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## THE LEVI PROBLEM IN THE BLOW-UP

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### Abstract

We prove that a locally Stein open subset of the blow-up of  $\mathbb{C}^n$  at a point is Stein if and only if it does not contain a subset of the form  $U \setminus A$  where  $A$  is the exceptional divisor and  $U$  is an open neighborhood of  $A$ . We also study an analogous statement for locally Stein open subsets of line bundles over  $\mathbb{P}^n$ .

### 1. Introduction

Let  $U \subset \mathbb{C}^n$  be an open set which is locally Stein. Then  $-\log d$  is a pluri-subharmonic function on  $U$  ( $d$  denotes here the distance to the boundary of  $U$ ) and therefore by Oka's theorem (see [6])  $U$  is Stein. A similar result holds (see Fujita [2]) if  $U$  is an open locally Stein subset of  $\mathbb{P}^n$ ,  $U \neq \mathbb{P}^n$ .

If  $U$  is a locally Stein open subset of  $\mathbb{P}^n \times \mathbb{C}$  (which can be identified with  $\mathcal{O}(0)$  over  $\mathbb{P}^n$ ) then it was proved in [1] that  $U$  is Stein if and only if  $U$  does not contain any compact fiber  $\mathbb{P}^n \times \{x\}$  for some  $x \in \mathbb{C}$ .

On the other hand, S.Yu. Nemirovskii studied in [7] the Levi problem for open sets  $U$  in  $\tilde{\mathbb{C}}^n$ , the blow-up of  $\mathbb{C}^n$  at a point, in the particular case when the intersection of  $U$  with the exceptional set is strongly pseudoconvex. In this particular case he uses the fact that there exists a smooth strongly plurisubharmonic function in a neighborhood of  $\bar{U}$  and therefore one can apply a theorem of A. Takeuchi [8] to deduce that  $U$  is Stein. In this context let us remark that if  $\pi: F \rightarrow C$  is a negative line bundle over a compact complex curve and  $D \Subset F$  is a smoothly bounded domain then  $C$  (identified with the zero section of  $F$ ) cannot be contained in  $\partial D$ , since otherwise  $F$  would be topologically trivial. Therefore if  $D$ , as above, is locally Stein it follows easily from this remark that it is Stein. It seems unknown what happens if  $D$  is locally Stein,  $\partial D$  is not smooth,  $C \subset \partial D$  and the genus of  $C$  is greater or equal to 1.

In this paper we consider locally Stein open subsets of the blow-up  $\tilde{\mathbb{C}}^n$  of  $\mathbb{C}^n$  at a point. However one can identify  $\tilde{\mathbb{C}}^n$  with  $\mathcal{O}(-1)$ , the holomorphic line bundle of degree  $-1$  over  $\mathbb{P}^{n-1}$  and one can consider the more general case  $X = \mathcal{O}(r)$ , for  $r < 0$  or  $r > 0$ . We want to decide under what additional geometrical conditions a locally Stein open subset of  $X$  is Stein. In this direction we prove:

**Theorem 1.** *Let  $X = \mathcal{O}(r)$  be the degree  $r$  line bundle on  $\mathbb{P}^n$  and let  $\Omega \subset X$  be a locally Stein open subset of  $X$ .*

- 1) *If  $r < 0$  then  $\Omega$  is Stein if and only if it does not contain an open subset of the form  $U \setminus A$  where  $A = \mathbb{P}^n$  is the zero section and  $U$  is an open neighborhood of  $A$ .*
- 2) *If  $r > 0$  then  $\Omega$  is Stein if and only if it does not contain a neighborhood of the section at infinity.*

**2. Proof of Theorem 1**

The proof of the theorem is based on the Lemmas 1 and 2 below.

**Lemma 1.** *Let  $F: Z \rightarrow Y$  be a holomorphic line bundle over a complex manifold  $Y$  and let  $Z_0$  be its zero section. If  $Z \setminus Z_0$  is Stein then  $Y$  is also Stein.*

This is a particular case of a more general result (see Theorem 5, p. 151 in [5]).

**DEFINITION 1.** Suppose that  $M$  is a complex manifold,  $A$  is an analytic subset of  $M$ ,  $\text{codim}(A) > 0$ , and  $D$  is an open subset of  $M \setminus A$ . A point  $z \in \partial D \cap A$  is called removable along  $A$  if there exists an open neighborhood  $U$  of  $z$  such that  $U \setminus A \subset D$ .

**Lemma 2.** *Let  $A$  be a closed analytic subset of a Stein manifold  $M$ ,  $\text{codim}(A) > 0$ , and  $D \subset M \setminus A$  be an open subset. If  $D$  is locally Stein at every point  $z \in \partial D \cap A$  and if no boundary point  $z \in \partial D \cap A$  is removable along  $A$  then  $D$  is Stein.*

This lemma is due to Grauert and Remmert [3] (see also Ueda [9]).

We begin now the proof of Theorem 1 and we start with a few notations. Let  $\pi: \mathcal{O}(r) \rightarrow \mathbb{P}^n$  be the vector bundle projection,  $z_0, z_1, \dots, z_n$  be the coordinate functions in  $\mathbb{C}^{n+1}$ , and, for  $k = 0, \dots, n$ , we let  $U_k = \{[z] = [z_0 : z_1 : \dots : z_n] \in \mathbb{P}^n : z_k \neq 0\}$ . We denote by  $\psi_k: \pi^{-1}(U_k) \rightarrow U_k \times \mathbb{C}$  the local trivializations. It follows that

$$(1) \quad (\psi_j \circ \psi_k^{-1})([z], \lambda) = \left( [z], \frac{z_k^r}{z_j^r} \lambda \right), \quad \forall [z] = [z_0 : z_1 : \dots : z_n] \in U_j \cap U_k.$$

We will define now a holomorphic map  $F: (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} \rightarrow \mathcal{O}(r)$  as follows. We set  $W_k = \{(z, \lambda) \in (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} : z_k \neq 0\}$  for  $k = 0, \dots, n$  and we define

$$F(z, \lambda) = \psi_k^{-1} \left( [z], \frac{\lambda}{z_k^r} \right), \quad \forall (z, \lambda) \in W_k.$$

We have to check of course that  $F$  is well defined. However it follows from (1) that  $(\psi_j \circ \psi_k^{-1})([z], \lambda/z_k^r) = ([z], \lambda/z_j^r)$  and hence  $\psi_k^{-1}([z], \lambda/z_k^r) = \psi_j^{-1}([z], \lambda/z_j^r)$ . As the map  $(z, \lambda) \in W_k \rightarrow ([z], \lambda/z_k^r) \in U_k \times \mathbb{C}$  is surjective, it follows that  $F|_{W_k}: W_k \rightarrow \pi^{-1}(U_k)$  is surjective as well. We claim that  $F$  is a local trivial fibration with fiber

$\mathbb{C}^*$  and the transition functions are linear on each fiber. In other words there exists a holomorphic line bundle  $\tilde{F}: Z \rightarrow \mathcal{O}(r)$  such that if we denote by  $Z_0$  its zero section, then  $Z \setminus Z_0 = (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}$  and  $F = \tilde{F}|_{(\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}}$ .

We define  $\Phi_k: W_k \rightarrow \pi^{-1}(U_k) \times \mathbb{C}^*$ ,  $\Phi(z, \lambda) = (F(z, \lambda), z_k)$ . We will show that  $\Phi_k$  is invertible and we will compute  $\Phi_j \circ \Phi_k^{-1}$ .

Note that if we set

$$\begin{aligned} \tilde{\Phi}_k: W_k &\rightarrow (U_k \times \mathbb{C}) \times \mathbb{C}^*, & \tilde{\Phi}_k(z, \lambda) &= \left( \left( [z], \frac{1}{z_k^r} \lambda \right), z_k \right), \\ \chi_k: (U_k \times \mathbb{C}) \times \mathbb{C}^* &\rightarrow \pi^{-1}(U_k) \times \mathbb{C}^*, & \chi_k((z, \lambda), \mu) &= (\psi_k^{-1}(z, \lambda), \mu) \end{aligned}$$

then  $\Phi_k = \chi_k \circ \tilde{\Phi}_k$ .

It is easy to see that  $\tilde{\Phi}_k$  is invertible and its inverse is

$$(2) \quad \tilde{\Phi}_k^{-1}([z], \lambda, \mu) = \left( \frac{\mu}{z_k} z, \lambda \mu^r \right).$$

Obviously  $\chi_k$  is invertible and its inverse is

$$(3) \quad \chi_k^{-1}(\zeta, \mu) = (\psi_k(\zeta), \mu).$$

Therefore  $\Phi_k$  is invertible and from (1), (2) and (3) we deduce that

$$\begin{aligned} \forall (\zeta, \mu) \in \pi^{-1}(U_j \cap U_k), \\ \text{if } \pi(\zeta) = [z] = [z_0 : \dots : z_n] \text{ then } (\Phi_j \circ \Phi_k^{-1})(\zeta, \mu) = \left( \zeta, \frac{z_j}{z_k} \mu \right) \end{aligned}$$

and our claim is proved.

Let  $\Omega$  be an open subset of  $\mathcal{O}(r)$  which is locally Stein but it is not Stein. It follows from Lemma 1 that, as an open subset of  $\mathbb{C}^{n+1} \times \mathbb{C}$ ,  $F^{-1}(\Omega)$  is locally Stein at every point of  $(\partial F^{-1}(\Omega)) \setminus (\{0\} \times \mathbb{C})$  and is not Stein. From Lemma 2 we conclude that there exists  $\lambda_0 \in \mathbb{C}$  such that  $(0, \lambda_0) \in (\partial F^{-1}(\Omega)) \cap (\{0\} \times \mathbb{C}) \subset \mathbb{C}^{n+1} \times \mathbb{C}$  is removable along  $\{0\} \times \mathbb{C}$  and therefore there exists  $\epsilon > 0$  such that

$$F^{-1}(\Omega) \supset \{(z, \lambda) \in (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} : |z_j| < \epsilon, \forall j = \overline{0, n} \text{ and } |\lambda - \lambda_0| < \epsilon\}.$$

PART 1:  $r > 0$ .

We have to show that  $\Omega$  contains a neighborhood of the section at infinity. This is the same thing as showing that for every  $[z] \in \mathbb{P}^n$  and every  $k \in \{0, 1, \dots, n\}$  such that  $[z] \in U_k$  there exists an open set,  $V$ , in  $\mathbb{P}^n$  and  $M \in (0, \infty)$  such that  $[z] \in V$  and  $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : |\lambda| > M\}$ .

Let  $[\tilde{z}]$  be a fixed point in  $U_k$  and let  $T$  be a real number such that  $T > \max\{|\tilde{z}_j|/|\tilde{z}_k| : j = \overline{0, n}\}$  (in particular  $T > 1$ ). We set  $V := \{[z] \in U_k : |z_j|/|z_k| < T\}$  which is an open neighborhood of  $[\tilde{z}]$ . Let  $\lambda_1 \in \mathbb{C}$  and  $M \in \mathbb{R}$  be such that  $\lambda_1 \neq 0$ ,  $|\lambda_1 - \lambda_0| < \epsilon$  and  $M > |\lambda_1| T^r / \epsilon^r$ .

We claim that  $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : |\lambda| > M\}$ . Indeed, let  $([w], \nu) \in V \times \{\lambda \in \mathbb{C} : |\lambda| > M\}$  and let  $\mu$  be a complex number such that  $\mu^r = \lambda_1/\nu$ . We set  $z := (\mu/w_k)w \in \mathbb{C}^{n+1} \setminus \{0\}$  ( $z$  depends only on  $[w]$  and not on a representative of this class). In particular  $z_k = \mu \neq 0$ . We note that, for every  $j \in \{0, \dots, n\}$ ,  $|z_j| = |\mu| |w_j|/|w_k| < |\mu|T$ . However  $|\mu| = (|\lambda_1|/|\nu|)^{1/r} < (|\lambda_1|/M)^{1/r} < \epsilon/T$  and therefore  $|z_j| < \epsilon$ . It follows that  $(z, \lambda_1) \in F^{-1}(\Omega)$  and hence  $F(z, \lambda_1) \in \Omega$ . As  $(z, \lambda_1) \in W_k$  we have that  $F(z, \lambda_1) = \psi_k^{-1}([z], \lambda_1/z_k^r)$  and hence  $\psi_k(F(z, \lambda_1)) = ([z], \lambda_1/z_k^r) = ([w], \nu)$ .

PART 2:  $r < 0$ .

We have to show that  $\Omega$  contains an open subset of the form  $U \setminus A$  where  $A$  is the zero section and  $U$  is an open neighborhood of  $A$ . That is, we must show that for every  $[z] \in \mathbb{P}^n$  and every  $k \in \{0, 1, \dots, n\}$  such that  $[z] \in U_k$  there exists an open set,  $V$ , in  $\mathbb{P}^n$  and  $\delta \in (0, \infty)$  such that  $[z] \in V$  and  $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$ .

Let  $[\tilde{z}]$  be a fixed point in  $U_k$  and let  $T$  be a real number such that  $T > \max\{|\tilde{z}_j|/|\tilde{z}_k| : j = \overline{0, n}\}$ . Let  $\lambda_1 \in \mathbb{C}$  and  $\delta \in \mathbb{R}$  be such that  $\lambda_1 \neq 0$ ,  $|\lambda_1 - \lambda_0| < \epsilon$  and  $\delta < (\epsilon/T)^{-r} |\lambda_1|$ . We claim that  $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$ . Indeed let  $([w], \nu) \in V \times \{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$ , and let  $\mu$  be a complex number such that  $\mu^{-r} = \nu/\lambda_1$  (hence  $\mu \neq 0$ ). It follows that  $|\mu|^{-r} < \delta/|\lambda_1| < (\epsilon/T)^{-r}$  and therefore  $|\mu| < \epsilon/T$ . Let  $z = (\mu/w_k)w$ . In particular  $z_k = \mu$ , hence  $z_k \neq 0$  and therefore  $(z, \lambda_1) \in W_k$ . For every  $j = 0, 1, \dots, n$ ,  $|z_j| = |\mu \cdot w_j|/|w_k| \leq |\mu|T < \epsilon$ . We deduce that  $(z, \lambda_1) \in F^{-1}(\Omega) \cap W_k$ . Therefore  $F(z, \lambda_1) \in \Omega \cap \pi^{-1}(U_k)$  and  $F(z, \lambda_1) = \psi_k^{-1}([z], \lambda_1/z_k^r)$ , which implies that  $\psi_k(F(z, \lambda_1)) = ([z], \lambda_1/z_k^r) = ([w], \lambda_1/\mu^r) = ([w], \nu)$ .

REMARK. 1) We used in the proof the fact that the pull-back of  $\mathcal{O}(r)$  on  $\mathbb{C}^{n+1} \setminus \{0\}$  is trivial, but we had to work with a fixed trivialization in order to apply the Grauert–Riemert removability theorem (Lemma 2).

2) The blow-up of a Stein manifold at a point is not infinitesimally homogeneous and therefore the results in [4] cannot be applied in our situation.

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