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## INDEX OF THE EXPONENTIAL MAP ON A COMPLEX SIMPLE LIE GROUP

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### 0. Introduction

Let  $\mathfrak{G}$  be a connected Lie group with Lie algebra  $G$ . Following Goto [2], for  $g \in \mathfrak{G}$ , we define the index (of the exponential map)  $\text{ind}(g)$  to be the smallest positive integer  $q$  such that  $g^q \in \exp G$ , if it exists, otherwise,  $\text{ind}(g) = \infty$ . The index  $\text{ind}(\mathfrak{G})$  of  $\mathfrak{G}$  is defined to be the least common multiple of all  $\text{ind}(g)$  ( $g \in \mathfrak{G}$ ).

Given a complex simple Lie algebra  $G$  with a Cartan subalgebra  $H$ , let  $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$  be the highest root of  $G$  with respect to  $H$  expressed in terms of a simple root system  $\{\alpha_1, \cdots, \alpha_l\}$ . Consider the center-free Lie group with Lie algebra  $G$ , which can be identified with the adjoint group of (all inner automorphisms of)  $G$ . In Lai [4], we proved the following theorem:

**Theorem.**  $\{\text{ind}(g); g \in \text{Ad}(G)\} = \{1, m_1, \cdots, m_l\} = \{d; d \text{ is a factor of some } m_j\}$ .

The main purpose of this paper is to generalize the above result to an arbitrary (always assumed to be connected) complex simple Lie group  $\mathfrak{G}$ .

**Theorem.** *Let  $\mathfrak{G}$  be a complex simple Lie group with Lie algebra  $G$ . We can find certain positive integers  $p_0, \cdots, p_l$  (depending on the center  $Z(\mathfrak{G})$  of  $\mathfrak{G}$ , to be defined in the next section) such that*

$$\{\text{ind}(g); g \in \mathfrak{G}\} = \{d; d \text{ is a factor of some } p_j m_j (0 \leq j \leq l) \text{ with } m_0 = 1\}.$$

The author would like to express his gratitude to Professor M. Goto for his generous help during the preparation of this paper.

### 1. Notation and definition of $p_j$ 's

Let  $G$  be a complex semisimple Lie algebra with a (fixed) Cartan subalgebra  $H$ . Let  $\Delta$  be the root system of  $G$  with respect to  $H$ ,  $\Pi = \{\alpha_1, \cdots, \alpha_l\}$  a fundamental root system of  $\Delta$ , and  $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$  be the highest root.

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1) Work partially supported by the National Science Council, Republic of China.

Let  $B$  be the Killing form on  $G$ . Then for each  $\alpha \in \Delta$ , we can find  $h_\alpha \in H$  with  $B(h, h_\alpha) = \alpha(h)$  for all  $h \in H$ , and  $e_\alpha \in G$  such that

$$G = H + \sum_{\alpha \in \Delta} \mathbb{C}e_\alpha$$

$$\begin{aligned} [h, e_\alpha] &= \alpha(h)e_\alpha, & [e_\alpha, e_\beta] &= N_{\alpha, \beta}e_{\alpha+\beta} & \text{if } \alpha + \beta \neq 0 \text{ is in } \Delta, \\ [e_\alpha, e_{-\alpha}] &= -h_\alpha, & [e_\alpha, e_\beta] &= 0 & \text{if } 0 \neq \alpha + \beta \notin \Delta. \end{aligned}$$

Let  $H_0 \subset H$  be the real vector space spanned by  $h_\alpha (\alpha \in \Delta)$ , then  $\beta|_{H_0}$  is real for any  $\beta \in \Delta$ . Since  $\Pi = \{\alpha_1, \dots, \alpha_l\}$  is linearly independent, we can choose  $h_1, \dots, h_l \in H_0$  such that  $\alpha_i(h_j) = \delta_{ij}$ ,  $1 \leq i, j \leq l$ . The lattice  $\Omega = \mathbb{Z}2\pi i h_1 + \dots + \mathbb{Z}2\pi i h_l \subset iH_0$  ( $i = \sqrt{-1}$ ) is the kernel of  $\exp|_H: H \rightarrow Ad(G)$ . On the other hand, let  $\mathfrak{G}$  be the simply connected Lie group with Lie algebra  $G$ , denoting  $2h_\alpha/B(h_\alpha, h_\alpha)$  by  $h_\alpha^*$ , the lattice  $\Omega^*$  generated by  $\{2\pi i h_\alpha^*; \alpha \in \Delta\}$  becomes the kernel of  $\exp|_H: H \rightarrow \mathfrak{G}$ ,  $\Omega^*$  is of finite index in  $\Omega$ . For simplicity, we identify  $\Delta$  with a subset of  $iH_0$  by the map  $\alpha \mapsto h_\alpha/2\pi i$ , and introduce an inner product in  $iH_0$  by  $(h, h') = -B(h, h')/(2\pi)^2$ . Then  $(\alpha, h) = \alpha(h)/2\pi i$  for  $\alpha \in \Delta, h \in iH_0$ .

If  $\mathfrak{G}$  is a connected Lie group with  $G$  as its Lie algebra. Let  $\Omega'$  be the kernel of  $\exp|_H: H \rightarrow \mathfrak{G}$ , then  $\Omega^* \subset \Omega' \subset \Omega$ , so that  $\Omega'$  is an additive subgroup of finite index in  $\Omega$ . For each  $h_j$ , let  $p_j$  be the smallest positive integer such that  $2\pi i p_j h_j \in \Omega' (j=1, \dots, l)$ . Denote by  $p_0$  the least common multiple of  $\{p_1, \dots, p_l\}$ , and  $m_0 = 1$ .

REMARK.  $p_0$  is the smallest positive integer such that  $g^{p_0} = 1$  for any element  $g$  in the center  $Z(\mathfrak{G})$  (which is equal to  $\exp(\Omega)$ ). In case  $G$  is simple, computation shows that  $p_0 = p_j$  for some  $j = 1, \dots, l$ . (For this, see, e.g. Goto-Grosshans [3] Chapter 5.)

Let  $Ad(\Delta)$  denote the Weyl group of  $\Delta$ . Any element  $S$  of  $Ad(\Delta)$ , regarded as a linear transformation on  $iH_0$ , can be extended to an inner automorphism of the Lie algebra  $G$ . Let  $T(\Omega^*)$  be the group of translations of the euclidean space  $iH_0$  induced by elements in  $\Omega^*$ . Then, if  $G$  is simple, the group  $Ad(\Delta) \cdot T(\Omega^*)$  acts transitively on the set of all cells, see Goto-Grosshans [3] Chapter 5. We summarize as follows:

**Proposition.** *Let  $G$  be a complex simple Lie algebra and  $C_0$  the fundamental cell:  $C_0 = \{h \in iH_0; (\alpha_1, h) > 0, \dots, (\alpha_l, h) > 0 \text{ and } (-\alpha_0, h) < 1\}$ . Let  $\bar{C}_0$  denote the closure of  $C_0$ . Then for any  $h$  in  $iH_0$ , we can find  $U \in Ad(\Delta) \cdot T(\Omega^*) = Afd(\Delta)$  such that  $h \in U\bar{C}_0$ .*

In the following, we assume  $\mathfrak{G}$  is a connected simple complex Lie group.

**2. Upper bound for  $\text{ind}(g)$**

**Theorem.** *For any  $g \in \mathfrak{G}$ ,  $\text{ind}(g)$  is a factor of  $p_j m_j$  for some  $j = 0, \dots, l$ .*

Any element  $g$  in  $\mathfrak{G}$  has a decomposition  $g=g_0 \cdot \exp N$  into semisimple part  $g_0$  and unipotent part  $\exp N$  such that  $g_0 \cdot \exp N = \exp N \cdot g_0$ . Let  $G(1, Ad g_0)$  denote the 1-eigenspace of  $Ad g_0$  in  $G$ . Then  $G(1, Ad g_0)$  is a subalgebra of  $G$  and  $N \in G(1, Ad g_0)$ .

By Gantmacher [1],  $g_0$  is conjugate to some element in  $\exp H$ . Hence, to prove our theorem, it suffices to consider elements  $g$  whose semisimple part lies in  $\exp H$ , i.e.,  $g = \exp h_0 \cdot \exp N$ ,  $h_0 \in H$  and  $N \in G(1, Ad \exp h_0)$ . Let  $\Delta(h_0) = \{\alpha \in \Delta; Ad \exp h_0 \cdot e_\alpha = e_\alpha\} = \{\alpha \in \Delta; \alpha(h_0) \in 2\pi i\mathbb{Z}\}$ . Then  $G(1, Ad \exp h_0) = H + \sum_{\alpha \in \Delta(h_0)} C e_\alpha$ , and  $\Delta(h_0)$  is a subsystem of  $\Delta$ , we can choose a simple root system  $\Pi(h_0) = \{\beta_1, \dots, \beta_r\}$  for  $\Delta(h_0)$ .

**Lemma 1.** *To find an upper bound for  $\text{ind}(g)$  ( $g \in \mathfrak{G}$ ), it suffices to consider elements with semisimple part  $\exp h_0$ , where  $h_0 \in iH_0$  and  $\Pi(h_0)$  has cardinality  $l = \text{rank of } G$ .*

*Proof.* Assume that  $h_0 = x_1 h_1 + \dots + x_r h_r$  for some complex numbers  $x_i$ . For each  $j = 1, \dots, r$ , since  $(Ad \exp h_0 - 1) \cdot e_{\beta_j} = 0$ , we have  $\beta_j(h_0) = 2\pi i k_j$  for some  $k_j \in \mathbb{Z}$ . If  $k_j$  are all zero, then  $[h_0, N] = 0$  for any  $N \in G(1, Ad \exp h_0)$ , so that  $\exp h_0 \cdot \exp N = \exp(h_0 + N)$ , and  $\text{ind}(\exp h_0 \cdot \exp N) = 1$ . So we assume that some  $k_j \neq 0$ , after this.

Since  $\exp h_0 = \exp(h_0 + \Omega')$ , if we can find a positive integer  $d$  and integers  $n_1, \dots, n_l$  such that for  $h = dh_0 + \sum_{j=1}^l 2\pi i n_j p_j h_j$ ,  $[h, dN] = 0$ , then  $\text{ind}(\exp h_0 \cdot \exp N)$  divides  $d$ . For this, it suffices to choose  $d$  and  $n_j$  with  $\alpha(h) = 0$  for all  $\alpha \in \Delta(h_0)$ , or equivalently, for all  $\alpha \in \Pi(h_0)$ . Therefore, the problem reduces to finding  $d$  so that  $\beta_i(\sum_{j=1}^l n_j p_j h_j) = -dk_i$  has integral solutions  $n_1, \dots, n_l$ .

Choose  $\beta_{r+1}, \dots, \beta_l \in \Delta$  so that  $\{\beta_1, \dots, \beta_l\}$  is a maximal linearly independent subset of  $\Delta$ . We write  $\beta_i = \sum_{j=1}^l q_{ij} \alpha_j$  where  $q_{ij} \in \mathbb{Z}$ . Consider the following system of linear equations:

$$\begin{aligned} q_{i1} p_1 n_1 + \dots + q_{il} p_l n_l &= -k_i & i &= 1, \dots, r; \\ q_{i1} p_1 n_1 + \dots + q_{il} p_l n_l &= 0 & i &= r+1, \dots, l. \end{aligned}$$

Since  $(q_{ij} p_j)$  is a nonsingular integral matrix with determinant  $p_1 \dots p_l \cdot \det(q_{ij})$  (which is not zero by the choice of  $\beta_j$ 's and the fact that  $p_j$  are positive), and  $k_i$  are integers, this has a rational solution, say,  $r_1, \dots, r_l$ .

Let  $h_0' = \sum_{j=1}^l 2\pi i r_j p_j h_j \in iH_0$ , then  $\beta_1, \dots, \beta_l \in \Delta(h_0')$ . Suppose we can find a positive integer  $d'$  and integers  $n_1', \dots, n_l'$  such that  $\beta(d' h_0' + \sum_{j=1}^l 2\pi i n_j' p_j h_j) = 0$  for all  $\beta \in \Delta(h_0')$ , then  $(n_1, \dots, n_l) = (n_1', \dots, n_l')$  is the solution for the following system of linear equations:

$$\begin{aligned} \sum_{j=1}^l q_{ij} p_j n_j &= -d' k_i & i &= 1, \dots, r; \\ \sum_{j=1}^l q_{ij} p_j n_j &= 0 & i &= r+1, \dots, l. \end{aligned}$$

Thus we can find  $n_j \in \mathbb{Z}$  such that  $\beta_i(\sum_{j=1}^l 2\pi i n_j p_j h_j) = -2\pi i d' k_i$  ( $i=1, \dots, r$ ). Hence for  $h = d'h_0 + \sum_{j=1}^l 2\pi i n_j p_j h_j$ , we have  $\beta_i(h) = 0$  ( $i=1, \dots, r$ ) and so  $\beta(h) = 0$  for all  $\beta \in \Delta(h_0)$ .

We have proved that  $\text{ind}(\exp h_0 \cdot \exp N)$  is a factor of  $\text{ind}(\exp h'_0 \cdot \exp N)$ . Therefore, we may replace  $h_0$  by  $h'_0$  which satisfies Lemma 1. ||

Let  $S$  be in the Weyl group  $Ad(\Delta)$ . Then  $S$  can be extended to an inner automorphism  $\sigma$  of the Lie algebra  $G$ , which can be extended to an inner automorphism of the Lie group  $\mathfrak{G}$ . Clearly  $\text{ind}(g) = \text{ind}(\sigma g)$ . Therefore, to find an upper bound for  $\text{ind}(g)$  ( $g \in \mathfrak{G}$ ), we may replace  $g$  (whose semisimple part is  $\exp h_0$ ) by an element whose semisimple part is  $\exp Sh_0$  ( $S \in Ad(\Delta)$ ).

On the other hand,  $\exp h_0 = \exp(h_0 + \Omega^*)$  (because  $\Omega^* \subset \Omega'$ ), so we may replace  $h_0$  by  $T(\Omega^*)h_0$ . We get the following lemma by applying the proposition we stated at the end of section 1.

**Lemma 2.** *Let  $-\alpha_0 = m_1\alpha_1 + \dots + m_l\alpha_l$  be the highest root. To find an upper bound for  $\text{ind}(g)$  ( $g \in \mathfrak{G}$ ), it suffices to consider elements whose semisimple part have the form  $\exp h$ ,  $h \in iH_0$  with  $0 \leq (\alpha_1, h), \dots, 0 \leq (\alpha_l, h)$  and  $(-\alpha_0, h) \leq 1$ .*

Let  $\tilde{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$  be the extended simple root system. The following two lemmas, proved in [4], being properties of simple Lie algebras, can be applied in the present case too. For a proof, please see [4] or Goto-Grosshans [3] Chapter 8.

**Lemma 3.** *Let  $h \in \bar{C}_0$  be an element satisfying Lemma 1, then  $\Pi' = \tilde{\Pi} \cap \Delta(h)$  is a simple root system for  $\Delta(h)$  with respect to a suitable ordering.*

Since  $\Pi(h) = \tilde{\Pi} \cap \Delta(h)$  has cardinality  $l$ . If  $\Pi(h) = \Pi$ , then  $\Delta(h) = \Delta$  and  $h \in \Omega$ , in this case,  $\text{ind}(\exp h \cdot \exp N)$  is a factor of  $p_0$  ( $= p_0 m_0$ ) because  $p_0 h \in \Omega'$ .

**Lemma 4.** *If  $\Pi(h) \neq \Pi$  has cardinality  $l$ , then  $h = 2\pi i h_j / m_j$  for some  $j = 1, \dots, l$  such that  $m_j > 1$ .*

In the case  $m_j = 1$ , we have  $\Pi(2\pi i h_j / m_j) = \Pi$ .

**Conclusion.** Let  $\mathfrak{G}$  be a connected complex simple Lie group. To find an upper bound for  $\{\text{ind}(g); g \in \mathfrak{G}\}$ , it suffices to consider elements  $g \in \mathfrak{G}$  whose semisimple part has the form  $\exp 2\pi i h_j / m_j$  for some  $j = 0, 1, \dots, l$ . i.e.  $g = \exp 2\pi i h_j / m_j \cdot \exp N$ .

Clearly,  $g^{p_j m_j} = \exp(p_j m_j N)$  because  $2\pi i p_j h_j \in \Omega'$ .

**Theorem.** *For any  $g \in \mathfrak{G}$ , there exists  $j$  ( $0 \leq j \leq l$ ) such that  $g^{p_j m_j} \in \exp G$ . In other words,  $\text{ind}(g)$  is a factor of some  $p_j m_j$  ( $0 \leq j \leq l$ ).*

### 3. Existence of elements with index exactly equal to $p_j m_j$

An element  $x$  in a semisimple Lie algebra  $G$  is said to be regular if the

centralizer  $z_G(x) = \{y \in G; [x, y] = 0\}$  of  $x$  has minimal dimension. If  $H$  is a Cartan subalgebra of  $G$  with root system  $\Delta$  and  $U = \sum_{\alpha > 0} \mathbb{C}e_\alpha$ , then  $B = H + U$  is a Borel subalgebra (i.e. a maximal solvable subalgebra). The following proposition is a consequence of the Lie algebra analogous of Theorem 1 and its corollary in Steinberg [5] (pp. 110–112).

**Proposition.** *If  $x = \sum_{\alpha > 0} c_\alpha e_\alpha \in U$  ( $c_\alpha \in \mathbb{C}$ ) is a nilpotent element in  $G$ , then  $x$  is regular if and only if  $c_\alpha \neq 0$  for any simple root  $\alpha$ . In such case,  $z_G(x) \subset U$ , in particular,  $z_G(x)$  consists only of nilpotent elements.*

Retaining the notation used in the previous sections, consider  $h_0 = 2\pi i h_j / m_j$ , ( $1 \leq j \leq l$ ). Then  $\Pi = \tilde{\Pi} - \{\alpha_j\}$  is a simple root system for  $\Delta(h_0)$  and  $G(1, Ad \exp h_0) = H + \sum_{\alpha \in \Delta(h_0)} \mathbb{C}e_\alpha$  is a semisimple subalgebra of  $G$ . Let  $N = \sum_{i=0, \dots, l; i \neq j} e_{\alpha_i}$ , then  $N$  is a regular element in  $G(1, Ad \exp h_0)$ , so that any element of  $G(1, Ad \exp h_0)$  which commutes with  $N$  must be nilpotent.

Let  $g = \exp h_0 \cdot \exp N$ , and  $\mathfrak{G}_1$  be the connected subgroup of  $\mathfrak{G}$  corresponding to the subalgebra  $G_1 = G(1, Ad g) = G(1, Ad \exp h_0)$ . Clearly,  $g \in \mathfrak{G}_1$  because  $h_0, N \in G_1$ . Therefore  $g^q \in \mathfrak{G}_1$  for any positive integer  $q$ .

If for certain  $q$ ,  $g^q = \exp x$  for some  $x \in G$ , then  $x$  lies in  $G_1$  (because  $G_1 = \{y \in G; \exp y \in \mathfrak{G}_1\}$ ). We know that  $x$  has a decomposition  $x = x_0 + N$ , where  $x_0$  is semisimple and  $[x_0, N] = 0$ . Since  $x, N \in G_1$ , we have  $x_0 \in G_1 = G(1, Ad \exp h_0)$ . But  $[x_0, N] = 0$ , the above argument implies that  $x_0$  is nilpotent. Thus  $x_0 = 0$  because  $x_0$  is also semisimple. This implies that  $\exp x_0 = \exp qh_0 = 1$ , or  $qh_0 \in \Omega'$ . This cannot happen if  $q < p_j m_j$ .

Therefore  $\text{ind}(g) = p_j m_j$ .

In case  $j = 0$ , let  $h_0 = \sum_{j=1}^l 2\pi i h_j$ , then  $qh_0 \notin \Omega'$  unless  $q$  is a multiple of  $p_0$ . Let  $N = \sum_{j=1}^l e_{\alpha_j}$ , which is regular in  $G$ . The same argument as above proves that  $\text{ind}(\exp h_0 \cdot \exp N) = p_0 = p_0 m_0$ . Q.E.D.

The results in sections 2 and 3 give the following:

**Theorem.** *Let  $\mathfrak{G}$  be a connected complex simple Lie group. Retaining the above notation. Then  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{q; q \text{ is a factor of some } p_j m_j, 0 \leq j \leq l\} = \{q; q \text{ is a factor of some } p_j m_j, 1 \leq j \leq l\}$ .*

**Corollary.**  $\text{ind}(\mathfrak{G})$  is the least common multiple of  $\{p_1 m_1, \dots, p_l m_l\}$ .

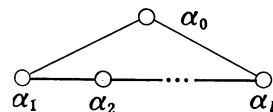
**4. List of  $\text{ind}(g)$  when  $\mathfrak{G}$  is simply connected**

In this case,  $p_j$  can be found by using the inverse matrix of Cartan matrix of  $G$ , please see e.g. Goto-Grosshans [3] Chapter 5.

(a)  $G$  is of type  $A_l$

The highest root is  $-\alpha_0 = \alpha_1 + \dots + \alpha_l$ .

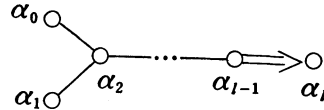
$p_1 = \dots = p_l = l + 1$ .



Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{q; q \text{ divides } l+1\}$  and  $\text{ind}(\mathfrak{G}) = l+1$ .

In fact, for any connected complex simple Lie group of type  $A$ ,  $\text{ind}(\mathfrak{G}) =$  order of the center  $Z(\mathfrak{G})$ .

(b)  $G$  is of type  $B_l$

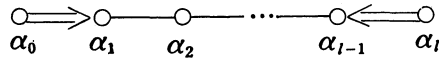


The highest root is  $-\alpha_0 = \alpha_1 + 2(\alpha_2 + \dots + \alpha_l)$ .

$p_j = 2$  when  $j$  is odd,  $p_j = 1$  when  $j$  is even.

Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2, 4\}$  in case  $l \geq 3$  and  $\text{ind}(\mathfrak{G}) = 4$ . And  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2\}$  in case  $l = 2$  and  $\text{ind}(\mathfrak{G}) = 2$ .

(c)  $G$  is of type  $C_l$

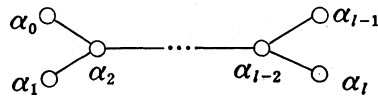


The highest root is  $-\alpha_0 = 2(\alpha_1 + \dots + \alpha_{l-1}) + \alpha_l$ .

$p_l = 2$  and  $p_j = 1$  when  $j < l$ .

Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2\}$  and  $\text{ind}(\mathfrak{G}) = 2$ .

(d)  $G$  is of type  $D_l$

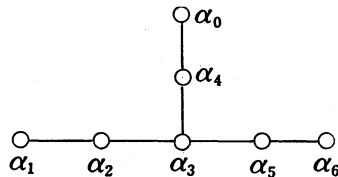


The highest root is  $-\alpha_0 = \alpha_1 + 2(\alpha_2 + \dots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$ .

**Case 1.** When  $l$  is even,  $p_j = 2$  if  $j \leq l-2$  is odd or  $j = l-1, l$ ;  $p_j = 1$  otherwise. Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2\}$  and  $\text{ind}(\mathfrak{G}) = 2$ .

**Case 2.** When  $l$  is odd,  $p_j = 2$  if  $j \leq l-2$  is odd,  $p_{l-1} = p_l = 4$ ;  $p_j = 1$  otherwise. Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2, 4\}$  and  $\text{ind}(\mathfrak{G}) = 4$ .

(e)  $G$  is of type  $E_6$

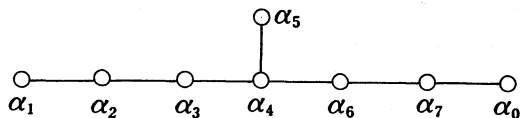


The highest root is  $-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ .

$p_1 = p_2 = p_5 = p_6 = 3$  and  $p_3 = p_4 = 1$ .

Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{1, 2, 3, 6\}$  and  $\text{ind}(\mathfrak{G}) = 6$ .

(f)  $G$  is of type  $E_7$



The highest root is  $-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$ .

$p_1 = p_3 = p_5 = 2$  and  $p_j = 1$  otherwise.

Hence  $\{\text{ind}(g); g \in \mathfrak{G}\} = \{\text{factors of } 12\}$  and  $\text{ind}(\mathfrak{G}) = 12$ .

Note that  $p_j = 1$  for any  $j$  in case  $G$  is of type  $E_8$ ,  $F_4$ , or  $G_2$ .

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