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On Principally Linear Elliptic Differential Equations of the Second Order.

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§ 0 Introduction

We use the notations $\underset{x_i}{\partial} u$ or $\underset{i}{\partial} u$ for $\dfrac{\partial u}{\partial x_i}$ and $\underset{x_i x_j}{\partial^2} u$ or $\underset{ij}{\partial^2} u$ for $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$. We write x for $x_1, \cdots, x_m$, $\underset{x}{\partial} u$ for $\underset{x_1}{\partial} u, \cdots \underset{x_m}{\partial} u$, and $\underset{x}{\partial^2} u$ for $\underset{ij}{\partial^2} u$ $(i, j = 1, \cdots, m)$.

In this note we shall consider principally linear partial differential equation[1] of elliptic type

$$(0) \qquad \sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} u = f(x, u, \underset{x}{\partial} u).$$

We assume once for all that the quadratic form $\sum_{i,j=1}^{m} a_{ij}(x)\xi_i \xi_j$ *is positive definite.* We denote by $C[A]$ the set of all continuous functions on A, and by $C^p[A]$ the set of all functions whose partial derivatives up to the p-th order are all continuous on A. Under a solution of (0) in the domain D we understand a function of $C^2[D]$ which satisfies (0) for $x \in D$.[2] We say that a solution $u(x)$ of (0) in D takes the boundary value $\beta(x)$, when $u(x) \in C[\bar{D}]$ and $u(x) = \beta(x)$ for $x \in \dot{D}$.[3]

We say a function $\omega(x)$ is a *quasi-supersolution* (*-subsolution*) of (0) in a domain D, if for every point $x_0 \in D$, there exist a neighborhood U of x_0 and a finite number of functions $\omega_\nu(x) \in C^2[U]$ $(\nu = 1, \cdots, n)$ such that

$$(0.1) \qquad \omega(x) = \underset{1 \leqq \nu \leqq n}{\mathrm{Min}}\, \omega_\nu(x) \quad (\underset{1 \leqq \nu \leqq n}{\mathrm{Max}}\, \omega_\nu(x)) \qquad \text{for} \quad x \in U$$

and

$$(0.2) \qquad \sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} \omega_\nu \leqq f(x, \omega_\nu, \underset{x}{\partial} \omega_\nu) \; (\geqq f(x, \omega_\nu, \underset{x}{\partial} \omega_\nu)).$$

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1) We say that a partial differential equation is principally linear, if it is linear in the terms of the highest derivatives with coeficients containing only independent variables.

2) D is a connected open set in the m-dimensional Euclidean space.

3) $\bar{D}$ means the closure of D, and $\dot{D}$ the boundary of D.

The purpose of this note is to give an existence theorem for the solution of the boundary value problem of the first kind regarding the equation (0), under adequate supplementary conditions, in such a way that the solution $u(x)$ is limited by the inequalities

$$\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x),$$

where $\bar{\omega}(x)$ and $\underline{\omega}(x)$ are quasi-supersolution and quasi-subsolution of (0) respectively. The main result of this note is given in § 6.

The argument of this note is based on the work of J. SCHAUDER: *Über lineare elliptische Differentialgleichungen zweiter Ordung.*[4] We define the distance of two points x and x' by $|x-x'| = (\sum_{i=1}^m (x_i - x_i')^2)^{1/2}$. We also define $|\underset{x}{\partial} f|$ and $|\underset{x}{\partial}^2 f|$ by

$$|\underset{x}{\partial} f| = (\textstyle\sum_{i=1}^m (\underset{i}{\partial} f)^2)^{1/2}, \quad |\underset{x}{\partial}^2 f| = (\textstyle\sum_{i,j=1}^m (\underset{ij}{\partial}^2 f)^2)^{1/2}.$$

A function $f(x)$ is said to be *H_α-continuous* $(0 < \alpha \leqq 1)$ on A, if there exists a constant C such that

$$|f(x) - f(x')| \leqq C|x-x'|^\alpha \qquad \text{for all} \quad x, x' \in A.$$

Then we define $H_A^\alpha(f)$ (the Hölder constant of f on A) as the least value of such C. We also use the notation

$$\|f\|_A^\alpha = \operatorname*{Max}_{x \in A} |f(x)| + H_A^\alpha(f) \tag{0.3}$$

and, if $f \in C^2(A)$,

$$\|f\|_A^{\alpha,2} = \|f\|_A^\alpha + \|\underset{x}{\partial} f\|_A^\alpha + \|\underset{x}{\partial}^2 f\|_A^\alpha. \tag{0.4}$$

Schauder proved the following theorems:

Theorem A. *Let D be a bounded domain, and let $a_{ij}(x)$ be $H_{\alpha+\varepsilon}$-continuous in D and subjected to the condition*

$$\det(a_{ij}) = 1 \quad \textit{and} \quad \|a_{ij}(x)\|_D^{\alpha+\varepsilon} \leqq \Lambda \qquad (0 < \alpha < 1,\ \varepsilon > 0). \tag{0.5}$$

Then there exists a constant C_Λ depending only on α, ε and Λ such that, for any compact set K in D and any solution $u(x)$ of

$$\textstyle\sum_{i,j=1}^m a_{ij}(x) \underset{ij}{\partial}^2 u = f(x) \tag{0.6}$$

such that $\|u\|_D^{\alpha,2} < +\infty$, holds the inequality

$$\|u(x)\|_K^{\alpha,2} \leqq C_\Lambda \delta^{-4}(\|f\|_D^\alpha + \operatorname*{Max}_D |u(x)|),$$

where $\delta = \operatorname{dist}(K, \dot{D})$.

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4) Math. Zeit. 38 (1938), 257-282.

Theorem B. *Let D be a bounded domain whose boundary $\dot{D}$ is of type Bh.*[5] *Let $a_{ij}(x)$ satisfy* (0.5) *and let $\beta(x)$ be a function of* $C^2[\bar{D}]$ *such that* $\|\beta\|_D^{\alpha,2} < +\infty$. *Then there exists a solution $u(x)$ of* (0.6) *in D with the boundary value $\beta(x)$ such that*

$$\|u\|_D^{\alpha,2} \leqq C(\|f\|_D^{\alpha} + \|\beta\|_D^{\alpha,2}),$$

where C is a constant depending only on D, α, ε and Λ.

REMARK. We can easily prove that there exists a constant Λ depending only on A and L such that (0.5) holds, if $a_{ij}(x)$ satisfies

$$A^{-1} \leqq \textstyle\sum_{i,j=1}^{m} a_{ij}(x)\,\xi_i\xi_j \leqq A \qquad \text{for } \textstyle\sum_{i=1}^{m}\xi_i^2 = 1 \ (A \leqq 1)$$

and

$$\{\textstyle\sum_{i,j=1}^{m}(a_{ij}(x') - a_{ij}(x))^2\}^{1/2} \leqq L\,|x'-x|^{\alpha+\varepsilon},$$

where A and L are positive constants.

§ 1 Limitation of $u(x)$

1. **Theorem 1.** *Let $\omega(x)$ be a quasi-supersolution (-subsolution) of the equation*

$$\Phi[u] \equiv \textstyle\sum_{i,j=1}^{m} a_{ij}(x)\,\underset{x}{\partial}^2 u - F(x, \underset{x}{\partial} u) = 0 \tag{1.1}$$

in the domain D, and let $v(x)$ be a function of $C^2[D]$ *with the following properties:*

$$\Phi[v] > 0 \ (<0) \quad \textit{for } x \textit{ such that } v(x) > \omega(x) \ (<\omega(x)) \tag{1.2}$$

and

$$\lim_{x\to\dot{x}} \{v(x) - \omega(x)\} \leqq 0 \ (\geqq 0) \qquad \textit{for} \quad \dot{x} \in \dot{D}. \tag{1.3}$$

Then

$$v(x) \leqq \omega(x) \ (\geqq \omega(x)) \qquad \textit{for} \quad x \in D.$$

Proof. If the conclusion was not true, there exist by (1.3) a positive constant α and a point $x_0 \in D$ such that

$$v(x_0) = \omega(x_0) + \alpha \quad \text{and} \quad v(x) \leqq \omega(x) + \alpha \quad \text{in } D.\text{[6]} \tag{1.4}$$

Then there exist a neighborhood U and a function $\omega_\nu(x) \in C^2[D]$ such that

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5) A l-dimentional manifold is said of type Bh, if it is locally representable in the form $x_i = \varphi_i(s_1, \dots, s_l)$ in such way that $\operatorname{Rank} \underset{s}{\partial}(\varphi) = l$ and $\underset{s}{\partial}^2\varphi$ is H_α-continuous $(0 < \alpha < 1)$.

6) $\alpha = \inf\{\lambda : \omega(x) + \lambda > v(x) \text{ for all } x \in D\}$.

(1.5) $$\omega_\nu(x_0)=\omega(x_0)\,,\quad \omega_\nu(x)\geqq\omega(x)\quad \text{in } U$$

and

(1.6) $$\Phi[\omega_\nu]\leqq 0 \qquad \text{in } U.$$

Thus,

(1.7) $$\omega_\nu(x_0)<v(x_0)$$

and, as $\omega_\nu(x)-v(x)$ is minimum for $x=x_0$ by (1.4) and (1.5), we have

(1.8) $$\underset{x}{\partial}\omega_\nu(x_0)=\underset{x}{\partial}v(x_0)$$

and

(1.9) $$\sum_{i,j=1}^m a_{ij}(x_0)\underset{ij}{\partial^2}\omega_\nu(x_0)\geqq\sum_{i,j=1}^m a_{ij}(x_0)\underset{ij}{\partial^2}v(x_0)\,.$$ [7]

Hence, from (1.8) and (1.9)

(1.10) $$\Phi[v]_{x=x_0}\leqq\Phi[\omega_\nu]_{x=x_0}\,.$$

On the other hand, from (1.7), (1.2) and (1.6) we get

$$\Phi[v]_{x=x_0}>0\geqq\Phi[\omega_\nu]_{x=x_0}\,,$$

which contradicts (1.10), q. e. d.

2. We say that a domain D has the *property* $((\sigma))$, when there exists a constant $\sigma>0$ with the following property: To any point p of $\dot{D}$ there corresponds a closed sphere S_p with radius σ such that $\bar{D}\frown S_p=(p)$.

Lemma 1. *Let D be a bounded domain with the property $((\sigma))$, and let d be the diameter of D. Let $a_{ij}(x)$ be subjected to the condition*

(2.1) $$A^{-1}\leqq\sum_{i,j=1}^m a_{ij}(x)\,\xi_i\xi_j\leqq A \qquad \textit{for } \sum_{i=1}^m \xi_i^2=1\,,$$

where A is a constant $\geqq 1$. Then there exists a constant $C_{A,\sigma,d}$ depending only on m, A, σ and d such that for the solution $u(x)$ of

(2.2) $$\sum_{i,j=1}^m a_{ij}(x)\underset{ij}{\partial^2}u=f(x)$$

with the boundary value $u=0$ $(x\in\dot{D})$, where $f(x)$ is bounded on D, holds the inequality

(2.3) $$|u(x)|\leqq C_{A,\sigma,d}\operatorname{dist}(x,\dot{D})\sup_D|f(x)|\,.$$

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7) By a linear transformation of coordinates we can bring the matrix $(a_{ij}(x_0))$ into the diagonal form $(\lambda_i\delta_{ij})$, where $\lambda_i>0$. Then $\sum\lambda_i\underset{ii}{\partial^2}\omega(x_0)\geqq\sum\lambda_i\underset{ii}{\partial^2}u(x_0)$, which is epuivalent to (1.9).

Proof. Let x_0 be any point of D and let p be a point of $\dot{D}$ such that $\operatorname{dist}(x_0, \dot{D}) = |x_0 - p|$. Let S_p be the closed sphere with radius σ such that $\bar{D} \frown S_p = (p)$, and let c be the center of S_p. If we put $\omega(x) = \varphi(r)$, where $r = |x-c|$, then

$$\sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} \omega = \alpha(x) \varphi'' + r^{-1}\{\sum_{i=1}^{m} a_{ii} - \alpha(x)\} \varphi', \tag{2.4}$$

where $$\alpha(x) = r^{-2} \sum_{i,j=1}^{m} a_{ij}(x)(x_i - c_i)(x_j - c_j).$$

Thus, if we define $\varphi(r)$ by

$$\varphi(r) = (mA)^{-1} F \int_{\sigma}^{r} \{(d+\sigma)^{mA^2} r^{-mA^2+1} - r\}\, dr, \tag{2.5}$$

where F is a constant $> \sup|f(x)|$, and as $\varphi'(r) > 0$, $\varphi''(r) < 0$ for $\sigma \leqq r < \delta + d$ and

$$\sum_{i=1}^{m} a_{ii}(x) \leqq mA, \qquad \alpha(x) \geqq A^{-1} \qquad (\text{by } (2.1)),$$

we have

$$\sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} \omega + F \leqq 0 \qquad \text{in } D \tag{2.6}$$

and $\omega(x) \geqq 0$ for $x \in \dot{D}$.

On the other hand, as $F > |f(x)|$ in D, we get from (2.2)

$$\sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} u + F > 0 \qquad \text{in } D \tag{2.7}$$

and $u(x) = 0 \leqq \omega(x)$ for $x \in \dot{D}$. Thus, by Theorem 1,

$$u(x) \leqq \omega(x) = \varphi(r) \qquad \text{in } D.$$

Then, as $\omega(p) = \varphi(\sigma) = 0$ and $\varphi'' < 0$,

$$u(x_0) \leqq \varphi'(\sigma)\,|x_0 - p|,$$

or from (2.5)

$$u(x_0) \leqq C_{A,\sigma,d} \operatorname{dist}(x_0, \dot{D})\, F,$$

where $$C_{A,\sigma,d} = (mA)^{-1}\{(d+\sigma)^{mA^2} \sigma^{-mA^2+1} - \sigma\}.$$

Similarly we obtain

$$u(x_0) \geqq -C_{A,\sigma,d} \operatorname{dist}(x_0, \dot{D})\, F.$$

Letting F tend to $\sup|f(x)|$ we get (2.3).

§ 2 Estimation of $\underset{x}{\partial} u$

3. Theorem 2. *Let D be a bounded domain, whose diameter is d. Let $a_{ij}(x)$ be subjected to the conditions* (2.1) *and*

(3. 1) $\quad (\sum_{i,j=1}^m \{a_{ij}(x')-a_{ij}(x)\}^2)^{1/2} \leqq L|x'-x| \qquad$ *for any* $x, x' \in D$,

Γ *and* $f(x, u, p)$ $(p=(p_1, \cdots, p_m))$ *satisfy the inequality*

(3. 2) $$|f(x, u, p)| \leqq B|p|^2+\Gamma$$

for $x \in D$, $\underline{\omega}(x) \leqq u \leqq \bar{\omega}(x)$ *and* $|p| < +\infty$. *Let* $u(x)$ *be any solution of* (0) *such that*

(3. 3) $$|u(x)| \leqq M \qquad \text{in } D, \text{ where } 16ABM < 1.$$

Then there exist constants $C^{(1)}$ *and* $C^{(2)}$, *depending only on* m, A, L, B, M Γ *and* d, *such that*

$$|\underset{x}{\partial}u(x)| \leqq C^{(1)}/\rho(x)^{-1} \operatorname*{Max}_{|x'-x| \leqq \rho(x)} \{|u(x')|\} + C^{(2)},$$

where $\rho(x) = \operatorname{dist}(x, \dot{D})$.

Proof. Let a be any point of D, and let $\sum_\kappa$ be a closed sphere defined by

$$\textstyle\sum_\kappa = \{x;\ |x-a| \leqq \kappa \operatorname{dist}(a, \dot{D})\} \qquad (0<\kappa<1).$$

We put also

(3. 4) $$\mu_\kappa = \operatorname*{Max}_{x \in \sum_\kappa} \{|\underset{x}{\partial}u| \rho_\kappa(x)\},$$

where $\rho_\kappa(x) = \operatorname{dist}(x, \dot{\sum_\kappa})$. Then there exists a point $x_0 \in \sum_\kappa$ such that

(3. 5) $$|\underset{x}{\partial}u(x_0)| \rho_\kappa(x_0) = \mu_\kappa \qquad (x_0 \in \textstyle\sum_\kappa).$$

Now let $Tx = x'$ be a linear transformation of coordinates such that

$$\textstyle\sum_{i,j=1}^m a_{ij}(x_0) \underset{ij}{\partial^2}u = \sum_{i=1}^m \underset{ii}{\partial^2}u',^{8)}$$

where $\qquad u'(x') = u(x)$ and $f(x, u, \underset{x}{\partial}u) = f'(x', u', \underset{x'}{\partial}u')$.

Then we have for (0)

(3. 6) $$\Delta u' = \textstyle\sum_{i,j=1}^m (\delta_{ij} - a'_{ij}(x')) \underset{ij}{\partial^2}u' + f'(x', u', \underset{x'}{\partial}u').$$

Let S_λ be a closed sphere in $T(D) = D'$ with the center $x_0' = T(x_0)$ and the radius $\lambda\rho_\kappa(x_0)$, where λ is a constant such that $0 < \lambda < A^{-1/2}/2$ and $S_\lambda \subset T(\sum_\kappa)$. Let $G(x', \xi)$ be the Green's function of the equation $\Delta u' = 0$ with respect to the domain S_λ so that from (3. 6)

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8) $\underset{i}{\partial}u'$ means $\partial_{x_i'}u'$.

$$u' = -\omega_m^{-1}\int_{S_\lambda} G(x', \xi)\{\textstyle\sum_{i,j=1}^m (\delta_{ij}-a'_{ij}(\xi))\,\underset{ij}{\partial^2} u'(\xi)\}\, d^m\xi^{9)}$$
$$-\omega_m^{-1}\int_{S_\lambda} G(x', \xi) f'(\xi, u'(\xi), \underset{\xi}{\partial} u'(\xi))\, d^m\xi + h(x'),$$

where $h(x')$ is the harmonic function which takes the same value as $u'(x')$ for $x'\in\dot{S}_\lambda$. Then

(3.7) $$|\underset{x'}{\partial} u'(x_0')| \leqq (\mathrm{I})+(\mathrm{II})+(\mathrm{III}),$$

where $$(\mathrm{I}) = |\omega_m^{-1}\int_{S_\lambda} \underset{x'}{\partial} G(x_0', \xi) f' d^m\xi|,$$

$$(\mathrm{II}) = |\omega_m^{-1}\int_{S_\lambda} \underset{x'}{\partial} G(x_0', \xi) \textstyle\sum_{ij} (\delta_{ij}-a'_{ij}(\xi))\, \underset{ij}{\partial^2} u'(\xi)\, d^m\xi|,$$

$$(\mathrm{III}) = |\underset{x'}{\partial} h(x_0')|.$$

Since by T the distance will be changed by the ratio between $A^{-1/2}$ and $A^{1/2}$, we have

(3.8) $$(\textstyle\sum_{i,j=1}^m \{a'_{ij}(\xi)-\delta_{ij}\}^2)^{1/2} \leqq A^{3/2}L|\xi-x_0'|, \quad (\sum_{ij}(\underset{\xi}{\partial} a'_{ij})^2)^{1/2} \leqq A^{3/2}L.$$

As $|\underset{x}{\partial} u(x)| \leqq (1-\lambda A^{1/2})^{-1}\rho_\kappa(x_0)^{-1}\mu_\kappa$ in $T^{-1}S_\lambda(\subset \sum_\kappa)$, we have, taking λ so small that $(1-\lambda A^{1/2})^{-2} < 2$,

(3.9) $$|\underset{\xi}{\partial} u'(\xi)| \leqq \sqrt{2}\, A^{1/2}\rho_\kappa(x_0)^{-1}\mu_\kappa \qquad \text{for} \quad \xi\in S_\lambda$$

and from (3.2)

(3.10) $$|f'(\xi, u'(\xi), \underset{\xi}{\partial} u'(\xi))| = |f(x, u, \underset{x}{\partial} u)| \leqq 2B\rho_\kappa(x_0)^{-2}\mu_\kappa^2 + \Gamma.$$

Then regarding (3.8), (3.9), (3.10), $|\underset{x'}{\partial} G(x_0', \xi)| \leqq 2|\xi - x_0'|^{-m+1}$ and $|\underset{x'\xi}{\partial^2} G(x_0', \xi)| \leqq (m+2)|\xi-x_0'|^{-m}$, we get

(3.11) $$(\mathrm{I}) \quad \leqq 4\lambda\rho_\kappa(x_0)^{-1}B\mu_\kappa^2 + 2\lambda\rho_\kappa(x_0)\Gamma,$$

$$(\mathrm{II}) \quad \leqq |\omega_m^{-1}\int_{\dot{S}_\lambda} \underset{x'}{\partial} G(x_0', \xi) \textstyle\sum_{i,j} (\delta_{ij}-a'_{ij}(\xi))\, \underset{i}{\partial} u'(\xi) \cos(\xi_i, n)\, d\sigma|^{10)}$$
$$+ |\omega_m^{-1}\int_{S_\lambda} \underset{x'}{\partial} G(x_0', \xi) \textstyle\sum_{i,j} \underset{\xi_j}{\partial} a'_{ij}(\xi)\, \underset{i}{\partial} u'(\xi) d^m\xi|$$
$$+ |\omega_m^{-1}\int_{S_\lambda} \textstyle\sum_{i,j} \underset{x'\xi_j}{\partial^2} G(x_0', \xi)\, (\delta_{ij}-a'_{ij}(\xi))\, \underset{i}{\partial} u'(\xi)\, d^m\xi|,$$

or

(3.12) $$(\mathrm{II}) \quad \leqq (m+6)\sqrt{2}\, AL\lambda\mu_\kappa$$

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9) ω_m means the surface measure of the m-dimensional unit sphere, and $d^m\xi = d\xi_1\cdots d\xi_m$.

10) $d\sigma$ means the infinitesimal surface element of S_λ and n is the normal of S_λ.

and (III) $\leqq \lambda^{-1}\rho_\kappa(x_0)^{-1}\operatorname{Max}\{|u'(x')|;\ |x'-x_0'|\leqq \lambda\rho_\kappa(x_0)\}$,

hence

$$\text{(3.13)} \qquad \text{(III)} \quad \leqq \lambda^{-1}\rho_\kappa(x_0)^{-1}\sup_{|x-a|\leqq\rho(a)}|u(x)|.$$

As
$$|\underset{x'}{\partial}u'(x)|\geqq A^{-1/2}|\underset{x}{\partial}u(x_0)|=A^{-1/2}\rho_\kappa(x_0)^{-1}\mu_\kappa$$

and $\rho_\kappa(x_0)<2\rho(a)\leqq d$, we get from (3.7), (3.11), (3.12) and (3.13),

$$\text{(3.14)} \qquad \lambda C_0\mu_\kappa^2-(1-\lambda C_1)\,\mu_\kappa+\lambda^{-1}C_2\geqq 0\,,$$

where
$$C_0=4A^{1/2}B,\quad C_1=\sqrt{2}\,(m+6)\,A^{5/2}Ld$$

and
$$C_2=A^{1/2}\sup_{|x-a|\leqq\rho(a)}|u(x)|+8\lambda\rho(a)^2A^{1/2}\Gamma\,.$$

Since $C_0C_2\leqq 4ABM+8\lambda d^2AB\Gamma$, by (3.3) we can take $\lambda>0$, depending only on m, A, L, B, Γ and d, so small that

$$\text{(3.15)} \qquad \lambda C_1<1/2 \quad\text{and}\quad (1-\lambda C_1)^2>4(\lambda C_0)\,(\lambda^{-1}C_2)\,.$$

Let R_1 and $R_2(R_1<R_2)$ be the distinct real roots of the equation in X

$$\text{(3.16)} \qquad \lambda C_0X^2-(1-\lambda C_1)\,X+\lambda^{-1}C_2=0\,.$$

Then we have from (3.14)

$$\mu_\kappa\leqq R_1 \quad\text{or}\quad \mu_\kappa\geqq R_2\ (R_1<R_2)\,.$$

But we can easily see from (3.4) that μ_κ depends on κ continuously for $0<\kappa<1$ and $\lim_{\kappa\to 0}\mu_\kappa=0$. Then we have only $\mu_\kappa\leqq R_1$. And, letting κ tend to 1, by the definition of μ_κ

$$\text{(3.17)} \qquad |\underset{x}{\partial}u(a)|\leqq R_1\rho(a)^{-1}.$$

As R_1 is the smaller root of (3.16) and $\lambda C_1<1/2$,

$$R_1<\frac{4C_2}{2\lambda(1-\lambda C_1)}<4\lambda^{-1}C_2\,.$$

Thus from (3.17)

$$|\underset{x}{\partial}u(a)|\leqq C^{(1)}\rho(a)^{-1}\sup_{|x-a|\leqq\rho(a)}|u(x)|+C^{(2)},$$

where $C^{(1)}=4\lambda^{-1}A^{1/2}$ and $C^{(2)}=16dA^{1/2}\Gamma$ depend only on m, A, L, B, M, Γ and d, q. e. d,

Corollary. *If we replace the condition* (3.2) *in Theorem* 2 *by*

$$\text{(3.19)} \qquad |f(x,u,p)|\leqq\Gamma\,,$$

and omit (3.3), *then there exists a constant* $C_{A,L,d}$ *depending only on* m, A, L *and* d, *such that*

$$|\underset{x}{\partial} u(x)| \leqq C_{A,L,d}\, \rho(x)^{-1} \sup_{|x'-x| \leqq \rho(x)} |u(x')| + 8A^{1/2}\rho(x)\, \Gamma .$$

where $\rho(x) = \mathrm{dist}\,(x, \dot{D})$.

Proof. We have instead of (3.14)

$$(1-\lambda C_1)\, \mu_\kappa \leqq \lambda^{-1} C_2 .$$

Then, putting $\lambda = C_1/2$, we get

$$\mu_\kappa \leqq 2C_1^{-1} A^{1/2} \sup_{|x-a| \leqq \rho(a)} |u(x)| + 8A^{1/2}\rho(a)^2\, \Gamma .$$

Thus we have from (3.4), letting κ tend to 1,

$$|\underset{x}{\partial} u(a)| \leqq C_{A,L,d}\, \rho(a)^{-1} \sup_{|x-a| \leqq \rho(a)} |u(x)| + 8A^{1/2}\rho(a)\, \Gamma ,$$

where $C_{A,L,d} = \sqrt{2}\,(m+6)\, A^3 L d$, q. e. d.

§ 3 Existence theorem for bounded $f(x, u, p)$

4. We say that $f(x, u, p)$ is *H_α-continuous in the finite part* of a $2m+1$-dimensional domain D^*, when there exists a constant $H_{M,N}$ depending on arbitrary positive numbers M and N such that

$$(4.1) \quad |f(x', u', p') - f(x, u, p)| \leqq H_{M,N} \{ |x'-x|^\alpha + |u'-u|^\alpha + |p'-p|^\alpha \}$$

for any (x, u, p), (x', u', p') with the restriction $|u|$, $|u'| \leqq M$ and $|p|$, $|p'| \leqq N$.

Theorem 3. *Let* D *be a bounded domain with the diameter* d, *the boundary* $\dot{D}$ *being a hypersurface of type Bh, and let* $a_{ij}(x)$ *be* H_1-*continuous in* $\bar{D}$. *Let* $f(x, u, p)$ *be* H_α-*continuous* $(0 < \alpha < 1)$ *in the finite part of*

$$D^* = \{(x, u, p)\,;\ x \in \bar{D},\ |u| < +\infty,\ |p| < +\infty\}$$

and bounded:

$$(4.2) \qquad |f(x, u, p)| \leqq \Gamma \qquad \text{in } D^*.$$

Then there exists a solution $u(x)$ *of* (0) *with the boundary value* $u = 0$ $(x \in \dot{D})$ *such that* $\| u(x) \|_D^{\alpha, 2} < +\infty$.

Proof. For fixed positive constants N and Λ, let $\mathfrak{F}_{N,\Lambda}$ be the set of functions $v(x) \in C^1[\bar{D}]$ with the following properties:

(4.3) $$v(x)=0 \qquad \text{for } x\in\dot{D},$$

(4.4) $$|\underset{x}{\partial}v(x)|\leqq N \qquad \text{in } D,$$

and

(4.5) $$|\underset{x}{\partial}v(x')-\underset{x}{\partial}v(x)|\leqq \Lambda|x'-x| \qquad \text{for } x, x'\in D.$$

Then

(4.6) $$|v(x)|\leqq Nd \qquad \text{for all } v(x)\in\mathfrak{F}_{N,\Lambda}.$$

$\mathfrak{F}_{N,\Lambda}$ is a compact convex set in $C^1[\bar{D}]$, where $C^1[\bar{D}]$ is a Banach space with the norm

$$||v|| = \underset{\bar{D}}{\mathrm{Max}}|v(x)| + \underset{\bar{D}}{\mathrm{Max}}|\underset{x}{\partial}v(x)|.$$

For convenience we write $f_{[v]}(x)=f(x, v(x), \underset{x}{\partial}v(x))$, then $f_{[v]}$ is H_α-continuous in D for $v\in\mathfrak{F}_{N\Lambda}$. Because, there exists a constant $\kappa\geqq 1$ such that any pair of points x and x' in D can be joined by a curve in D with length $\leqq\kappa|x-x'|$, hence from (4.4)

$$|v(x')-v(x)|\leqq\kappa N|x'-x| \qquad \text{for all } v\in\mathfrak{F}_{N,\Lambda}.$$

Thus by (4.1), (4.5) and (4.6)

(4.7) $$H^\alpha_D(f_{[v]})\leqq H_{Nd,N}(1+(\kappa N)^\alpha+\Lambda^\alpha).$$

Now by Schauder's Theorem B, for any $v\in\mathfrak{F}_{N\Lambda}$, there exists the solution $u(x)$ of

(4.8) $$\sum_{i,j=1}^m a_{ij}(x)\,\underset{ij}{\partial^2}u = f_{[v]}(x)$$

with the boundary value $u=0$ $(x\in\dot{D})$, which satisfies

(4.9) $$|\underset{x}{\partial^2}u|+H^\alpha_D(\underset{x}{\partial^2}u)\leqq C^{(1)}\{\underset{\bar{D}}{\mathrm{Max}}|f_{[v]}|+H^\alpha_D(f_{[v]})\},$$

where $C^{(1)}$ depend only on D, A and L, as there exist constants A and L such that (2.1) and (3.1) hold.

Since D has the property $((\sigma))$ for certain $\sigma>0$, we have by (4.2) and Lemma 1,

(4.10) $$|u(x)|\leqq C_{A,\sigma,d}\,\rho(x)\Gamma, \quad \text{where} \quad \rho(x)=\mathrm{dist}\,(x,\dot{D}).$$

Then from Corollary in §3, by (4.10),

(4.11) $$|\underset{x}{\partial}u(x)|\leqq C^*\Gamma,$$

where C^* is a constant depending only on m, A, L, σ and d. Now we put

(4.12) $$N = C^*\Gamma = N_0.$$

Then from (4.7) and (4.9), for any $v \in \mathfrak{F}_{N_0, \Lambda}$,

$$|\underset{x}{\partial^2} u| \leqq C^{(1)}\{\Gamma + H_0(1 + (\kappa N_0)^\alpha + \Lambda^\alpha)\} \qquad (H_0 = H_{N_0 d, N_0}),$$

hence

(4.13) $$|\underset{x}{\partial} u(x') - \underset{x}{\partial} u(x)| \leqq \kappa C^{(1)}\{\Gamma + H_0(1 + (\kappa N_0)^\alpha + \Lambda^\alpha)\}\, |x' - x|.$$

Since $0 < \alpha < 1$, we can choose Λ_0 so large that

$$\kappa C^{(1)}\{\Gamma + H_0(1 + (\kappa N_0)^\alpha + \Lambda_0^\alpha)\} \leqq \Lambda_0.$$

Then by (4.13)

(4.14) $$|\underset{x}{\partial} u(x') - \underset{x}{\partial} u(x)| \leqq \Lambda_0 |x' - x|.$$

If we denote by Φ the transformation of $v \in \mathfrak{F}_{N_0, \Lambda_0}$ into the solution u of (4.8) with the boundary value $u = 0$ $(x \in \dot{D})$:

$$u = \Phi[v],$$

such that $\| u \|_D^{\alpha, 2} < +\infty$, then (4.11), (4.12) and (4.14) show that

(4.15) $$\Phi(\mathfrak{F}_{N_0, \Lambda_0}) \subset \mathfrak{F}_{N_0, \Lambda_0}.$$

The mapping Φ of $\mathfrak{F}_{N_0, \Lambda_0}$ into itself is continuous in $C^1[\bar{D}]$. Because, if $v, v' \in \mathfrak{F}_{N_0, \Lambda_0}$

(4.16) $$|f_{[v]} - f_{[v']}| \leqq H_0(|v' - v|^\alpha + |\underset{x}{\partial} v' - \underset{x}{\partial} v|^\alpha) \leqq 2H_0(\| v' - v \|)^\alpha.$$

And for $u = \Phi[v]$ and $u' = \Phi[v']$

$$\textstyle\sum_{i,j=1}^m a_{ij}(x)\, \underset{ij}{\partial^2}(u - u') = f_{[v]}(x) - f_{[v']}(x) \qquad \text{in } D$$

and $u(x) - u'(x) = 0$ for $x \in \dot{D}$. Thus by Lemma 1 and Corollary in § 2, replacing Γ by $2H_0(\| v' - v \|)^\alpha$ in (4.10) and (4.11), we get

$$|u(x) - u'(x)| \leqq 2C_{\Lambda, \sigma, d}\, dH_0(\| v - v' \|)^\alpha$$

and $$|\underset{x}{\partial} u - \underset{x}{\partial} u'| \leqq 2C^* H_0(\| v - v' \|)^\alpha.$$

These show the continuity of Φ. Then from (4.15), by the fixed point theorem in functional space,[11] there exists a $u_0 \in \mathfrak{F}_{N_0, \Lambda_0}$ such that

$$\Phi[u_0] = u_0.$$

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11) Tychonoff: *Ueber einen Fixpuktsatz*, Math. Ann. 111.

Then $u_0(x)$ is a solution of (0) with the boundary value $u=0$, q. e. d.

§ 4 Existence theorem for regular boundary condition

5. Lemma 2. *Let D be a bounded domain with the property* $((\sigma))$ *and the diameter d. Let $a_{ij}(x)$ be subjected to the conditions* (2.1) *and* (3.1), *and $f(x, u, p)$ to the condition* (3.2) *Let $u(x)$ be a solution of* (0) *with the boundary value $u=0$ $(x \in \dot{D})$ and satisfy* (3.3). *Then there exists a constant $C^\#$ depending only on m, A, L, B, Γ, M, σ and d, such that*

$$|\underset{x}{\partial} u(x)| \leqq C^\# .$$

Proof. First we shall prove the existence of a constant C^* depending only on m, A, L, B, Γ, M and σ such that for the solution of (0), which vanishes on D and satisfy (3.3), holds the inequality

$$|u(x)| \leqq C^* \operatorname{dist}(x, \dot{D}) . \tag{5.1}$$

Let x_0 be any point of D and let p be a point of $\dot{D}$ such that $|x_0 - p| = \operatorname{dist}(x_0, \dot{D})$. Let S_p be a closed sphere with the radius σ such that $S_p \frown \bar{D} = (p)$, and c be the center of S_p. Then the function

$$\omega(x) = M \log [(r-\sigma')/(\sigma-\sigma')] , \tag{5.2}$$

where $\qquad r = |x-c|, \quad \sigma' = (1-\varepsilon^2)\sigma ,$

satisfies the inequality

$$\Phi[\omega] \equiv \textstyle\sum_{i,j=1}^m a_{ij}(x) \underset{ij}{\partial^2} \omega + B|\underset{x}{\partial} \omega|^2 + \Gamma' \leqq 0 \tag{5.3}$$

for $\sigma \leqq |x-c| \leqq \sigma(1+\varepsilon)$, where $\Gamma' > \Gamma > 0$ (for example $\Gamma' = \Gamma + 1$) and

$$\varepsilon = \operatorname{Min} \{(2mA^2)^{-1}, \ \sigma^{-1}(M/8A\Gamma')^{1/2}\} . \tag{5.4}$$

In fact, as $(r-\sigma')r^{-1} \leqq \varepsilon < 1/4$ for $\sigma \leqq r \leqq \sigma\ (1+\varepsilon)$, $2ABM < 1/2$, $\sum_{i=1}^m a_{ii} \leqq mA$ and

$$\begin{aligned} r^{-2} \textstyle\sum_{i,j=1}^m a_{ij}(x)\,(x_i-c_i)\,(x_j-c_j) &\equiv \alpha(x) \leqq A^{-1}, \\ \Phi[\omega] = M(r-\sigma')^{-2}\{(r-\sigma')\,r^{-1} \textstyle\sum_i a_{ii} &- (1+(r-\sigma')\,r^{-1})\,\alpha(x)\} \\ &+ BM^2(r-\sigma')^{-2} + \Gamma' \\ &\leqq M(r-\sigma')^{-2}(\varepsilon mA - A^{-1} + BM) + \Gamma' \\ &\leqq \Gamma' - M(1-2ABM)/2A(r-\sigma')^2 \\ &< \Gamma' - M/8A\varepsilon^2\sigma^2 \leqq 0 \qquad \text{(by (5.4))} . \end{aligned}$$

We have also, as $\log(1+\varepsilon^{-1}) > \log 5 > 1$,

$$(5.5)\qquad \omega(x) > M \qquad \text{for } |x-c| = (1+\varepsilon)\sigma .$$

Let D_ε be the part of D defined by

$$D_\varepsilon = \{x\,;\ x \in D,\ |x-c| < (1+\varepsilon)\sigma\}\ ,$$

then $\omega(x)$ is a quasi-supersolution of $\Phi = 0$ in D_ε. But $u(x)$ satisfies the inequalities

$$\Phi[u] < 0 \quad \text{in } D_\varepsilon, \qquad u(x) \leqq \omega(x) \quad \text{for} \quad x \in \dot{D}_\varepsilon .$$

Thus, by Theorem 1,

$$u(x) \leqq \omega(x) = M \log\left[(|x-c| - \sigma')/(\sigma - \sigma')\right] \qquad \text{for } x \in D_\varepsilon .$$

We get a similar inequality, if we replace $u(x)$ by $-u(x)$. Then

$$(5.6)\qquad |u(x)| \leqq M \log\left[(|x-c| - \sigma')/(\sigma-\sigma')\right] \qquad \text{for } x \in D_\varepsilon .$$

As $\log[(r-\sigma')/(\sigma-\sigma')] \leqq (\sigma-\sigma')^{-1}(r-\sigma) = (\sigma\varepsilon^2)^{-1}(r-\sigma)$ for $r \geqq \sigma$ and $r - \sigma = \operatorname{dist}(x_0, \dot{D})$ for $x = x_0$, we get from (5.6)

$$|u(x_0)| \leqq (\sigma\varepsilon^2)^{-1} M \operatorname{dist}(x_0,\ D) \qquad \text{for } x_0 \in D_\varepsilon .$$

But this inequality holds also for $x_0 \in D - D_\varepsilon$. (5.1) is thus proved.

Now by Theorem 2 we have

$$(5.7)\qquad |\underset{x}{\partial} u(x)| \leqq C^{(1)} \rho(x)^{-1} \operatorname*{Max}_{|x'-x| \leqq \rho(x)} |u(x')| + C^{(2)},$$

where $\rho(x) = \operatorname{dist}(x, \dot{D})$, and $C^{(1)}$ and $C^{(2)}$ depend only on m, A, L, B, Γ, M and d. And, as by (5.1)

$$|u(x')| \leqq 2C^* \rho(x) \qquad \text{for } |x'-x| \leqq \rho(x)\,,$$

we get from (5.7)

$$|\underset{x}{\partial} u(x)| \leqq 2C^* C^{(1)} + C^{(2)} = C^\#\,, \quad \text{q.e.d.}$$

6. Theorem 4. *Let D be a bounded domain with the boundary of type Bh. Let $a_{ij}(x)$ be H_1-continuous in $\bar{D}$, and let $f(x, u, p)$ be H_α-continuous* $(0 < \alpha < 1)$ *in the finite part of*

$$D^* = \{(x, u, p)\,;\ x \in \bar{D},\quad \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x),\ |p| < +\infty\}\ ,$$

where $\bar{\omega}(x)$ and $\underline{\omega}(x)$ are quasi-supersolution and quasi-subsolution of (0) *respectively such that*

$$|\underline{\omega}(x)| \leqq M, \quad |\bar{\omega}(x)| \leqq M, \quad \underline{\omega}(x) < \bar{\omega}(x)\,.$$

And there is a finite set $\{U_j\}_{j=1}^n$ *such that* (0.1) *and* (0.2) *hold in each* $U=U_j$ *and* $\bigcup_{j=1}^n U_j \supset \bar{D}$. *Let* $a_{ij}(x)$ *be also subjected to the condition* (2.1) *and* $f(x, u, p)$ *to the condition* (3.2), *where* (3.3) *holds. Then there exists a solution of* (0) *with the boundary value* $\beta(x)$ $(x \in \dot{D})$ *such that*

$$\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x) \quad \textit{and} \quad \| u(x) \|_D^{\alpha, 2} < +\infty ,$$

where $\beta(x)$ *is a given function of* $C^3[\bar{D}]$ *such that*

$$\underline{\omega}(x) < \beta(x) < \bar{\omega}(x) \qquad \textit{in } D .$$

Proof. Without loss of generality we can assume that $\beta(x)=0$. Then we put

$$(6.1) \qquad N_0 = \operatorname{Max} \{\operatorname*{Max}_{U_j} |\partial_x \bar{\omega}_\nu|, \ \operatorname*{Max}_{U_j} |\partial_x \underline{\omega}_\nu|, \ C^\#_{(\Gamma+1)}\} ,$$[12]

where $C^\#_{(\Gamma+1)}$ is the constant given in Lemma 2 but Γ is replaced by $\Gamma+1$. We define $f^*(x, u, p)$ by

$$(6.2) \quad f^*(x, u, p) = \begin{cases} f(x, u, p) & \text{if } |p| \leqq N_0 , \\ f(x, u, N_0 |p|^{-1} p) & \text{if } |p| > N_0 , \end{cases}$$

and then $f^\#(x, u, p)$ by

$$(6.3) \quad f^\#(x, u, p) = \begin{cases} f^*(x, \bar{\omega}(x), p) + \dfrac{u-\bar{\omega}(x)}{1+u-\bar{\omega}(x)} & \text{for } u > \bar{\omega}(x) , \\ f^*(x, u, p) & \text{for } \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x) , \\ f^*(x, \underline{\omega}(x), p) + \dfrac{u-\underline{\omega}(x)}{1+\underline{\omega}(x)-u} & \text{for } u < \underline{\omega}(x) . \end{cases}$$

We can easily prove that $f^\#(x, u, p)$ is bounded and H_α-continuous in

$$D^\# = \{(x, u, p)\,;\ x \in \bar{D}, \ \ |u| < +\infty, \ |p| < +\infty\} .$$

Then by Theorem 3 there exists a solution $u(x)$ of

$$(6.4) \qquad \sum_{i,j=1}^m a_{ij}(x)\, \partial_{ij}^2 u = f^\#(x, u, \partial_x u)$$

vanishing on $\dot{D}$ such that $\| u(x) \|_D^{\alpha, 2} < +\infty$. $u(x)$ must satisfy the inequality

$$(6.5) \qquad \underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x) .$$

In fact, as $f^\#(x, \bar{\omega}_\nu(x), \partial_x \bar{\omega}_\nu(x)) = f(x, \bar{\omega}_\nu(x), \partial_x \bar{\omega}_\nu(x))$ for x and ν such that $\bar{\omega}(x) = \bar{\omega}_\nu(x)$, $\bar{\omega}(x)$ is a quasi-supersolution of the equation

12) We can assume that U_j are bounded and closed.

$$\Phi[u] \equiv \sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} u - f^{\#}(x, \bar{\omega}(x), \underset{x}{\partial} u) = 0 .$$

But, as $f^{\#}(x, \bar{\omega}(x), p) = f^*(x, \bar{\omega}(x), p)$, $u(x)$ satisfies

$$\Phi[u] = \frac{u - \bar{\omega}(x)}{1 + u - \bar{\omega}(x)} > 0$$

for x such that $u(x) > \bar{\omega}(x)$, and $u(x) = 0 \leqq \bar{\omega}(x)$ for $x \in \dot{D}$. Then by Theorem 1 we get

$$u(x) \leqq \bar{\omega}(x) \qquad \text{in } D .$$

Similarly we obtain $u(x) \geqq \underline{\omega}(x)$ in D.

Now $f^{\#}$ satisfies the condition

$$|f^{\#}(x, u, p)| \leqq B|p|^2 + \Gamma + 1 ,$$

and for $u(x)$ holds $|u(x)| \leqq M$, and $16ABM < 1$. Then by Lemma 2 we have

$$|\underset{x}{\partial} u(x)| \leqq C^{\#}_{(\Gamma+1)} \leqq N_0 . \tag{6.6}$$

(6.5) and (6.6) show that $u(x)$ is a solution of (0), q. e. d.

§ 5 Preparation for the general boundary condition.

7. Lemma 3. *Let D be a bounded domain. Let $a_{ij}(x)$ be subjected to the conditions* (2.1) *and* (3.1), *and $f(x, u, p)$ to the conditions* (3.2) *and* (4.1) *in*

$$D^* = \{(x, u, p) ;\ x \in \bar{D},\ \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x),\ |p| < +\infty\} ,$$

where $\underline{\omega}(x)$ and $\bar{\omega}(x)$ are continuous functions on D such that

$$|\underline{\omega}(x)| \leqq M,\ |\bar{\omega}(x)| \leqq M \quad \textit{and } 16\, ABM < 1 .$$

Let $\mathfrak{F}$ be the set of all solutions $u(x)$ of (0) *such that*

$$\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x) \quad \textit{and } \|u\|_D^{\alpha,2} < +\infty .$$

Then, for any closed sphere S in D, there exist constants C_S^{i}, C_S^{ii} and C_S^{iii} such that for all $u \in \mathfrak{F}$

$$|\underset{x}{\partial} u(x)| \leqq C_S^{\mathrm{i}},\ |\underset{x}{\partial^2} u(x)| \leqq C_S^{\mathrm{ii}} \quad \textit{for } x \in S$$

and

$$H_S^{\alpha}(u) \leqq C_S^{\mathrm{iii}} .$$

Proof. Let δ be the distance between S and $\dot{D}$. Then by Theorem 3

$$|\underset{x}{\partial} u(x)| \leqq C^{(1)} M \delta^{-1} + C^{(2)} \equiv C_S^{\mathrm{i}} \quad \text{for} \quad x \in S , \tag{7.1}$$

Now let S' be the sphere concentric with S such that $\mathrm{rad}(S') = \mathrm{rad}(S) + \delta/2$. We put

$$\mu = \operatorname*{Max}_{x \in S'} \{ |\underset{x}{\partial}^2 u(x)| \cdot \rho(x)^k \}, \tag{7.2}$$

where $\rho(x) = \mathrm{dist}(x, \dot{S}')$ and k is a positive constant to be defined afterwards. Then there exists a point $x_0 \in S'$ such that

$$|\underset{x}{\partial}^2 u(x_0)| \cdot \rho(x_0)^k = \mu \tag{7.3}$$

Let $\sum$ be the closed sphere with the center x_0 and the radius $\rho(x_0)/2$. Then, as $\rho(x) \geqq \rho(x_0)/2$ for $x \in \sum$, we have from (7.2)

$$|\underset{x}{\partial}^2 u(x)| \leqq 2^k \rho(x_0)^{-k} \mu \quad \text{for} \quad x \in \textstyle\sum .$$

Hence

$$|\underset{x}{\partial} u(x') - \underset{x}{\partial} u(x)| \leqq 2^k \rho(x_0)^{-k} \mu |x' - x| \quad \text{for} \quad x, x' \in \textstyle\sum . \tag{7.4}$$

From (7.1) we get also

$$|u(x') - u(x)| \leqq C^{\mathrm{i}}_{S'} |x' - x| \quad \text{for} \quad x, x' \in \textstyle\sum (\subset S') . \tag{7.5}$$

Then by (4.1) we obtain for $x, x' \in \sum$

$$|f_{[u]}(x') - f_{[u]}(x)| \leqq H_1 (1 + (C^{\mathrm{i}}_{S'})^\alpha + 2^{\alpha k} \rho_0^{-\alpha k} \mu^\alpha) |x' - x|^\alpha,$$

where $f_{[u]}(x) = f(x, u(x), \underset{x}{\partial} u(x))$, $H_1 = H_{M, C^{\mathrm{i}}_{S'}}$ and $\rho_0 = \rho(x_0)$, or

$$H^\alpha_\Sigma (f_{[u]}) \leqq C^{(1)}_S + C^{(2)}_S \rho_0^{-\alpha k} \mu^\alpha, \tag{7.6}$$

where $\quad C^{(1)}_S = H_1 (1 + (C^{\mathrm{i}}_{S'})^\alpha), \; C^{(2)}_S = 2^{\alpha k} H_1 .$

By Schauder's Theorem A we have

$$\begin{aligned} |\underset{x}{\partial}^2 u(x_0)| &\leqq C_{(A, L)} (\rho_0/2)^{-4} \{ H^\alpha_\Sigma (f_{[u]}) + \operatorname*{Max}_\Sigma |f_{[u]}| + \operatorname*{Max}_\Sigma |u| \} \\ &\leqq 16 C_{(A, L)} \rho_0^{-4} \{ H^\alpha_\Sigma (f_{[u]}) + B (C^{\mathrm{i}}_{S'})^2 + \Gamma + M \} . \end{aligned}$$

Then by (7.6), putting $k = 4(1 - \alpha)^{-1}$, we get

$$\rho_0^k |\underset{x}{\partial}^2 u(x_0)| \leqq C^{(3)}_S \rho_0^{k-4} + C^{(4)}_S \mu^\alpha,$$

where $C^{(3)}_S$ and $C^{(4)}_S$ are positive constants depending on S. Thus from (7.3), as $k < 4$ and $\rho_0 \leqq \mathrm{rad}(S')$,

$$\mu \leqq C^{(3)}_S \; \mathrm{rad}(S')^{k-4} + C^{(4)}_S \mu^\alpha . \tag{7.7}$$

But, since $0 < \alpha < 1$, we obtain from (7.7)

$$\mu \leqq C^{(5)}_S , \tag{7.8}$$

where $C_S^{(5)}$ is a positive constant depending on S. Then, as $\rho(x) \geqq \delta/2$ for $x \in S$, from (7.2) and (7.8)

$$(7.9) \qquad |\underset{x}{\partial^2} u(x)| \leqq 2^k C_S^{(5)} \delta^{-k} \equiv C_S^{\text{ii}} \quad \text{in } S \text{ for all } u \in \mathfrak{F}.$$

Now we get easily from (4.1), (7.5) and (7.9), replacing S by S',

$$(7.10) \qquad H_{S'}^{\alpha}(f_{[u]}) \leqq H_1\{1 + (C_{S'}^{\text{i}})^{\alpha} + (C_{S'}^{\text{ii}})^{\alpha}\} .$$

But by Schauder's Theorem A

$$\begin{aligned} H_S^{\alpha}(\underset{x}{\partial^2} u) &\leqq C_{(A,L)}(\delta/2)^{-4}\{H_{S'}^{\alpha}(f_{[u]}) + \underset{S'}{\operatorname{Max}} |f_{[u]}| + \underset{S'}{\operatorname{Max}} |u|\} \\ &\leqq 16 C_{(A,L)} \delta^{-4}\{H_{S'}^{\alpha}(f_{[u]}) + B(C_{S'}^{\text{i}})^2 + \Gamma + M\} . \end{aligned}$$

Thus by (7.10) there exists a constant C_S^{iii} such that

$$H_S^{\alpha}(\underset{x}{\partial^2} u) \leqq C_S^{\text{iii}} \quad \text{for all} \quad u \in \mathfrak{F}, \quad \text{q.e.d.}$$

8. Now we assume the existence of a sequence of domains $\{D_n\}$ such that $\bar{D}_{n-1} \subset D_n$, $\bigcup_{n=1}^{\infty} D_n = D$ and $\dot{D}_n$ is of type Bh. We can prove the existence of such sequence $\{D_n\}$ for any open domain D, but we will not enter into it here.

Theorem 5. *Let $a_{ij}(x)$ be H_1-continuous in each $\bar{D}_n$ and satisfy* (2.1) *in D. Let $f(x, u, p)$ be H_α-continuous* $(0 < \alpha < 1)$ *in the finite part of each*

$$D_n^* = \{(x, u, p)\,;\ x \in \bar{D}_n,\ \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x),\ |p| < +\infty\} ,$$

where $\underline{\omega}(x)$ and $\bar{\omega}(x)$ are bounded continuous functions such that

$$|\underline{\omega}(x)| \leqq M, \quad |\bar{\omega}(x)| \leqq M,$$

and

$$|f(x, u, p)| \leqq B|p|^2 + \Gamma_n \quad in\ D_n^* ,$$

where B and Γ_n are positive constants such that $16ABM < 1$. *Let $\{\bar{\omega}_\gamma(x)\}$ and $\{\underline{\omega}_\gamma(x)\}$* $(\gamma \in \Omega)$ *be systems of quasi-supersolutions and quasi-subsolutions of* (0) *respectively such that*

$$\underline{\omega}(x) \leqq \underline{\omega}_\gamma(x) < \bar{\omega}_{\gamma'}(x) \leqq \bar{\omega}(x) \quad in\ D \quad (\gamma, \gamma' \in \Omega) .$$

Then there exists a solution $u(x)$ of (0) *such that*

$$\sup_{\gamma \in \Omega} \underline{\omega}_\gamma(x) \leqq u(x) \leqq \inf_{\gamma \in \Omega} \bar{\omega}_\gamma(x) \quad in\ D.$$

Proof. First we consider a fixed $\gamma \in \Omega$. Let $\beta_n(x)$ be a function of $C^3[\bar{D}]$ such that $\underline{\omega}_\gamma(x) < \beta_n(x) < \bar{\omega}_\gamma(x)$ in D_n. Then by Theorem 3 there exists a solution $u_n(x)$ of (0) such that $u_n(x)=\beta_n(x)$ for $x\in\dot{D}_n$,

$$\underline{\omega}_\gamma(x)\leqq u_n(x)\leqq\bar{\omega}_\gamma(x) \quad \text{in } D_n, \quad \text{and } \|u_n\|_{D_n}^{\alpha,2}<+\infty .$$

Let S be any closed sphere in D, then $S\subset D_i$ for sufficiently large i. By Lemma 3 the sequences $\{u_n(x)\}, \{\underset{x}{\partial}u_n(x)\}$ and $\{\underset{x}{\partial}^2 u_n(x)\}$ are all uniformly bounded and equi-continuous in S. Then, as S is an arbitrary closed sphere in D, we can choose a sequence of natural numbers $\{n(\nu)\}(n(\nu+1)>n(\nu))$ in such a way that the sequences

$$\{u_{n(\nu)}(x)\}, \ \{\underset{x}{\partial}u_{n(\nu)}(x)\} \text{ and } \{\underset{x}{\partial}^2 u_{n(\nu)}(x)\}$$

converge uniformly in D in the generalised sense. Then we can easily see that

$$\lim_{\nu\to\infty} u_{n(\nu)}(x)=u(x)$$

is also a solution of (0) such that

$$\underline{\omega}_\gamma(x)\leqq u(x)\leqq\bar{\omega}_\gamma(x) \quad \text{in} \quad D.$$

Now let $\mathfrak{F}_\gamma$ be the set of all solutions of (0) such that

$$\underline{\omega}_\gamma(x)\leqq u(x)\leqq\bar{\omega}_\gamma(x) \quad \text{in } D \quad \text{and } \|u\|_{D_n}^{\alpha,2}<+\infty.$$

By Lemma 3 $\mathfrak{F}_\gamma$ is compact in $C^2[D]$, where $C^2[D]$ is a linear topological space with the pseudo-norm

$$\|u\|_n=\underset{D_n}{\text{Max}}|u(x)|+\underset{D_n}{\text{Max}}|\underset{x}{\partial}u|+\underset{D_n}{\text{Max}}|\underset{x}{\partial}^2u|.$$

If $\gamma_1,\dots,\gamma_n$ are any finite number of $\gamma\in\Omega$, we see easily that

$$\underset{1\leqq i\leqq n}{\text{Min}} \ \{\bar{\omega}_{\gamma_i}(x)\}=\bar{\omega}_*(x) \text{ is a quasi-supersolution of (0)}$$

$$\underset{1\leqq i\leqq n}{\text{Max}} \ \{\underline{\omega}_{\gamma_i}(x)\}=\underline{\omega}_*(x) \text{ is a quasi-subsolution of (0),}$$

such that $\underline{\omega}(x)\leqq\underline{\omega}_*(x)<\bar{\omega}_*(x)\leqq\bar{\omega}(x)$ in D, and $\bigwedge_{i=1}^n \mathfrak{F}_{\gamma_i}$ is the set of all solutions of (0) such that

$$\underline{\omega}_*(x)\leqq u(x)\leqq\bar{\omega}_*(x) \quad \text{in} \quad D, \qquad \|u\|_{Dn}^{\alpha,2}<+\infty .$$

Then by the first part of the proof $\bigwedge_{i=1}^n \mathfrak{F}_{\gamma_i}$ is not empty. Thus

$$\bigwedge\nolimits_{\gamma\in\Omega} \mathfrak{F}_\gamma \neq 0 ,$$

by the intersection property of compact sets, q. e. d.

§6 Main existence theorem.

9. We say that a domain D satisfies the *condition of Poincaré*, if for each point c of $\dot{D}$ there exists a cone of one nappe K with the vertex c such that, in a sufficiently small neighbourhood of c, K lies outside of D. Now we shall prove the main existence theorem:

Theorem 6. *Let D be a bounded domain satisfying the condition of Poincaré. Let $a_{ij}(x)$ be H_1-continuous in $\bar{D}$ and satisfy*

$$(9.1) \qquad A^{-1} \leqq \sum_{ij=1}^{m} a_{ij}(x)\xi_i\xi_j \leqq A \quad for \ \sum_{i=1}^{m} \xi_i^2 = 1 \ (A \geqq 1).$$

Let $f(x, u, p)$ be H_α-continuous $(0 < \alpha < 1)$ in the finite part of

$$D^* = \{(x, u, p);\ x \in \bar{D},\ \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x),\ |p| < +\infty\},$$

where $\bar{\omega}(x)$ and $\underline{\omega}(x)$ are quasi-supersolution and quasi-subsolution of (0) *respectively such that*

$$(9.2) \qquad |\bar{\omega}(x)| \leqq M, \ |\underline{\omega}(x)| \leqq M \quad in\ D.$$

$f(x, u, p)$ satisfies also

$$(9.3) \qquad |f(x, u, p)| \leqq B|p|^2 + \Gamma.$$

where B and Γ are positive constants such that

$$(9.4) \qquad 16\,ABM < 1.$$

Then there exists a solution $u(x)$ of (0) *such that*

$$|\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x) \quad in\ D$$

with the boundary value $\beta(x)\ (x \in \dot{D})$, where $\beta(x)$ is a given continuous function on $\bar{D}$ such that $\underline{\omega}(x) < \beta(x) < \bar{\omega}(x)$ in D.

Proof. Let c be any point of $\dot{D}$, and K be a cone of one nappe with the vertex c, which lies outside of D for $|x-c| \leqq \delta_0\ (\delta_0 > 0)$. By a suitable linear transformation of coordinates, we can assume

$$\sum_{i,j=1}^{m} a_{ij}(c)\partial_{ij}^2 u(c) = \sum_{i=1}^{m} \partial_{ii}^2 u(c).$$

But (9.3) must be replaced by

$$(9.3') \qquad |f(x, u, p)| \leqq AB|p|^2 + \Gamma.$$

We assume also that the axis of the cone K is the x_1-axis with the positive sence directed into D. Let us introduce the new coordinates $r, \theta, \xi_2, \dots, \xi_m$ by

$$|x-c| = r, \quad x_1 - c_1 = r\cos\theta, \quad x_i - c_i = r\sin\theta\cdot\xi_i \quad (i \geqq 2).$$

And we assume that K is represented by

$$(K) \qquad \pi-\varepsilon_0 \leqq \theta \leqq \pi \quad (0<\varepsilon_0<\pi/2)\,.$$

Now we shall construct a quasi-supersolution $\omega_c(x)$ of (0) of the form

$$\omega_c(x) = r^\gamma\varphi(\theta)+\beta(c)+\varepsilon \quad (\varepsilon>0)$$

in a neighbourhood of c. Then we have

$$\underset{i}{\partial}\omega_c = \gamma r^{\gamma-2}(x_i-c_i)\varphi(\theta)+r^\gamma\underset{i}{\partial}\theta\varphi'(\theta)\,,$$
$$\underset{ij}{\partial^2}\omega_c = r^{\gamma-2}\{(\gamma(\gamma-2)r^{-2}(x_i-c_i)(x_j-c_j)+\delta_{ij})\varphi$$
$$+(\gamma(x_i-c_i)\underset{j}{\partial}\theta+\gamma(x_j-c_j)\underset{i}{\partial}\theta+r^2\underset{ij}{\partial^2}\theta)\varphi'+r^2\underset{i}{\partial}\theta\underset{j}{\partial}\theta\varphi''\}\,,$$

where

$$\underset{i}{\partial}\theta = \begin{cases} -r^{-1}\sin\theta & \text{if } i=1\,,\\ r^{-1}\cos\theta\cdot\xi_i & \text{if } i\geqq 2\,,\end{cases}$$
$$\underset{ij}{\partial^2}\theta = \begin{cases} r^{-2}\sin 2\theta & \text{if } i=j=1\,,\\ -r^{-2}\cos 2\theta\cdot\xi_i & \text{if } i=1,\ j\geqq 2\,,\\ r^{-2}[\cot\theta\cdot(\delta_{ij}-\xi_i\xi_j)-\sin 2\theta] & \text{if } i,j\geqq 2\,.\end{cases}$$

Thus, assuming $0<\gamma<1$ and $0<\delta<1$, we get for $r=|x-c|<\delta$

$$(9.5)\qquad \begin{cases} \Delta\omega_c \leqq r^{\gamma-2}\{\varphi''+(m-2)\cot\theta\cdot\varphi'+\gamma(m-1)|+|\varphi|\}\,,\\ |\underset{x}{\partial}\omega_c|^2\leqq r^{\gamma-2}(|\varphi'|+\gamma|\varphi|)^2\,,\\ \sum_{i,j=1}^m(a_{ij}-\delta_{ij})\underset{ij}{\partial^2}\omega_c \leqq kr^{\gamma-2}\delta\{\gamma|\varphi|+(1+|\cot\theta|)|\varphi'|+|\varphi''|\}\,,\end{cases}$$

where k is a fixed constant. Then, assuming $0<\delta<\mathrm{Min}\{1, k^{-1}\}$, we get from (9.5) for $r<\delta$

$$(9.6)\qquad \Phi[\omega_c] \equiv \sum_{i,j=1}^m a_{ij}(x)\underset{ij}{\partial^2}\omega_c+AB|\underset{x}{\partial}\omega_c|^2+\Gamma$$
$$\leqq r^{\gamma-2}\{(\varphi''+k\delta|\varphi''|)+(m-2)\cot\theta\cdot\varphi'$$
$$+(2AB|\varphi|+1+|\cot\theta|)|\varphi'|+m\gamma|\varphi|+\delta\Gamma\}\,.$$

Now we put

$$(9.7)\qquad \varphi(\theta) = \lambda^{-1}\mu|\theta|+(2AB)^{-1}\log\{(1+\lambda^2/2AB\mu)-e^{\lambda|\theta|}\}+C\,,$$

where $\qquad C = 6M-(2AB)^{-1}\log(\lambda^2/2AB\mu)\,,$

$$(9.8)\qquad \lambda = 2((m-1)\cot\varepsilon_0+12ABM+1)\,,$$

and $\qquad \mu = \lambda^2(2AB)^{-1}(1-e^{-4ABM})(e^{\lambda\pi}-1)^{-1}\,.$

Then $\varphi(\theta)$ ($\in C^2[|\theta|\leqq\pi]$) satisfies, for $|\theta|\leqq\pi$, the inequalities

$$(9.9)\qquad \varphi'(\theta)\cdot\theta\leqq 0\,,\quad \varphi''(\theta)<0,\quad 4M\leqq\varphi(\theta)\leqq 6M\,,$$

and

$$\varphi''+\lambda|\varphi'|+2AB\varphi'^2+\mu<0. \tag{9.10}$$

Thus, assuming

$$\begin{cases} 0<\delta<\mathrm{Min}\,\{\delta_0,\, 1,\, (2k)^{-1},\ \mu/4\Gamma\}\,, \\ 0<\gamma<\mathrm{Min}\,\{1,\, \mu/24mM,\ \log 2/\log(1/\delta)\}\,, \end{cases} \tag{9.11}$$

from (9.6), (9.8), (9.9) and (9.10) we obtain

$$\Phi[\omega_c]<0 \quad \text{for} \quad 0<|x-c|=r\leqq\delta\,,\quad |\theta|\leqq\pi-\varepsilon_0 \tag{9.12}$$

and, as $\delta^\gamma>1/2$, $\varphi(\theta)\geqq 4M$ and $\beta(c)\geqq -M$,

$$\omega_c(x)>M \quad \text{for} \quad |x-c|=\delta,\ |\theta|\leqq\pi-\varepsilon_0\,. \tag{9.13}$$

Hence $\omega_c=r^\gamma\varphi(\theta)+\beta(c)+\varepsilon$ is a quasi-supersolution of (0) in $D\cap\{x;\ |x-c|\leqq\delta\}$. Then

$$\bar{\omega}_{(c,\varepsilon)}(x)=\begin{cases} \bar{\omega}(x) & \text{for } |x-c|>\delta\,, \\ \mathrm{Min}\{\bar{\omega}(x),\ \omega_c(x)\} & \text{for } |x-c|\leqq\delta\,, \end{cases}$$

is a quasi-supersolution of (0) such that

$$\bar{\omega}_{(c,\varepsilon)}(x)>\beta(x) \quad \text{in} \quad D\,,$$

if we take $\delta=\delta(\varepsilon)>0$ so small that (9.11) and $|\beta(x)-\beta(c)|<\varepsilon$ for $|x-c|\leqq\delta$. Similarly

$$\underline{\omega}_{(c,\varepsilon)}(x)=\begin{cases} \underline{\omega}(x) & |x-c|>\delta\,, \\ \mathrm{Max}\{\underline{\omega}(x)\ \ \omega'_c(x)\} & \text{for } |x-c|\leqq\delta\,, \end{cases}$$

where $\omega'_c=-r^\gamma\varphi(\theta)+\beta(c)-\varepsilon$, is a quasi-subsolution of (0) such that

$$\bar{\omega}_{(c,\varepsilon)}(x)<\beta(x) \quad \text{in} \ \ D\,.$$

Then by Theorem 5 there exists a solution $u(x)$ of (0) such that, for all $c\in\dot{D}$ and $\varepsilon>0$,

$$\underline{\omega}_{(c,\varepsilon)}(x)\leqq u(x)\leqq\bar{\omega}_{(c,\varepsilon)}(x) \quad \text{in} \ \ D\,. \tag{9.14}$$

Letting ε tend to 0, we obtain from (9.14)

$$\lim_{x\to c} u(x)=\beta(c) \qquad \text{for any} \quad c\in\dot{D}$$

and

$$\underline{\omega}(x)\leqq u(x)\leqq\bar{\omega}(x) \quad \text{in} \ \ D, \ \ \text{q. e. d.}$$

REMARK. The condition imposed on the boundary of D can be weakened for the case $m=2$, while the calculations in the proof will be much simplified.

10. Really we have the conjecture: *the restriction* (9.4) *in Theorem* 6 *may be removed.* But now we shall only show that the condition (9.3) can not be replaced by

$$|f(x, u, p)| \leqq B|p|^{\kappa}+\Gamma$$

where κ is any constant >2. For this, we consider the following example:

$$(10.1)\quad \Delta u = -(m-1)\sum_{i=1}^{m} x_i \underset{i}{\partial} u/(\sum_{i=1}^{m} x_i^2) + u\{1+\sum_{i=1}^{m}(\underset{i}{\partial} u)^2\}^{1+\varepsilon}\quad (\varepsilon>0),$$

and D is the domain

$$\text{(D)}\qquad a^2 < \sum_{i=1}^{m} x < b^2 \quad (0<a<b).$$

(10.1) has the form $\Delta u = f(x, u, \underset{x}{\partial} u)$, where f is strictly increasing with u. Then, as we can easily prove, (10.1) has at most one solution under the boundary condition

$$(10.2)\qquad u=0 \quad \text{for } \sum x_i^2 = a^2, \qquad u=h \ (h>0) \quad \text{for } \sum x_i^2 = b^2.$$

Since (10.1) is invariant under any orthogonal transformation of independent variables (rotation about the origin), the unique solution of (10.1) under (10.2) is a function of $r=(\sum_{i=1}^{m} x_i^2)^{1/2}$ only: $u=u(r)$. Hence $u(r)$ satisfies the ordinary differential equation

$$(10.3)\qquad u'' = u(+u'^2)^{1+\varepsilon}.$$

The solution u of (10.3) satisfies

$$(1+u'^2)^{-\varepsilon} = \varepsilon(C-u^2) \quad (C=\text{const.}).$$

Then $0 \leqq u^2 < C = c^2$ for $a<x<b$, and

$$- < \varepsilon^{1/2\varepsilon}(c^2-u^2)^{1/2\varepsilon}u' < 1.$$

Thus, as $u(a)=0$ and $u(b)=h\leqq c$,

$$\varepsilon^{1/2\varepsilon}\int_0^c (c^2-u^2)^{1/2\varepsilon}du < b-a,$$

or

$$\gamma(\varepsilon)c^{1+1/\varepsilon} < b-a,$$

where

$$\gamma(\varepsilon) = 2^{1/\varepsilon}\Gamma(1/2\varepsilon+1)^2/\Gamma(1/\varepsilon+2).$$

Hence

$$0<h\leqq c<\gamma_1(\varepsilon)(b-a)^{\varepsilon/(1+\varepsilon)},$$

where $\gamma_1(\varepsilon)$ is a constant depending only on ε.

Therefore, if

$$h \geqq \gamma_1(\varepsilon)(b-a)^{\varepsilon/(1+\varepsilon)}, \tag{10.4}$$

there exists no solution of (10. 1) under (10. 2), although

$$\bar{\omega}(x) = M \leqq h\,(>0) \quad \text{and} \quad \underline{\omega}(x) = 0$$

are quasi-supersolution and quasi-subsolution of (10. 1) respectively in D. And for $x \in D$ and $0 \leqq u \leqq M$ holds the inequality

$$|f(x, u, p)| \leqq B|p|^{2(1+\varepsilon)} + \Gamma,$$

only if $B = (1+\varepsilon)h$, and Γ is sufficiently large. $ABM = (1+\varepsilon)h^2$ may also be arbitrarily small, if $b-a$ is so small that (10. 4) holds.

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