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Supplementary Remarks on Frobeniusean Algebras II

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An algebra A over a field F is quasi-Frobeniusean[1] if and only if the dualities

$$\alpha_1) \quad l(r(\mathfrak{l})) = \mathfrak{l}, \qquad \alpha_2) \quad r(l(\mathfrak{r})) = \mathfrak{r}$$

between left and right ideals with respect to annihilation hold, where $\mathfrak{l}$ and $\mathfrak{r}$ are left and right ideals respectively, and $l(*)$ and $r(*)$ denote left and right annihilators in A respectively. Further A is Frobeniusean if and only if besides the annihilation dualities $\alpha_1)$ and $\alpha_2)$ also the rank relations

$$\beta_1) \ (\mathfrak{l} : F) + (r(\mathfrak{l}) : F) = (A : F), \quad \beta_2) \ (\mathfrak{r} : F) + (l(\mathfrak{r}) : F) = (A : F)$$

are valid.[2]

The notion of Frobeniusean and quasi-Frobeniusean algebras as well as these duality criteria have been extended to general rings with minimum condition. In this note we shall study some properties of a ring with the duality $\alpha_1)$ for left ideals, and show, among others, that only the duality relations $\alpha_1)$ and $\beta_1)$ or $\alpha_1)$ are sufficient for an algebra to be Frobeniusean or quasi-Frobeniusean respectively.

Let A be a ring with minimum condition for left and right ideals. (We shall understand by a ring always such a ring.) Let N be the radical of A, $A/N = \bar{A} = \bar{A}_1 + \bar{A}_2 + \dots + \bar{A}_k$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals $\bar{A}_\kappa$ and let $f_{(\kappa)}$, $e_{\kappa,i}$, $e_\kappa = e_{\kappa,1}$, $c_{\kappa,ij}$ and $E_\kappa = \sum_{i=1}^{f(\kappa)} e_{\kappa,i}$ $(\kappa = 1, \dots, k)$ have the same meaning as in S. I or Fr. I §1; namely $e_{\kappa,i}$ $(\kappa = 1, \dots, k;\ i = 1, \dots, f_{(\kappa)})$ are mutually

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1) See Part I, Proc. Jap. Acad. (1949) (referred to SI) or Nakayama, On Frobeniusean algebras I, II, III, Ann. Math. 40 (1939), 42 (1941), Jap. J. Math 18 (1942) (referred to Fr I, II, III)

2) Of course $\alpha_1)$, $\alpha_2)$ follow respectively from $\beta_1)$, $\beta_2)$, since $l(r(\mathfrak{l})) \supseteqq \mathfrak{l}$, $r(l(\mathfrak{r})) \supseteqq \mathfrak{r}$, $r(l(r(\mathfrak{l}))) = r(\mathfrak{l})$ and $l(r(l(\mathfrak{r}))) = l(\mathfrak{r})$ always. The same is the case with the modified rank relation $\beta')$ in Fr II, Theorem 7; namely $\beta')$ implies $\alpha_1)$ and $\alpha_2)$, and is sufficient, by itself, to secure that a ring A is Frobeniusean, provided A has composition series for left and right ideals.

$$C_l(\mathfrak{l}) = C_r(A/r(\mathfrak{l})) \quad C_r(\mathfrak{r}) = C_l(A/l(\mathfrak{r}))$$

with C_l and C_r denoting left and right composition lengths, are sufficient (and necessary) in order that an algebra or a ring be quasi-Frobeniusean.

orthogonal primitive idempotent elements whose sum is a principal idempotent element E of A, whence $A=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}+l(E)$ $(=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}e_{\kappa i}A+r(E))$ is the direct decomposition of A into directly indecomposable left (right) ideals $Ae_{\kappa,i}$ $(e_{\kappa,i}A)$ and $l(E)$ $(r(E))$; here $Ae_{\kappa,i}$ $(e_{\kappa,i}A)$ and $Ae_{\lambda,j}$ $(e_{\lambda,j}A)$ are operator isomorphic if and only if $\kappa=\lambda$. $C_{\kappa,ij}$ $(i=1,\dots,f(\kappa)$; $j=1,\dots,f(\kappa))$ are matric units and $C_{\kappa,ii}=e_{\kappa,i}$, for each κ, i, and $C_{\kappa,ij}C_{\lambda,l,m}=\delta_{\kappa,\lambda}\delta_{j,l}C_{\kappa,i,m}$. The residue class $\bar{E}_\kappa$ of E_κ mod. N is, for each κ, the unit element of simple ring $\bar{A}_\kappa$.

Theorem 1. Let A be a ring with the dualily α_1). Then A possesses a unit element, and there exists a permutation π of $(1, 2, \dots, k)$ such that for each κ,

(i) The largest completely reducible right subideal $\mathfrak{r}_\kappa$ of $e_\kappa A$ is a direct sum of simple right ideals isomorphic to $e_{\pi(\kappa)}A/e_{\pi(\kappa)}N$, and

(ii) $Ae_{\pi(\kappa)}$ has a unique simple left subideal $\mathfrak{l}_{\pi(\kappa)}$ and $\mathfrak{l}_{\pi(\kappa)}$ is isomorphic to Ae_κ/Ne_κ.

Proof.

We proceed stepwise:

1) By the duality α_1) we have $l(r(0))=0=l(A)$.

2) We denote $l(N)$ by R. Then $r(Re_{\kappa,i})\geqq N\cup(E-e_{\kappa,i})A$. Here the right hand side is a maximal right ideal of A, whence $r(Re_{\kappa,i})$ coincides with either $N\cup(E-e_{\kappa,i})A$ or A. If $r(Re_{\kappa,i})=A$, then, $Re_{\kappa,i}=l(A)=0$, by the duality. If $r(Re_{\kappa,i})=N\cup(E-e_{\kappa,i})A$ and $Re_{\kappa,i}$ contains a proper left subideal $\mathfrak{l}$ then $r(\mathfrak{l})\supset r(Re_{\kappa,i})$ by the duality, and so $r(\mathfrak{l})=A$, $\mathfrak{l}=l(A)=0$. Hence $Re_{\kappa,i}$ is either a simple left ideal or zero.

3) $R=RE+(R\cap l(E))$ and $A=EA+r(E)$. Here $l(E)$ and $r(E)$ are contained in N; hence $(R\cap l(E))A=(R\cap l(E))(EA+r(E))=0$. Therefore, by 1) $R\cap l(E)=0$, whence $R=RE$.

4) $R=RE=\sum_{\kappa,i}Re_{\kappa,i}$, and $Re_{\kappa,i}$ is either a simple left ideal or zero. So $NR=0$, namely $R\subseteqq r(N)$. We denote $r(N)$ by L.

5) RE_κ is a two-sided ideal. For, $A=\sum_\lambda E_\lambda AE_\lambda\cup N$, whence $RE_\kappa A=RE_\kappa(\sum_\lambda E_\lambda AE_\lambda\cup N)\subseteqq RE_\kappa$. Moreover $r(RE_\kappa)\supseteqq N\cup(E-E_\kappa)A$, and here the right hand side is a maximal two-sided ideal. Hence RE_κ is either a simple two-sided ideal or zero, as we see in a similar way as in 2).

6) $E_\kappa R$ is a (not only right, but) two-sided ideal. For, $A=\sum_\lambda E_\lambda AE_\lambda\cup N$, $R\subseteqq L$, and so $AE_\kappa R=(\sum_\lambda E_\lambda AE_\lambda\cup N)E_\kappa R\subseteqq E_\kappa R$. $E_\kappa R$ is a non-zero ideal for each K, since $e_{\kappa,i}R$ is the largest com-

pletely reducible right subideal of $e_{\kappa,i}A$.

7) $R = RE = \sum_{\kappa} RE_{\kappa} = ER + (R \cap r(E)) = \sum_{\kappa} E_{\kappa}R + (R \cap r(E))$. $R \cap r(E)$ is a (not only right, but) two-sided ideal, because $A = EAE \cup N$, $R \subseteqq L$ and so $A(R \cap r(E)) = (EAE \cup N)(R \cap r(E)) = 0$. Hence the extreme right member of the above equation is a direct decomposition into k non-zero two-sided ideals $E_{\kappa}R$ and $R \cap r(E)$, while the third is a direct decomposition into at most k simple two-sided ideals RE_{κ}. This shows that $E_{\kappa}R$ is a simple two-sided ideal, $R \cap r(E) = 0$ and RE_{κ} is a non-zero two-sided ideal, for each κ.

8) $R \cap r(E) = 0$ implies $r(E) = 0$. Hence $A = EA + r(E) = EA$ and $l(E) = l(EA) = l(A) = 0$ according to 1). So $A = AE + l(E) = AE$, and E is a unit element of A.

9) $E_{\kappa}L$ is a (not only right, but) two-sided ideal. For, $A = \sum E_{\lambda}AE_{\lambda} \cup N$ whence $AE_{\kappa}L = (\sum_{\lambda} E_{\lambda}AE_{\lambda} \cup N)\, E_{\kappa}L \subseteqq E_{\kappa}L$. $l(E_{\kappa}L) \supseteqq N \cup A(1-E_{\kappa})$. Since the right hand side is a maximal two-sided ideal and $l(E_{\kappa}L)$ is also a two-sided ideal, $l(E_{\kappa}L)$ is equal either to $N \cup A(1-E_{\kappa})$ or to A. If $l(E_{\kappa}L) = A$, then $r(A) = r(l(E_{\kappa}L)) \supseteqq E_{\kappa}L$. But the left hand side is zero, since A possesses unit element, and so $E_{\kappa}L$ must vanish too. If $l(E_{\kappa}L) = N \cup A(1-E_{\kappa})$, however, then $r(l(E_{\kappa}L)) = r(N \cup A(1-E_{\kappa})) = E_{\kappa}L$. Now we consider a composition series

$$A \supset N \cup A(1-E_{\kappa}) \supset \mathfrak{Z}_1 \supset \mathfrak{Z}_2 \supset \cdots \supset 0$$

of two-sided ideals of A. If a left ideal $\mathfrak{l}'$ is a proper subideal of a left ideal $\mathfrak{l}$, then $r(\mathfrak{l})$ is a proper subideal of $r(\mathfrak{l}')$ by the duality for left ideals. Hence the series

$$r(A) = 0 \subset r(N \cup A(1-E_{\kappa})) \subset r(\mathfrak{Z}_1) \subset r(\mathfrak{Z}_2) \subset \cdots \subset r(0) = A$$

has the same length as the above series, and, therefore this is also a composition series of two-sided ideals of A, and $r(N \cup A(1-E_{\kappa})) = E_{\kappa}L$ is a simple two-sided ideal. Thus $E_{\kappa}L$ is either a simple two-sided ideal or zero.

Therefore $E_{\kappa}L \cdot N = 0$ for every κ, and L is contained in R, which gives, when combined with 4), $L = R$. We denote this by M.

10) According to 7) we have two direct decompositions of M into simple two-sided ideals:

$$M = \sum_{\kappa} ME_{\kappa} = \sum_{\kappa} E_{\kappa}M$$

There exists then a permutation π of $(1, 2, \dots, k)$ such that $ME_{\pi(\kappa)} = E_{\kappa}M$ for each κ. This shows that $Me_{\pi(\kappa),i}$, which is by 2) and $R =$

$L=M$ the unique simple left subideal of $Ae_{\pi(\kappa),i}$, is isomorphic to Ae_κ/Ne_κ, while the largest completely reducible right subideal $e_{\kappa,i}M$ of $e_{\kappa,i}A$ is a direct sum of simple subideals isomorphic to $e_{\pi(\kappa)}A/e_{\pi(\kappa)}N$. This completes our proof.

In the case of algebras, we have, on combining this theorem and Theorem 1 of SI, the following

Theorem 2. An algebra A, with a finite rank over a field F, is quasi-Frobeniusean if (and only if) α_1) $l(r(\mathfrak{l}))=\mathfrak{l}$ for every left ideal $\mathfrak{l}$ in A. A is further Frobeniusean if (and only if) besides α_1)

β_1) $(\mathfrak{l}:F)+(r(\mathfrak{l}):F)=(A:F)$

is valid for every left ideal $\mathfrak{l}$ in A.[3]

The assertion of Theorem 2 is not valid for rings (with minimum condition) in general.[4] But a ring A with the duality α_1) possesses a unit element, and, satisfies so the maximum condition also; hence A possesses composition series of either right ideals and left ideals.

Theorem 3. Let A be a ring (with minimum condition) with the duality α_1) for left ideals. A is quasi-Frobeniusean if (and only if) A has the same composition lengths for right and left ideals. Further A is Frobeniusean if (and only if) $f_{(\kappa)}=f_{(\pi(\kappa))}$ for each κ besides the above condition.

Proof. Assume that A has the same composition lengths for right ideals and left ideals. By Theorem 1 $Me_{\kappa,i}$ is a simple left ideal and $e_{\kappa,i}M$ is a direct sum of simple subideal isomorphic to $e_{\pi(\kappa)}A/e_{\pi(\kappa)}N$. The left annihilator of $e_{\kappa,i}M$ contains a maximal left ideal $N\cup A(1-e_{\kappa,i})$, and is so equal either to A or to $N\cup A(1-e_{\kappa,i})$. But if $l(e_{\kappa,i}M)=A$ then $r(A)=r(l(e_{\kappa,i}M))\supseteqq e_{\kappa,i}M$, which gives a contradiction, since A has unit element, $r(A)=0$, while $e_{\kappa,i}M\neq 0$. Therefore $l(e_{\kappa,i}M)=N\cup A(1-e_{\kappa,i})$ and $r(l(e_{\kappa,i}M))=r(N\cup A(1-e_{\kappa,i}))=e_{\kappa,i}M$. Consider now a composition series:

$$A\supset N\cup A(1-e_{\kappa,i})\supset\mathfrak{l}_1\supset\mathfrak{l}_2\supset\cdots\supset 0$$

of left ideals. By the duality for left ideals, the series

$$r(A)=0\subset r(N\cup A(1-e_{\kappa,i}))=e_{\kappa,i}M\subset r(\mathfrak{l}_1)\subset r(\mathfrak{l}_2)\subset\cdots\subset r(0)=A$$

has the same length as the above series. Then by our assumption this series must be a composition series of right ideals of A. Therefore $e_{\kappa,i}M$ is a simple right ideal. This shows that A is quasi-Frobeniusean.

3) The last assertion is a rather immediate consequence of the first. For α_1), α_2) and β_1) together imply β_2) as we see readily.

4) See SI.

In our above proof of Theorem 1 we used only $l(A)=l(r(0))=0$ and that $\mathfrak{l}\supset\mathfrak{l}_0$, with left ideals $\mathfrak{l}$ and $\mathfrak{l}_0$, implies $r(\mathfrak{l})\subset r(\mathfrak{l}_0)$. So we may restate our theorems in the following forms refined in this sense:

Theorem 4. If A is a ring in which $\mathfrak{l}\supset\mathfrak{l}_0$, with left ideals $\mathfrak{l}$ and $\mathfrak{l}_0$, then the duality α_1) holds for every left ideal $\mathfrak{l}$ in A, and conversely.

Proof. Let $\mathfrak{l}$ be a left ideal of A. If the above condition is satisfied and $l(r(\mathfrak{l}))\supset\mathfrak{l}$, then $r(l(r(\mathfrak{l})))=r(\mathfrak{l})\subset r(\mathfrak{l})$, which is absurd.

Corollary. Let A be a ring in which $\mathfrak{l}\supset\mathfrak{l}_0$, $\mathfrak{r}\supset\mathfrak{r}_0$ imply $r(\mathfrak{l})\subset r(\mathfrak{l}_0)$, $l(\mathfrak{r})\subset l(\mathfrak{r}_0)$ respectively, with left ideals $\mathfrak{l}$, $\mathfrak{l}_0$ and right ideals $\mathfrak{r}$, $\mathfrak{r}_0$. Then A is quasi-Frobeniusean. [5]

We have further

Theorem 5. A ring A satisfies the duality α_1) for left ideals if (and only if) the duality α_1) holds for every nilpotent left ideal. [6]

Proof. Assume that the duality α_1) holds for every nilpotent left ideal in A. Let $\mathfrak{l}$ be a left ideal generated by an idempotent e; $\mathfrak{l}=Ae$. Then $r(\mathfrak{l})=r(e)$, and $r(e)$ is the set of elements $a-ea\ (a\in A)$. If c is an element of $l(r(\mathfrak{l}))$, then $c(a-ea)=(c-ce)a=0$ for all elements a of A. But $l(A)=l(r(0))=0$ by our assumption. Hence $c-ce=0$ and, $c\in Ae=\mathfrak{l}$. This shows $l(r(\mathfrak{l}))$, and thus the duality α_1) holds for $\mathfrak{l}$.

As in our proof of Theorem 1, $r(Re_{\kappa,i})$ is equal either to A or to $N\cup(E-e_{\kappa,i})A$. If $r(Re_{\kappa,i})=A$, then $l(A)=0=l(r(Re_{\kappa,i}))\supseteqq Re_{\kappa,i}$, hence $Re_{\kappa,i}$ is zero. If $r(Re_{\kappa,i})=N\cup(E-e_{\kappa,i})A$ and $Re_{\kappa,i}$ contains a simple left ideal $\mathfrak{l}_1$, then[7] $r(\mathfrak{l}_1)\supset r(Re_{\kappa,i})=N\cup(E-e_{\kappa,i})A$ and, therefore, $r(\mathfrak{l}_1)=A$, $\mathfrak{l}_1=0$. This shows that $Re_{\kappa,i}$ is either a simple left ideal or zero. And 3), 4), 5), 6), 7), 8), all remain valid under the present weaker assumption. Thus A possesses, in particular, a unit element. Let $\mathfrak{l}_2$ be a left ideal. We may express it as $\mathfrak{l}_2=Ae_2+\mathfrak{l}'_2$, where e_2 is an idempotent element and $\mathfrak{l}'_2$ is contained in N. $r(\mathfrak{l}_2)=r(Ae_2)\cap r(\mathfrak{l}'_2)=(1-e_2)A\cap r(\mathfrak{l}'_2)=(1-e_2)\ r(\mathfrak{l}'_2)$, and $l(r(\mathfrak{l}_2))\supseteqq\mathfrak{l}_2=Ae_2+\mathfrak{l}'_2$. Let z be an element of $l(r(\mathfrak{l}_2))$. Then $z=ze_2+z(1-e_2)$ and $z\cdot r(\mathfrak{l}_2)=z(1-e_2)\ r(\mathfrak{l}'_2)=0$, whence $z(1-e_2)\in l(r(\mathfrak{l}'_2))$. This shows $l(r(\mathfrak{l}_2))=Ae_2\cup l(r(\mathfrak{l}'_2))$. Since $\mathfrak{l}'_2$ is a nilpotent left ideal, $l(r(\mathfrak{l}'_2))=\mathfrak{l}'_2$. Thus $l(r(\mathfrak{l}_2))=Ae_2+\mathfrak{l}'_2=\mathfrak{l}_2$.

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5) Also in our privious criteria of Frob. and quasi-Frob. rings we did not assume the annihilation dualities α_1) and α_2) fully. See Fr. I, Theorem 3 (and Theorem 7) and Fr II, Theorem 6 (and Theorems 7, 10).

6) See foot note 5).

7) The duality α_1) holds for our $\mathfrak{l}_1$. For, $\mathfrak{l}_1$ is either nilpotent or idempotent. And, if $\mathfrak{l}_1$ is nilpotent, then the duality for $\mathfrak{l}_1$ is valid by our assumption. If $\mathfrak{l}_1$ is idempotent, then it is generated by an idempotent element and the duality holds for $\mathfrak{l}_1$.

Next we have

Lemma. A ring A is directly decomposable into a bound ring and a semi-simple ring; here a ring is called a bound ring if $R \cap L \subseteqq N$.[8]

Now we can refine Theorem 2 as follows:

Theorem 6. An algebra A, with a finite rank over a field F, is quasi-Frobeniusean if (and only if) the duality α_1) holds for every nilpotent left ideal. A is further Frobeniusean if (and only if) besides α_1) the rank relation β_1) is valid for every nilpotent simple left ideal.[9]

Proof. The first half is an immediate consequence of Theorems 2 and 5. To prove the second, we observe first that $A = A_1 \oplus A_2$ is Frobeniusean if and only if subrings A_1 and A_2 are so, and that if the conditions of the second part are satisfied in A, then they are also satisfied in A_1 and A_2. So we may, by virtue of the above lemma, restrict ourselves to the case of a bound algebra A, satisfying our conditions.

A is quasi-Frobeniusean, any how, and so $Me_{\pi(\kappa)}$ is a simple left ideal isomorphic to Ae_κ/Ne_κ. Hence $(Me_{\pi(\kappa)} : F) = (Ae_\kappa/Ne_\kappa : F) = f_{(\kappa)}(e_\kappa Ae_\kappa/e_\kappa Ne_\kappa : F)$. Since A is a bound algebra, $Me_{\pi(\kappa)}$ is a nilpotent simple left ideal. So

$$\begin{aligned}(Me_{\pi(\kappa)} : F) &= (A : F) - (r(Me_{\pi(\kappa)}) : F) = (A : F) - (N \cup (1 - e_{\pi(\kappa)})A : F) \\ &= (e_{\pi(\kappa)}A/e_{\pi(\kappa)}N : F) = f_{(\pi(\kappa))}(e_{\pi(\kappa)}Ae_{\pi(\kappa)}/e_{\pi(\kappa)}Ne_{\pi(\kappa)} : F)\,.\end{aligned}$$

But $e_\kappa Ae_\kappa/e_\kappa Ne_\kappa \cong e_{\pi(\kappa)}Ae_{\pi(\kappa)}/e_{\pi(\kappa)}Ne_{\pi(\kappa)}$ as was shown in the proof of Theorem 3 in Fr. I, hence $f_{(\kappa)} = f_{(\pi(\kappa))}$ is valid, for each κ. This shows that A is Frobeniusean.

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8) See M. HALL, The position of the radical in an algebra, Trans. Amer. Math. Soc. 48 (1940), Theorem 2.

9) See footnote 5).