 0, 1, \dots, l$. i.e. $g = \exp 2\pi i h_j / m_j \cdot \exp N$.

Clearly, $g^{p_j m_j} = \exp(p_j m_j N)$ because $2\pi i p_j h_j \in \Omega'$.

Theorem. *For any $g \in \mathfrak{G}$, there exists j ($0 \leq j \leq l$) such that $g^{p_j m_j} \in \exp G$. In other words, $\text{ind}(g)$ is a factor of some $p_j m_j$ ($0 \leq j \leq l$).*

3. Existence of elements with index exactly equal to $p_j m_j$

An element x in a semisimple Lie algebra G is said to be regular if the

centralizer $z_G(x) = \{y \in G; [x, y] = 0\}$ of x has minimal dimension. If H is a Cartan subalgebra of G with root system Δ and $U = \sum_{\alpha > 0} \mathbb{C}e_\alpha$, then $B = H + U$ is a Borel subalgebra (i.e. a maximal solvable subalgebra). The following proposition is a consequence of the Lie algebra analogous of Theorem 1 and its corollary in Steinberg [5] (pp. 110–112).

Proposition. *If $x = \sum_{\alpha > 0} c_\alpha e_\alpha \in U$ ($c_\alpha \in \mathbb{C}$) is a nilpotent element in G , then x is regular if and only if $c_\alpha \neq 0$ for any simple root α . In such case, $z_G(x) \subset U$, in particular, $z_G(x)$ consists only of nilpotent elements.*

Retaining the notation used in the previous sections, consider $h_0 = 2\pi i h_j / m_j$, ($1 \leq j \leq l$). Then $\Pi = \tilde{\Pi} - \{\alpha_j\}$ is a simple root system for $\Delta(h_0)$ and $G(1, \text{Ad exp } h_0) = H + \sum_{\alpha \in \Delta(h_0)} \mathbb{C}e_\alpha$ is a semisimple subalgebra of G . Let $N = \sum_{i=0, \dots, l; i \neq j} e_{\alpha_i}$, then N is a regular element in $G(1, \text{Ad exp } h_0)$, so that any element of $G(1, \text{Ad exp } h_0)$ which commutes with N must be nilpotent.

Let $g = \exp h_0 \cdot \exp N$, and \mathfrak{G}_1 be the connected subgroup of \mathfrak{G} corresponding to the subalgebra $G_1 = G(1, \text{Ad } g) = G(1, \text{Ad exp } h_0)$. Clearly, $g \in \mathfrak{G}_1$ because $h_0, N \in G_1$. Therefore $g^q \in \mathfrak{G}_1$ for any positive integer q .

If for certain q , $g^q = \exp x$ for some $x \in G$, then x lies in G_1 (because $G_1 = \{y \in G; \exp y \in \mathfrak{G}_1\}$). We know that x has a decomposition $x = x_0 + N$, where x_0 is semisimple and $[x_0, N] = 0$. Since $x, N \in G_1$, we have $x_0 \in G_1 = G(1, \text{Ad exp } h_0)$. But $[x_0, N] = 0$, the above argument implies that x_0 is nilpotent. Thus $x_0 = 0$ because x_0 is also semisimple. This implies that $\exp x_0 = \exp qh_0 = 1$, or $qh_0 \in \Omega'$. This cannot happen if $q < p_j m_j$.

Therefore $\text{ind}(g) = p_j m_j$.

In case $j=0$, let $h_0 = \sum_{i=1}^l 2\pi i h_j$, then $qh_0 \notin \Omega'$ unless q is a multiple of p_0 . Let $N = \sum_{j=1}^l e_{\alpha_j}$, which is regular in G . The same argument as above proves that $\text{ind}(\exp h_0 \cdot \exp N) = p_0 = p_0 m_0$. Q.E.D.

The results in sections 2 and 3 give the following:

Theorem. *Let \mathfrak{G} be a connected complex simple Lie group. Retaining the above notation. Then $\{\text{ind}(g); g \in \mathfrak{G}\} = \{q; q \text{ is a factor of some } p_j m_j, 0 \leq j \leq l\} = \{q; q \text{ is a factor of some } p_j m_j, 1 \leq j \leq l\}$.*

Corollary. $\text{ind}(\mathfrak{G})$ is the least common multiple of $\{p_1 m_1, \dots, p_l m_l\}$.

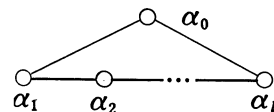
4. List of $\text{ind}(g)$ when \mathfrak{G} is simply connected

In this case, p_j can be found by using the inverse matrix of Cartan matrix of G , please see e.g. Goto-Grosshans [3] Chapter 5.

(a) G is of type A_l

The highest root is $-\alpha_0 = \alpha_1 + \dots + \alpha_l$.

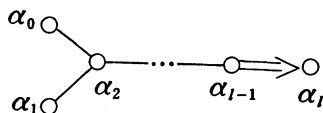
$p_1 = \dots = p_l = l + 1$.



Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{q; q \text{ divides } l+1\}$ and $\text{ind}(\mathfrak{G}) = l+1$.

In fact, for any connected complex simple Lie group of type A , $\text{ind}(\mathfrak{G}) =$ order of the center $Z(\mathfrak{G})$.

(b) G is of type B_l

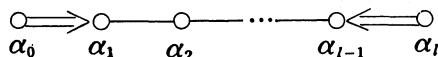


The highest root is $-\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l)$.

$p_j = 2$ when j is odd, $p_j = 1$ when j is even.

Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2, 4\}$ in case $l \geq 3$ and $\text{ind}(\mathfrak{G}) = 4$. And $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2\}$ in case $l = 2$ and $\text{ind}(\mathfrak{G}) = 2$.

(c) G is of type C_l

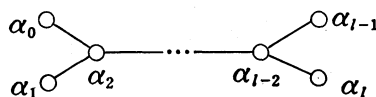


The highest root is $-\alpha_0 = 2(\alpha_1 + \cdots + \alpha_{l-1}) + \alpha_l$.

$p_l = 2$ and $p_j = 1$ when $j < l$.

Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2\}$ and $\text{ind}(\mathfrak{G}) = 2$.

(d) G is of type D_l

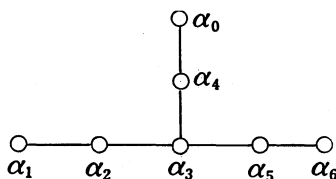


The highest root is $-\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$.

Case 1. When l is even, $p_j = 2$ if $j \leq l-2$ is odd or $j = l-1, l$; $p_j = 1$ otherwise. Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2\}$ and $\text{ind}(\mathfrak{G}) = 2$.

Case 2. When l is odd, $p_j = 2$ if $j \leq l-2$ is odd, $p_{l-1} = p_l = 4$; $p_j = 1$ otherwise. Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2, 4\}$ and $\text{ind}(\mathfrak{G}) = 4$.

(e) G is of type E_6

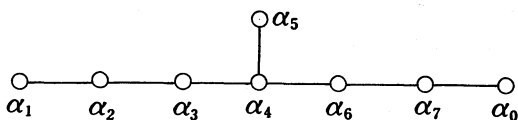


The highest root is $-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$.

$p_1 = p_2 = p_5 = p_6 = 3$ and $p_3 = p_4 = 1$.

Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2, 3, 6\}$ and $\text{ind}(\mathfrak{G}) = 6$.

(f) G is of type E_7



The highest root is $-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$.

$p_1 = p_3 = p_5 = 2$ and $p_j = 1$ otherwise.

Hence $\{\text{ind}(g); g \in \mathfrak{G}\} = \{\text{factors of } 12\}$ and $\text{ind}(\mathfrak{G}) = 12$.

Note that $p_j = 1$ for any j in case G is of type E_8 , F_4 , or G_2 .

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