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INDEX OF THE EXPONENTIAL MAP ON A COMPLEX SIMPLE LIE GROUP

HENG-LUNG LAI

(Received November 21, 1977)

0. Introduction

Let $\mathcal{G}$ be a connected Lie group with Lie algebra $G$. Following Goto [2], for $g \in \mathcal{G}$, we define the index (of the exponential map) $\text{ind}(g)$ to be the smallest positive integer $q$ such that $g^q \in \exp G$, if it exists, otherwise, $\text{ind}(g) = \infty$. The index $\text{ind}(\mathcal{G})$ of $\mathcal{G}$ is defined to be the least common multiple of all $\text{ind}(g)$ $(g \in \mathcal{G})$.

Given a complex simple Lie algebra $G$ with a Cartan subalgebra $H$, let $-\alpha_0 = m_1 \alpha_1 + \cdots + m_l \alpha_l$ be the highest root of $G$ with respect to $H$ expressed in terms of a simple root system $\{\alpha_1, \cdots, \alpha_l\}$. Consider the center-free Lie group with Lie algebra $G$, which can be identified with the adjoint group of (all inner automorphisms of) $G$. In Lai [4], we proved the following theorem:

**Theorem.** $\{\text{ind}(g); g \in \text{Ad}(G)\} = \{1, m_1, \cdots, m_l\} = \{d; d \text{ is a factor of some } m_j\}$.

The main purpose of this paper is to generalize the above result to an arbitrary (always assumed to be connected) complex simple Lie group $\mathcal{G}$.

**Theorem.** Let $\mathcal{G}$ be a complex simple Lie group with Lie algebra $G$. We can find certain positive integers $p_0, \cdots, p_l$ (depending on the center $Z(\mathcal{G})$ of $\mathcal{G}$, to be defined in the next section) such that

$$\{\text{ind}(g); g \in \mathcal{G}\} = \{d; d \text{ is a factor of some } p_j m_j (0 \leq j \leq l) \text{ with } m_0 = 1\}.$$ 

The author would like to express his gratitude to Professor M. Goto for his generous help during the preparation of this paper.

1. Notation and definition of $p_j$'s

Let $G$ be a complex semisimple Lie algebra with a (fixed) Cartan subalgebra $H$. Let $\Delta$ be the root system of $G$ with respect to $H$, $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ a fundamental root system of $\Delta$, and $-\alpha_0 = m_1 \alpha_1 + \cdots + m_l \alpha_l$ be the highest root.

1) Work partially supported by the National Science Council, Republic of China.
Let $B$ be the Killing form on $G$. Then for each $\alpha \in \Delta$, we can find $h_\alpha \in H$ with $B(h, h_\alpha) = \alpha(h)$ for all $h \in H$, and $e_\alpha \in G$ such that

$$G = H + \sum_{\alpha \in \Delta} C e_\alpha$$

$$[h, e_\alpha] = \alpha(h)e_\alpha, \quad [e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha + \beta} \quad \text{if } \alpha + \beta \in \Delta,$$

$$[e_\alpha, e_\alpha] = -h_\alpha, \quad [e_\alpha, e_\beta] = 0 \quad \text{if } 0 \not= \alpha + \beta \in \Delta.$$

Let $H_0 \subset H$ be the real vector space spanned by $h_\alpha (\alpha \in \Delta)$, then $\beta | H_0$ is real for any $\beta \in \Delta$. Since $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is linearly independent, we can choose $h_1, \ldots, h_l \in H_0$ such that $\alpha_i(h_j) = \delta_{ij}$, $1 \leq i, j \leq l$. The lattice $\Omega = \mathbb{Z} 2\pi i h_1 + \cdots + \mathbb{Z} 2\pi i h_l \subset i H_0$ ($i = \sqrt{-1}$) is the kernel of $\exp|_H: H \to Ad(G)$. On the other hand, let $\tilde{G}$ be the simply connected Lie group with Lie algebra $G$, denoting $2\pi i h_\alpha B(h_\alpha, h_\alpha)$ by $h_\alpha^*$, the lattice $\Omega^*$ generated by $\{2\pi i h_\alpha^*; \alpha \in \Delta\}$ becomes the kernel of $\exp|_{\tilde{H}}: \tilde{H} \to \tilde{G}$, $\Omega^*$ is of finite index in $\Omega$. For simplicity, we identify $\Delta$ with a subset of $i H_0$ by the map $h \mapsto h_\alpha^*/2\pi i$, and introduce an inner product in $i H_0$ by $(h, h') = B(h, h')/(2\pi)^2$. Then $(\alpha, h) = \alpha(h)/2\pi i$ for $\alpha \in \Delta, h \in i H_0$.

If $\tilde{G}$ is a connected Lie group with $G$ as its Lie algebra. Let $\Omega'$ be the kernel of $\exp|_{\tilde{H}}: \tilde{H} \to \tilde{G}$, then $\Omega' \subset \Omega \subset \Omega'$, and $\Omega'$ is an additive subgroup of finite index in $\Omega$. For each $h_j$, let $p_j$ be the smallest positive integer such that $2\pi p_j h_j \in \Omega'$ ($j = 1, \ldots, l$). Denote by $p_0$ the least common multiple of $\{p_1, \ldots, p_l\}$, and $m_0 = 1$.

**Remark.** $p_0$ is the smallest positive integer such that $g^{p_0} = 1$ for any element $g$ in the center $Z(\tilde{G})$ (which is equal to $\exp(\Omega)$). In case $G$ is simple, computation shows that $p_0 = p_j$ for some $j = 1, \ldots, l$. (For this, see, e.g. Goto-Grosshans [3] Chapter 5.)

Let $Ad(\Delta)$ denote the Weyl group of $\Delta$. Any element $S$ of $Ad(\Delta)$, regarded as a linear transformation on $i H_0$, can be extended to an inner automorphism of the Lie algebra $G$. Let $T(\Omega^*)$ be the group of translations of the euclidean space $i H_0$ induced by elements in $\Omega^*$. Then, if $G$ is simple, the group $Ad(\Delta) \cdot T(\Omega^*)$ acts transitively on the set of all cells, see Goto-Grosshans [3] Chapter 5. We summarize as follows:

**Proposition.** Let $G$ be a complex simple Lie algebra and $C_0$ the fundamental cell: $C_0 = \{h \in i H_0; (\alpha_1, h) > 0, \ldots, (\alpha_l, h) > 0 \text{ and } (-\alpha_0, h) < 1\}$. Let $\bar{C}_0$ denote the closure of $C_0$. Then for any $h$ in $i H_0$, we can find $U \in Ad(\Delta) \cdot T(\Omega^*) = Af(G)$ such that $h \in U C_0$.

In the following, we assume $\tilde{G}$ is a connected simple complex Lie group.

2. **Upper bound for $\text{ind}(g)$**

**Theorem.** For any $g \in \tilde{G}$, $\text{ind}(g)$ is a factor of $p_j m_j$ for some $j = 0, \ldots, l$. 
Any element \( g \in \mathfrak{g} \) has a decomposition \( g = g_0 \cdot \exp N \) into semisimple part \( g_0 \) and unipotent part \( \exp N \) such that \( g_0 \cdot \exp N = \exp N \cdot g_0 \). Let \( G(1, \text{Ad} g_0) \) denote the 1-eigenspace of \( \text{Ad} g_0 \) in \( G \). Then \( G(1, \text{Ad} g_0) \) is a subalgebra of \( G \) and \( N \subseteq G(1, \text{Ad} g_0) \).

By Gantmacher [1], \( g_0 \) is conjugate to some element in \( \exp H \). Hence, to prove our theorem, it suffices to consider elements \( g \) whose semisimple part lies in \( \exp H \), i.e., \( g = \exp h_0 \cdot \exp N, \ h_0 \in H \ and \ N \subseteq G(1, \text{Ad} h_0) \). Let \( \Delta(h_0) = \{ \alpha \in \Delta; \text{Ad} \exp h_0 \cdot e_\alpha = e_\alpha \} = \{ \alpha \in \Delta; \alpha(h) \in 2\pi i \mathbb{Z} \} \). Then \( G(1, \text{Ad} h_0) = H + \sum_{\alpha \in \Delta(h_0)} C e_\alpha \) and \( \Delta(h_0) \) is a subsystem of \( \Delta \), we can choose a simple root system \( \Pi(h_0) = \{ \beta_1, \ldots, \beta_r \} \) for \( \Delta(h_0) \).

**Lemma 1.** To find an upper bound for \( \text{ind}(g) (g \in \mathfrak{g}) \), it suffices to consider elements with semisimple part \( \exp h_0 \), where \( h_0 \in i H_0 \) and \( \Pi(h_0) \) has cardinality \( l = \text{rank of } G \).

Proof. Assume that \( h_0 = x_1 h_1 + \cdots + x_r h_r \) for some complex numbers \( x_i \). For each \( j = 1, \ldots, r \), since \( (\text{Ad} \exp h_0 - 1) \cdot e_{\beta_j} = 0 \), we have \( \beta_j(h_0) = 2\pi ik_j \) for some \( k_j \in \mathbb{Z} \). If \( k_j \) are all zero, then \( [h_0, N] = 0 \) for any \( N \in G(1, \text{Ad} h_0) \), so that \( \exp h_0 \cdot \exp N = \exp (h_0 + N) \), and \( \text{ind}(\exp h_0 \cdot \exp N) = 1 \). So we assume that some \( k_j \neq 0 \), after this.

Since \( \exp h_0 = \exp (h_0 + \Omega') \), if we can find a positive integer \( d \) and integers \( n_1, \ldots, n_r \) such that for \( h = dh_0 + \sum_{j=1}^r 2\pi in_j p_j h_j \), \( [h, dN] = 0 \), then \( \text{ind}(\exp h_0 \cdot \exp N) \) divides \( d \). For this, it suffices to choose \( d \) and \( n_j \) with \( \alpha(h) = 0 \) for all \( \alpha \in \Delta(h_0) \), or equivalently, for all \( \alpha \in \Pi(h_0) \). Therefore, the problem reduces to finding \( d \) so that \( \beta_j(\sum_{j=1}^r n_j p_j h_j) = -dk_j \) has integral solutions \( n_1, \ldots, n_r \).

Choose \( \beta_{r+1}, \ldots, \beta_r \in \Delta \) so that \( \{ \beta_1, \ldots, \beta_r \} \) is a maximal linearly independent subset of \( \Delta \). We write \( \beta_i = \sum_{j=1}^{r+1} q_{ij} \alpha_j \) where \( q_{ij} \in \mathbb{Z} \). Consider the following system of linear equations:

\[
q_{i1} p_1 n_1 + \cdots + q_{ir} p_r n_r = -k_i \quad i = 1, \ldots, r; \\
q_{i1} p_1 n_1 + \cdots + q_{ir} p_r n_r = 0 \quad i = r+1, \ldots, l.
\]

Since \( (q_{ij}, p_j) \) is a nonsingular integral matrix with determinant \( p_1 \cdots p_r \cdot \det (q_{ij}) \) (which is not zero by the choice of \( \beta_j \)'s and the fact that \( p_j \) are positive), and \( k_i \) are integers, this has a rational solution, say, \( r_1, \ldots, r_l \).

Let \( h'_0 = \sum_{j=1}^{l'} 2\pi i r_j p_j h_j \in i H_0 \), then \( \beta_1, \ldots, \beta_{l'} \in \Delta(h'_0) \). Suppose we can find a positive integer \( d' \) and integers \( n'_1, \ldots, n'_{l'} \) such that \( \beta(d'h'_0 + \sum_{j=1}^{l'} 2\pi in'_j p_j h_j) = 0 \) for all \( \beta \in \Delta(h'_0) \), then \( n_1, \ldots, n_r = (n'_1, \ldots, n'_{l'}) \) is the solution for the following system of linear equations:

\[
\sum_{j=1}^{r} q_{ij} p_i n_i = -d'k_i \quad i = 1, \ldots, r; \\
\sum_{j=1}^{r} q_{ij} p_i n_i = 0 \quad i = r+1, \ldots, l.
\]
Thus we can find $n_j \in \mathbb{Z}$ such that $\beta(\sum_{i=1}^{j} 2\pi in_j p_j h_j) = -2\pi id'k_i$ ($i = 1, \ldots, r$). Hence for $h = a'h_0 + \sum_{j=1}^{r} 2\pi in_j p_j h_j$, we have $\beta(h) = 0$ ($i = 1, \ldots, r$) and so $\beta(h) = 0$ for all $\beta \in \Delta(h_0)$.

We have proved that $\text{ind}(\exp h_0 \cdot \exp N)$ is a factor of $\text{ind}(\exp h_0' \cdot \exp N)$. Therefore, we may replace $h_0$ by $h_0'$ which satisfies Lemma 1. ||

Let $S$ be in the Weyl group $Ad(\Delta)$. Then $S$ can be extended to an inner automorphism $\sigma$ of the Lie algebra $G$, which can be extended to an inner automorphism of the Lie group $G$. Clearly $\text{ind}(g) = \text{ind}(\sigma g)$. Therefore, to find an upper bound for $\text{ind}(g)$ ($g \in G$), we may replace $g$ (whose semisimple part is $\exp h_0$) by an element whose semisimple part is $\exp S_0(S \in Ad(\Delta))$.

On the other hand, $\exp h_0 = \exp (h_0 + \Omega^*)$ (because $\Omega^* \subset \Omega'$), so we may replace $h_0$ by $T(\Omega^*)h_0$. We get the following lemma by applying the proposition we stated at the end of section 1.

**Lemma 2.** Let $-\alpha_0 - m_1 \alpha_1 - \cdots - m_l \alpha_l$ be the highest root. To find an upper bound for $\text{ind}(g)$ ($g \in G$), it suffices to consider elements whose semisimple part have the form $\exp h$, $h \in iH_0$ with $0 \leq (\alpha_i, h)$, $0 \leq (\alpha_i, h)$ and $(-\alpha_0, h) \leq 1$.

Let $\tilde{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ be the extended simple root system. The following two lemmas, proved in [4], being properties of simple Lie algebras, can be applied in the present case too. For a proof, please see [4] or Goto-Grosshans [3] Chapter 8.

**Lemma 3.** Let $h \in C_0$ be an element satisfying Lemma 1, then $\Pi' = \tilde{\Pi} \cap \Delta(h)$ is a simple root system for $\Delta(h)$ with respect to a suitable ordering.

Since $\Pi(h) = \tilde{\Pi} \cap \Delta(h)$ has cardinality $l$. If $\Pi(h) = \Pi$, then $\Delta(h) = \Delta$ and $h \in \Omega$, in this case, $\text{ind}(\exp h \cdot \exp N)$ is a factor of $p_0$ ($= p_0m_0$) because $p_0h \in \Omega'$.

**Lemma 4.** If $\Pi(h) \neq \Pi$ has cardinality $l$, then $h = 2\pi i h_j m_j$ for some $j = 1, \ldots, l$ such that $m_j > 1$.

In the case $m_j = 1$, we have $\Pi(2\pi i h_j m_j) = \Pi$.

**Conclusion.** Let $G$ be a connected complex simple Lie group. To find an upper bound for $\{\text{ind}(g); g \in G\}$, it suffices to consider elements $g \in G$ whose semisimple part has the form $\exp 2\pi i h_j m_j$ for some $j = 0, 1, \ldots, l$. i.e. $g = \exp 2\pi i h_j m_j \cdot \exp N$.

Clearly, $g^{\rho_{m_j}} = \exp (p_j m_j N)$ because $2\pi i p_j h_j \in \Omega'$.

**Theorem.** For any $g \in G$, there exists $j$ ($0 \leq j \leq l$) such that $g^{\rho_{m_j}} \in \exp G$. In other words, $\text{ind}(g)$ is a factor of some $p_j m_j$ ($0 \leq j \leq l$).

3. **Existence of elements with index exactly equal to $p_j m_j$**

An element $x$ in a semisimple Lie algebra $G$ is said to be regular if the
centralizer \( z_o(x) = \{ y \in G; [x, y] = 0 \} \) of \( x \) has minimal dimension. If \( H \) is a Cartan subalgebra of \( G \) with root system \( \Delta \) and \( U = \sum_{a > 0} C e_a \), then \( B = H + U \) is a Borel subalgebra (i.e. a maximal solvable subalgebra). The following proposition is a consequence of the Lie algebra analogous of Theorem 1 and its corollary in Steinberg [5] (pp. 110–112).

**Proposition.** If \( x = \sum_{a > 0} c_a e_a \in U (c_a \in C) \) is a nilpotent element in \( G \), then \( x \) is regular if and only if \( c_a \neq 0 \) for any simple root \( \alpha \). In such case, \( z_o(x) \cup U \), in particular, \( z_o(x) \) consists only of nilpotent elements.

Retaining the notation used in the previous sections, consider \( h_0 = 2\pi i h_j / m, \) \((1 \leq j \leq l)\). Then \( \Pi = \Pi - \{ \alpha_j \} \) is a simple root system for \( \Delta(h_0) \) and \( G(1, Ad \exp h_0) = H + \sum_{a \in \Delta(h_0)} C e_a \) is a semisimple subalgebra of \( G \). Let \( N = \sum_{i=0, \ldots, l; i \neq j} e_a \), then \( N \) is a regular element in \( G(1, Ad \exp h_0) \), so that any element of \( G(1, Ad \exp h_0) \) which commutes with \( N \) must be nilpotent.

Let \( g = \exp h_0 \cdot \exp N \), and \( G_i \) be the connected subgroup of \( G \) corresponding to the subalgebra \( G_i = G(1, Ad g) = G(1, Ad h_0) \). Clearly, \( g \in G_i \) because \( h_0, N \in G_1 \). Therefore \( g^q \in G_i \) for any positive integer \( q \).

If for certain \( q_i, g^q = \exp x \) for some \( x \in G \), then \( x \) lies in \( G_1 \) (because \( G_1 = \{ y \in G; \exp y \in G_1 \}) \). We know that \( x \) has a decomposition \( x = x_0 + N \), where \( x_0 \) is semisimple and \( [x_0, N] = 0 \). Since \( x, N \in G_1 \), we have \( x_0 \in G_1 = G(1, Ad \exp h_0) \). But \( [x_0, N] = 0 \), the above argument implies that \( x_0 \) is nilpotent. Thus \( x_0 = 0 \) because \( x_0 \) is also semisimple. This implies that \( \exp x_0 = \exp q h_0 = 1 \), or \( q h_0 \in \Omega' \). This cannot happen if \( q < p_j m_j \).

Therefore \( \text{ind}(g) = p_j m_j \).

In case \( j = 0 \), let \( h_0 = \sum_{j=1}^{l} 2\pi i h_j \), then \( q h_0 \in \Omega' \) unless \( q \) is a multiple of \( p_0 \). Let \( N = \sum_{j=1}^{l} e_{a_j} \), which is regular in \( G \). The same argument as above proves that \( \text{ind}(\exp h_0 \cdot \exp N) = p_0 = p_0 m_0 \). \( \Box \).

The results in sections 2 and 3 give the following:

**Theorem.** Let \( \mathfrak{g} \) be a connected complex simple Lie group. Retaining the above notation. Then \( \{ \text{ind}(g); g \in \mathfrak{g} \} = \{ q; q \) is a factor of some \( p_j m_j, 0 \leq j \leq l \} \) = \( \{ q; q \) is a factor of some \( p_j m_j, 1 \leq j \leq l \} \).

**Corollary.** \( \text{ind}(\mathfrak{g}) \) is the least common multiple of \( \{ p_j m_j, \ldots, p_j m_j \} \).

4. **List of \( \text{ind}(g) \) when \( \mathfrak{g} \) is simply connected**

In this case, \( p_j \) can be found by using the inverse matrix of Cartan matrix of \( G \), please see e.g. Goto-Grosshans [3] Chapter 5.

(a) \( G \) is of type \( A \)

The highest root is \( -\alpha_0 = -\alpha_1 + \cdots + \alpha_l \).

\( p_1 = \cdots = p_l + 1 \).
Hence \{\text{ind}(g); g \in \mathfrak{g}\} = \{q; q \text{ divides } l+1\} and \text{ind}(\mathfrak{g}) = l+1.

In fact, for any connected complex simple Lie group of type \(A\), \text{ind}(\mathfrak{g}) = \text{order of the center } Z(\mathfrak{g}).

(b) \(G\) is of type \(B_l\)

The highest root is \(-\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l)\).
\(p_j = 2\) when \(j\) is odd, \(p_j = 1\) when \(j\) is even.
Hence \{\text{ind}(g); g \in \mathfrak{g}\} = \{1, 2, 4\} in case \(l \geq 3\) and \text{ind}(\mathfrak{g}) = 4.\ And \{\text{ind}(g); g \in \mathfrak{g}\} = \{1, 2\} in case \(l = 2\) and \text{ind}(\mathfrak{g}) = 2.

(c) \(G\) is of type \(C_l\)

The highest root is \(-\alpha_0 = 2(\alpha_1 + \cdots + \alpha_{l-1}) + \alpha_l\).
\(p_l = 2\) and \(p_j = 1\) when \(j < l\).
Hence \{\text{ind}(g); g \in \mathfrak{g}\} = \{1, 2\} and \text{ind}(\mathfrak{g}) = 2.

(d) \(G\) is of type \(D_l\)

The highest root is \(-\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l\).

Case 1. When \(l\) is even, \(p_j = 2\) if \(j \leq l-2\) is odd or \(j = l-1, l\); \(p_j = 1\) otherwise.
Hence \{\text{ind}(g); g \in \mathfrak{g}\} = \{1, 2\} and \text{ind}(\mathfrak{g}) = 2.

Case 2. When \(l\) is odd, \(p_j = 2\) if \(j \leq l-2\) is odd, \(p_{l-1} = p_l = 4\); \(p_j = 1\) otherwise.
Hence \{\text{ind}(g); g \in \mathfrak{g}\} = \{1, 2, 4\} and \text{ind}(\mathfrak{g}) = 4.

(e) \(G\) is of type \(E_6\)

The highest root is \(-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\).
\(p_1 = p_2 = p_3 = p_6 = 3\) and \(p_3 = p_4 = 1\).
Hence \{\text{ind}(g); g \in \mathfrak{g}\} = \{1, 2, 3, 6\} and \text{ind}(\mathfrak{g}) = 6.

(f) \(G\) is of type \(E_7\)
The highest root is $-\alpha_5 = -\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$.

$p_1 = p_3 = p_5 = 2$ and $p_j = 1$ otherwise.

Hence $\{\text{ind}(g); g \in \mathcal{G}\} = \{\text{factors of 12}\}$ and $\text{ind}(\mathcal{G}) = 12$.

Note that $p_j = 1$ for any $j$ in case $G$ is of type $E_8$, $F_4$, or $G_2$.

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