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## FORMATION OF SINGULARITIES FOR NONLINEAR HYPERBOLIC $2 \times 2$ SYSTEMS WITH PERIODIC DATA

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### 1. Introduction

In this work we study the formation of singularities in the solutions of the following *system of two first order partial differential equations*

$$\begin{cases} z_t + az_x + bw_x = 0 \\ w_t + cz_x + dw_x = 0. \end{cases}$$

We assume that the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  has smooth entries depending on  $(z, w)$  and moreover we suppose that, for  $(z, w)$  in a neighborhood of the origin, the matrix  $A$  has two distinct real eigenvalues, i. e. the system is *strictly hyperbolic*. It is well known that this system can be written in the following form

$$(S) \quad \begin{cases} u_t + \lambda(u, v)u_x = 0 \\ v_t + \mu(u, v)v_x = 0, \end{cases}$$

where  $\lambda(u, v) < \mu(u, v)$ . The function  $u$  and  $v$  are the *Riemann invariants* (see [12, p. 94]). Supposing that  $u(x, 0)$  and  $v(x, 0)$  are smooth functions, *we ask if the system (S) may have a nontrivial solution which remains smooth for all time.*

This question has an almost complete answer in the case of initial data with compact support (see [10], [5, Ch. 1], [9] and [1, Ch. 4]). Also for  $N \times N$  systems the case of initial data with compact support has been widely studied (see [6], [11], [5, Ch. 1] and [1, Ch. 4]). For  $N \times N$  systems, but only in diagonal form, some results are known when initial data have a limit for  $x \rightarrow +\infty$  and for  $x \rightarrow -\infty$  (see [4]).

Here we are interested in the case of *periodic initial data*. By the works of Lax [10] and Glimm and Lax [3] we know that if the system (S) is *genuinely nonlinear*, i. e.  $\lambda_u(0, 0) \neq 0$  and  $\mu_v(0, 0) \neq 0$ , then smooth nonconstant periodic initial data which are “sufficiently small” give rise to the *blow-up* of the solution, i. e. the first

derivatives of  $u$  or  $v$  becomes infinite for certain  $(x, t)$  with  $t > 0$ . Consequently there exists no nontrivial solution to (S) that remains smooth for all time if initial data are “small” .

This paper is devoted to show a similar result when the genuine nonlinearity is replaced by the following weaker condition: there exists a neighborhood  $W$  of the origin such that  $\lambda_u$  is not identically equal to zero on any open subset of  $W$  and  $\lambda_u$  doesn't assume both positive and negative values on  $W$ , and the same is true for  $\mu_v$ .

Our starting point has been the work of Klainerman and Majda [9] in which the authors consider a particular system of type (S) obtained from the hyperbolic equation

$$(E) \quad w_{tt} + k(w_x)w_{xx} = 0,$$

with  $k(0) = -1$ . They prove that if there exists a positive integer  $p > 1$  such that  $k^{(j)}(0) = 0$  for  $j = 1, \dots, p - 1$  and  $k^{(p)}(0) \neq 0$ , then the blow-up of the solution takes place for all nonconstant periodic initial data whose  $C^1$  norm is sufficiently small. It is easy to see that in the system (S) obtained from (E) we have  $-\lambda(u, v) = \mu(u, v) = h(u - v)$  where  $h$  is a certain function linked to  $k$  in such a way that  $h(0) = 1$ ,  $h^{(j)}(0) = 0$  for  $j = 1, \dots, p - 1$  and  $h^{(p)}(0) \neq 0$ .

Our work improves Klainerman and Majda's result in the case of  $p$  odd, and the main idea of our proof is a development of Klainerman and Majda's one. In particular the formation of singularities is shown by the following “geometric” argument (see also [8, p. 415]): we associate to the solution of (S) with initial data  $(u_0, v_0)$  two families of characteristic curves defined by

$$\begin{aligned} \frac{dx_1}{dt} &= \lambda(u, v)(x_1, t), & x_1(0, \alpha) &= \alpha, \\ \frac{dx_2}{dt} &= \mu(u, v)(x_2, t), & x_2(0, \beta) &= \beta. \end{aligned}$$

Functions  $u$  and  $v$  are constant along  $x_1$  and  $x_2$  respectively. Suppose that  $\bar{\alpha} \in \mathbb{R}$  and  $u'_0(\bar{\alpha}) \neq 0$ . If there exists  $\bar{t} > 0$  such that  $\frac{dx_1}{d\alpha}$  evaluated in  $(\bar{t}, \bar{\alpha})$  is less than zero then the characteristic curve  $x_1(\alpha, t)$  starting near  $\bar{\alpha}$  intersect each other before the time  $\bar{t}$ , and, as the function  $u$  is constant along these curves and assumes different values on different curves,  $u$  ceases to be continuous before time  $\bar{t}$ , i.e. blow-up occurs.

It is interesting to remark that when the system is genuinely nonlinear the fact that  $dx_1/d\alpha$  will become less than zero on the  $x_1$ -characteristic starting from  $\bar{\alpha}$  depends only on the value of  $u'_0(\bar{\alpha})$ ; on the contrary, when the system is not genuinely nonlinear the value of  $dx_1/d\alpha$  along the  $x_1$ -characteristic depends on the value of the functions  $u_0$  and  $v_0$  on the whole domain  $\mathbb{R}$ .

Let us end saying that as in the work of Klainerman and Majda, the blow-up result for the system (S) can be used to deduce a similar result for the hyperbolic

equation (E), also in the case of the Dirichlet problem. This is the content of the Corollaries 1 and 2. In particular we show that in the Dirichlet initial boundary value problem for (E) with  $k$  monotone and nonconstant there are no solutions which are time-periodic, if the initial data are sufficiently small and non zero.

Other results improving Klainerman and Majda's one can be found in [2].

## 2. Results and Remarks

**Theorem.** *Consider the Cauchy problem for the following nonlinear hyperbolic  $2 \times 2$  system of first order partial differential equations, written in terms of the Riemann invariants,*

$$(1) \quad \begin{cases} u_t + \lambda(u, v)u_x = 0 \\ v_t + \mu(u, v)v_x = 0 \end{cases}$$

with  $C^1$  periodic initial data  $(u(x, 0), v(x, 0)) = (u_0^\varepsilon(x), v_0^\varepsilon(x)) = (\varepsilon u_0(x), \varepsilon v_0(x))$ , where  $u_0, v_0$  are both nonconstant and have period  $\sigma > 0$ ;  $\lambda$  and  $\mu$  are smooth functions with  $\lambda(0, 0) < \mu(0, 0)$ .

Suppose that there exists an open neighborhood  $W$  of the origin of  $\mathbb{R}^2$  such that

$$(H) \quad \begin{aligned} &\lambda_u \text{ is not identically zero on any open subset of } W \text{ and} \\ &\lambda_u(u, v) \geq 0 \text{ for all } (u, v) \in W \text{ or } \lambda_u(u, v) \leq 0 \text{ for all } (u, v) \in W, \\ &\text{or} \\ &\mu_v \text{ is not identically zero on any open subset of } W \text{ and} \\ &\mu_v(u, v) \geq 0 \text{ for all } (u, v) \in W \text{ or } \mu_v(u, v) \leq 0 \text{ for all } (u, v) \in W. \end{aligned}$$

Then there exists  $\varepsilon_0 > 0$  such that the  $C^1$  solution of (1) with initial data  $(u_0^\varepsilon, v_0^\varepsilon)$  develops a singularity in the first derivatives in finite time, for all  $\varepsilon \in ]0, \varepsilon_0]$ .

**REMARK 1.** If one of the initial data is constant, e. g.  $v_0(x) = a$ , the system is in fact reduced to the single equation  $u_t + \lambda(u, a)u_x = 0$ ; in this case if the function  $u \mapsto \lambda(u, a)$  is nonconstant on every open interval then there is a development of singularities in finite time for all nonconstant and periodic  $u_0$  (see [7, p. 4]).

**REMARK 2.** It is not necessary that the functions  $\lambda$  and  $\mu$  are smooth functions; actually only  $C^1$  regularity is needed.

**REMARK 3.** As it will be clear by the proof, the constant  $\varepsilon_0$  depends only on the functions  $\lambda$  and  $\mu$ , on the period  $\sigma$  and on the  $C^1$  norm of the initial data. It is possible to give an estimate for the lifespan of the classical solution in term of the

parameter  $\varepsilon$ . The result is the following: let  $\varepsilon \in ]0, \varepsilon_0]$  and consider the functions

$$L(\varepsilon) = \min_{\alpha \in [0, \sigma]} \varepsilon u'_0(\alpha) \int_0^\sigma \lambda_u(\varepsilon u_0(\alpha), \varepsilon v_0(\beta)) d\beta$$

$$R(\varepsilon) = \min_{\beta \in [0, \sigma]} \varepsilon v'_0(\beta) \int_0^\sigma \mu_v(\varepsilon u_0(\alpha), \varepsilon v_0(\beta)) d\alpha.$$

Easily  $L(\varepsilon) \leq 0$ ,  $R(\varepsilon) \leq 0$  and  $(L(\varepsilon))^2 + (R(\varepsilon))^2 > 0$ . Denoting by  $T_\varepsilon$  the lifespan of the classical solution with initial data  $(u_0^\varepsilon, v_0^\varepsilon)$  we have

$$T_\varepsilon \leq \frac{C}{\max\{-L(\varepsilon), -R(\varepsilon)\}},$$

where  $C$  does not depend on  $\varepsilon$ .

The Theorem has an easy corollary in the case of a particular second order nonlinear hyperbolic equation.

**Corollary 1.** *Consider the Cauchy problem for the following wave equation*

$$(2) \quad u_{tt} + k(u_x)u_{xx} = 0$$

with periodic initial data  $u(x, 0) = u_0^\varepsilon(x) = \varepsilon u_0(x)$  and  $u_t(x, 0) = v_0^\varepsilon(x) = \varepsilon v_0(x)$ , where  $(u_0, v_0) \in \mathcal{C}^2 \times \mathcal{C}^1$  is nonconstant and has period  $\sigma > 0$ ;  $k$  is a smooth function, with  $k(0) = -1$ .

*Suppose that there exists an open neighborhood  $U$  of the origin of  $\mathbb{R}$  such that*

$$(H') \quad \begin{array}{l} k \text{ is not constant on any open interval contained in } U, \\ \text{and} \\ k \text{ is monotone in } U \end{array}$$

*Then there exists  $\varepsilon_0 > 0$  such that the classical  $\mathcal{C}^2$  solution of (2) with initial data  $(u_0^\varepsilon, v_0^\varepsilon)$  develops a singularity in the second derivatives in finite time, for all  $\varepsilon \in ]0, \varepsilon_0]$ .*

REMARK 4. The result of Corollary 1 when the function  $k'$  has a zero with finite even order in 0 was already known by the work of Klainerman and Majda ([9, Th. 1]) in which the case of function  $k'$  with a zero with finite order in 0, without any monotonicity condition, is treated. In that paper also an estimate of the lifespan of the classical solution is given. It is possible to recover an estimate similar to Klainerman and Majda's one in the case of finite even order zero for  $k'$ , using the computation of Remark 3.

We can use Corollary 1 to prove a result on the Dirichlet problem for the equation (2). This is the content of the following statement.

**Corollary 2.** *Consider the Dirichlet initial boundary value problem in  $[0, L] \times [0, +\infty[$  for (2) with Dirichlet boundary condition  $u(0, t) = u(L, t) = 0$  for  $t > 0$  and with non zero initial data  $u(x, 0) = u_0^\varepsilon(x) = \varepsilon u_0(x)$  and  $u_t(x, 0) = v_0^\varepsilon(x) = \varepsilon v_0(x)$ . Let  $(u_0, v_0) \in C^2([0, L]) \times C^1([0, L])$  satisfy the following compatibility conditions*

$$u_0(0) = u_0(L) = 0; \quad v_0(0) = v_0(L) = 0; \quad u_0''(0) = u_0''(L) = 0.$$

*Suppose that the condition (H') is satisfied.*

*Then there exists  $\varepsilon_0 > 0$  such that the classical  $C^2$  solution of this problem with initial data  $(u_0^\varepsilon, v_0^\varepsilon)$  develops a singularity in the second derivatives in finite time, for all  $\varepsilon \in ]0, \varepsilon_0]$ .*

**REMARK 5.** We prove that given the function  $k$  and the value  $L$  there exists  $\delta_0 > 0$  such that if the  $C^1$  norm of  $(u'_0, v_0)$  is less than  $\delta_0$  then there will be the development of singularities in the second derivatives of the solution of the Dirichlet initial boundary value problem considered in Corollary 2; as a consequence, there exists a ball in  $C^2([0, L] \times [0, +\infty[)$  such that the only solution to this Dirichlet problem which is in the ball and it is time-periodic is the null function.

### 3. Proof of the Theorem

The proof of the Theorem is inspired to that one of Theorem 1' in [9]. The main difference is that in our case the initial data are periodic with the same period  $\sigma$  and this will allow us to perform *twice* a cancellation of terms in the computation of the quantities  $I_1(\alpha_0, t_N)$  and  $I_2(\beta_0, t_N)$  (see Lemmas 1 and 2 below).

Without loss of generality we can suppose that  $\lambda(0, 0) = -1$  and  $\mu(0, 0) = 1$ . Moreover there exist  $\delta > 0$  such that  $] -2\delta, 2\delta[ \times ] -2\delta, 2\delta[ = W' \subset W$ , and if  $|l|, |r| \leq \delta$  then

$$-\frac{5}{4} \leq \lambda(l, r) \leq -\frac{3}{4} \quad \text{and} \quad \frac{3}{4} \leq \mu(l, r) \leq \frac{5}{4}.$$

Let  $l_0, r_0$  be two real  $C^1$  functions defined on  $\mathbb{R}$ , nonconstant and periodic with period  $\sigma > 0$ , with  $\|l_0\|_\infty, \|r_0\|_\infty \leq \delta$ . Consider the Cauchy problem

$$(3) \quad \begin{cases} l_t + \lambda(l, r)l_x = 0 \\ r_t + \mu(l, r)r_x = 0 \\ l(x, 0) = l_0(x) \\ r(x, 0) = r_0(x). \end{cases}$$

We claim that there exists  $\delta' \in ]0, 1[$  such that if  $\|l'_0\|_\infty, \|r'_0\|_\infty \leq \delta'$  then (3) cannot have a  $C^1$  solution defined on  $\mathbb{R} \times [0, +\infty[$ . We argue by contradiction. Let  $(l, r)$  be

a  $C^1$  solution to (3) on  $\mathbb{R} \times [0, +\infty[$ . We introduce the characteristic curves  $x_1(\alpha, t)$  and  $x_2(\beta, t)$ , relative to the solution  $(l, r)$ ; we have

$$(4) \quad \begin{cases} \frac{dx_1}{dt} = \lambda(l(x_1, t), r(x_1, t)) \\ x_1(\alpha, 0) = \alpha, \end{cases}$$

and

$$(5) \quad \begin{cases} \frac{dx_2}{dt} = \mu(l(x_2, t), r(x_2, t)) \\ x_2(\beta, 0) = \beta. \end{cases}$$

Consequently  $l(x_1(\alpha, t), t) = l_0(\alpha)$  and  $r(x_2(\beta, t), t) = r_0(\beta)$  for all  $t \geq 0$ .

We define

$$(6) \quad h_1(l, r) = \int_0^r \frac{\lambda_s(l, s)}{(\lambda - \mu)(l, s)} ds, \quad h_2(l, r) = \int_0^l \frac{\mu_s(s, r)}{(\mu - \lambda)(s, r)} ds,$$

so that  $h_1, h_2$  are smooth functions on  $W'$ . Next we consider

$$(7) \quad k_1(\alpha, t) = \frac{e^{h_1(l_0(\alpha), r(x_1(\alpha, t), t))}}{e^{h_1(l_0(\alpha), r_0(\alpha))}}, \quad k_2(\beta, t) = \frac{e^{h_2(l(x_2(\beta, t), t), r_0(\beta))}}{e^{h_2(l_0(\beta), r_0(\beta))}}$$

As for all  $(\alpha, t) \in \mathbb{R} \times [0, +\infty[$  we have that  $(l_0(\alpha), r(x_1(\alpha, t), t)) \in [-\delta, \delta] \times [-\delta, \delta]$  and for all  $(\beta, t) \in \mathbb{R} \times [0, +\infty[$  we have that  $(l(x_2(\beta, t), t), r_0(\beta)) \in [-\delta, \delta] \times [-\delta, \delta]$ , there exists  $d > 1$  such that

$$(8) \quad \frac{1}{d} \leq k_1(\alpha, t) \leq d, \quad \frac{1}{d} \leq k_2(\beta, t) \leq d,$$

for all  $(\alpha, t) \in \mathbb{R} \times [0, +\infty[$  and for all  $(\beta, t) \in \mathbb{R} \times [0, +\infty[$  respectively.

A central role in the proof will be played by the following identities for the waves infinitesimal compression ratio which can be obtained by an easy computation (see [12, p. 102]),

$$(9) \quad \frac{dx_1}{d\alpha}(\alpha, t) = k_1(\alpha, t) \left[ 1 + l'_0(\alpha) \int_0^t \lambda_l(l_0(\alpha), r(x_1(\alpha, s), s)) k_1^{-1}(\alpha, s) ds \right],$$

and

$$(10) \quad \frac{dx_2}{d\beta}(\beta, t) = k_2(\beta, t) \left[ 1 + r'_0(\beta) \int_0^t \mu_r(l(x_2(\beta, s), s), r_0(\beta)) k_2^{-1}(\beta, s) ds \right].$$

Let us denote by  $I_1(\alpha, t)$ ,  $I_2(\beta, t)$  the quantities

$$(11) \quad I_1(\alpha, t) = l'_0(\alpha) \int_0^t \lambda_l(l_0(\alpha), r(x_1(\alpha, s), s)) k_1^{-1}(\alpha, s) ds,$$

and

$$(12) \quad I_2(\beta, t) = r'_0(\beta) \int_0^t \mu_r(l(x_2(\beta, s), s), r_0(\beta)) k_2^{-1}(\beta, s) ds.$$

Next we define the functions  $f(\alpha, \beta)$  and  $g(\alpha, \beta)$  in the following way

$$(13) \quad f(\alpha, \beta) = \frac{\lambda_l(l_0(\alpha), r_0(\beta))}{(\mu - \lambda)(l_0(\alpha), r_0(\beta))} \frac{e^{h_2(l_0(\alpha), r_0(\beta))}}{e^{h_1(l_0(\alpha), r_0(\beta))}},$$

and

$$(14) \quad g(\alpha, \beta) = \frac{\mu_r(l_0(\alpha), r_0(\beta))}{(\mu - \lambda)(l_0(\alpha), r_0(\beta))} \frac{e^{h_1(l_0(\alpha), r_0(\beta))}}{e^{h_2(l_0(\alpha), r_0(\beta))}}.$$

For each  $\alpha, \beta \in \mathbb{R}$  with  $\alpha \geq \beta$  we denote by  $s(\alpha, \beta)$  the unique solution  $s$  to the equation

$$(15) \quad x_1(\alpha, s) = x_2(\beta, s).$$

The proof of the Theorem will be deduced by the following two lemmas.

**Lemma 1.** *Let  $\alpha_0$  be any real number. For every  $N \in \mathbb{N}$  there exists  $t_N > 0$  such that*

$$(16) \quad \begin{aligned} I_1(\alpha_0, t_N) = & l'_0(\alpha_0) e^{h_1(l_0(\alpha_0), r_0(\alpha_0))} \left[ N \int_{\alpha_0 - \sigma}^{\alpha_0} f(\alpha_0, \beta) e^{-h_2(l_0(\beta), r_0(\beta))} d\beta \right. \\ & + N \int_{\alpha_0 - \sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\ & + \sum_{n=1}^N \sum_{j=1}^{n-1} \int_{\alpha_0 - \sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0 - \sigma}^{\alpha_0} g(\alpha, \beta) l'_0(\alpha) \int_{\beta}^{\alpha_0} f(\alpha, \gamma) r'_0(\gamma) \\ & \left. \int_{s(\alpha_0 - j\sigma, \gamma - (n-1)\sigma)}^{s(\alpha - (j-1)\sigma, \gamma - (n-1)\sigma)} (\mu_r e^{-h_2})(l(x_2(\gamma, s), s), r_0(\gamma)) ds d\gamma d\alpha d\beta \right] \end{aligned}$$

**Lemma 2.** *Let  $\beta_0$  be any real number. For every  $N \in \mathbb{N}$  there exists  $t_N > 0$  such that*



$$\begin{aligned}
 I_2(\beta_0, t_N) &= r'_0(\beta_0) e^{h_2(l_0(\beta_0), r_0(\beta_0))} \left[ N \int_{\beta_0}^{\beta_0+\sigma} g(\alpha, \beta_0) e^{-h_1(l_0(\alpha), r_0(\alpha))} d\alpha \right. \\
 (17) \quad &+ N \int_{\beta_0}^{\beta_0+\sigma} g(\alpha, \beta_0) l'_0(\alpha) \int_0^{s(\alpha, \beta_0)} (\lambda_l e^{-h_1})(l_0(\alpha), r(x_1(\alpha, s), s)) ds d\alpha \\
 &+ \sum_{n=1}^N \sum_{j=1}^{n-1} \int_{\beta_0}^{\beta_0+\sigma} g(\alpha, \beta_0) l'_0(\alpha) \int_{\beta_0}^{\beta_0+\sigma} f(\alpha, \beta) r'_0(\beta) \int_{\beta_0}^{\alpha} g(\gamma, \beta) l'_0(\gamma) \\
 &\quad \int_{s(\gamma-(n-1)\sigma, \beta_0+j\sigma)}^{s(\gamma-(n-1)\sigma, \beta_0+(j-1)\sigma)} (\lambda_l e^{-h_1})(l_0(\alpha), r(x_1(\alpha, s), s)) ds d\gamma d\beta d\alpha
 \end{aligned}$$

Let us show how from Lemma 1 and Lemma 2 we obtain the conclusion of the Theorem. Suppose for instance that  $\lambda_l$  is not identically zero on any open subset of  $W$  and  $\lambda_l \leq 0$  on  $W$ ; in this case only the result of Lemma 1 is needed. Keeping in mind that a contradiction is obtained when  $dx_1/d\alpha$  assumes the value 0, we start to consider  $I_1(\alpha_0, t_N)$ : if there exist  $\alpha_0$  and  $t_N$  such that this quantity is less than  $-1$ , the Theorem is proved.

First we remark that if  $\beta \leq \alpha$  and  $\alpha - \beta \leq \sigma$ , then

$$(18) \quad s(\alpha, \beta) \leq \frac{2}{3}\sigma.$$

Similarly, using the fact that the solution  $(l, r)$  is periodic in  $x$  of period  $\sigma$  and consequently the characteristic curves  $x_1(\alpha, t)$  and  $x_2(\beta, t)$  are periodic in  $\alpha$  and  $\beta$  respectively, of period  $\sigma$ , we have that, for all  $\alpha, \beta, \gamma \in \mathbb{R}$  with  $\beta \leq \gamma \leq \alpha$  and  $\alpha - \gamma \leq \sigma$ ,

$$(19) \quad s(\alpha, \beta) - s(\gamma, \beta) \leq \frac{2}{3}\sigma.$$

Next we observe that there exist  $M_1, M_2 > 0$  such that

$$(20) \quad |\mu_r(l(x_2(\beta, t), t), r_0(\beta)) e^{-h_2(l(x_2(\beta, t), t), r_0(\beta))})| \leq M_1,$$

for all  $\beta \in \mathbb{R}, t > 0$ ;

$$(21) \quad |f(\alpha, \beta)| \leq M_2, \quad \text{and} \quad |g(\alpha, \beta)| \leq M_2,$$

for all  $\alpha, \beta \in \mathbb{R}$ ; moreover there exists  $c > \max\{1, \sigma M_2\}$  such that

$$(22) \quad \frac{1}{c} \leq e^{h_1(l_0(\alpha), r_0(\alpha))} \leq c, \quad \frac{1}{c} \leq e^{h_2(l_0(\alpha), r_0(\alpha))} \leq c,$$

for all  $\alpha \in \mathbb{R}$ . It is worthy to say that, after having fixed  $\delta$ , the constants  $c, M_1, M_2$  depends only on the functions  $\lambda$  and  $\mu$  and on the value of  $\sigma$ .

Let  $\tilde{\alpha}$  be any real number and consider the function

$$\alpha \mapsto l'_0(\alpha) \int_{\tilde{\alpha}-\sigma}^{\tilde{\alpha}} f(\alpha, \beta) d\beta;$$

this function is periodic of period  $\sigma$ , its value does not depend on  $\tilde{\alpha}$  and

$$\int_0^\sigma l'_0(\alpha) \int_{\tilde{\alpha}-\sigma}^{\tilde{\alpha}} f(\alpha, \beta) d\beta d\alpha = 0.$$

Consequently

$$\inf_{\alpha \in [0, \sigma]} l'_0(\alpha) \int_{\tilde{\alpha}-\sigma}^{\tilde{\alpha}} f(\alpha, \beta) d\beta = \min_{\alpha \in [0, \sigma]} l'_0(\alpha) \int_{\tilde{\alpha}-\sigma}^{\tilde{\alpha}} f(\alpha, \beta) d\beta < 0.$$

In fact if this infimum is  $\geq 0$  then  $l'_0(\alpha) \int_{\tilde{\alpha}-\sigma}^{\tilde{\alpha}} f(\alpha, \beta) d\beta \geq 0$  for all  $\alpha$ . As  $f(\alpha, \beta) \leq 0$  for all  $(\alpha, \beta)$ , we get that if, for some  $\tilde{\alpha}$ ,  $l'_0(\tilde{\alpha}) > 0$ , then  $f(\tilde{\alpha}, \beta) = 0$  for all  $\beta$ . As  $l'_0$  is a continuous function taking positive value for some  $\tilde{\alpha}$ , we obtain that  $f(\alpha, \beta)$  is zero on an open rectangle, and  $\lambda_l$  would be zero on an open rectangle of  $W$ , against the hypothesis.

We fix  $\alpha_0$  in such a way that this minimum is attained. We denote by  $a$  the the value

$$a = l'_0(\alpha_0) \int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta,$$

and we remark that  $|a| \leq \sigma M_2$  and

$$a = \min_{\alpha \in [0, \sigma]} l'_0(\alpha) \int_{\tilde{\alpha}-\sigma}^{\tilde{\alpha}} f(\alpha, \beta) d\beta,$$

for all  $\tilde{\alpha} \in \mathbb{R}$ .

From (16) we have that

$$\begin{aligned} I_1(\alpha_0, t_N) = & e^{h_1((l_0(\alpha_0), r_0(\alpha_0)))} \left[ Na \frac{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) e^{-h_2((l_0(\beta), r_0(\beta)))} d\beta}{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta} \right. \\ & + Na \frac{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(\dots) ds d\beta}{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta} \\ & + a^2 \sum_{n=1}^N \sum_{j=1}^{n-1} \\ & \left. \frac{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0-\sigma}^{\alpha_0} g(\alpha, \beta) l'_0(\alpha) \int_{\beta}^{\alpha_0} f(\alpha, \gamma) r'_0(\gamma) \int_{\dots} \mu_r e^{-h_2} ds d\gamma d\alpha d\beta}{a \int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta} \right] \end{aligned}$$

Now, from (22), we obtain

$$(23) \quad \frac{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) e^{-h_2(l_0(\beta), r_0(\beta))} d\beta}{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta} > \frac{1}{c};$$

recalling (18), (20) and the fact that  $\|r'_0\|_\infty \leq \delta'$ , we deduce

$$(24) \quad \left| \frac{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(\dots) ds d\beta}{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta} \right| \leq \frac{2}{3} \sigma M_1 \delta'.$$

We remark now that

$$\begin{aligned} \int_{\alpha_0-\sigma}^{\alpha_0} |l'_0(\alpha)| \int_{\beta}^{\alpha_0} |f(\alpha, \gamma)| d\gamma d\alpha &\leq \int_{\alpha_0-\sigma}^{\alpha_0} |l'_0(\alpha)| \int_{\alpha_0-\sigma}^{\alpha_0} |f(\alpha, \gamma)| d\gamma d\alpha \\ &\leq \int_{\alpha_0-\sigma}^{\alpha_0} |l'_0(\alpha)| \int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha, \gamma) d\gamma d\alpha \leq 2\sigma |a|. \end{aligned}$$

From this, together with (19), (20) and (21), we have

$$(25) \quad \begin{aligned} &\left| \frac{\int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0-\sigma}^{\alpha_0} g(\alpha, \beta) l'_0(\alpha) \int_{\beta}^{\alpha_0} f(\alpha, \gamma) r'_0(\gamma) \int_{\dots} \mu_r e^{-h_2} ds d\gamma d\alpha d\beta}{a \int_{\alpha_0-\sigma}^{\alpha_0} f(\alpha_0, \beta) d\beta} \right| \leq \\ &\left| \frac{\int_{\alpha_0-\sigma}^{\alpha_0} |f(\alpha_0, \beta)| |r'_0(\beta)| \int_{\alpha_0-\sigma}^{\alpha_0} |g(\alpha, \beta)| |l'_0(\alpha)| \int_{\beta}^{\alpha_0} |f(\alpha, \gamma)| |r'_0(\gamma)| \int_{\dots} |\mu_r e^{-h_2}| ds d\gamma d\alpha d\beta}{|a| \int_{\alpha_0-\sigma}^{\alpha_0} |f(\alpha_0, \beta)| d\beta} \right| \\ &\leq \left| \frac{\int_{\alpha_0-\sigma}^{\alpha_0} |f(\alpha_0, \beta)| \delta' M_2 2\sigma |a| \delta' \frac{2}{3} \sigma M_1 d\beta}{|a| \int_{\alpha_0-\sigma}^{\alpha_0} |f(\alpha_0, \beta)| d\beta} \right| \leq \frac{4}{3} \sigma^2 M_1 M_2 (\delta')^2. \end{aligned}$$

From (23), (24) and (25) we deduce that

$$I_1(\alpha_0, t_N) \leq \frac{Na}{c^2} + \frac{2}{3} cN |a| \delta' \sigma M_1 + \frac{2}{3} cN^2 a^2 (\delta')^2 \sigma^2 M_1 M_2.$$

Choose now  $N$  such that  $Na/c^2 \in [-3, -2]$ ; if

$$\delta' \leq \min \left\{ \frac{1}{4\sigma M_1 c^3 + 12\sigma^2 M_1 M_2 c^5}, 1 \right\},$$

then  $I_1(\alpha_0, t_N) < -1$  and this is enough for concluding the proof of the Theorem. To obtain the estimate in Remark 3 we note that if  $I_1(\alpha_0, t_N) < -1$  then  $t_N$  is an

upper bound for the lifespan of the solution. On the other hand  $t_N \leq 2N\sigma/5$  and  $N < 3c^2/|a|$ . Recalling that, by (13), there exists  $c_1, c_2 > 0$  such that

$$c_1 \lambda_l(l_0(\alpha), r_0(\beta)) \leq f(\alpha, \beta) \leq c_2 \lambda_l(l_0(\alpha), r_0(\beta)),$$

we conclude that

$$\frac{1}{|a|} \leq C(-\min_{\alpha \in [0, \sigma]} l'_0(\alpha) \int_0^\sigma \lambda_l(l_0(\alpha), r_0(\beta)) d\beta)^{-1}.$$

To end we prove Lemma 1, the proof of Lemma 2 being similar. We start computing  $I_1(\alpha_0, t_N)$ . For a fixed  $\alpha_0$  we consider the function  $\beta \mapsto s(\alpha_0, \beta)$  and we use  $\beta$  as a new variable in the integral in (11). Differentiating the identity (15) we have

$$\frac{dx_1}{dt}(\alpha_0, s(\alpha_0, \beta)) \frac{ds}{d\beta}(\alpha_0, \beta) = \frac{dx_2}{d\beta}(\beta, s(\alpha_0, \beta)) + \frac{dx_2}{dt}(\beta, s(\alpha_0, \beta)) \frac{ds}{d\beta}(\alpha_0, \beta),$$

from this and (4), (5) we get

$$\frac{ds}{d\beta}(\alpha_0, \beta) = \frac{\frac{dx_2}{d\beta}(\beta, s(\alpha_0, \beta))}{(\lambda - \mu)(l_0(\alpha_0), r_0(\beta))}.$$

Calling  $t_N$  the solution  $s$  of the equation  $x_1(\alpha_0, s) = x_2(\alpha_0 - N\sigma, s)$ , from (11) we obtain

$$I_1(\alpha_0, t_N) = l'_0(\alpha_0) e^{h_1(l_0(\alpha_0), r_0(\alpha_0))} \times \int_{\alpha_0 - N\sigma}^{\alpha_0} \frac{\lambda_l(l_0(\alpha), r_0(\beta))}{(\mu - \lambda)(l_0(\alpha), r_0(\beta))} \frac{\frac{dx_2}{d\beta}(\beta, s(\alpha_0, \beta))}{e^{h_1(l_0(\alpha_0), r_0(\beta))}} d\beta.$$

Using (10) and (13) we have

$$(26) \quad I_1(\alpha_0, t_N) = l'_0(\alpha_0) e^{h_1(l_0(\alpha_0), r_0(\alpha_0))} \left[ \int_{\alpha_0 - N\sigma}^{\alpha_0} f(\alpha_0, \beta) e^{-h_2(l_0(\beta), r_0(\beta))} d\beta + \int_{\alpha_0 - N\sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \right].$$

As the function  $f(\alpha_0, \beta) e^{-h_2(l_0(\beta), r_0(\beta))}$  is periodic in  $\beta$  with period  $\sigma$ , the former term in (26) can be written as

$$(27) \quad N l'_0(\alpha_0) e^{h_1(l_0(\alpha_0), r_0(\alpha_0))} \int_{\alpha_0 - \sigma}^{\alpha_0} f(\alpha_0, \beta) e^{-h_2(l_0(\beta), r_0(\beta))} d\beta.$$

We consider the latter term in (26). Again using the periodicity of the functions involved we can write

(28)

$$\begin{aligned}
 & \int_{\alpha_0 - N\sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\
 &= \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\
 &= \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0 - (n-1)\sigma, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\
 &\quad + \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{s(\alpha_0 - (n-1)\sigma, \beta)}^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\
 &= N \int_{\alpha_0 - \sigma}^{\alpha_0} f(\alpha_0, \beta) r'_0(\beta) \int_0^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\
 &\quad + \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{s(\alpha_0 - (n-1)\sigma, \beta)}^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta
 \end{aligned}$$

We work now on the last term of (28). Keeping  $\beta$  fixed we consider the function  $\alpha \mapsto s(\alpha, \beta)$  and we use it to change the variable in the inner integral. As

$$\frac{ds}{d\alpha}(\alpha, \beta) = \frac{\frac{dx_1}{d\alpha}(\alpha, s(\alpha, \beta))}{(\mu - \lambda)(l_0(\alpha), r_0(\beta))},$$

we obtain

(29)

$$\begin{aligned}
 A &= \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{s(\alpha_0 - (n-1)\sigma, \beta)}^{s(\alpha_0, \beta)} (\mu_r e^{-h_2})(l(x_2(\beta, s), s), r_0(\beta)) ds d\beta \\
 &= \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0 - (n-1)\sigma}^{\alpha_0} \frac{\mu_r (l_0(\alpha), r_0(\beta))}{(\mu - \lambda)(l_0(\alpha), r_0(\beta))} \frac{\frac{dx_1}{d\alpha}(\alpha, s(\alpha, \beta))}{e^{h_2(l_0(\alpha), r_0(\beta))}} d\alpha d\beta;
 \end{aligned}$$

Suppose now that  $0 \leq t_1 \leq t_2$ ; from (9) we easily deduce that

$$\begin{aligned}
 (30) \quad \frac{dx_1}{d\alpha}(\alpha, t_2) &= \frac{k_1(\alpha, t_2)}{k_1(\alpha, t_1)} \frac{dx_1}{d\alpha}(\alpha, t_1) \\
 &\quad + k_1(\alpha, t_2) l'_0(\alpha) \int_{t_1}^{t_2} \lambda_l(l_0(\alpha), r(x_1(\alpha, s), s)) k_1^{-1}(\alpha, s) ds.
 \end{aligned}$$

We write (30) with  $t_1 = s(\alpha, \alpha_0 - (n - 1)\sigma)$  and  $t_2 = s(\alpha, \beta)$ , and we replace  $(dx_1/d\alpha)(\alpha, s(\alpha, \beta))$  in the second term of (29) ; using (14) we have

$$\begin{aligned}
 (31) \quad A &= \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0 - (n-1)\sigma}^{\alpha_0} g(\alpha, \beta) \frac{\frac{dx_1}{d\alpha}(\alpha, s(\alpha, \alpha_0 - (n - 1)\sigma))}{e^{h_1(l_0(\alpha), r_0(\alpha_0 - (n-1)\sigma))}} d\alpha d\beta \\
 &+ \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0 - (n-1)\sigma}^{\alpha_0} g(\alpha, \beta) l'_0(\alpha) \\
 &\int_{s(\alpha, \alpha_0 - (n-1)\sigma)}^{s(\alpha, \beta)} (\lambda_l e^{-h_1})(l_0(\alpha), r(x_1(\alpha, s), s)) ds d\alpha d\beta.
 \end{aligned}$$

interchanging the order of integration we see that the first term in (31) is zero. In the second term we perform once again a change of variable in the inner integral, obtaining

$$\begin{aligned}
 (32) \quad A &= \sum_{n=1}^N \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0 - (n-1)\sigma}^{\alpha_0} g(\alpha, \beta) l'_0(\alpha) \\
 &\int_{\beta}^{\alpha_0 - (n-1)\sigma} \frac{\lambda_l(l_0(\alpha), r_0(\gamma))}{(\mu - \lambda)(l_0(\alpha), r_0(\gamma))} \frac{\frac{dx_2}{d\beta}(\gamma, s(\alpha, \gamma))}{e^{h_1(l_0(\alpha), r_0(\gamma))}} d\gamma d\alpha d\beta \\
 &= \sum_{n=1}^N \sum_{j=1}^{n-1} \int_{\alpha_0 - n\sigma}^{\alpha_0 - (n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0 - j\sigma}^{\alpha_0 - (j-1)\sigma} g(\alpha, \beta) l'_0(\alpha) \\
 &\int_{\beta}^{\alpha_0 - (n-1)\sigma} \frac{\lambda_l(l_0(\alpha), r_0(\gamma))}{(\mu - \lambda)(l_0(\alpha), r_0(\gamma))} \frac{\frac{dx_2}{d\beta}(\gamma, s(\alpha, \gamma))}{e^{h_1(l_0(\alpha), r_0(\gamma))}} d\gamma d\alpha d\beta.
 \end{aligned}$$

Arguing as before we compute

$$\begin{aligned}
 \frac{dx_2}{d\beta}(\gamma, s(\alpha, \gamma)) &= \frac{k_2(\gamma, s(\alpha, \gamma))}{k_2(\gamma, s(\alpha_0 - j\sigma, \gamma))} \frac{dx_2}{d\beta}(\gamma, s(\alpha_0 - j\sigma, \gamma)) \\
 &+ k_2(\gamma, s(\alpha, \gamma)) r'_0(\gamma) \int_{s(\alpha_0 - j\sigma, \gamma)}^{s(\alpha, \gamma)} \mu_r(l(x_2(\gamma, s), s), r_0(\gamma)) k_2^{-1}(\gamma, s) ds.
 \end{aligned}$$

Replacing this in the second term of (32) we have

$$\begin{aligned}
 (33) \quad A = & \sum_{n=1}^N \sum_{j=1}^{n-1} \int_{\alpha_0-n\sigma}^{\alpha_0-(n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0-j\sigma}^{\alpha_0-(j-1)\sigma} g(\alpha, \beta) l'_0(\alpha) \\
 & \int_{\beta}^{\alpha_0-(n-1)\sigma} f(\alpha, \gamma) \frac{\frac{dx_2}{d\beta}(\gamma, s(\alpha_0 - j\sigma, \gamma))}{e^{h_2(l_0(\alpha_0-j\sigma), r_0(\gamma))}} d\gamma d\alpha d\beta \\
 & + \sum_{n=1}^N \sum_{j=1}^{n-1} \int_{\alpha_0-n\sigma}^{\alpha_0-(n-1)\sigma} f(\alpha_0, \beta) r'_0(\beta) \int_{\alpha_0-j\sigma}^{\alpha_0-(j-1)\sigma} g(\alpha, \beta) l'_0(\alpha) \\
 & \int_{\beta}^{\alpha_0-(n-1)\sigma} f(\alpha, \gamma) r'_0(\gamma) \int_{s(\alpha_0-j\sigma, \gamma)}^{s(\alpha, \gamma)} (\mu_r e^{-h_2})(\dots) ds d\gamma d\alpha d\beta.
 \end{aligned}$$

The first term in (33) is zero : to see this it is sufficient to interchange the order of integration between the variables  $\gamma$  and  $\alpha$ . From (27), (28) and (33), using the periodicity we obtain (16) ; the proof is complete.

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