A characterization of the Fuchsian locus of $\mathrm{PSL}_n\mathbb{R}$-Hitchin components

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Abstract. In this thesis, I characterize $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations in Hitchin components, using the Bonahon-Dreyer parameterization.

The Hitchin component $H_n(S)$ is the component of the $\mathrm{PSL}_n\mathbb{R}$-character variety of a closed surface S of negative Euler characteristic which contains the discrete faithful representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ via an irreducible representation. The images of discrete faithful representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ in $H_n(S)$ are called $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations.

Bonahon-Dreyer ([**BD14**], [**BD17**]) gave a parameterization of $H_n(S)$ by the triangle invariants and the shearing-type invariants fixing an arbitrary maximal geodesic lamination on S, so that the Hitchin component is a cone in a Euclidean space.

During doctoral program, I first calculated the parameters of Bonahon and Dreyer for $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations of a pair of pants. In this calculation, I had an explicit parameterization of Fuchsian representations of pants, and of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations via the Bonahon-Dreyer parameterization.

Next, I generalized the result for a pair of pants, to more general surfaces in my second paper. In particular, I proved that, for arbitrary compact orientable surfaces of negative Euler characteristics, the triangle invariants of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations are equal to zero and the shearing-type invariants are equal to the shearing parameters of hyperbolic structures. This explicit characterization implies the set of the $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations is an affine slice.

This thesis explains these two studies, the calculation for a pair of pants, and the general properties of the Bonahon-Dreyer parameters of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations.

Contents

CHAPTER 1

Introduction

1. Motivation

Let S be a closed oriented surface of negative Euler characteristics. The Hitchin component of S is a special connected component $H_n(S)$ of the $\mathrm{PSL}_n\mathbb{R}$-character variety $X_n(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_n\mathbb{R})/\mathrm{PSL}_n\mathbb{R}$, the space of conjugacy classes of representations. This component was introduced by Hitchin in [**Hi92**]. When $n = 2$, the Hitchin component $H_2(S)$ is the Teichmüller space $\mathscr{T}(S)$ of S, which is the deformation space of hyperbolic structures on S. For general $n \geq 2$, $H_n(S)$ is, by definition, the connected component of $X_n(S)$ which contains elements induced from holonomy representations of hyperbolic structures on S via the irreducible representation $\iota_n \colon \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_n\mathbb{R}$. The Hitchin component has many properties which the Teichmüller space has, and it is a higher dimensional analog of the Teichmüller space in the sense of the rank of Lie groups. It is natural to consider the relation between the Teichmüller space and the Hitchin component.

In this thesis, we characterize $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations in the Hitchin component. We call the elements of $H_n(S)$ the *Hitchin representation*, and call Hitchin representations induced from holonomy representations of hyperbolic structures the $\mathrm{PSL}_n\mathbb{R}$-*Fuchsian representation*. The locus of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations in $H_n(S)$ is called the *Fuchsian locus*. To characterize $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations, we use the parameterization of the Hitchin component given by Bonahon and Dreyer ([**BD14**], [**BD17**]). The Hitchin component is parameterized by two kinds of invariants, the triangle invariant and the shearing-type invariant along maximal geodesic laminations on S. Through an observation of the invariants of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations, we show that, a Hitchin representation is a $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation if and only if the triangle invariants are equal to zero, and the shearing-type invariants are equal to the shearing parameters of hyperbolic structures.

2. Main results

Let λ be an arbitrary maximal geodesic lamination on S, which yields an ideal triangulations of S. Given a representation in $H_n(S)$, the triangle invariants are defined for ideal triangles of this triangulation, and the shearing-type invariants are defined for leaves of λ.

The Bonahon-Dreyer parameterization is different depending on whether λ consists of finitely many geodesics, or contains an irrational sublamination. Although the triangle invariants are defined in the same way, the shearing-type invariants are defined in different ways. In particular, the former case is more combinatorial. In this thesis, we characterize, indeed, the parameters for $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations in the both cases.

When λ consists of finitely many leaves, letting $\chi(S)$ be the Euler characteristic, we set $\lambda = \{C_1, \cdots, C_k, B_1, \cdots, B_{3|\chi(S)|}\}$ where $C_1, \cdots, C_k$ is a closed geodesic ($1 \leq k \leq 3g-3$), and B_i is a bi-infinite geodesic. We denote the ideal triangles which are complementary regions of λ by $T_1, \cdots, T_{2|\chi(S)|}$. Let s_0^i, s_1^i, s_2^i be the spikes of the ideal triangle T_i. In this case, Bonahon and Dreyer introduced the invariants, called the triangle invariants, the shearing invariants, and twist invariants to define the parameterization of $H_n(S)$. Given $\rho \in H_n(S)$,

- the triangle invariant $\tau_{pqr}(s_j^i, \rho)$ is defined for spikes s_j^i of the ideal triangles T_i,
- the shearing invariant $\sigma_b(B_i, \rho)$ is defined for the bi-infinite leaves B_i,
- the twist invariant $\theta_c(C_i, \rho)$ is defined for the closed leaves C_i,

where the indices p, q, r, b, c are positive integers with $p+q+r = n$, and $1 \leq b, c \leq n-1$. In this setting, the Bonahon-Dreyer parameterization $\Phi_\lambda \colon H_n(S) \to \mathbb{R}^N$ is defined by

$$\Phi_\lambda(\rho) = (\tau_{pqr}(s_j^i, \rho), \cdots, , \sigma_b(B_i, \rho), \cdots, \theta_c(C_i, \rho), \cdots),$$

where $N = 6|\chi(S)|\binom{n-1}{2} + 3|\chi(S)|(n-1) + k(n-1)$. The image of Φ_λ, denoted by $\mathcal{P}_\lambda$, is the interior of a certain polyhedron of $\mathbb{R}^N$ ([**BD14**]). The following is our main theorem.

THEOREM 1.1. *Let S be a closed oriented surface of negative Euler characteristics, and λ be a maximal geodesic lamination on S consisting of finitely many leaves. Then a Hitchin representation $\rho \in H_n(S)$ is $\mathrm{PSL}_n\mathbb{R}$-Fuchsian if and only if all triangle invariants are zero, and the shearing, and twist invariants are constants depending only on ρ, i.e.*

$$\tau_{pqr}(s_j^i, \rho) = 0, \quad \sigma_b(B_i, \rho) = \sigma_{b'}(B_i, \rho), \quad \theta_c(C_i, \rho) = \theta_{c'}(C_i, \rho)$$

for all possible $i, j, p, q, r, b, b', c, c'$. Moreover, the shearing invariant of a $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation is equal to the shearing parameter associated to the Fuchsian representation, i.e. for any Fuchsian representation $\eta \in \mathscr{T}(S)$,

$$\sigma_b(B_i, \iota_n \circ \eta) = \sigma^\eta(B_i)$$

for all b and i.

This theorem characterizes the $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations in the Hitchin component by the conditions of the triangle, shearing, and twist invariants.

In the case of general laminations, we use the shearing classes instead of shearing, and twist invariants. In [**BD17**], Bonahon and Dreyer defined the twisted tangent cycle relative to slits for maximal geodesic laminations, which was a vector valued cocycle defined on the set of oriented arcs transverse to λ. The shearing class is a twisted tangent cycle relative to slits defined by Hitchin representations. The Bonahon-Dreyer parametrization in this case is a parameterization defined by the triangle invariant and the shearing class. We denote this parameterization by $\Phi_\lambda \colon H_n(S) \to Z(\lambda, \mathrm{slits}; \mathbb{R}^{n-1}) \times \mathbb{R}^{6|\chi(S)|\binom{n-1}{2}}$, where $Z(\lambda, \mathrm{slits}; \mathbb{R}^{n-1})$ is the vector space of the twisted tangent cycles relative to slits. The image $\mathcal{P}_\lambda$ of Φ_λ is the interior of a convex polyhedron in $Z(\lambda, \mathrm{slits}; \mathbb{R}^{n-1}) \times \mathbb{R}^{6|\chi(S)|\binom{n-1}{2}}$ ([**BD17**]). We show that the shearing classes $\sigma^{\iota_n \circ \rho}$ of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations $\iota_n \circ \rho$ are determined only by the shearing cocycle σ^ρ.

THEOREM 1.2. *Suppose that λ is an arbitrary maximal geodesic lamination. Then a Hitchin representation $\rho \in H_n(S)$ is* $\mathrm{PSL}_n\mathbb{R}$*-Fuchsian if and only if all triangle invariants are equal to zero, and, for any oriented arc k tightly transverse to λ, the shearing class is of the form $(\sigma(k), \cdots, \sigma(k))^t$ where σ is a transverse cocycle of λ, i.e. $\sigma \in Z(\lambda; \mathbb{R})$.*

Theorem 1.2 generalizes Theorem 1.1, in the following sense. Let λ be an oriented maximal geodesic lamination which consists of finitely many leaves. For a bi-infinite leave B_i of λ, we pick an oriented arc k transverse to B_i so that k intersects to B_i only once from left to right. Then the shearing class $\sigma^\rho(k)$ associated to k is the vector whose entries are the shearing invariants $\sigma_b(B_i, \rho)$, *i.e.* $\sigma^\rho(k) = (\sigma_1(B_i, \rho), \cdots, \sigma_{n-1}(B_i, \rho))$. Since Theorem 1.2 implies that all entries of shearing classes are equal to each other for $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations, Theorem 1.2 proves the statement with bi-infinite leaves in Theorem 1.1.

3. Structure of this paper

Chapter 2: We give a preliminaries. Section 1 explains the hyperbolic geometry on surfaces, the Teichmüller spaces, Shearing coordinate of Teichmüller spaces. The Shearing coordinate is the origin of the Bonahon-Dreyer parameterization. In Theorem 2.6, we give an improved version of the shearing coordinate using train tracks. Section 2 explains the definition of Hitchin components. Hyperconvex curves in Subsction 2.3 is essentially used in the computation of ratios. Section 3 explains the Bonahon-Dreyer parameterization.

Chapter 3: In this chapter, we calculate the ratios of Veronese flag curves, which play an important role in the proof of the main theorem. Section 1 gives an example of calculation. Using parameters of Fuchsian representations of a pair of pants, we explicitly calculate the ratios in the case of a pair of pants. In Section 2, we prove that the triple ratio of Veronese flag curves is always equal to 1, and the double ratio is reduced to the cross ratio.

Chapter 4: We show the main results of this paper. Theorem 4.1 and Theorem 4.3 give the sufficiency of the main theorems. Theorem 4.1 and Theorem 4.3 characterize the triangle invariants and the shearing-type invariants of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations. Theorem 4.2 and Theorem 4.4 imply the necessity of the main theorems. In the proof of these theorems, for any parameters which satisfy the condition for invariants, we construct $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations whose Bonahon-Dreyer parameters are equal to given parameters. We remark the case of surfaces with boundary in Section 3.

4. Acknowledgements

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CHAPTER 2

Preliminaries

1. Hyperbolic geometry of surface

1.1. Hyperbolic structures on surfaces. Let S be a closed oriented surface of negative Euler characteristics. A *hyperbolic metric* on S is a complete Riemannian metric on S of constant curvature -1. A *hyperbolic structure* on S is an isometric class of a hyperbolic metric on S.

We denote, by $\mathbb{H}^2$, the hyperbolic plane of the upper-half plane model with the orientation induced by the framing $\langle e_1, e_2\rangle$, where $e_1 = (1,0)^t, e_2 = (0,1)^t$. The group of orientation-preserving isometries $\mathrm{Isom}^+(\mathbb{H}^2)$ is isomorphic to $\mathrm{PSL}_2\mathbb{R}$, where $\mathrm{PSL}_2\mathbb{R}$ acts on $\mathbb{H}^2$ as linear fractional transformations.

If S is endowed with a hyperbolic metric, we obtain an isometry $f\colon \widetilde{S} \to \mathbb{H}^2$ with respect to the metric on $\widetilde{S}$ induced from the hyperbolic structure on S. Then, there exists a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ so that f is $(\pi_1(S), \rho)$-equivariant, *i.e.* for any $x \in \widetilde{S}$ and $\gamma \in \pi_1(S)$, $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$. This representation ρ is discrete, faithful and unique up to conjugacy of $\mathrm{PSL}_2\mathbb{R}$. We call a discrete faithful representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ a *Fuchsian representation*. The above isometry $f\colon \widetilde{S} \to \mathbb{H}^2$ with the equivariance for a Fuchsian representation ρ is called the *developing map* associated to ρ. In this paper, we denote, by f_ρ, the developing map associated to ρ.

The correspondence between hyperbolic structures and conjugacy classes of Fuchsian representations is one to one. In fact, for a Fuchsian representation ρ, we have the universal covering $\mathbb{H}^2 \to S$ with the covering transformation group $\rho(\pi_1(S))$. This covering map defines the hyperbolic metric on S, which is unique up to isometry.

1.2. Teichmüller space. The *Teichmüller space* $\mathscr{T}(S)$ of S is defined by

$$\mathscr{T}(S) = \{\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R} \mid \text{Fuchsian, } f_\rho \text{ is orientation-pres.}\}/\mathrm{PSL}_2\mathbb{R}$$

where the quotient is defined by the conjugate action of $\mathrm{PSL}_2\mathbb{R}$. The topology of $\mathscr{T}(S)$ is the quotient topology of the compact open topology which is defined on the set of representations.

We remark an equivalent definition of the Teichmüller space via hyperbolic structures of S. Let $\mathrm{Hyp}(S)$ be the set of hyperbolic metrics on S, and $\mathrm{Diff}_0(S)$ be the group of diffeomorphisms isotopic to the identity. The group $\mathrm{Diff}_0(S)$ acts on $\mathrm{Hyp}(S)$ by the pull-back. Then the Teichmüller space is also defined by $\mathscr{T}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}_0(S)$.

Two definitions above are equivalent via the one to one correspondence between hyperbolic structures and Fuchsian representations. There are another equivalent definitions of $\mathscr{T}(S)$, see [**IT**].

1.3. Geodesic laminations. Fix a hyperbolic metric on S. A *geodesic lamination* is a closed subset of S which is a disjoint union of simple complete geodesics, called *leaves*. Geodesic laminations consist of closed geodesic, called *closed leaves*, and bi-infinite geodesics, called *bi-infinite leaves*.

The concept of geodesics depends on a hyperbolic metric on S. We remark that, for different hyperbolic metrics g_1 and g_2 on S, there exists a natural bijection between the set of g_1-geodesic laminations and the set of g_2-geodesic laminations. In particular, for any hyperbolic metric g and any simple curve c on S, there is a g-geodesic c_g which is isotopic to c.

The bi-infinite geodesics on the universal covering $\widetilde{S}$ are characterized their ideal end points. Especially, there exists a bijection between the space $G(\widetilde{S})$ of bi-infinite geodesics on $\widetilde{S}$ and $(\partial\widetilde{S} \times \partial\widetilde{S} - \Delta)/\mathbb{Z}_2$, where Δ denotes the diagonal and where $\mathbb{Z}_2$ acts by exchanging the two factors. The metric structure and the Hölder structure on $G(\widetilde{S})$ (used in Section 3.3.2 in this chapter) is given by an (arbitrary) metric structure on $(\partial\widetilde{S} \times \partial\widetilde{S} - \Delta)/\mathbb{Z}_2$ via this bijection.

A geodesic lamination is *oriented* if each leaf is oriented. We may choose the orientation of each leaf independently.

For a geodesic lamination λ of S, the preimage $\widetilde{\lambda}$ of λ in $\widetilde{S}$ gives a geodesic lamination of $\mathbb{H}^2$. A connected component of the closure of $\mathbb{H}^2 \setminus \widetilde{\lambda}$ is called a *plaque*.

A geodesic lamination is said to be *maximal* if it is properly contained in no other geodesic lamination. This property is equivalent to the condition that the complementary regions of λ consists of ideal triangles. Hence, a maximal geodesic lamination induces an ideal triangulation on S.

Given maximal oriented geodesic lamination λ with finitely many leaves, we often use the bridge system for closed leaves as an additional data, which is used in [**SZ**], [**SWZ**]. Let C be a (oriented) closed leaf of λ. Since λ consists of finitely many leaves, in both sides of C, some bi-infinite leaves and ideal triangles spiral to C. A *bridge* J_C along C is a pair of ideal triangles $\{T^L, T^R\}$ where T^L spirals to C from left, and T^R spirals to C from right. A *bridge system* of λ is $\mathcal{J} = \{J_C \mid C \text{ is a closed leaf}\}$, an association of bridges to closed leaves. We denote, by $\lambda_{\mathcal{J}}$, the lamination λ with a bridge system $\mathcal{J}$. The bridge system in this paper plays a role of the system of short arcs in [**BD14**].

1.4. Hyperbolic structures on surfaces with boundary. Let S be a compact oriented surface of negative Euler characteristics, which has non empty boundary. A hyperbolic metric on S is a complete Riemannian metric of constant curvature -1 which makes the boundary components totally geodesic. A hyperbolic structure on S is an isometric class of hyperbolic metrics on S. Geodesic laminations on S is similarly defined. In the case of compact surfaces with boundary, we require that maximal geodesic laminations must contain all of the boundary components as closed leaves. A hyperbolic structure on the surface with boundary also uniquely corresponds, up to conjugacy, to a representation with the following properties:

(i) ρ is a discrete and faithful representation, and
(ii) if $\gamma \in \pi_1(S)$ is the homotopy class of a boundary component, then $\rho(\gamma)$ is a hyperbolic element in $\mathrm{PSL}_2\mathbb{R}$.

In this paper, we call such a representation a *hyperbolic Fuchsian representation*. We can associated to a hyperbolic Fuchsian representation ρ the developing map $f_\rho\colon \widetilde{S} \to \mathbb{H}^2$. The image of f_ρ is a convex domain of $\mathbb{H}^2$, which does not coincides with $\mathbb{H}^2$ in general.

We define the Teichmüller space $\mathscr{T}(S)$ of S by

$$\mathscr{T}(S) = \{\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R} \mid \rho \text{ is hyperbolic Fuchsian, } f_\rho \text{ is orientation-pres. }\}$$

This Teichmüller space is also identified with the deformation space of hyperbolic structures as in the case of closed surfaces.

1.5. Shearing parameterization of Teichmüller spaces.

1.5.1. *The space of transverse cocycles.* We recall transverse cocycles. Let S be a compact oriented surface of negative Euler characteristics, and λ be an (arbitrary) maximal geodesic lamination on S. An ($\mathbb{R}$-valued) *transverse cocycle* σ for λ is a map associating a real number $\sigma(k) \in \mathbb{R}$ to each (unoriented) arc k transverse to λ which satisfies that

(i) (*Additivity*) if k is cut into the union of two subarcs at an interior point of $k \setminus \lambda$ so that $k = k_1 \cup k_2$, then $\sigma(k) = \sigma(k_1) + \sigma(k_2)$, and
(ii) (*Homotopy invariance*) if k and k' are homotopic respecting to λ, then $\sigma(k) = \sigma(k')$.

We denote the space of transverse cocycles for λ by $Z(\lambda)$.

The space $Z(\lambda)$ is parameterized by the train track neighborhood ([**Bo97**]). The *train track neighborhood* N_λ of λ is a family of finitely many "long" rectangles $e_1, \cdots, e_l$, called *edges*, so that the union of e_i contains λ. Two rectangles intersect only along their short sides, and every point of the short side of a rectangle is contained in another short side of the rectangles. We require that the complementary region of N_λ contains no component which is a disc with 0, 1, or 2 spikes, or an annulus with no spikes. Transverse cocycles $\sigma \in Z(\lambda)$ associate a real number to each e_i as follows. Each e_i is foliated by the arcs parallel to the short sides of e_i. We call the leaves of this foliation *ties*. We pick a tie k_i for the edge e_i, which is transverse to λ. Given $\sigma \in Z(\lambda)$, we define $\sigma(e_i)$ by the value $\sigma(k_i)$. The homotopy invariance of σ implies that $\sigma(e_i)$ is independent of the choice of k_i.

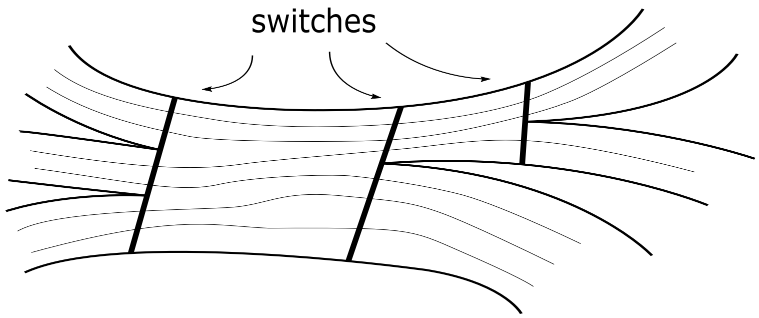


FIGURE 1. Train track neighborhood.

THEOREM 2.1. *(*[**Bo97**, Theorem 11]*) Let λ be a geodesic lamination, and let N_λ be an train track neighborhood of λ consisting of the edges $e_1, \cdots, e_\ell$. Then, the mapping $Z(\lambda) \to \mathbb{R}^l$, which sends transverse cocycles σ to the point $(\sigma(e_1), \cdots, \sigma(e_l))$, is a bijection onto the image. The image is defined by the switch relation.*

Let us recall the switch relation. *Switches* of N_λ are ties, which are short sides of edges. Suppose that $e_1^L, \cdots, e_p^L$ and $e_1^R, \cdots, e_q^R$ intersect along a switch s such that $e_1^L, \cdots, e_p^L$ are the edges adjacent to the one side of s, and $e_1^R, \cdots, e_q^R$ are the edges adjacent to the other side. The switch relation at s is the equation $e_1^L + \cdots + e_p^L = e_1^R + \cdots + e_q^R$. All possible switch relations define the range of the above parameterization of $Z(\lambda)$. The topology and the analytic structure of $Z(\lambda)$ is defined by the structure of the Euclidean space $\mathbb{R}^l$ via the mapping in Theorem 2.1.

1.5.2. *shearing cocycles and a parameterization of Teichmüller spaces.* Given $\rho \in \mathscr{T}(S)$, we construct the *shearing cocycle* $\sigma^\rho \in Z(\lambda)$ of ρ, which is the transverse cocycle associated to ρ. Fix a universal covering $\mathbb{H}^2 \to S$ associated to ρ. To define $\sigma^\rho(k)$ for an arbitrary arc k transverse to λ, we lift k to $\widetilde{k}$, which is transverse to the preimage $\widetilde{\lambda} \subset \mathbb{H}^2$ of λ. Then the endpoints of $\widetilde{k}$ are contained in different plaques. We denote these plaques by P and Q, and consider the set $\mathscr{P}$ of plaques which separate P and Q. Let g (resp. h) be the boundary leaf P

(resp. Q) which is nearest to Q (resp. P). On g (resp. h), there is a canonical base point which is the orthogonal projection of the third vertex of P (resp. Q). We call this point the *base point* of g (resp. h). Each plaque in $\mathscr{P}$ is partially foliated by the horocyclic flow. Then, we can construct a foliation which joins g and h. Along this foliation, we carry the base point of g to a point in h.

We define $\sigma^\rho(k)$ by the signed length between the carried point and the base point of h. Here the sign of the length is defined by the parameterization of h by $\mathbb{R}$ as follows. The orientation of S defines an orientation of the boundary of Q, so of h. Then we can take an isometric parameterization $\mathbb{R} \to h$ so that it is compatible with the orientation of h and maps 0 to the base point of h. The value $\sigma^\rho(k)$ is independent of the choice of $\widetilde{k}$, and we finish the construction of the shearing cocycle σ^ρ of ρ.

For an arc k which is transverse to a bi-infinite leaf B of λ only once, there is a simple formula of the value $\sigma^\rho(k)$. To explain this, we recall the cross ratio on the boundary $\partial\mathbb{H}^2$.

Definition 2.2. *Let $a, b, c, d \in \partial\mathbb{H}^2$ be a quadruple of distinct points of the ideal boundary $\partial\mathbb{H}^2$. The cross ratio* $\mathrm{cr}(a,b,c,d)$ *is the ratio*

$$\mathrm{cr}(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

We respectively lift k and B to $\widetilde{k}$ and $\widetilde{B}$ on the universal covering so that they intersect. There are two plaques P, Q which contains the endpoints of $\widetilde{k}$. In particular, since λ is maximal, these plaques are adjacent ideal triangles along $\widetilde{B}$. We denote, by x, y, z^L, z^R, the ideal vertices of P, Q by the following rules : (i) x and y are the endpoints of $\widetilde{B}$, (ii) x, z^L, y, z^R are in counterclockwise order. By direct computations, we obtain the following relation. Let us write $\sigma^\rho(k)$ by $\sigma^\rho(B)$.

Lemma 2.3.

$$\sigma^\rho(B) = \log[-\mathrm{cr}(x, y, z^L, z^R)].$$

The shearing cocycle is applied to parameterize the Teichmüller spaces.

THEOREM 2.4. *([**Bo96**, Theorem A]) There is a real analytic homeomorphism $\phi_\lambda\colon \mathscr{T}(S) \to Z(\lambda) : \rho \mapsto \sigma^\rho$ onto an open convex cone bounded by finitely many faces in $Z(\lambda)$.*

This parameterization is called the *shearing parameterization*. The image of ϕ_λ is characterized by a certain intersection form on $Z(\lambda)$, defined along train tracks. A train track neighborhood is called *generic* if all switches are trivalent as Figure 2. We can always choose a generic train track neighborhood for all geodesic laminations. Fix a generic train track N_λ of λ. At each switch s of N_λ, a single edge "comes" to the switch s, and two edges "leave" the switch. We denote, by e_s^L, the edge which leaves to the left of the incoming edge, and, by e_s^R, the edge which leaves to the right. For $\sigma, \eta \in Z(\lambda)$, the intersection form τ is defined by

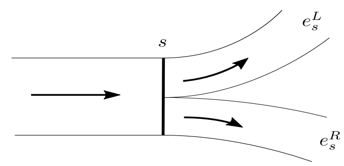


FIGURE 2. A generic switch.

$$\tau(\sigma, \eta) = \frac{1}{2} \sum_s \left(\sigma(e_s^R)\eta(e_s^L) - \sigma(e_s^L)\eta(e_s^R) \right),$$

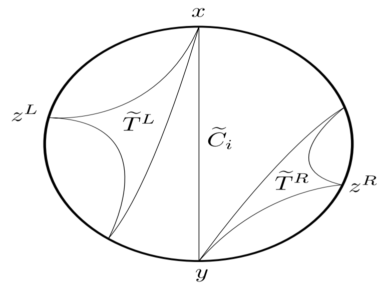
where s ranges over all switches of N_λ. The following theorem determines the image of ϕ_λ.

THEOREM 2.5. *([**Bo96**, Theorem 20]) For every non-zero transverse measures $\mu \in Z(\lambda)$ and for every shearing cocycles σ^ρ, $\tau(\mu, \sigma^\rho) > 0$.*

Note that this theorem follows for all generic train track neighborhoods of λ, hence the positivity of intersection numbers is independent of the choice of N_λ.

1.5.3. *Shearing parameterization along train tracks.* We arrange Theorem 2.4 by the weights on the edges of the train track neighborhood and the twist parameters along closed leaves of λ. Let us define the twist parameter. Let $C_1, \cdots, C_k$ be closed leaves of λ, contained in the interior of S. Under the ideal triangulation by λ, some ideal triangles spiral to C_i from the both sides. Choose an ideal triangle T^L in the one side, and an ideal triangle T^R in the other side.

We respectively lift C_i, T^L, and T^R to $\widetilde{C}_i$, $\widetilde{T}^L$, and $\widetilde{T}^R$ so that $\widetilde{T}^L$ and $\widetilde{T}^R$ have a common end point with $\widetilde{C}_i$. We denote, by x and y, the endpoints of C_i so that x is on the left from $\widetilde{T}^L$. Two edges of $\widetilde{T}^L$ are asymptotic to $\widetilde{C}_i$. In particular, one of these edges separates $\widetilde{S}$ so that $\widetilde{C}_i$, $\widetilde{T}^L$, and $\widetilde{T}^R$ are contained in the same component. We denote, by z^L, the end point of the edge, which is different from x or y. Similarly we take the ideal vertex z^R for $\widetilde{T}^R$. Note that the points x, z^L, y, z^R are in counterclockwise order. We define the *twist parameter* $\theta^\rho(C_i)$ by $\log[-\mathrm{cr}(f_\rho(x), f_\rho(y), f_\rho(z^L), f_\rho(z^R))]$.



We distinguish the edges of generic train track neighborhoods as follows. We call that an edge is *internal* if the edge intersects to no closed leaves, and we call the other edges *non-internal*. In addition, we call that a switch is *internal* if it is a short side of three internal edges, and we call other switches *non-internal*. In other words, the internal switch is a switch which intersects to no closed leaves.

The following version of Theorem 2.4 is used in the proof of Theorem 4.2.

THEOREM 2.6. *Let S be a compact oriented surface of negative Euler characteristics, and λ be an arbitrary maximal geodesic lamination on S, which has closed leaves $C_1, \cdots, C_k$ in the interior of S. Fix a generic train track neighborhood N_λ. We denote, by $e_1, \cdots, e_l$, the internal edges of N_λ. Then, the following map is an analytic embedding of the Teichmüller space $\mathscr{T}(S)$.*

$$\widetilde{\phi}_\lambda \colon \mathscr{T}(S) \to \mathbb{R}^{l+k} \colon \rho \mapsto (\sigma^\rho(e_1), \cdots, \sigma^\rho(e_l), \theta^\rho(C_1), \cdots, \theta^\rho(C_k)).$$

To prove this, we determine the range of the mapping $\widetilde{\phi}_\lambda$ by three conditions as follows.

(I) The parameters $\sigma^\rho(e_1), \cdots, \sigma^\rho(e_l)$ satisfy the switch relations at all internal switches by Theorem 2.1. This is the first condition which defines the image of $\widetilde{\phi}_\lambda$.

(II) Next, we focus on the spiraling of bi-infinite leaves along closed leaves. Let us introduce the signature of the spiraling of bi-infinite leaves. When the spiraling occurs in the direction opposite to the orientation of S, we call this spiraling *positive spiraling*. See Figure 3. Similarly, we call the spiraling in Figure 4 *negative spiraling*.

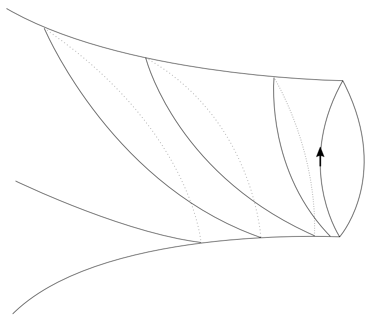


FIGURE 3. Positive spiraling.

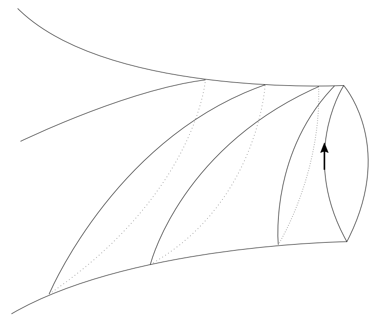


FIGURE 4. Negative spiraling.

We refer to the following proposition.

PROPOSITION 2.7. ([**Th**, Proposition 3.4.21]) *Let F be a compact oriented surface of negative Euler characteristics with boundary. Fix $\rho \in \mathscr{T}(F)$, and a maximal geodesic lamination λ on F. Let $B_1, \cdots, B_l$ be the bi-infinite leaves of λ spiral to a boundary component C of F. Then, if the spiraling of B_j is positive,*

$$l_\rho(C) = \sum_{j=1}^{l} \sigma^\rho(B_j),$$

and if the spiraling of B_j is negative,

$$l_\rho(C) = -\sum_{j=1}^{l} \sigma^\rho(B_j).$$

For each C_i, let $B_1^{i,L}, \cdots, B_{l_L}^{i,L}$ be bi-infinite leaves spiraling to C_i from the one side, and $B_1^{i,R}, \cdots, B_{l_R}^{i,R}$ be bi-infinite leaves spiraling to

C_i from the other side. Then, Proposition 2.7 gives us the following relation

$$\text{sign} \cdot \sum_{k=1}^{l_L} \sigma^\rho(B_k^{i,L}) = \text{sign} \cdot \sum_{k=1}^{l_R} \sigma^\rho(B_k^{i,R}) > 0 \qquad \cdots (*).$$

The symbol "sign" means the signature of each spiraling along C_i. Similarly, for each boundary component C of S, letting $B_1^C, \cdots, B_{l_C}^C$ be the bi-infinite leaves spiraling to C, it follows that

$$\text{sign} \cdot \sum_{k=1}^{l_C} \sigma^\rho(B_k^C) > 0 \qquad \cdots (**)$$

by Proposition 2.7, where "sign" also means the signature of the spiraling along C.

By definition of $\sigma^\rho(e_i)$, $(*)$ and $(**)$ give the relation between the parameters $\sigma^\rho(e_1), \cdots, \sigma^\rho(e_l)$, which is the second condition.

(III) The final condition is given by Theorem 2.5, which implies that $\tau(\mu, \sigma^\rho) > 0$ for every non-zero transverse measure μ. For switches s, set $\tau_s(\mu, \sigma^\rho) = \mu(e_s^R)\sigma^\rho(e_s^L) - \mu(e_s^L)\sigma^\rho(e_s^R)$. The non-internal switches correspond to the spiraling of bi-infinite leaves to closed leaves. Depending the signature of the spiraling, two types of the branches at non-internal switches occur as the following figures.

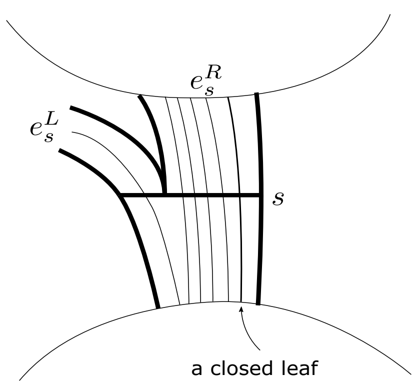


FIGURE 5. Positive case.

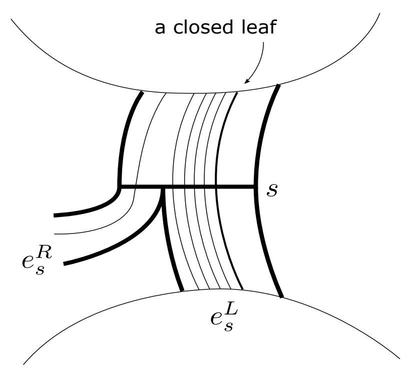


FIGURE 6. Negative case.

If s is given by the positive spiraling (Figure 5), then $\tau_s(\mu, \sigma^\rho) = \mu(e_s^R)\sigma^\rho(e_s^L)$, since the support of μ contains no isolated bi-infinite leaves, so $\mu(e_s^L) = 0$. Similarly if s is given by the negative spiraling (Figure 6), then $\tau_s(\mu, \sigma^\rho) = -\mu(e_s^L)\sigma^\rho(e_s^R)$. Hence, $\tau(\mu, \sigma^\rho) > 0$ implies that

$$\sum_s \tau_s(\mu, \sigma^\rho) + \sum_{s'} \left(\mu(e_{s'}^R)\sigma^\rho(e_{s'}^L)\right) - \sum_{s''} \left(\mu(e_{s''}^L)\sigma^\rho(e_{s''}^R)\right) > 0,$$

where s ranges over the internal switches, s' (resp. s'') ranges over the non-internal switches which correspond to the positive (resp. negative) spiraling.

If λ is uncountable, then we can take transverse measures μ such that μ associates 0 to the non-internal edges. Hence, for all such μ,

$$\sum_s \tau_s(\mu, \sigma^\rho) = \sum_s \left(\mu(e_s^R)\sigma^\rho(e_s^L) - \mu(e_s^L)\sigma^\rho(e_s^R)\right) > 0,$$

where s ranges only over internal switches. Note that e_s^L and e_s^R are internal edges, and this inequality is a relation between the parameters $\sigma^\rho(e_1), \cdots, \sigma^\rho(e_\ell)$.

If λ consists of finitely many leaves, all bi-infinite leaves are isolated. Then $\mu(e_1) = \cdots = \mu(e_\ell) = 0$ since the support of μ contains no isolated bi-infinite leaves. Hence we obtain

$$\sum_{s'} \left(\mu(e_{s'}^R)\sigma^\rho(e_{s'}^L)\right) + \left(-\sum_{s''} \left(\mu(e_{s''}^L)\sigma^\rho(e_{s''}^R)\right)\right) > 0.$$

However this inequality follows from the condition (II) since $\mu(e_{s'}^R)$ and $\mu(e_{s''}^L)$ are positive, so it gives no new conditions.

We summarize these conditions (I), (II), and (III).

Proposition 2.8. *The parameters $\sigma^\rho(e_1), \cdots, \sigma^\rho(e_\ell)$ satisfy the following three conditions:*

(I) *The switch relations at all internal switches.*

(II) *The equality and inequality obtained from the condition $(*)$ and $(**)$ along each closed leaf.*

(III) *The positivity $\sum_s \tau_s(\mu, \sigma^\rho) > 0$, where μ is an arbitrary transverse measure which associates 0 to the non-internal edges, and s ranges over the internal switches.*

Now we prove Theorem 2.6. The analyticity is obtained from the argument of [**Bo96**] and [**BD14**]. Hence it suffices to give an inverse mapping of $\widetilde{\phi}_\lambda$. In particular, we reconstruct a Fuchsian representation of S from the parameters which satisfy the conditions (I), (II), and (III) in Proposition 2.8.

Proof. (Theorem 2.6) Given parameter $(x_1, \cdots, x_l, y_1, \cdots, y_k)$ where x_i is the $\sigma^\rho(e_i)$-entry and y_i is the $\theta^\rho(C_i)$-entry , we construct a Fuchsian representation which has this parameter. To construct this, cut the surface S along the closed leaves C_i of λ. Then S is separated to finitely many (compact) surfaces with boundary.

First we construct a Fuchsian representation of each separated component. Let F be a connected component which is obtained in the above separation. Then the lamination λ (resp. the train track neighborhood N_λ) are restricted to the lamination λ_F (resp. the train track neighborhood N_{λ_F}) on F. We denote the internal edges of N_{λ_F} by $e^F_{i_1}, \cdots, e^F_{i_p}$, and denote the $\sigma^\rho(e^F_{i_j})$-parameter by $x^F_{i_j}$. The switch condition (I) implies the existence of a transverse cocycle in $Z(\lambda_F)$ which sends $e^F_{i_j}$ to $x^F_{i_j}$ (Theorem 2.1), and the condition (II) and (III) implies that its transverse cocycle satisfies the positivity in Theorem 2.5 for all non-zero transverse measures μ of λ_F. Thus, applying Theorem 2.4 to λ_F, we obtain a Fuchsian representation $\rho_F \in \mathscr{T}(F)$.

We can glue these representations ρ_F on each component F to obtain a Fuchsian representation ρ on S. Indeed, the condition (II) implies that the glued boundaries have the same length. By the construction, $\sigma^\rho(e_i)$ of ρ is equal to the given parameter x_i.

Now we deform the Fuchsian representation ρ to a representation η by the twist deformation along each closed leaf C_i to realize that $\theta^\eta(C_i) = y_i$. For the universal covering $\pi\colon \widetilde{S} \to S$, we set $\mathscr{C}_i = \pi^{-1}(C_i)$, which is an geodesic lamination on the universal covering. In the definition of $\theta^\rho(C_i)$, we fix a geodesic $\widetilde{C}_i$, ideal triangles $\widetilde{T}^L, \widetilde{T}^R$, and ideal vertices x, y, z^L, z^R. We orient $\widetilde{C}_i$ in the direction from y to x, and orient the leaves of $\mathscr{C}_i$ so that, for all $\ell \in \mathscr{C}_i$, $\pi(\ell)$ and $\pi(\widetilde{C}_i)$ are oriented in the same direction.

Let f_ρ be the developing map associated to ρ. The twist deformation of ρ along C_i is lifted onto the universal covering as follows. Each leaf $\ell \in \mathscr{C}_i$ cuts $\widetilde{S}$ into two components P and Q, where P is on the left of ℓ. We consider these ℓ, P, Q in the hyperbolic plane $\mathbb{H}^2$ via f_ρ. Let h_t^ℓ be the hyperbolic translation along ℓ whose translation length is t. Here the direction of h_t^ℓ is determined by the orientation of ℓ. Then we define a mapping g_t^ℓ by h_t^ℓ on $P \setminus \ell$, and the identity on Q. The iteration of such an action via g_t^ℓ for all $\ell \in \mathscr{C}_i$ gives a new universal covering of S, and the associated Fuchsian representation is a twist deformation of ρ.

We consider the variation of $\theta^\rho(C_i)$ under the twist deformation along C_i. Let $\ell \in \mathscr{C}_i$, and let P (resp. Q) be the left (resp. right) side of ℓ. If ℓ is different from $\widetilde{C}_i$, the ideal verices x, y, z^L, z^R are in the common side for ℓ. Hence, in the both cases,

$$\mathrm{cr}(g_t^\ell \circ f_\rho(x), g_t^\ell \circ f_\rho(y), g_t^\ell \circ f_\rho(z^L), g_t^\ell \circ f_\rho(z^R)) = \mathrm{cr}(f_\rho(x), f_\rho(y), f_\rho(z^L), f_\rho(z^R)).$$

If $\ell = \widetilde{C}_i$, z^L is on P and z^R is on Q. Then, via the translation g_t^ℓ, only z^L moves on the interval between x and y, and the other vertices x, y, z^R are fixed. In particular, the point z^L goes to x when $t \to \infty$ and goes to y when $t \to -\infty$. Hence, we obtain the following variation of the cross ratio.

$$\begin{aligned}
&\mathrm{cr}(g_t^\ell \circ f_\rho(x), g_t^\ell \circ f_\rho(y), g_t^\ell \circ f_\rho(z^L), g_t^\ell \circ f_\rho(z^R)) \\
&= \mathrm{cr}(f_\rho(x), f_\rho(y), g_t^\ell \circ f_\rho(z^L), f_\rho(z^R)) \\
&\to \begin{cases} 0 & (t \to \infty) \\ -\infty & (t \to -\infty). \end{cases}
\end{aligned}$$

This proves the next lemma.

LEMMA 2.9. *For any negative real numbers $r < 0$, there exists a twist deformation η_i of ρ along C_i such that*

$$\mathrm{cr}(f_{\eta_i}(x), f_{\eta_i}(y), f_{\eta_i}(z^L), f_{\eta_i}(z^R)) = r.$$

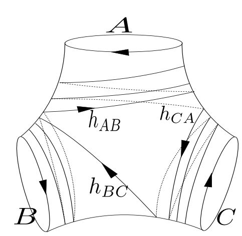
Applying Lemma 2.9 as $r = -e^{y_i}$, we complete the twist deformation η_i of ρ along the leaf C_i to obtain a Fuchsian representation η_i such that $\theta^{\eta_i}(C_i) = y_i$. We note that this twist deformation preserves the other twist parameters $\theta^\rho(C_j)$ for $i \neq j$. Since the closed leaf C_j does not intersect to C_i, the geodesic laminations $\mathscr{C}_i$ and $\mathscr{C}_j$ are disjoint. Moreover, C_i is asymptotic to some bi-infinite leaves, but does not intersect to bi-infinite leaves transversally. Thus the points x, y, z^L, z^R, which define the twist parameter along C_j, belong to a common plaque of $\mathscr{C}_i$. Hence, under the twist deformation along C_i, it holds that $\theta^\rho(C_j) = \theta^{\eta_i}(C_j)$. Similarly, the twist deformation preserves the shearing parameters, i.e. $\sigma^\rho(e_1) = \sigma^{\eta_i}(e_1), \cdots, \sigma^\rho(e_\ell) = \sigma^{\eta_i}(e_\ell)$. Therefore, twisting ρ along all closed leaves $C_1, \cdots, C_k$, we obtain a Fuchsian representation η of S such that $\theta^\eta(C_1) = y_1, \cdots, \theta^\eta(C_k) = y_k$. For this η, the shearing parameter does not change from one of the original representation ρ. We finish the reconstruction of Fuchsian representations. □

Finally, we make remarks about the case of laminations λ consisting of finitely many leaves. In this case, letting $B_1, \cdots, B_{3|\chi(S)|}$ be bi-infinite leaves of λ, and $C_1, \cdots, C_k$ be closed leaves in the interior of S, we can take a simple generic train track neighborhood N_λ which satisfies that the internal edges of N_λ are only $3|\chi(S)|$ edges $e_1, \cdots, e_{3|\chi(S)|}$ such that e_i intersects only to B_i and has no intersections with other leaves. Then the parameterization $\widetilde{\phi}_\lambda$ along N_λ is defined by

$$\widetilde{\phi}_\lambda(\rho) = (\sigma^\rho(e_1), \cdots, \sigma^\rho(e_{3|\chi(S)|}), \theta^\rho(C_1), \cdots, \theta^\rho(C_k)).$$

Note that there are no internal switches of N_λ. Thus, Proposition 2.8 implies that the range of $\widetilde{\phi}_\lambda$ is determined only by the condition (II). This parameterization is used in the proof of Theorem 4.2.

1.6. Example: the Teichmüller space of a pair of pants. Let P be a pair of pants. We denote the boundary components of P by A, B, and C as the right figure. The hyperbolic structure of P is uniquely determined by the boundary length.



Proposition 2.10. *The following mapping is a diffeomorphism.*

$$\mathscr{T}(P) \to \mathbb{R}^3_{>0} : \quad \rho \mapsto (l_\rho(A), l_\rho(B), l_\rho(C)).$$

Proof. We construct an inverse mapping. Set $a = \frac{1}{2} l_\rho(A)$, $b = \frac{1}{2} l_\rho(B)$, $c = \frac{1}{2} l_\rho(C)$. We refer the following fact.

LEMMA 2.11. ([**Ra**, Theorem 3.5.13]) *Set the lengths of sides of a right-angled hyperbolic convex hexagon* $\mathrm{V}_0\mathrm{V}_1\mathrm{V}_2\mathrm{V}_3\mathrm{V}_4\mathrm{V}_5$ *as* $\mathrm{V}_0\mathrm{V}_1 = a$, $\mathrm{V}_2\mathrm{V}_3 = b$, $\mathrm{V}_4\mathrm{V}_5 = c$, $\mathrm{V}_1\mathrm{V}_2 = c'$. *Then*

$$\cosh c' = \frac{\cosh a \cosh b + \cosh c}{\sinh a \sinh b}.$$

This lemma implies that a, b, c determine a unique right-angled hyperbolic convex hexagon $H_{a,b,c}$. (We can uniquely draw such a hexagon in the hyperbolic plane.) Let $H'_{a,b,c}$ be an isometric copy of $H_{a,b,c}$. Glue these hexagons $H_{a,b,c}$, $H'_{a,b,c}$ along the opposite sides of a, b, c to obtain a pair of pants P. Then the boundary lengths of P are equal to $2a, 2b, 2c$, and we obtain an inverse mapping. □

We parameterize Fuchsian representations of $\pi_1(P)$. Let λ the oriented maximal geodesic lamination $\lambda = \{h_{AB}, h_{BC}, h_{CA}, A, B, C\}$ which is described in the previous figure . This lamination induces an ideal triangulation of P and we denote by T_0 and T_1 two triangles given by the triangulation. We fix a presentation of $\pi_1(P)$ as

$$\pi_1(P) = \langle a, b, c \mid abc = 1\rangle$$

where a, b and c are the homotopy classes of A, B and C.

PROPOSITION 2.12. *Let* (l_A, l_B, l_C) *be a triple of the hyperbolic lengths of the boundary components* A, B, C. *Then we can take a representative* ρ *in the conjugacy class of the Fuchsian representation associated to* (l_A, l_B, l_C) *such that the developing map associated to* ρ *is described as Figure 7 and Figure 8, and the attracting point of the axis of* $\rho(b)$ *equals to* 0 *in* $\partial_\infty \mathbb{H}^2$. *In particular, the biinfinite leaves* h_{AB}, h_{BC} *and* h_{CA} *can lift to the geodesics* $\widetilde{h}_{AB}, \widetilde{h}_{BC}$ *and* $\widetilde{h}_{CA}$ *in Figure 7 and Figure 8, whose terminal points are* $\infty, 1$ *and* 0 *in* $\partial_\infty \mathbb{H}^2$ *respectively. Moreover we can write such a representative* ρ *concretely as follows.*

$$\rho(a) = \begin{bmatrix} \alpha & \alpha\beta\gamma + \alpha^{-1} \\ 0 & \alpha^{-1} \end{bmatrix},\ \rho(b) = \begin{bmatrix} \gamma & 0 \\ -\beta^{-1} - \gamma^{-1} & \gamma^{-1} \end{bmatrix},$$

where $\alpha, \beta, \gamma : \mathbb{R}^3_{>0} \to \mathbb{R}_{>0}$ *are defined by*

$$\alpha(l_A, l_B, l_C) = e^{l_A/2},\ \beta(l_A, l_B, l_C) = e^{(l_C - l_A)/2},\ \gamma(l_A, l_B, l_C) = e^{-l_B/2}$$

with the conditions $\alpha > 1, 1 > \gamma > 0, \beta > 0$.

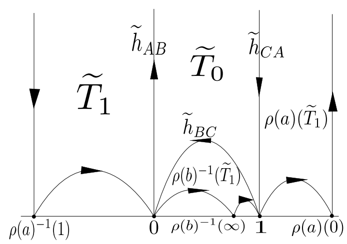


FIGURE 7. Upper half plane model.

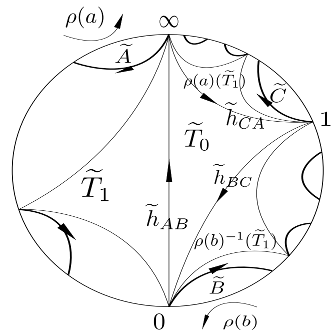


FIGURE 8. Poincaré disk model.

PROOF. The first assertion follows by the normalization of Fuchsian groups. See [**IT**]. We can set the Fuchsian representation ρ which satisfies the condition of the fixed points of $\rho(a)$ and $\rho(b)$ by

$$\rho(a) = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \ \rho(b) = \begin{bmatrix} \gamma & 0 \\ \delta & \gamma^{-1} \end{bmatrix}.$$

We compute the parameter δ so that $\rho(c) = \rho(b)^{-1}\rho(a)^{-1}$ fixes $1 \in \partial_\infty \mathbb{H}^2$. Since

$$\rho(c) = \rho(b)^{-1}\rho(a)^{-1} = \begin{bmatrix} \alpha^{-1}\gamma^{-1} & -\beta\gamma^{-1} \\ -\alpha^{-1}\delta & \beta\delta + \alpha\gamma \end{bmatrix},$$

the condition of the fixed points of $\rho(c)$ implies that

$$\rho(c)(1) = \frac{\alpha^{-1}\gamma^{-1} - \beta\gamma^{-1}}{-\alpha^{-1}\delta + \beta\delta + \alpha\gamma} = 1.$$

Thus we can show that $-\alpha^{-1} + \beta \neq 0$ and obtain

$$\delta = -\gamma^{-1} + \frac{-\alpha\gamma}{-\alpha^{-1} + \beta}.$$

Replacing the parameter β with a parameter β' satisfying $\beta = \alpha\gamma\beta' + \alpha^{-1}$, we deform the equation above as

$$\delta = -\gamma^{-1} - \beta'^{-1}.$$

We denote β' by β again. The representation ρ is given by the following form.

$$\rho(a) = \begin{bmatrix} \alpha & \alpha\beta\gamma + \alpha^{-1} \\ 0 & \alpha^{-1} \end{bmatrix}, \ \rho(b) = \begin{bmatrix} \gamma & 0 \\ -\beta^{-1} - \gamma^{-1} & \gamma^{-1} \end{bmatrix}.$$

Note that $\rho(a)$ and $\rho(b)$ are elements in $\mathrm{PSL}_2(\mathbb{R})$, so we can assume $\alpha, \gamma > 0$ by the multiple of -1. Since the attracting point of $\rho(b)$ is $0 \in \partial_\infty \mathbb{H}^2$, it holds that $\gamma < 1$.

Next we consider inequalities among parameters α, β and γ which are given by positional relations of fixed points at infinity. We can check the following easily:

$$\begin{aligned} \mathrm{Fix}(\rho(a)) &= \{\infty, \frac{\alpha^2\beta\gamma + 1}{1-\alpha^2}\}, \\ \mathrm{Fix}(\rho(b)) &= \{0, \frac{\gamma - \gamma^{-1}}{-\beta^{-1} - \gamma^{-1}}\}, \\ \mathrm{Fix}(\rho(c)) &= \{1, \frac{\alpha\beta + \alpha^{-1}\gamma^{-1}}{\alpha^{-1}\gamma^{-1} + \alpha{-}1\beta^{-1}}\}. \end{aligned}$$

By Figure 6, the following inequalities hold:

$$\frac{\alpha^2\beta\gamma + 1}{1-\alpha^2} < 0, \tag{1}$$

$$0 < \frac{\gamma - \gamma^{-1}}{-\beta^{-1} - \gamma^{-1}} < 1, \tag{2}$$

$$1 < \frac{\alpha\beta + \alpha^{-1}\gamma^{-1}}{\alpha^{-1}\gamma^{-1} + \alpha{-}1\beta^{-1}}. \tag{3}$$

Noting that $1 > \gamma > 0$ which implies that $\gamma < \gamma^{-1}$, these inequalities are deformed as follows:

$$\begin{aligned} (2) &\Leftrightarrow \beta^{-1} + \gamma^{-1} > 0, \beta^{-1} + \gamma > 0, \qquad (4) \\ (3), (4) &\Leftrightarrow \alpha^2\beta > \beta^{-1}, \qquad (5) \\ (1), (4) &\Leftrightarrow \alpha > 1. \qquad (6) \end{aligned}$$

We can deduce $\beta > 0$ from these inequalities since if β is negative,

$$
\begin{aligned}
(4) \quad &\Leftrightarrow \quad \beta + \gamma < 0, \beta < -\frac{1}{\gamma}, \\
(5) \quad &\Leftrightarrow \quad -\frac{1}{\alpha} < \beta < \frac{1}{\alpha}
\end{aligned}
$$

and, by $-1/\gamma < -1/\alpha$, a contradiction occurs.

In general, the hyperbolic length l of a simple closed geodesic which is covered by the axis of a hyperbolic isometry $M \in \mathrm{PSL}_2(\mathbb{R})$ is given by the following formula:

$$|tr(M)| = 2\cosh(l/2).$$

Under the conditions above of α, β and γ, the data of the hyperbolic lengths of A and B detect α and γ.

$$\alpha = \exp(l_A/2),\ \gamma = \exp(-l_B/2).$$

We consider an equation

$$|tr(\rho(c))| = 2\cosh(l_C/2)$$

which implies that

$$
\begin{aligned}
& |-\alpha\beta - \alpha^{-1}\beta^{-1}| = 2(\cosh(l_C/2)) \\
\Leftrightarrow\ & \alpha\beta + \alpha^{-1}\beta^{-1} = 2\cosh(l_C/2)\ (\alpha, \beta > 0) \\
\Leftrightarrow\ & \beta = \frac{\cosh(l_C/2) \pm \sinh(l_C/2)}{\alpha}.
\end{aligned}
$$

The inequality (5) gives us the following condition

$$(5) \Leftrightarrow \alpha^2\beta^2 > 1$$

and then β is uniquely determined by

$$\beta = \frac{\cosh(l_C/2) + \sinh(l_C/2)}{\alpha}.$$

$\square$

2. Hitchin representations, Anosov property, and flag curves

2.1. Representation varieties and Character varieties. Let Γ be a finitely generated group, presented by $\Gamma = \langle g_1, \cdots, g_k \mid \gamma_\lambda \ (\lambda \in \Lambda)\rangle$. The $\mathrm{PSL}_n\mathbb{R}$-*representation variety* $R_n(\Gamma)$ of Γ is the set of group homomorphisms $R_n(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{PSL}_n(\mathbb{R}))$ with the compact open topology. The representation variety $R_n(\Gamma)$ has an algebraic structure induced by the Lie group $\mathrm{PSL}_n\mathbb{R}$. Through the following map,

$$R_n(\Gamma) \to (\mathrm{PSL}_n\mathbb{R})^k : \quad \rho \mapsto (\rho(g_1), \rho(g_2), \cdots, \rho(g_k))$$

$R_n(\Gamma)$ is embedded into $(\mathrm{PSL}_n\mathbb{R})^k$. The image of this embedding is detected by the relation of Γ. Indeed, $\rho(\gamma_\lambda)$ is represented by a product of $\rho(g_1), \rho(g_2), \cdots, \rho(g_k)$, hence $R_n(\Gamma)$ is identified with a subvariety of $(\mathrm{PSL}_n\mathbb{R})^k$. Note that the relative topology of $R_n(\Gamma)$, induced from $(\mathrm{PSL}_n\mathbb{R})^k$, coincides with the compact open topology.

$\mathrm{PSL}_n\mathbb{R}$ acts on $R_n(\Gamma)$ by conjugation. The quotient space $X_n(\Gamma) = R_n(\Gamma)/\mathrm{PSL}_n(\mathbb{R})$ is called the $\mathrm{PSL}_n(\mathbb{R})$-*character variety*. Character varieties are often defined by the GIT quotient. We do not require the algebraic property of them, so we define $X_n(\Gamma)$ via the natural quotient. Note that our character variety $X_n(\Gamma)$ is not Hausdorff in general. An example of non-separable orbits is given in [**Go84**]. Let Γ be a fundamental group of the closed surface of genus 2, and let $n = 2$. Then Γ is generated as $\Gamma = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2]\rangle$. We define a representation ρ_g for $g \in \mathrm{PSL}_2\mathbb{R}$ by

$$\rho_g(a_1) = \rho_(b_2) = g, \quad \rho(a_2) = \rho(b_1) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix},$$

where $a > 1$. Set $g_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $g_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then, by a direct computation, we can check that ρ_{g_1}, ρ_{g_2} are not conjugate. Hence, they are projected to different points in $X_2(\Gamma)$.

Consider $g_n = \begin{bmatrix} (1 + a^{-2n})^{\frac{1}{2}} & a^{-2n} \\ 1 & (1 + a^{-2n})^{\frac{1}{2}} \end{bmatrix}$. Since $g_n \to g_1$ as $n \to \infty$, ρ_{g_n} converges to ρ_{g_1}. On the other hand, setting $D_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$, the conjugacy $D_a^n \rho_{g_n} D_a^{-n}$ converges to ρ_{g_2}. In $X_2(\Gamma)$, ρ_{g_n} and $D_a^n \rho_{g_n} D_a^{-n}$

are the same point for each n, hence ρ_{g_1}, ρ_{g_2} are not separable since ρ_{g_n} is closed to both ρ_{g_1}, ρ_{g_2} if n is sufficiently large.

2.2. Hitchin components. Let S be a closed surface of genus $g \geq 2$. Set $X_n(S) = X_n(\pi_1(S))$. We focus on the connected components of $X_n(S)$. First, we consider the case of $n = 2$. In this case, the number of connected components was computed by Goldman using a certain characteristic class ([**Go88**]). Given $\rho \in X_2(S)$, we can construct the associated flat bundle $P_\rho = \widetilde{S} \times \mathbb{R}^2/\pi_1(S)$ over S, where $\pi_1(S)$ acts on the first factor by the covering translations, and acts on the second factor through ρ. Let $e(\rho) \in \mathbb{Z}$ be the Euler class of P_ρ. Through a computation of the upper and lower bound of $e(\rho)$, Goldman proved the following theorem.

THEOREM 2.13. *([**Go88**]) The Euler class is bounded as $|e(\rho)| \leqq 2g - 2$, and there is a one-to-one correspondence between $e(\rho)$ and connected components of $X_2(S)$. Hence, the number of connected components of $X_2(S)$ is equal to $4g - 3$. Moreover $|e(\rho)| = 2g - 2$ if and only if ρ is discrete.*

This theorem implies that $X_2(S)$ contains two connected components which diffeomorphic to $\mathscr{T}(S)$. The Teichmüller space is often called the *Teichmüller component* of $X_2(S)$.

The case of $n \geq 3$, Hitchin detected the number of connected components of $X_n(S)$. Using Higgs bundle techniques, he proved the following theorem.

THEOREM 2.14. *([**Hi92**]) Let $n \geq 3$. The number of connected components of $X_n(S)$ is equal to 3 if n is odd, and 6 if n is even.*

In addition, He found the special component in $X_n(S)$ associated to $\mathscr{T}(S)$. Let us consider an irreducible representation $\mathrm{SL}_2\mathbb{R} \to \mathrm{SL}_n\mathbb{R}$ which is unique up to conjugacy. This representation is obtained by the symmetric power of the natural representation $(\mathrm{SL}_2\mathbb{R}, \mathbb{R}^2)$. We denote its projectivization $\mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_n\mathbb{R}$ by ι_n. The representation ι_n induces a map between character varieties $(\iota_n)_*\colon X_2(S) \to X_n(S)$ by the correspondence $\rho \mapsto \iota_n \circ \rho$. Since ι_n is a group homomorphism, this induced map is well-defined.

DEFINITION 2.15. *The* ($\mathrm{PSL}_n\mathbb{R}$-) *Hitchin component* $H_n(S)$ *is the connected component of* $X_n(S)$ *which contains the image* $F_n(S) = (\iota_n)_*(\mathscr{T}(S))$.

We call the image $F_n(S)$ of $\mathscr{T}(S)$ the *Fuchsian locus* of $H_n(S)$. *Hitchin representations* are representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ whose conjugacy class belongs to $H_n(S)$. A Hitchin representation ρ is $\mathrm{PSL}_n\mathbb{R}$-*Fuchsian* if ρ is contained in $F_n(S)$, *i.e.* there is a Fuchsian representation $\rho_0\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ such that $\rho = \iota_n \circ \rho_0$.

The diffeomorphic type of Hitchin components is as follows.

THEOREM 2.16 (Hitchin [**Hi92**]). *The Hitchin component* $H_n(S)$ *is diffeomorphic to* $\mathbb{R}^{(2g-2)(n^2-1)}$.

Moreover, $H_n(S)$ consists of faithful discrete representations. This fact was shown by Labourie [**La06**] from the Anosov property of Hitchin representations, see Section 2.4 in this chapter.

2.3. Hyperconvex property. The projective special linear group $\mathrm{PSL}_n\mathbb{R}$ acts on the projective space $\mathbb{RP}^{n-1} = P(\mathbb{R}^n)$ by the projectivization of the linear action of $\mathrm{SL}_n\mathbb{R}$ on $\mathbb{R}^n$. We define the hyperconvexity of projective linear representations of $\pi_1(S)$. Let $\partial\pi_1(S)$ be the ideal boundary of $\pi_1(S)$ which is the visual boundary of a Cayley graph of $\pi_1(S)$. Note that $\partial\pi_1(S)$ is homeomorphic to $\partial\widetilde{S}$ through a hyperbolic structure of S. Therefore, in this paper, we identify $\partial\pi_1(S)$ with $\partial\widetilde{S}$ by using the reference hyperbolic structure of S.

DEFINITION 2.17. *A representation* $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ *is said to be hyperconvex if there exists a* $(\pi_1(S), \rho)$*-equivariant continuous map* $\xi_\rho\colon \partial\pi_1(S) \to \mathbb{RP}^{n-1}$ *such that* $\xi_\rho(x_1) + \cdots + \xi_\rho(x_n)$ *is direct for any pairwise distinct points* $x_1, \cdots, x_n \in \partial\pi_1(S)$.

The associated curve ξ_ρ is called the *hyperconvex curve* of ρ. Labourie showed that Hitchin representations are hyperconvex by the Anosov property which is explained in the next subsection. The converse result was shown by Guichard in [**Gu08**], so

THEOREM 2.18 (Guichard [**Gu08**], Labourie [**La06**]). *A representation* $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ *is Hitchin if and only if* ρ *is hyperconvex.*

In addition, Labourie showed the following theorem.

THEOREM 2.19 ([**La06**]). *Let* $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ *be a hyperconvex representation with the hyperconvex curve* $\xi_\rho\colon \partial\pi_1(S) \to \mathbb{RP}^{n-1}$. *Then there exists a unique curve* $\xi_\rho^k\colon \partial\pi_1(S) \to \mathrm{Gr}^k(\mathbb{R}^n)$ *with the properties from (i) to (iv) below.*

(i) $\xi^p(x) \subset \xi^{p+1}(x)$ *for any* $x \in \partial\pi_1(S)$.
(ii) $\xi^1(x) = \xi_\rho(x)$ *for any* $x \in \partial\pi_1(S)$.
(iii) *If* $n_1, \cdots, n_l$ *are positive integers such that* $\sum n_i \le n$, *then* $\xi^{n_1}(x_1) + \cdots + \xi^{n_l}(x_l)$ *is direct for any pairwise distinct points* $x_1, \cdots, x_l \in \partial\pi_1(S)$.
(iv) *If* $n_1, \cdots, n_l$ *are positive integers such that* $p = \sum n_i \le n$, *then*

$$\lim_{(y_1,\cdots,y_l)\to x;\ y_i\, distinct} \xi^{n_1}(y_1) + \cdots + \xi^{n_l}(y_l) \to \xi^p(x)$$

This theorem implies that any hyperconvex curves are extended to curves in the flag manifold. (See Section 3.1 in this chapter for the precise definition of flags.) The map $(\xi^1, \cdots, \xi^{n-1})\colon \partial\pi_1(S) \to \mathrm{Flag}(\mathbb{R}^n)$ is called the *(osculating) flag curve* of the hyperconvex curve ξ_ρ.

We can explicitly describe the hyperconvex curve of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations. Let $\rho_n = \iota_n \circ \rho$ be a $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation. Recall that the irreducible representation ι_n is defined by symmetric power of the representation $(\mathrm{SL}_2\mathbb{R}, \mathbb{R}^2)$. We identify $\mathbb{R}^n$ with $Sym^{n-1}(\mathbb{R}^2)$. Consider the Veronese embedding $\nu\colon \mathbb{RP}^1 \to \mathbb{RP}^{n-1}$ defined by sending $[a:b]$ to $[a^{n-1} : a^{n-2}b : \cdots : b^{n-1}]$. Then the composition $\nu \circ f_\rho$ of the Veronese embedding with the developing map gives the hyperconvex curve of ρ_n. Using homogeneous polynomials, the flag is also described explicitly. The symmetric power $Sym^{n-1}(\mathbb{R}^2)$, which is identified with $\mathbb{R}^n$, is also identified with the vector space

$$\mathrm{Poly}_n(X,Y) = \{a_1X^{n-1} + a_2X^{n-2}Y + \cdots + a_nY^{n-1} \mid a_i \in \mathbb{R}\}$$

of homogeneous polynomials of degree $n-1$. If we denote the canonical basis of $Sym^{n-1}(\mathbb{R}^2)$ by $e_1^{n-1}, e_1^{n-2}\cdot e_2, \cdots, e_2^{n-1}$, where e_1, e_2 are the canonical basis of $\mathbb{R}^2$, the identification is defined by mapping the vector $e_1^i \cdot e_2^{n-1-i}$ to $\binom{n-1}{i}X^iY^{n-1-i}$. Then the one dimensional subspace $\nu([a:b])$ is equal to $\mathbb{R}\{(aX+bY)^{n-1}\}$ in the vector space $\mathrm{Poly}_n(X,Y)$. In addition, the d-dimensional subspace of the flag curve associated to

ν, which is again denoted by ν, is defined by

$$\{P(X,Y) \in \mathrm{Poly}_n(X,Y) \mid P(X,Y) \text{ can be divided by } (aX+bY)^{n-d}\}.$$

We call the flag curve ν the *Veronese flag curve*. The composition $\nu \circ f_\rho$ of the Veronese flag curve with the developing map is just the flag curve of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations.

2.4. Anosov property. The existence of the flag curve of Hitchin representations follows from the Anosov property. Let G be a semisimple Lie group, and P be a parabolic subgroup, that is, the stabilizer of a point of the visual boundary of the Riemannian symmetric space G/K. A representation $\rho\colon \pi_1(S) \to G$ is said to be P-Anosov if there exists a continuous ρ-equivariant map $\xi_\rho\colon \partial_\infty \pi_1(S) \to G/P$ with a certain dynamical property with respect to the action of $\rho(\pi_1(S))$. In general case, the dynamical property is defined by the Cartan projection of G, and the definition is not short. However, in the case of $G = \mathrm{PSL}_n\mathbb{R}$, we can define the Anosov property more simply and more explicitly. Let $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ be the singular values of $A \in \mathrm{PSL}_n\mathbb{R}$. For $1 \leq k \leq \frac{n}{2}$, a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ is said to be P_k-Anosov if there exist constants $A, C > 0$ such that $s_k(\rho(\gamma))/s_{k+1}(\rho(\gamma)) \geq A \exp C|\gamma|$.

Especially, when ρ is P_k-Anosov for all k, ρ is called Borel-Anosov. In [**La06**], Labourie showed Hitchin representations are Borel-Anosov. Moreover it was also shown by the Borel-Anosov property that Hitchin representations are faithful, discrete, irreducible and purely-loxodromic.

For a Borel-Anosov representation ρ, through the argument with the action on the symmetric space G/K and its Furstenberg boundary G/B, we obtain a continuous ρ-equivariant map $\xi\colon \partial_\infty \pi_1(S) \to G/B$, called the boundary map of ρ. The boundary maps are the analog of the limit set of discrete subgroups of rank 1. In particular, when $G = \mathrm{PSL}_n\mathbb{R}$, the boundary G/B is the (complete) flag manifold $\mathrm{Flag}(\mathbb{R}^n)$. Recall that Hitchin representations are hyperconvex, and they have the osculating flag curves. These osculating flag curves are just equal to the boundary maps of the Borel-Anosov property.

REMARK 2.20. *For general references of Anosov representations/ Anosov subgroups, see Guéritaud-Guichard-Kassel-Wienhard* [**GGKW17**], *Kapovich-Leeb-Porti* [**KLP17**]. *The original definition was given by*

Labourie [**La06**] *for surface groups, and by Guichard-Wienhard* [**GW12**] *for Gromov hyperbolic groups.*

3. The Bonahon-Dreyer parameterization

3.1. Projective invariants. We define projective invariants of tuples of flags. A (complete) *flag* in $\mathbb{R}^n$ is a sequence of nested vector subspaces of $\mathbb{R}^n$

$$\{0\} = F^0 \subset F^1 \subset F^2 \subset \cdots \subset F^n = \mathbb{R}^n,$$

where dim $F^d = d$. The *flag manifold* of $\mathbb{R}^n$ is a set of flags in $\mathbb{R}^n$. We denoted the flag manifold by $\mathrm{Flag}(\mathbb{R}^n)$. Note that $\mathrm{Flag}(\mathbb{R}^n)$ is diffeomorphic to a homogeneous space $\mathrm{PSL}_n\mathbb{R}/B$, where B is a Borel subgroup of $\mathrm{PSL}_n\mathbb{R}$, and $\mathrm{PSL}_n\mathbb{R}$ naturally acts on the flag manifold. A *generic* tuple of flags is a tuple $(F_1, F_2, \cdots, F_k)$ of a finite number of flags $F_1, F_2, \cdots, F_k \in \mathrm{Flag}(\mathbb{R}^n)$ such that if $n_1, \cdots, n_k$ are nonnegative integers satisfying $n_1 + \cdots + n_k = n$, then $F_1^{n_1} \cap \cdots \cap F_k^{n_k} = \{0\}$.

Let (E, F, G) be a generic triple of flags, and $p, q, r \geq 1$ integers with $p + q + r = n$. For each $d = 1, \cdots, n$, choose a basis "e^d, f^d, g^d" of the wedge products "$\bigwedge^d E^d, \bigwedge^d F^d, \bigwedge^d G^d$", respectively. We fix an identification between $\bigwedge^n \mathbb{R}^n$ with $\mathbb{R}$. Then we can regard $e^{d_1} \wedge f^{d_2} \wedge g^{d_3}$ as an element of $\mathbb{R}$ when $d_1 + d_2 + d_3 = n$. In particular $e^{d_1} \wedge f^{d_2} \wedge g^{d_3}$ is not equal to 0 since (E, F, G) is generic.

Definition 2.21. *The (p, q, r)-th triple ratio $T_{pqr}(E, F, G)$ is defined by*

$$T_{pqr}(E, F, G) = \frac{e^{p+1} \wedge f^q \wedge g^{r-1} \cdot e^p \wedge f^{q-1} \wedge g^{r+1} \cdot e^{p-1} \wedge f^{q+1} \wedge g^r}{e^{p-1} \wedge f^q \wedge g^{r+1} \cdot e^p \wedge f^{q+1} \wedge g^{r-1} \cdot e^{p+1} \wedge f^{q-1} \wedge g^r} \in \mathbb{R}.$$

The value of $T_{pqr}(E, F, G)$ is independent of the identification $\bigwedge^n \mathbb{R}^n \cong \mathbb{R}$ and the choice of elements e^d, f^d, g^d. If one of exponents of e^d, f^d, g^d is equal to 0, then we ignore the corresponding terms. For example, $e^0 \wedge f^q \wedge g^{n-q} = f^q \wedge g^{n-q}$. The triple ratio is invariant under the action of $\mathrm{PSL}_n\mathbb{R}$.

For permutations of (E, F, G), the triple ratio behaves as follows.

Proposition 2.22. *For every generic triples (E, F, G) of flags,*

$$T_{pqr}(E, F, G) = T_{qrp}(F, G, E) = T_{qpr}(F, E, G)^{-1}.$$

Let (E, F, G, G') be a generic quadruple of flags, and b be an integer with $1 \leq b \leq n-1$. We choose nonzero elements "e^d, f^d, g^d, g'^d" of "$\bigwedge^d E^d, \bigwedge^d F^d, \bigwedge^d G^d, \bigwedge^d G'^d$" respectively. We fix an identification $\bigwedge^n \mathbb{R}^n \cong \mathbb{R}$ again. Then, $e^{d_1} \wedge f^{d_2} \wedge g^{d_3}$ and $e^{d_1} \wedge f^{d_2} \wedge g'^{d_3}$ are also regarded as real values when $d_1 + d_2 + d_3 = n$.

DEFINITION 2.23. *The b-th double ratio $D_b(E, F, G, G')$ is defined by*

$$D_b(E, F, G, G') = -\frac{e^b \wedge f^{n-b-1} \wedge g^1 \cdot e^{b-1} \wedge f^{n-b} \wedge g'^1}{e^b \wedge f^{n-b-1} \wedge g'^1 \cdot e^{b-1} \wedge f^{n-b} \wedge g^1} \in \mathbb{R}.$$

This is well-defined since the ratio is independent of the choice of $\bigwedge^n \mathbb{R}^n \cong \mathbb{R}$ and e^d, f^d, g^d, g'^d. The double ratio is also invariant under the action of $\mathrm{PSL}_n\mathbb{R}$.

3.2. The Bonahon-Dreyer parameterization for finite laminations.

3.2.1. *Construction of invariants.* We define three kinds of invariants of Hitchin representations, *triangle invariant*, *shearing invariant*, and *twist invariant* for an oriented maximal geodesic lamination which consists of finitely many leaves with a bridge system. We fix a hyperbolic metric on S, and an oriented maximal geodesic lamination λ on S. We suppose that λ consists only of closed leaves $C_1, \cdots, C_k$ and bi-infinite leaves $B_1, \cdots, B_{3|\chi(S)|}$. In addition, we fix a bridge system $\mathcal{J} = \{J_{C_i}\}_i$ of λ. The lamination λ induces an ideal triangulation of S by ideal triangles $T_1, \cdots, T_{2|\chi(S)|}$. Each ideal triangle T_i has three spikes. We denote these spikes by s_0^i, s_1^i, s_2^i so that, in a lift $\widetilde{T}_i$ of T_i, their corresponding ideal vertices of $\widetilde{T}_i$ are in clockwise order. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ be a Hitchin representation and $\xi_\rho\colon \partial\pi_1(S) \to \mathrm{Flag}(\mathbb{R}^n)$ the associated flag curve.

Let T_i be an ideal triangle, and choose a spike s_j^i of T_i. Fix a lift $\widetilde{T}_i$ of T_i. We denote the ideal vertex corresponding to s_j^i by v. In addition, we denote the other vertices of $\widetilde{T}_i$ by v', v'' so that v, v', v'' are in clockwise order. Let p, q, r be integers such that $p, q, r \geq 1$ and $p + q + r = n$.

DEFINITION 2.24. *The (p, q, r)-th triangle invariant $\tau_{pqr}(s_j^i, \rho)$ of a Hitchin representation ρ associated to the spike s_j^i of the ideal triangle*

T_i is defined by

$$\tau_{pqr}(s^i_j, \rho) = \log T_{pqr}(\xi_\rho(v), \xi_\rho(v'), \xi_\rho(v'')).$$

The triangle invariant is independent of a choice of the lift $\widetilde{T}_i$ since flag curves are ρ-equivariant and the triple ratio is invariant under the $\mathrm{PSL}_n\mathbb{R}$-action. By Proposition 2.22, we have the relation between triangle invariants:

$$\tau_{pqr}(s^i_0, \rho) = \tau_{qrp}(s^i_1, \rho) = \tau_{rpq}(s^i_2, \rho).$$

This relation is called the *rotation condition*, and is going to be used to define the parameter space.

A bi-infinite leaf $B_i \in \lambda_{\mathcal{J}}$ is a side of two ideal triangles. Let T^L (resp. T^R) be the ideal triangle which is on the left (resp. right) side with respect to the orientation of B_i. We lift B_i to a geodesic $\widetilde{B}_i$ in $\widetilde{S}$, and we also lift T^L and T^R to two ideal triangles $\widetilde{T}^L$ and $\widetilde{T}^R$ so that they are adjacent along $\widetilde{B}_i$. We denote the repelling point and the attracting point of $\widetilde{B}_i$ by y and x, and denote the other vertices of $\widetilde{T}^L$ (resp. $\widetilde{T}^R$) by z^L (resp. z^R). Let b be an integer with $1 \leq b \leq n-1$.

Definition 2.25. *The b-th shearing invariant of a Hitchin representation ρ along B_i is defined by*

$$\sigma_b(B_i, \rho) = \log D_b(\xi_\rho(x), \xi_\rho(y), \xi_\rho(z^L), \xi_\rho(z^R)).$$

This invariant is also well-defined for a choice of lifts by the same reason with the case of triangle invariants.

Consider a closed (oriented) leaf $C_i \in \lambda_{\mathcal{J}}$. By the bridge system $\mathcal{J}$, we have a bridge $J_{C_i} = \{T^L_i, T^R_i\}$ associated to C_i. Here T^L_i spirals to C_i from the left, and T^R_i spirals to C_i from the right. Lift C_i, T^L_i and T^R_i to $\widetilde{C}_i$, $\widetilde{T}^L_i$ and $\widetilde{T}^R_i$ in the universal covering so that the ideal triangles $\widetilde{T}^L_i$, $\widetilde{T}^R_i$ have a common ideal vertex with $\widetilde{C}_i$. We denote, by x, the attracting point of $\widetilde{C}_i$ and, by y, the repelling point of $\widetilde{C}_i$. Let us define the vertex z^L, z^R of ideal triangles $\widetilde{T}^L_i, \widetilde{T}^R_i$ as follows. Note that two sides of $\widetilde{T}^L_i$ are asymptotic to $\widetilde{C}_i$. One of these sides cuts the universal cover $\widetilde{S}$ such that an ideal triangle $\widetilde{T}^L_i$ and the geodesic $\widetilde{C}_i$ is contained in the same connected component. The ideal vertex z^L is the end point of such a geodesic side of $\widetilde{T}^L_i$ other from the ideal point x or y. We define z^R for $\widetilde{T}_i^R$ similarly. Let c be an integer with $1 \leq c \leq n-1$.

Definition 2.26. *The c-th twist invariant of a Hitchin representation ρ along C_i is defined by*

$$\theta_c(C_i, \rho) = \log D_c(\xi_\rho(x), \xi_\rho(y), \xi_\rho(z^L), \xi_\rho(z^R)).$$

The invariants above are well-defined on Hitchin components *i.e.* these three invariants are independent of representatives of conjugacy classes of Hitchin representations.

3.2.2. *The Bonahon-Dreyer parameterization.* Set $N = 6|\chi(S)|\binom{n-1}{2} + 3|\chi(S)|(n-1) + k(n-1)$. Bonahon and Dreyer showed that Hitchin representations are parameterized by the all triangle invariants, shearing invariants, and twist invariants we can define.

Theorem 2.27 (Bonahon-Dreyer [**BD14**]). *The map $\Phi_{\lambda_{\mathcal{J}}} \colon H_n(S) \to \mathbb{R}^N$ defined by*

$$\Phi_{\lambda_{\mathcal{J}}}(\rho) = (\tau_{pqr}(s_j^i, \rho), \cdots, \sigma_b(B_i, \rho), \cdots, \theta_c(C_i, \rho), \cdots)$$

is an analytic homeomorphism onto the image. Moreover the image of this map is the interior $\mathcal{P}_{\lambda_{\mathcal{J}}}$ of a convex polyhedron.

We denote the coordinate of the image by

$$(\tau_{pqr}(s_j^i), \cdots, \sigma_b(B_i), \cdots, \theta_c(C_i), \cdots).$$

3.2.3. *The parameter space $\mathcal{P}_{\lambda_{\mathcal{J}}}$.* The range $\mathcal{P}_{\lambda_{\mathcal{J}}}$ is defined by the rotation condition referred after Definition 20, and the closed leaf condition defined as follows. This condition is given by the equality and the inequality of triangle, shearing invariants associated to closed leaves C. Let C be a closed oriented leaf of the lamination λ. We focus on the right side of C with respect to the orientation of C. Let $B_1, \cdots, B_l$ be the bi-infinite leaves spiraling to C from the right, and $T_1, \cdots, T_l$ be the ideal triangles spiraling to C from the right. Suppose that these leaves and ideal triangles spiral to C in the direction (resp. the opposite direction) of the orientation of C. Let s_i is the spike of T_i which is asymptotic to C. Define $\overline{\sigma}_b(B_i)$ by $\sigma_b(B_i)$ if B_i is oriented toward C, and by $\sigma_{n-b}(B_i)$ otherwise. We define

$$R_b(C) = \sum_{i=1}^{l} \overline{\sigma}_b(B_i) + \sum_{i=1}^{l} \sum_{q+r=n-b} \tau_{bqr}(s_i)$$

in the former case, and

$$R_b(C) = -\sum_{i=1}^{l} \overline{\sigma}_{n-b}(B_i) - \sum_{i=1}^{l} \sum_{q+r=b} \tau_{(n-b)qr}(s_i)$$

in the latter case.

When we focus on the left side of C, we can similarly define $L_b(C)$ by

$$L_b(C) = -\sum_{i=1}^{l} \overline{\sigma}_b(B_i) - \sum_{i=1}^{l} \sum_{q+r=n-b} \tau_{bqr}(s_i)$$

if the spiraling is in the direction, and

$$L_b(C) = \sum_{i=1}^{l} \overline{\sigma}_{n-b}(Bi) + \sum_{i=1}^{l} \sum_{q+r=b} \tau_{(n-b)qr}(s_i)$$

if the spiraling is in the opposite direction.

The closed leaf equality for C is the equality $L_b(C) = R_b(C)$, and the closed leaf inequality for C is the inequality $L_b(C), R_b(C) > 0$. The rotation condition for all spikes and the closed leaf condition for all closed leaves define $\mathcal{P}_{\lambda_{\mathcal{J}}}$. See [**BD14**] for details.

3.3. The Bonahon-Dreyer parameterization for general laminations. In the previous subsection, we recall the Bonahon-Dreyer parameterization for laminations with finitely many leaves, which is a higher dimensional analog of Theorem 2.6. In this subsection, we recall the Bonahon-Dreyer parameterization for general laminations, which is a higher dimensional analog of Theorem 2.4. In the following, we fix a maximal geodesic lamination λ on S which may contain an irrational lamination.

3.3.1. *Relative tangent cycles.* A relative $\mathbb{R}^{n-1}$-valued tangent cycle is roughly a twisted transverse cocycle of λ, which is an association of vectors in $\mathbb{R}^{n-1}$ to oriented tightly transverse arcs. A *tightly* transverse arc k of λ is an arc transverse to λ with the following properties:

(i) k is contained in a fixed small train track neighborhood of λ, and
(ii) if a component d of $k \setminus \lambda$ contains no end points of k, then d cuts only one spike.

Here "spike" means a spike of an ideal triangle, which is a complementary region of λ. We denote the set of such spikes by $\mathfrak{s}_\lambda$. The tightness of a transverse arc k implies that every components of $k \setminus \lambda$, which contains no end points of k, pass near a spike $s \in \mathfrak{s}_\lambda$.

A relative $\mathbb{R}^{n-1}$-valued tangent cycle α for λ is an assignment of a vector $\alpha(k) \in \mathbb{R}^{n-1}$ to each oriented arc k tightly transverse to λ with the homotopy invariance respecting λ, and the quasi-additivity defined below.

Consider the splitting of k to k_1 and k_2 at an interior point of a component d of $k \setminus \lambda$, where d has no end points of k. Let $s \in \mathfrak{s}_\lambda$ be a spike, which corresponds to d. Then we require that there exists a vector $\partial\alpha(s) \in \mathbb{R}^{n-1}$ such that

$$\alpha(k) = \alpha(k_1) + \alpha(k_2) - \partial\alpha(s)$$

if k passes in counterclockwise direction for s, and

$$\alpha(k) = \alpha(k_1) + \alpha(k_2) + \partial\alpha(s)$$

if k passes in clockwise direction for s. This property is called the *quasi-additivity*. We call the correspondence $\partial\alpha\colon \mathfrak{s}_\lambda \to \mathbb{R}^{n-1}$ the *boundary* of α. We denote the space of relative $\mathbb{R}^{n-1}$-valued tangent cycles of λ by $Z(\lambda, \mathrm{slits}; \mathbb{R}^{n-1})$ following [**BD17**]. We remark that, in this paper, "slits" simply means $\mathfrak{s}_\lambda$.

3.3.2. *Slithering maps and shearing classes.* Let ρ be a Hitchin representation and ξ_ρ be the associated flag curve. We denote, by $\widetilde{\lambda}$, the preimage of λ into the universal covering of S. The slithering map is a family of elements $\Sigma_{gg'} \in \mathrm{SL}_n\mathbb{R}$ associated to all pairs of leaves of $\widetilde{\lambda}$ which is uniquely determined by the following conditions:

(i) $\Sigma_{gg} = \mathrm{Id}_{\mathbb{R}^n}, \Sigma_{g'g} = \Sigma_{gg'}^{-1}$, and $\Sigma_{gg''} = \Sigma_{gg'} \circ \Sigma_{g'g''}$ if g, g', g'' are leaves of $\widetilde{\lambda}$ such that g, g'' are separated by g',
(ii) $\Sigma_{gg'}$ depends locally Hölder continuously on g and g',
(iii) if g and g' have a common ideal vertex, then $\Sigma_{gg'}$ naturally sends the associated line decomposition of g' to the line decomposition of g.

In the condition (iii), the line decomposition associated to a leaf g is defined as follows. Fix an orientation of g. Let x be its attracting point, and y be its repelling point. We set $F^+ = \xi_\rho(x)$ and $F^- = \xi_\rho(y)$. By the hyperconvexity, the intersection $L_b(g) = (F^+)^b \cap (F^-)^{n-b+1}$ are one dimensional subspaces for every $b = 1, \cdots, n$, and give a decomposition of $\mathbb{R}^n = \oplus_{b=1}^n L_b(g)$. If two geodesics g, g' have a common vertex x, we orient g, g' so that x is the attracting point with respect to the orientation. The condition (iii) says that $\Sigma_{gg'}$ is a unipotent special linear transformation which sends $L_b(g')$ to $L_b(g)$ for all $b = 1, 2, \cdots, n$.

The shearing class σ^ρ of a Hitchin representation ρ is one of relative $\mathbb{R}^{n-1}$-valued tangent cycles defined by the flag curve ξ_ρ. Let k be a tightly transverse oriented arc of λ. We define $\sigma_b^\rho(k)$ $(1 \leq b \leq n-1)$ as follows. Consider two plaques which contains the endpoints of a lift $\widetilde{k}$ of k in the universal covering. Note that $\widetilde{k}$ is also oriented from the orientation of k. We denote the plaque containing the starting (resp. terminal) point of $\widetilde{k}$ by P (resp. Q). Let g (resp. g') be the side of P (resp. Q) which are nearest to Q (resp. P). Let x, y, z be the ideal vertices of P and z' be the ideal vertex defined as Figure 4. Then, for $b = 1, 2, \cdots, n-1$, we define

$$\sigma_b^\rho(k) = \log[D_b(\xi_\rho(x), \xi_\rho(y), \xi_\rho(z), \Sigma_{gg'}\xi_\rho(z'))].$$

Combining these, we define $\sigma^\rho(k) = \left(\sigma_b^\rho(k)\right)_b \in \mathbb{R}^{n-1}$. We call this vector valued cocycle σ^ρ the *shearing class* of a Hitchin representation ρ. The shearing class has the homotopy invariance respecting to λ, and has the quasi-additivity, so this is a relative tangent cycle. For more details, see [**BD17**, Section 5].

3.3.3. *The Bonahon-Dreyer parameterization for general laminations.* In general cases, the Hitchin components are parameterized by the shearing classes and the triangle invariants. By the maximality, λ

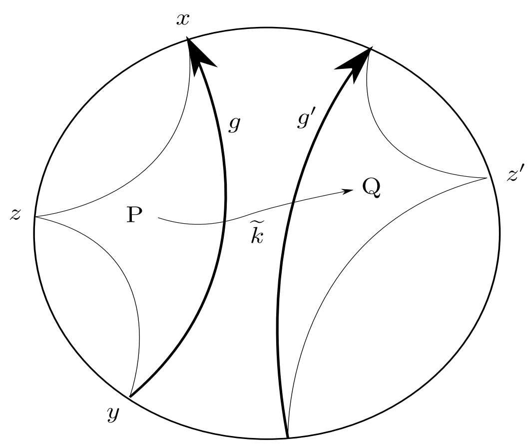


FIGURE 9. Ideal vertices x, y, z, z'.

induces an ideal triangulation of S. Let T_i be ideal triangles obtained by the ideal triangulation of λ, and $s^i_j \in \mathfrak{s}_\lambda$ be its spikes.

THEOREM 2.28 ([**BD17**]). *The map* $\Phi_\lambda \colon H_n(S) \to Z(\lambda, \mathrm{slits}; \mathbb{R}^{n-1}) \times \mathbb{R}^{6|\chi(S)|\binom{n-1}{2}}$ *defined by* $\Phi_\lambda(\rho) = (\sigma^\rho, \tau_{pqr}(s^i_j, \rho))$ *is a homeomorphism onto the interior* $\mathcal{P}_\lambda$ *of a convex polyhedron.*

3.3.4. *The parameter space* $\mathcal{P}_\lambda$. The image of Φ_λ is determined by three conditions, the rotation condition, the shearing cycle boundary condition, and the positive intersection condition. The rotation condition is the same as the case of laminations with finitely many leaves. The shearing cycle boundary condition is given by the following relation. For every spikes $s \in \mathfrak{s}_\lambda$,

$$\partial\sigma^\rho_b(s) = \sum_{q+r=n-b} \tau_{bqr}(s, \rho),$$

where $\partial\sigma^\rho_b$ is the boundary of the b-th entry of the coordinate σ^ρ.

The positive intersection condition is defined by a homological interpretation of relative tangent cycles. Each entry σ^ρ_b of relative tangent cycles σ^ρ of λ can be translated to relative homology classes defined from train track neighborhoods of λ. The positive intersection condition is the inequality

$$\mu \cdot \sigma_b^\rho > 0$$

for all non-trivial transverse measures μ. Here the intersection number is defined in the homological sense.

These three conditions define the range $\mathcal{P}_\lambda$ of the Bonahon-Dreyer parameterization. See [**BD17**, Section 8] for more details.

CHAPTER 3

The ratios of the Veronese flag curves

1. An observation of the Bonahon-Dreyer parameters of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations for a pair of pants

1.1. Formulae. Here we try to calculate the triangle invariants and shearing invariants of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations for a pair of pants, using λ, and the parameter α, β, γ in Proposition 2.12. Let ρ be a Fuchsian representation defined by

$$\rho(a) = \begin{bmatrix} \alpha & \alpha\beta\gamma + \alpha^{-1} \\ 0 & \alpha^{-1} \end{bmatrix}, \ \rho(b) = \begin{bmatrix} \gamma & 0 \\ -\beta^{-1} - \gamma^{-1} & \gamma^{-1} \end{bmatrix}$$

It suffices to calculate the following invariants.

- $\tau_{pqr}^{\rho_n}(T_0, \infty) = \log T_{pqr}(\nu(\infty), \nu(1), \nu(0))$
- $\tau_{pqr}^{\rho_n}(T_1, \infty) = \log T_{pqr}(\nu(\infty), \nu(0), \nu(-\beta\gamma))$
- $\sigma_b^{\rho_n}(h_{AB}) = \log D_b(\nu(\infty), \nu(0), \nu(\beta\gamma), \nu(1))$
- $\sigma_b^{\rho_n}(h_{BC}) = \log D_b(\nu(0), \nu(1), \nu(\beta/\beta + \gamma), \nu(\infty))$
- $\sigma_b^{\rho_n}(h_{CA}) = \log D_b(\nu(1), \nu(\infty), \nu(\alpha^2\beta\gamma + 1), \nu(0))$

The direct calculation, the above invariants are formulated as follows. See [**In2**] for the detail of computations.

- Shearing invariant $\sigma_p^{\rho_n}(h_{AB}) (1 \leq p \leq n-1)$

$$\sigma_p^{\rho_n}(h_{AB}) = \log -\frac{Y_{h_{AB}}(p)}{Y'_{h_{AB}}(p)} \cdot \frac{Y'_{h_{AB}}(p-1)}{Y_{h_{AB}}(p-1)}$$

  where

$$Y_{h_{AB}}(p) = \binom{n-1}{p} (\beta\gamma)^{n-p-1},$$

$$Y'_{h_{AB}}(p) = (-1)^{n-p-1} \binom{n-1}{p}.$$

- Shearing invariant $\sigma_p^{\rho_n}(h_{BC})(1 \leq p \leq n-1)$

$$\sigma_p^{\rho_n}(h_{BC}) = \log -\frac{Y_{h_{BC}}(p)}{Y'_{h_{BC}}(p)} \cdot \frac{Y'_{h_{BC}}(p-1)}{Y_{h_{BC}}(p-1)}$$

where

$$Y_{h_{BC}}(p) = (-1)^{(n-p)p} \begin{vmatrix} \binom{p+1}{0} & \cdots & \binom{p+1}{-n+p+2} & \binom{n-1}{0}(\frac{\beta}{\beta+\gamma})^{n-1} \\ \vdots & \vdots & \vdots & \\ \binom{p+1}{n-p-1} & \cdots & \binom{p+1}{1} & \binom{n-1}{n-p-1}(\frac{\beta}{\beta+\gamma})^{p} \end{vmatrix}$$

if $p \neq n-1$ and $Y_{h_{BC}}(n-1) = (-1)^{n-1}\binom{n-1}{0}(\frac{\beta}{\beta+\gamma})^{n-1}$,

$$Y'_{h_{BC}}(p) = (-1)^{np+n+1} \begin{vmatrix} \binom{p+1}{1} & \cdots & \binom{p+1}{-n+p+3} \\ \vdots & & \vdots \\ \binom{p+1}{n-p-1} & \cdots & \binom{p+1}{1} \end{vmatrix}$$

if $p \neq n-1$ and $Y'_{h_{BC}}(n-1) = (-1)^{n-1}$.

- Shearing invariant $\sigma_p^{\rho_n}(h_{CA})(1 \leq p \leq n-1)$

$$\sigma_p^{\rho_n}(h_{CA}) = \log -\frac{Y_{h_{CA}}(p)}{Y'_{h_{CA}}(p)} \cdot \frac{Y'_{h_{CA}}(p-1)}{Y_{h_{CA}}(p-1)}$$

where

$$Y_{h_{CA}}(p) = (-1)^{np} \begin{vmatrix} \binom{n-p}{n-p-1} & \cdots & \binom{n-p}{n-2p} & \binom{n-1}{n-p-1}(\alpha^2\beta\gamma+1)^{p} \\ \vdots & & \vdots & \vdots \\ \binom{n-p}{n-1} & \cdots & \binom{n-p}{n-p} & \binom{n-1}{n-1}(\alpha^2\beta\gamma+1)^{0} \end{vmatrix}$$

if $p \neq 0$ and $Y_{h_{CA}}(0) = 1$,

$$Y'_{h_{CA}}(p) = (-1)^{np} \begin{vmatrix} \binom{n-p}{n-p-1} & \cdots & \binom{n-p}{n-2p} \\ \vdots & \vdots & \vdots \\ \binom{n-p}{n-2} & \cdots & \binom{n-p}{n-p-1} \end{vmatrix}$$

if $p \neq 0$ and $Y'_{h_{CA}}(0) = 1$.

- Triangle invariant $\tau^{\rho_n}_{pqr}(\widetilde{T}_0,\infty)(p,q,r\geq 1\ s.t.\ p+q+r=n)$

$$\tau^{\rho_n}_{pqr}(\widetilde{T}_0,\infty)=\log\frac{X_{T_0}(p+1,q,r-1)}{X_{T_0}(p-1,q,r+1)}\cdot\frac{X_{T_0}(p,q-1,r+1)}{X_{T_0}(p,q+1,r-1)}\cdot\frac{X_{T_0}(p-1,q+1,r)}{X_{T_0}(p+1,q-1,r)}$$

where

$$X_{T_0}(p,q,r)=\begin{vmatrix}\binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1}\\ \vdots & \vdots & \vdots\\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p}\end{vmatrix}$$

if $q\neq 0$ and $X_{T_0}(p,0,r)=1$ for all p,r.

- Triangle invariant $\tau^{\rho_n}_{pqr}(\widetilde{T}_1,\infty)(p,q,r\geq 1\ s.t.\ p+q+r=n)$

$$\tau^{\rho_n}_{pqr}(\widetilde{T}_1,\infty)=\log\frac{X_{T_1}(p+1,q,r-1)}{X_{T_1}(p-1,q,r+1)}\cdot\frac{X_{T_1}(p,q-1,r+1)}{X_{T_1}(p,q+1,r-1)}\cdot\frac{X_{T_1}(p-1,q+1,r)}{X_{T_1}(p+1,q-1,r)}$$

where

$$X_{T_1}(p,q,r)=(-1)^{q(r+1)}\begin{vmatrix}\binom{p+q}{p}(-\beta\gamma)^q & \cdots & \binom{p+q}{p-r+1}(-\beta\gamma)^{q+r-1}\\ \vdots & \vdots & \vdots\\ \binom{p+q}{p+r-1}(-\beta\gamma)^{q-r+1} & \cdots & \binom{p+q}{p}(-\beta\gamma)^q\end{vmatrix}$$

if $r\neq 0$ and $X_{T_1}(p,q,0)=(-1)^q$ for all p,q.

The shearing invariant $\sigma^{\rho_n}_p(h_{AB})$ can be reduce to $\log(1/\beta\gamma)$. However, other invariants seem to be difficult to reduce into more simple values.

1.2. Example. Let us apply the above formula in the cases $n=3,4$.

$\boxed{\tau^{\rho_3}_{111}(T_0,\infty)}$

$X(2,1,0)=\binom{2}{2}=1$, $X(0,1,2)=\binom{2}{0}=1$, $X(1,0,2)=X(2,0,1)=1$, and

$$X(0,2,1)=\begin{vmatrix}\binom{1}{0} & \binom{1}{-1}\\ \binom{1}{1} & \binom{1}{0}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}=1$$

$$X(1,2,0)=\begin{vmatrix}\binom{1}{1} & \binom{1}{0}\\ \binom{1}{2} & \binom{1}{1}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}=1$$

Hence,

$$\tau_{111}^{\rho_3}(T_0,\infty)=\log\frac{X_{T_0}(2,1,0)}{X_{T_0}(0,1,2)}\cdot\frac{X_{T_0}(1,0,2)}{X_{T_0}(1,2,0)}\cdot\frac{X_{T_0}(0,2,1)}{X_{T_0}(2,0,1)}=0.$$

$\boxed{\tau_{121}^{\rho_4}(T_1,\infty)}$

$X(2,2,0)=(-1)^2=1$, $X(1,3,0)=(-1)^3=-1$, $X(0,3,1)=\binom{3}{0}(-\beta^3\gamma^3)=-\beta^3\gamma^3$, $X(2,1,1)=\binom{3}{2}(-\beta\gamma)=-3\beta\gamma$, and

$$X(1,1,2)=-\begin{vmatrix}\binom{2}{1}(-\beta\gamma) & \binom{2}{0}\beta^2\gamma^2\\ \binom{2}{2} & \binom{2}{1}(-\beta\gamma)\end{vmatrix}=-\begin{vmatrix}-2\beta\gamma & \beta^2\gamma^2\\ 1 & 2\beta\gamma\end{vmatrix}=-3\beta^2\gamma^2,$$

$$X(0,2,2)=-\begin{vmatrix}\binom{2}{0}\beta^2\gamma^2 & \binom{2}{-1}(-\beta^3\gamma^3)\\ \binom{2}{1}(-\beta\gamma) & \binom{2}{0}\beta^2\gamma^2\end{vmatrix}=-\begin{vmatrix}\beta^2\gamma^2 & 0\\ -2\beta\gamma & \beta^2\gamma^2\end{vmatrix}=\beta^4\gamma^4$$

Hence,

$$\tau_{121}^{\rho_4}(T_1,\infty)=\log\frac{1\cdot(-3\beta^2\gamma^2)\cdot(-\beta^3\gamma^3)}{\beta^4\gamma^4\cdot(-1)\cdot(-3\beta\gamma)}=0.$$

In this way, we observe that the triangle invariants are always equal to zero.

$\boxed{\sigma_1^{\rho_3}(h_{CA})}$ $Y'(1)=(-1)^3\binom{2}{1}=-2$, $Y(0)=Y'(0)=1$, and

$$Y(1)=-\begin{vmatrix}\binom{2}{1} & \binom{2}{1}(\alpha^2\beta\gamma+1)\\ \binom{2}{2} & \binom{2}{2}\end{vmatrix}=-\begin{vmatrix}2 & 2(\alpha^2\beta\gamma+1)\\ 1 & 1\end{vmatrix}=2\alpha^2\beta\gamma.$$

Hence.

$$\sigma_1^{\rho_3}(h_{CA})=\log-\frac{2\alpha^2\beta\gamma\cdot 1}{-2\cdot 1}=\log\alpha^2\beta\gamma.$$

$\boxed{\sigma_2^{\rho_3}(h_{CA})}$ By the above computation, $Y(1)=2\alpha^2\beta\gamma$ and $Y'(1)=-2$. Moreover,

$$Y(2)=\begin{vmatrix}\binom{1}{0} & \binom{1}{-1} & \binom{2}{0}(\alpha^2\beta\gamma+1)^2\\ \binom{1}{1} & \binom{1}{0} & \binom{2}{1}(\alpha^2\beta\gamma+1)\\ \binom{1}{2} & \binom{1}{1} & \binom{2}{2}\end{vmatrix}=\begin{vmatrix}1 & 0 & (\alpha^2\beta\gamma+1)^2\\ 1 & 1 & 2(\alpha^2\beta\gamma+1)\\ 0 & 1 & 1\end{vmatrix}$$

$$=\begin{vmatrix}1 & 2(\alpha^2\beta\gamma+1)\\ 1 & 1\end{vmatrix}-\begin{vmatrix}0 & (\alpha^2\beta\gamma+1)^2\\ 1 & 1\end{vmatrix}=1-2(\alpha^2\beta\gamma+1)+(\alpha^2\beta\gamma+1)^2=\alpha^4\beta^2\gamma^2,$$

$$Y'(2)=\begin{vmatrix}\binom{1}{0} & \binom{1}{-1}\\ \binom{1}{1} & \binom{1}{0}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}=1.$$

Hence,

$$\sigma_2^{\rho_3}(h_{CA}) = \log -\frac{\alpha^4\beta^2\gamma^2\cdot(-2)}{1\cdot 2\alpha^2\beta\gamma} = \log\alpha^2\beta\gamma.$$

$\boxed{\sigma_1^{\rho_4}(h_{BC}), \sigma_2^{\rho_4}(h_{BC}), \sigma_3^{\rho_4}(h_{BC})}$ Calculate $Y(p)$ and $Y'(p)$ as follows.

$$Y(0) = \begin{vmatrix} \binom{1}{0} & \binom{1}{-1} & \binom{1}{-2} & \binom{3}{0}\left(\frac{\beta}{\beta+\gamma}\right)^3 \\ \binom{1}{1} & \binom{1}{0} & \binom{1}{-1} & \binom{3}{1}\left(\frac{\beta}{\beta+\gamma}\right)^2 \\ \binom{1}{2} & \binom{1}{1} & \binom{1}{0} & \binom{3}{2}\left(\frac{\beta}{\beta+\gamma}\right) \\ \binom{1}{3} & \binom{1}{2} & \binom{1}{1} & \binom{3}{3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \left(\frac{\beta}{\beta+\gamma}\right)^3 \\ 1 & 1 & 0 & 3\left(\frac{\beta}{\beta+\gamma}\right)^2 \\ 0 & 1 & 1 & 3\left(\frac{\beta}{\beta+\gamma}\right) \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 3\left(\frac{\beta}{\beta+\gamma}\right)^2 \\ 1 & 1 & 3\left(\frac{\beta}{\beta+\gamma}\right) \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & \left(\frac{\beta}{\beta+\gamma}\right)^3 \\ 1 & 1 & 3\left(\frac{\beta}{\beta+\gamma}\right) \\ 0 & 1 & 1 \end{vmatrix} = \left(1-\frac{\beta}{\beta+\gamma}\right)^3 = \frac{\gamma^3}{(\beta+\gamma)^3}$$

$$Y'(0) = -\begin{vmatrix} \binom{1}{1} & \binom{1}{0} & \binom{1}{-1} \\ \binom{1}{2} & \binom{1}{1} & \binom{1}{0} \\ \binom{1}{3} & \binom{1}{2} & \binom{1}{1} \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$$Y(1) = -\begin{vmatrix} \binom{2}{0} & \binom{2}{-1} & \binom{3}{0}\left(\frac{\beta}{\beta+\gamma}\right)^3 \\ \binom{2}{1} & \binom{2}{0} & \binom{3}{1}\left(\frac{\beta}{\beta+\gamma}\right)^2 \\ \binom{2}{2} & \binom{2}{1} & \binom{3}{2}\left(\frac{\beta}{\beta+\gamma}\right) \end{vmatrix} = -\begin{vmatrix} 1 & 0 & \left(\frac{\beta}{\beta+\gamma}\right)^3 \\ 2 & 1 & 3\left(\frac{\beta}{\beta+\gamma}\right)^2 \\ 1 & 2 & 3\left(\frac{\beta}{\beta+\gamma}\right) \end{vmatrix}$$

$$= -\frac{3\beta}{\beta+\gamma}\left(\frac{\beta}{\beta+\gamma}-1\right)^2 = -\frac{3\beta\gamma^2}{(\beta+\gamma)^3}$$

$$Y'(1) = -\begin{vmatrix} \binom{2}{1} & \binom{2}{0} \\ \binom{2}{2} & \binom{2}{1} \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$$

$$Y(2) = \begin{vmatrix} \binom{3}{0} & \binom{3}{0}\left(\dfrac{\beta}{\beta+\gamma}\right)^3 \\ \binom{3}{1} & \binom{3}{1}\left(\dfrac{\beta}{\beta+\gamma}\right)^2 \end{vmatrix} = \begin{vmatrix} 1 & \left(\dfrac{\beta}{\beta+\gamma}\right)^3 \\ 3 & 3\left(\dfrac{\beta}{\beta+\gamma}\right)^2 \end{vmatrix} = \frac{3\beta^2\gamma}{(\beta+\gamma)^3}$$

$$Y'(2) = -\binom{3}{1} = -3$$

$$Y(3) = (-1)\left(\frac{\beta}{\beta+\gamma}\right)^3$$

$$Y'(3) = -1$$

Hence

$$\sigma_1^{\rho_4}(h_{BC}) = \log\left[-\left(\frac{3\beta\gamma^2}{(\beta+\gamma)^3}\cdot(-1)\right)/\left(-3\cdot\left(-\frac{\gamma^3}{(\beta+\gamma)^3}\right)\right)\right] = \log\frac{\beta}{\gamma},$$

$$\sigma_2^{\rho_4}(h_{BC}) = \log\left[-\left(\frac{3\beta^2\gamma}{(\beta+\gamma)^3}\cdot(-3)\right)/\left(-3\cdot\left(-\frac{3\beta\gamma^2}{(\beta+\gamma)^3}\right)\right)\right] = \log\frac{\beta}{\gamma},$$

$$\sigma_3^{\rho_4}(h_{BC}) = \log\left[-\left((-1)\left(\frac{\beta}{\beta+\gamma}\right)^3\cdot(-3)\right)/\left(-1\cdot\frac{3\beta^2\gamma}{(\beta+\gamma)^3}\right)\right] = \log\frac{\beta}{\gamma}.$$

The above computation gives the relation $\sigma_1^{\rho_3}(h_{CA}) = \sigma_2^{\rho_3}(h_{CA})$, and $\sigma_1^{\rho_4}(h_{BC}) = \sigma_2^{\rho_4}(h_{BC}) = \sigma_3^{\rho_4}(h_{BC})$. In particular, these values are equal to the shearing parameter of ρ. Therefore we expect that the shearing invariants are independent of their indices, and equal to the shear parameter of Fuchsian representations.

2. Computations of ratios for the Veronese flag curve

2.1. Triple ratios. In this section, we compute the triple ratio and the double ratio of the Veronese flag curves. Let $\nu : \mathbb{RP}^1 \to \mathbb{RP}^{n-1}$ be the Veronese flag curve. First we show that all triple ratios of ν are equal to 1.

PROPOSITION 3.1. *For any triples (x, y, z) of clockwise ordered points in $\mathbb{PR}^1$, an integer $n \geq 2$, and positive integers p, q, r with $p + q + r = n$, $T_{pqr}(\nu(x), \nu(y), \nu(z)) = 1$.*

PROOF. Given (x,y,z), we can take a transformation $A \in \mathrm{PSL}_2\mathbb{R}$ such that $A(x)=\infty, A(y)=1$, and $A(z)=0$. Using this normalization, we have

$$\begin{aligned} T_{pqr}(\nu(x),\nu(y),\nu(z)) &= T_{pqr}(\nu(A^{-1}(\infty)),\nu(A^{-1}(1)),\nu(A^{-1}(0)) \\ &= T_{pqr}(\iota_n(A)^{-1}\nu(\infty),\iota_n(A)^{-1}\nu(1),\iota_n(A)^{-1}\nu(0)) \\ &= T_{pqr}(\nu(\infty),\nu(1),\nu(0)). \end{aligned}$$

Thus it is enough to consider the value $T_{pqr}(\nu(\infty),\nu(1),\nu(0))$.

Recall that the flag $\nu([a:b]) = \{V_d\}_d$ for $[a:b] \in \mathbb{RP}^1$ consists of the nested vector space V_d of dimension $d = 0,1,\cdots,n$ defined by

$$V_d = \{P(X,Y) \in \mathrm{Poly}_n(X,Y) \mid P(X,Y) \text{ can be divided by } (aX+bY)^{n-d}\}.$$

For example, the d-dimensional vector space $\nu(0)^d$ is

$$\begin{aligned} \nu(0)^d &= \{P(X,Y) \mid \exists Q(X,Y) \; s.t. \; P(X,Y) = Y^{n-d}Q(X,Y)\} \\ &= \{(k_1X^{d-1} + k_2X^{d-2}Y + \cdots + k_dY^{d-1})Y^{n-d} \mid k_1,\cdots k_d \in \mathbb{R}\} \\ &= \mathrm{Span}\{X^{d-1}Y^{n-d}, X^{d-2}Y^{n-d+1},\cdots,Y^{n-1}\}. \end{aligned}$$

Similarly,

$$\begin{aligned} \nu(\infty)^d &= \mathrm{Span}\{X^{n-1}, X^{n-2}Y,\cdots,X^{n-d}Y^{d-1}\}, \\ \nu(1)^d &= \mathrm{Span}\{(X+Y)^{n-d}X^{d-1},(X+Y)^{n-d}X^{d-2}Y,\cdots,(X+Y)^{n-d}Y^{d-1}\}. \end{aligned}$$

To compute the triple ratio, first we choose a basis of $\bigwedge^d \nu(0)^d, \bigwedge^d \nu(1)^d, \bigwedge^d \nu(\infty)^d$ as follows:

$$t_0^d = X^{d-1}Y^{n-d} \wedge X^{d-2}Y^{n-d+1} \wedge \cdots \wedge Y^{n-1} \in \bigwedge^d \nu(0)^d,$$

$$t_\infty^d = X^{n-1} \wedge X^{n-2}Y \wedge \cdots \wedge X^{n-d}Y^{d-1} \in \bigwedge^d \nu(\infty)^d,$$

$$t_1^d = (X+Y)^{n-d}X^{d-1} \wedge (X+Y)^{n-d}X^{d-2}Y \wedge \cdots \wedge (X+Y)^{n-d}Y^{d-1} \in \bigwedge^d \nu(1)^d.$$

Then $T_{pqr}(\nu(\infty),\nu(1),\nu(0))$ is precisely equal to

$$\frac{t_\infty^{p+1} \wedge t_1^q \wedge t_0^{r-1} \cdot t_\infty^p \wedge t_1^{q-1} \wedge t_0^{r+1} \cdot t_\infty^{p-1} \wedge t_1^{q+1} \wedge t_0^r}{t_\infty^{p-1} \wedge t_1^q \wedge t_0^{r+1} \cdot t_\infty^p \wedge t_1^{q+1} \wedge t_0^{r-1} \cdot t_\infty^{p+1} \wedge t_1^{q-1} \wedge t_0^r},$$

so we should verify the values of wedge products $t_\infty^p \wedge t_1^q \wedge t_0^r$ for integers p, q, r with $0 \leq p, q, r \leq n$ and $p + q + r = n$. (There is abuse of notations p, q, r which appeared in the statement of Proposition 3.1.) The following formula is shown by easy linear algebra.

LEMMA 3.2. *Let V be an n-dimensional vector space with a basis $\{b_1, \cdots, b_n\}$ and $\{v_1, \cdots, v_n\}$ be arbitrary vectors in V. We set $v_i = \sum_{i=1}^n v_{ij} b_j$ with $v_{ij} \in \mathbb{R}$. Then*

$$v_1 \wedge \cdots \wedge v_n = \mathrm{Det}((v_{ij})) b_1 \wedge \cdots \wedge b_n.$$

We fix a basis of $\mathrm{Poly}_n(X, Y)$ by $b_1 = X^{n-1}, b_2 = X^{n-2}Y, \cdots, b_n = Y^{n-1}$, and we may choose an identification $\bigwedge^n \mathrm{Poly}_n(X, Y) \to \mathbb{R}$ so that $b_1 \wedge b_2 \wedge \cdots \wedge b_n$ is identified with 1. Then, using this basis,

$$\begin{aligned}
t_\infty^p \wedge t_1^q \wedge t_0^r = {} & X^{n-1} \wedge X^{n-2}Y \wedge \cdots \wedge X^{n-p}Y^{p-1} \wedge \\
& (X+Y)^{n-q}X^{q-1} \wedge (X+Y)^{n-q}X^{q-2}Y \wedge \cdots \wedge (X+Y)^{n-q}Y^{q-1} \wedge \\
& X^{r-1}Y^{n-r} \wedge X^{r-2}Y^{n-r+1} \wedge \cdots \wedge Y^{n-1} \\
= {} & b_1 \wedge b_2 \wedge \cdots b_p \wedge \\
& \sum_{i=1}^{n-q} \binom{n-q}{i} b_{i+1} \wedge \sum_{i=1}^{n-q} \binom{n-q}{i} b_{i+2} \wedge \cdots \wedge \sum_{i=1}^{n-q} \binom{n-q}{i} b_{i+q} \wedge \\
& b_{n-r+1} \wedge b_{n-r+2} \wedge \cdots \wedge b_n.
\end{aligned}$$

By Lemma 3.2 and a computation of determinants of matrices, if $q \neq 0$, then

$$t_\infty^p \wedge t_1^q \wedge t_0^r = \begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \vdots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix},$$

and if $q = 0$, then $t_\infty^p \wedge t_1^0 \wedge t_0^r = 1$. We may suppose $q \neq 0$. In this determinant, we consider an extended binomial coefficient which is defined by

$$\binom{n}{p} = \begin{cases} \dfrac{n!}{p!(n-p)!} & (0 \leq p \leq n) \\ 0 & (otherwise). \end{cases}$$

Hence many zero entries may appear in the determinant above.

LEMMA 3.3. *The determinant*

$$\begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \vdots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix}$$

is equal to

$$(-1)^{\frac{(q-1)q}{2}} \frac{(n-q)!\ (n-q+1)! \cdots (n-1)!\ 1!\ 2! \cdots (q-1)!}{(n-r-q)!\ (n-r-q+1)! \cdots (n-r-1)!\ r!\ (r+1)! \cdots (r+q-1)!}.$$

PROOF OF LEMMA 3.3. The following formulae still hold for extended binomial coefficients.

$$\binom{n}{p} = \binom{n}{n-p}, \tag{7}$$

$$\binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}. \tag{8}$$

By elemental transformations of matrices, adding the second row to the first row, the third row to the second row, ... and then the qth row to the $(q-1)$th row, and applying the formula (8), we obtain

$$\begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \vdots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix} = \begin{vmatrix} \binom{p+r+1}{p+1} & \cdots & \binom{p+r+1}{p-q+2} \\ \binom{p+r+1}{p+2} & \cdots & \binom{p+r+1}{p-q+3} \\ \binom{p+r+1}{p+3} & \cdots & \binom{p+r+1}{p-q+4} \\ \vdots & \vdots & \vdots \\ \binom{p+r+1}{p+q-2} & \cdots & \binom{p+r+1}{p-1} \\ \binom{p+r+1}{p+q-1} & \cdots & \binom{p+r+1}{p} \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix}.$$

Next, by adding the second row to the first row, the third row to the second row, ... and then the $(q-1)$th row to the $(q-2)$th row and using (8),

$$\begin{vmatrix} \binom{p+r+1}{p+1} & \cdots & \binom{p+r+1}{p-q+2} \\ \binom{p+r+1}{p+2} & \cdots & \binom{p+r+1}{p-q+3} \\ \binom{p+r+1}{p+3} & \cdots & \binom{p+r+1}{p-q+4} \\ \vdots & \vdots & \vdots \\ \binom{p+r+1}{p+q-2} & \cdots & \binom{p+r+1}{p-1} \\ \binom{p+r+1}{p+q-1} & \cdots & \binom{p+r+1}{p} \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix} = \begin{vmatrix} \binom{p+r+2}{p+2} & \cdots & \binom{p+r+2}{p-q+3} \\ \binom{p+r+2}{p+3} & \cdots & \binom{p+r+2}{p-q+4} \\ \binom{p+r+2}{p+4} & \cdots & \binom{p+r+2}{p-q+5} \\ \vdots & \vdots & \vdots \\ \binom{p+r+2}{p+q-1} & \cdots & \binom{p+r+2}{p} \\ \binom{p+r+1}{p+q-1} & \cdots & \binom{p+r+1}{p} \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix}.$$

Iterating such a deformation, we get

$$\begin{vmatrix} \binom{p+r}{p} & \cdots & \binom{p+r}{p-q+1} \\ \vdots & \vdots & \vdots \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix} = \begin{vmatrix} \binom{p+r+q-1}{p+q-1} & \cdots & \binom{p+r+q-1}{p} \\ \binom{p+r+q-2}{p+q-1} & \cdots & \binom{p+r+q-2}{p} \\ \binom{p+r+q-3}{p+q-1} & \cdots & \binom{p+r+q-3}{p} \\ \vdots & \vdots & \vdots \\ \binom{p+r+2}{p+q-1} & \cdots & \binom{p+r+2}{p} \\ \binom{p+r+1}{p+q-1} & \cdots & \binom{p+r+1}{p} \\ \binom{p+r}{p+q-1} & \cdots & \binom{p+r}{p} \end{vmatrix} = \begin{vmatrix} \binom{n-1}{p+q-1} & \cdots & \binom{n-1}{p} \\ \binom{n-2}{p+q-1} & \cdots & \binom{n-2}{p} \\ \binom{n-3}{p+q-1} & \cdots & \binom{n-3}{p} \\ \vdots & \vdots & \vdots \\ \binom{n-q+2}{p+q-1} & \cdots & \binom{n-q+2}{p} \\ \binom{n-q+1}{p+q-1} & \cdots & \binom{n-q+1}{p} \\ \binom{n-q}{p+q-1} & \cdots & \binom{n-q}{p} \end{vmatrix}.$$

Note that $p+q+r=n$ for the last equality. We consider a similar deformation for columns. By adding the second column to the first column, the third column to the second column, ..., and the qth column to the $(q-1)$th column, and using the formula (8), the determinant is deformed to

$$\begin{vmatrix} \binom{n}{p+q-1} & \binom{n}{p+q-2} & \binom{n}{p+q-3} & \cdots & \binom{n}{p+2} & \binom{n}{p+1} & \binom{n-1}{p} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \binom{n-q+1}{p+q-1} & \binom{n-q+1}{p+q-2} & \binom{n-q+1}{p+q-3} & \cdots & \binom{n-q+1}{p+2} & \binom{n-q+1}{p+1} & \binom{n-q}{p} \end{vmatrix}.$$

By adding the second column to the first column, the third column to the second column, ..., and the $(q-1)$th column to the $(q-2)$th column, and using the formula (8), the determinant is again deformed to

$$\begin{vmatrix} \binom{n+1}{p+q-1} & \binom{n+1}{p+q-2} & \binom{n+1}{p+q-3} & \cdots & \binom{n+1}{p+2} & \binom{n}{p+1} & \binom{n-1}{p} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \binom{n-q+2}{p+q-1} & \binom{n-q+2}{p+q-2} & \binom{n-q+2}{p+q-3} & \cdots & \binom{n-q+2}{p+2} & \binom{n-q+1}{p+1} & \binom{n-q}{p} \end{vmatrix}.$$

By iterating such a deformation, the determinant is deformed to:

$$\begin{vmatrix} \binom{n+q-2}{p+q-1} & \binom{n+q-3}{p+q-2} & \binom{n+q-4}{p+q-3} & \cdots & \binom{n+1}{p+2} & \binom{n}{p+1} & \binom{n-1}{p} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \binom{n-1}{p+q-1} & \binom{n-2}{p+q-2} & \binom{n-3}{p+q-3} & \cdots & \binom{n-q+2}{p+2} & \binom{n-q+1}{p+1} & \binom{n-q}{p} \end{vmatrix}.$$

Using $p+q+r=n$ and replacing columns and rows, the determinant is deformed as follows.

$$\begin{vmatrix} \binom{n+q-2}{p+q-1} & \binom{n+q-3}{p+q-2} & \cdots & \binom{n}{p+1} & \binom{n-1}{p} \\ \binom{n+q-3}{p+q-1} & \binom{n+q-4}{p+q-2} & \cdots & \binom{n-1}{p+1} & \binom{n-2}{p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{p+q-1} & \binom{n-1}{p+q-2} & \cdots & \binom{n-q+2}{p+1} & \binom{n-q+1}{p} \\ \binom{n-1}{p+q-1} & \binom{n-2}{p+q-2} & \cdots & \binom{n-q+1}{p+1} & \binom{n-q}{p} \end{vmatrix}$$

$$= (-1)^{\frac{q(q-1)}{2}} \begin{vmatrix} \binom{n-1}{p+q-1} & \binom{n-2}{p+q-2} & \cdots & \binom{n-q+1}{p+1} & \binom{n-q}{p} \\ \binom{n}{p+q-1} & \binom{n-1}{p+q-2} & \cdots & \binom{n-q+2}{p+1} & \binom{n-q+1}{p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n+q-3}{p+q-1} & \binom{n+q-4}{p+q-2} & \cdots & \binom{n-1}{p+1} & \binom{n-2}{p} \\ \binom{n+q-2}{p+q-1} & \binom{n+q-3}{p+q-2} & \cdots & \binom{n}{p+1} & \binom{n-1}{p} \end{vmatrix}$$

$$= (-1)^{\frac{q(q-1)}{2}} \cdot (-1)^{\frac{q(q-1)}{2}} \begin{vmatrix} \binom{n-q}{p} & \binom{n-q+1}{p+1} & \cdots & \binom{n-2}{p+q-2} & \binom{n-1}{p+q-1} \\ \binom{n-q+1}{p} & \binom{n-q+2}{p+1} & \cdots & \binom{n-1}{p+q-2} & \binom{n}{p+q-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-2}{p} & \binom{n-1}{p+1} & \cdots & \binom{n+q-4}{p+q-2} & \binom{n+q-3}{p+q-1} \\ \binom{n-1}{p} & \binom{n}{p+1} & \cdots & \binom{n+q-3}{p+q-2} & \binom{n+q-2}{p+q-1} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{n-q}{n-r-q} & \binom{n-q+1}{n-r-q+1} & \cdots & \binom{n-2}{n-r-2} & \binom{n-1}{n-r-1} \\ \binom{n-q+1}{n-r-q} & \binom{n-q+2}{n-r-q+1} & \cdots & \binom{n-1}{n-r-2} & \binom{n}{n-r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-2}{n-r-q} & \binom{n-1}{n-r-q+1} & \cdots & \binom{n+q-4}{n-r-2} & \binom{n+q-3}{n-r-1} \\ \binom{n-1}{n-r-q} & \binom{n}{n-r-q+1} & \cdots & \binom{n+q-3}{n-r-2} & \binom{n+q-2}{n-r-1} \end{vmatrix}. \quad \cdots (\dagger)$$

Lemma 3.3 is obtained by applying the following lemma. The determinant $\Diamond(n,k,l)$ below corresponds to a rhombus in Pascal's triangle.

The entries of $\diamondsuit(n,k,l)$ are usual binomial coefficients, so positive integers. We can apply the formula in Lemma 3.4 to compute (†) by replacing n, k, l to $n-q, n-r-q, q-1$, and we obtain Lemma 3.3. $\square$

LEMMA 3.4. *Let $n, l \in \mathbb{N}$ and $0 \le k \le n$. The determinant*

$$\diamondsuit(n,k,l) = \begin{vmatrix} \binom{n}{k} & \binom{n+1}{k+1} & \cdots & \binom{n+l}{k+l} \\ \binom{n+1}{k} & \binom{n+2}{k+1} & \cdots & \binom{n+l+1}{k+l} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{n+l}{k} & \binom{n+l+1}{k+1} & \cdots & \binom{n+2l}{k+l} \end{vmatrix}$$

is equal to

$$\frac{n!\,(n+1)!\cdots(n+l)!}{k!\,(k+1)!\cdots(k+l)!\,(n-k)!\cdots(n-k+l)!}\cdot(-1)^{\frac{l(l+1)}{2}}\;1!\cdots l!.$$

PROOF OF LEMMA 3.4. First, we deform $\diamondsuit(n,k,l)$ as follows.

$$\diamondsuit(n,k,l) = \begin{vmatrix} \frac{n!}{k!(n-k)!} & \frac{(n+1)!}{(k+1)!(n-k)!} & \cdots & \frac{(n+l)!}{(k+l)!(n-k)!} \\ \frac{(n+1)!}{k!(n-k+1)!} & \frac{(n+2)!}{((k+1)!(n-k+1)!} & \cdots & \frac{(n+l+1)!}{(k+l)!(n-k+1)!} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(n+l)!}{k!(n-k+l)!} & \frac{(n+l+1)!}{(k+1)!(n-k+l)!} & \cdots & \frac{(n+2l)!}{(k+l)!(n-k+l)!} \end{vmatrix}$$

$$= C\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (n+1) & (n+2) & \cdots & (n+l+1) \\ \vdots & \vdots & \vdots & \vdots \\ (n+1)\cdots(n+l) & (n+2)\cdots(n+l+1) & \cdots & (n+l+1)\cdots(n+2l) \end{vmatrix},$$

where

$$C = \frac{n!\,(n+1)!\cdots(n+l)!}{k!\,(k+1)!\cdots(k+l)!\,(n-k)!\cdots(n-k+l)!}.$$

We add the $(-l+1)$ times of the l-th row to the $(l+1)$-th row, the $(-l+2)$ times of the $(l-1)$-th row to the l-th row, ..., and (-1) times of the second row to the third row:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (n+1) & (n+2) & \cdots & (n+l+1) \\ \vdots & \vdots & \vdots & \vdots \\ (n+1)\cdots(n+l) & (n+2)\cdots(n+l+1) & \cdots & (n+l+1)\cdots(n+2l) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (n+1) & (n+2) & \cdots & (n+l+1) \\ \vdots & \vdots & \vdots & \vdots \\ (n+1)^2\cdots(n+l) & (n+2)^2\cdots(n+l+1) & \cdots & (n+l+1)^2\cdots(n+2l) \end{vmatrix}.$$

The iteration of such a deformation gives us the following determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (n+1) & (n+2) & \cdots & (n+l+1) \\ \vdots & \vdots & \vdots & \vdots \\ (n+1)^l & (n+2)^l & \cdots & (n+l+1)^l \end{vmatrix}.$$

Using the formula of Vandermonde's determinant, we can expand this as follows.

$$\begin{aligned} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (n+1) & (n+2) & \cdots & (n+l+1) \\ \vdots & \vdots & \vdots & \vdots \\ (n+1)^l & (n+2)^l & \cdots & (n+l+1)^l \end{vmatrix} &= (-1)^l l! \cdot (-1)^{l-1}(l-1)! \cdots (-1) \\ &= (-1)^{l+(l-1)+\cdots+1} l!\ (l-1)! \cdots 1 \\ &= (-1)^{\frac{l(l+1)}{2}} 1! \cdots l!. \end{aligned}$$

Thus

$$\Diamond(n,k,l) = \frac{n!\ (n+1)!\cdots(n+l)!}{k!\ (k+1)!\cdots(k+l)!\ (n-k)!\cdots(n-k+l)!} \cdot (-1)^{\frac{l(l+1)}{2}} 1! \cdots l!.$$

□

Finally, applying Lemma 3.3, we can check that the value of the triple ratio $T_{pqr}(\nu(\infty), \nu(1), \infty(0))$ is equal to 1. We finish the proof of Proposition 3.1. □

2.2. Double ratios.

PROPOSITION 3.5. *Let (x, z, y, z') be a quadruple of counterclockwise ordered points in $\mathbb{RP}^1$. The b-th double ratio $D_b(\nu(x), \nu(y), \nu(z), \nu(z'))$ is equal to $-r$ for all integers b with $1 \leq b \leq n-1$, where r is the cross ratio $r = cr(x, y, z, z')$.*

PROOF. Let $A \in \mathrm{PSL}_2\mathbb{R}$ be a transformation which sends x,y,z' to $\infty, 0, 1$. Then the transformation A maps z to r^{-1}, where $r = cr(x, y, z, z')$. Then, by the same computation with the case of triple ratio,

$$D_b(\nu(x), \nu(y), \nu(z), \nu(z')) = D_b(\nu(\infty), \nu(0), \nu(r^{-1}), \nu(1)).$$

The flags $\nu(\infty), \nu(0), \nu(r^{-1}), \nu(1)$ are defined by the following vector spaces:

$$\begin{aligned} \nu(\infty)^d &= \mathrm{Span}\{b_1, b_2, \cdots, b_d\}, \\ \nu(0)^d &= \mathrm{Span}\{b_{n-d+1}, b_{n-d+2}, \cdots, b_n\}, \\ \nu(1)^1 &= \mathbb{R}\sum_{i=0}^{n-1} \binom{n-1}{i} b_{i+1}, \\ \nu(r^{-1})^1 &= \mathbb{R}\sum_{i=0}^{n-1} \binom{n-1}{i} r^{-(n-1-i)} b_{i+1}, \end{aligned}$$

where $b_1, \cdots, b_n$ are the basis of $\mathrm{Poly}_n(X, Y)$ we used in the proof of Proposition 3.1. We choose bases of $\bigwedge^d \nu(\infty)^d, \bigwedge^d \nu(0)^d, \nu(1)^1, \nu(r^{-1})^1$ as follows:

$$\begin{aligned} t_\infty^d &= b_1 \wedge b_2 \wedge \cdots \wedge b_d \in \bigwedge^d \nu(\infty)^d, \\ t_0^d &= b_{n-d+1} \wedge b_{n-d+2} \wedge \cdots \wedge b_n \in \bigwedge^d \nu(0)^d, \\ t_1^1 &= \sum_{i=0}^{n-1} \binom{n-1}{i} b_{i+1} \in \nu(1)^1, \\ t_{r^{-1}}^1 &= \sum_{i=0}^{n-1} \binom{n-1}{i} r^{-(n-1-i)} b_{i+1} \in \nu(r^{-1})^1. \end{aligned}$$

By the definition of the double ratio,

$$D_b(\nu(\infty),\nu(0),\nu(r^{-1}),\nu(1)) = -\frac{t_\infty^b \wedge t_0^{n-b-1} \wedge t_{r^{-1}}^1 \cdot t_\infty^{b-1} \wedge t_0^{n-b} \wedge t_1^1}{t_\infty^b \wedge t_0^{n-b-1} \wedge t_1^1 \cdot t_0^{b-1} \wedge t_0^{n-b} \wedge t_{r^{-1}}^1}$$

Compute each factor of this fraction.

$$\begin{aligned} t_\infty^b \wedge t_0^{n-b-1} \wedge t_{r^{-1}}^1 &= \begin{vmatrix} \mathrm{Id}_b & 0 & \binom{n-1}{0} r^{-(n-1)} \\ & & \binom{n-1}{1} r^{-(n-2)} \\ & & \vdots \\ 0 & \mathrm{Id}_{n-b-1} & \binom{n-1}{n-1} (r^{-1})^0 \end{vmatrix} \\ &= (-1)^{n-b-1} \binom{n-1}{b} r^{-(n-b-1)}, \end{aligned}$$

$$\begin{aligned} t_\infty^b \wedge t_0^{n-b-1} \wedge t_1^1 &= \begin{vmatrix} \mathrm{Id}_b & 0 & \binom{n-1}{0} \\ & & \binom{n-1}{1} \\ & & \vdots \\ 0 & \mathrm{Id}_{n-b-1} & \binom{n-1}{n-1} \end{vmatrix} \\ &= (-1)^{n-b-1} \binom{n-1}{b}. \end{aligned}$$

Hence

$$\begin{aligned} D_a(\nu(\infty),\nu(0),\nu(r^{-1}),\nu(1)) &= -\frac{(-1)^{n-b-1}\binom{n-1}{b} r^{-(n-b-1)} \cdot (-1)^{n-(b+1)-1}\binom{n-1}{b+1}}{(-1)^{n-b-1}\binom{n-1}{b} \cdot (-1)^{n-(b+1)-1}\binom{n-1}{b+1} r^{-(n-(b+1)-1)}} \\ &= -r \end{aligned}$$

□

CHAPTER 4

A characterization of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations.

1. The case of finite laminations

Let S be a closed oriented hyperbolic surface, and λ be an oriented maximal geodesic lamination consisting of finitely many leaves. We denote bi-infinite (resp. closed) leaves of λ by B_i (resp. C_i). The maximal geodesic lamination λ gives an ideal triangulation of S. We denote ideal triangles of the ideal triangulation by T_i. In addition, we fix a bridge system system $\mathcal{J}$ for λ. Recall that the Bonahon-Dreyer parameterization $\Phi_{\lambda_{\mathcal{J}}}\colon H_n(S) \to \mathbb{R}^N$ associated to $\lambda_{\mathcal{J}}$ is defined by

$$\Phi_{\lambda_{\mathcal{J}}}(\rho) = (\tau_{pqr}(s^i_j, \rho), \cdots, \sigma_b(B_i, \rho), \cdots, \theta_c(C_i, \rho), \cdots)$$

and the coordinate of $\mathbb{R}^N$ is represented by $(\tau_{pqr}(s^i_j), \cdots, \sigma_b(B_i), \cdots, \theta_c(C_i), \cdots)$. Set $\mathcal{P}_{\lambda_{\mathcal{J}}} = \mathrm{Image}(\Phi_{\lambda_{\mathcal{J}}})$, which is the interior of a convex polyhedron in $\mathbb{R}^N$.

THEOREM 4.1. *If $\rho_n = \iota_n \circ \rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ is a $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation, then*

(i) *all triangle invariants $\tau_{pqr}(s^i_j, \rho_n)$ are equal to 0, and*
(ii) *all shearing invariants $\sigma_b(B_i, \rho_n)$, and all twist invariants $\theta_c(C_i, \rho_n)$ are constants depending only on the Fuchsian representation ρ, and are independent of their indices b, c.*

Moreover, the shearing invariant of ρ_n along B_i is equal to the shearing parameter of ρ along B_i, i.e. $\sigma_b(B_i, \rho_n) = \sigma^\rho(B_i)$

PROOF. (i) Recall the definition of triangle invariants. Fix a spike s^i_j of the ideal triangle T_i, and a lift $\widetilde{T}_i$ of T_i. Let $x, y, z \in \partial\pi_1(S)$ be the the vertices of $\widetilde{T}_i$, where x corresponds to s^i_j and they are in clockwise order. Then $\tau_{pqr}(s^i_j, \rho_n) = \log[T_{pqr}(\xi_{\rho_n}(x), \xi_{\rho_n}(y), \xi_{\rho_n}(z))]$. Since ξ_{ρ_n} is of the Veronese type, its triple ratio is equal to 1 by Proposition 3.1. Hence $\tau_{pqr}(s^i_j, \rho_n) = 0$.

(ii) Let $\widetilde{B}_i$ be a lift of a bi-infinite leaf B_i. We denote the left ideal triangle with the side $\widetilde{B}_i$ by $\widetilde{T}^L_i$, and the right ideal triangle by $\widetilde{T}^R_i$. Respecting the orientation of $\widetilde{B}_i$, we label x, y, z^L, z^R on the ideal vertices of $\widetilde{T}^L_i, \widetilde{T}^R_i$ as in Section 3.2 in Chapter 2. Then the quadruple (x, z^L, y, z^R) is counterclockwise ordered, so by Proposition 3.5,

$$\begin{aligned}\sigma_b(B_i, \rho_n) &= \log D_b(\xi_{\rho_n}(x), \xi_{\rho_n}(y), \xi_{\rho_n}(z^L), \xi_{\rho_n}(z^R)) \\ &= \log[-\mathrm{cr}(f_\rho(x), f_\rho(y), f_\rho(z^L), f_\rho(z^R))].\end{aligned}$$

Especially, the shearing invariant is independent of the index b, and is equal to the shearing parameter of ρ by Lemma 2.9. We can similarly show the case of twist invariants. The differences are only in the choice of ideal triangles and a quadruple of ideal vertices which are used in the definition of the twist invariants. □

We define an affine slice $\mathcal{S}_{\lambda_\mathcal{J}}$ of $\mathcal{P}_{\lambda_\mathcal{J}}$ by $\tau_{pqr}(s^i_j) = 0$, $\sigma_b(B_i) = \sigma_{b'}(B_i)$, and $\theta_c(C_i) = \theta_{c'}(C_i)$ for all possible indices.

THEOREM 4.2. *The restriction* $\Phi_{\lambda_\mathcal{J}}|_{F_n(S)} \colon F_n(S) \to \mathcal{S}_{\lambda_\mathcal{J}}$ *is surjective.*

PROOF. A point $x \in \mathcal{S}_{\lambda_\mathcal{J}}$ is represented by the following coordinate

$$(0, \cdots, 0, z_1, \cdots, z_1, \cdots, z_{3|\chi(S)|}, \cdots, z_{3|\chi(S)|}, w_1, \cdots, w_1, \cdots, w_k, \cdots, w_k),$$

where 0 is the $\tau_{pqr}(s^i_j)$-coordinate, z_i is the $\sigma_b(B_i)$-coordinate, and w_i is the $\theta_c(C_i)$-coordinate. It suffices to show that, for such $z_i, w_i \in \mathbb{R}$, there exists a Fuchsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ such that the associated $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation satisfies that $\sigma_b(B_i, \iota_n \circ \rho) = z_i$ and $\theta_c(C_i, \iota_n \circ \rho) = w_i$ for all i.

We see that the closed leaf condition of the Bonahon-Dreyer parameterization implies that the parameter $(z_1, \cdots, z_{3|\chi(S)|}, w_1, \cdots, w_k)$ is contained in the range of the shearing parameterization $\widetilde{\phi}_\lambda$ in Theorem 2.6. Here we define $\widetilde{\phi}_\lambda$ along the simple train track neighborhood N_λ (see the end of Section 1.5.3 in Chapter 2). To define the twist parameter of $\widetilde{\phi}_\lambda$, we require to choose two spiraling ideal triangles for each closed leaf C_i on both sides. As these two ideal triangles, we choose the bridge $J_{C_i} = \{T^L, T^R\}$ from the bridge system $\mathcal{J}$. Then the parameterization $\widetilde{\phi}_\lambda\colon \mathscr{T}(S) \to \mathbb{R}^{3|\chi(S)|+k}$ is defined by

$$\widetilde{\phi}_\lambda(\rho) = (\sigma^\rho(e_1), \cdots, \sigma^\rho(e_{3|\chi(S)|}), \theta^\rho(C_1), \cdots, \theta^\rho(C_k)).$$

Note that $\sigma^\rho(e_i)$ is defined by $\sigma^\rho(B_i)$.

It is enough to check only the condition (II) of Proposition 2.8 by the final remark in Chapter 2, Section 1.5.3. Let $B_1^{i,L}, \cdots, B_{l_L}^{i,L}$ be bi-infinite leaves spiraling to C_i from left and $B_1^{i,R}, \cdots, B_{l_R}^{i,R}$ be bi-infinite leaves spiraling to C_i from the right. We denote, by $z_j^{i,L}$, the $\sigma_b(B_j^{i,L})$-coordinate of x. Since $x \in \mathcal{S}_{\lambda_{\mathcal{J}}}$, it satisfies the closed leaf condition. Note that $B_j^{i,L}$ spirals to C_i from the left with respect to the orientation of C_i. In addition, we remark that $B_j^{i,L}$ spirals to C_i in the direction (resp. the opposite direction) of the orientation of C_i if and only if the sign of this spiraling is negative (resp. positive). (See Figure 3 and Figure 4.) Hence, using the condition that all $\tau_{pqr}(s_j^i)$-coordinates are equal to 0, the closed leaf inequality implies that

$$L_b(C_i) = -\sum_{j=1}^{l_L} \overline{\sigma}_b(B_j^{i,L}) = -\sum_{j=1}^{l_L} z_j^{i,L} > 0$$

if the spiraling is negative, and

$$L_b(C) = \sum_{j=1}^{l_L} \overline{\sigma}_{n-b}(B_j^{i,L}) = \sum_{j=1}^{l_L} z_j^{i,L} > 0$$

if the spiraling is positive. Thus, we have $L_b(C_i) = \text{sign} \cdot \sum_{j=1}^{l_L} z_j^{i,L} > 0$.

We give a similar observation for the bi-infinite leaves $B_j^{i,R}$. Let $z_j^{i,R}$ be the $\sigma_b(B_j^{i,R})$-coordinate of x. Since $B_j^{i,R}$ spiral to C_i from the right, $B_j^{i,L}$ spirals to C_i in the direction (resp. the opposite direction) of the orientation of C_i if and only if the sign of this spiraling is positive (resp. negative). Hence, the closed leaf inequality implies that

$$R_b(C_i) = \sum_{j=1}^{l_R} \overline{\sigma}_b(B_j^{i,R}) = \sum_{j=1}^{l_R} z_j^{i,R} > 0$$

if the spiraling is positive, and

$$R_b(C_i) = -\sum_{j=1}^{l_R} \overline{\sigma}_{n-b}(B_j^{i,R}) = -\sum_{j=1}^{l_R} z_j^{i,R} > 0$$

if the spiraling is negative. Thus, we have $R_b(C_i) = \text{sign}\cdot\sum_{j=1}^{l_R} z_j^{i,R} > 0$.

Finally, the closed equality $L_b(C_i) = R_b(C_i)$ gives us the following condition

$$\text{sign}\cdot\sum_{j=1}^{l_L} z_j^{i,L} = \text{sign}\cdot\sum_{j=1}^{l_R} z_j^{i,R} > 0.$$

This implies that the parameters z_i and w_i satisfy the condition (II). Hence, $(z_1, \cdots, z_{|3\chi(S)|}, w_1, \cdots, w_k)$ is contained in the range of $\widetilde{\phi}_\lambda$.

Using the reconstruction of the Fuchsian representations in Theorem 2.6, we obtain a Fuchsian representation $\rho \in \mathscr{T}(S)$ such that $\sigma^\rho(B_i) = \sigma^\rho(e_i) = z_i$ and $\theta^\rho(C_i) = w_i$. For this Fuchsian representation ρ, we have $\theta_c(C_i, \iota_n \circ \rho) = \theta^\rho(C_i) = w_i$ by Proposition 3.5, and $\sigma_b(B_i, \iota_n \circ \rho) = \sigma^\rho(B_i) = z_i$ by Theorem 4.1. Hence we finish the proof. □

2. The case of general laminations

The Fuchsian locus is a slice even in the case of general laminations. Let S be a closed oriented hyperbolic surface, and λ be an arbitrary maximal geodesic lamination on S. In this case, the Bonahon-Dreyer parameterization $\Phi_\lambda\colon H_n(S) \to Z(\lambda, \text{slits}; \mathbb{R}^n) \times \mathbb{R}^{6|\chi(S)|\binom{n-1}{2}}$ is defined by

$$\Phi_\lambda(\rho) = (\sigma^\rho, \tau_{pqr}(s_j^i, \rho)).$$

Let $\rho_n = \iota_n \circ \rho \in H_n(S)$ be a $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation.

THEOREM 4.3. *We denote, by $\sigma_b^{\rho_n}$, the b-th entry of σ^{ρ_n}. Let k be a tightly transverse arc of λ. Then, for all $b = 1, 2, \cdots, n-1$, $\sigma_b^{\rho_n}(k) = \sigma^\rho(k)$, where σ^ρ is the shearing cocycle associated to ρ.*

PROOF. Recall the definition of the shearing class. For a tightly transverse arc k, we take the plaques P, Q, the ideal vertices x, y, z, z', and the boundary leaves g, g' as we prepared in Section 3.3.2 of Chapter 2. Then, the value of the shearing class $\sigma_b^{\rho_n}(k)$ is defined by

$$\sigma_b^{\rho_n}(k) = \log[D_b(\nu \circ f_\rho(x), \nu \circ f_\rho(y), \nu \circ f_\rho(z), \Sigma_{gg'}^{\rho_n} \nu \circ f_\rho(z'))].$$

In the $\mathrm{PSL}_n\mathbb{R}$-Fuchsian case, the slithering map $\Sigma_{gg'}^{\rho_n}$ is equal to $\iota_n(\Sigma_{gg'}^{\rho})$ since the linear map $\iota_n(\Sigma_{gg'}^{\rho})$ satisfies the properties which define $\Sigma_{gg'}^{\rho_n}$. Indeed, the first property holds since $\Sigma_{gg'}$ is the slithering map and ι_n is a group homomorphism. The second property follows since ι_n is Hölder continuous with respect to the operator norm. In particular, by definition of ι_n, the image $\iota_n(A)$ has entries which are polynomials of the entries of A. In the definition of $\Sigma_{gg'}^{\rho_n}$, we consider the flag curve of Veronese type. Since the Veronese flag curve ν is ι_n-equivariant, $\iota_n(\Sigma_{gg'}^{\rho})$ satisfies the third property. Thus we obtain $\Sigma_{gg'}^{\rho_n} = \iota_n(\Sigma_{gg'}^{\rho})$ by the uniqueness.

Using this equality and Proposition 3.5, we can calculate the shearing class as follows.

$$\begin{aligned}
\sigma_b^{\rho_n}(k) &= \log[D_b(\nu \circ f_\rho(x), \nu \circ f_\rho(y), \nu \circ f_\rho(z), \Sigma_{gg'}^{\rho_n} \nu \circ f_\rho(z'))] \\
&= \log[D_b(\nu \circ f_\rho(x), \nu \circ f_\rho(y), \nu \circ f_\rho(z), \iota_n(\Sigma_{gg'}^{\rho}) \nu \circ f_\rho(z'))] \\
&= \log[D_b(\nu \circ f_\rho(x), \nu \circ f_\rho(y), \nu \circ f_\rho(z), \nu \circ \Sigma_{gg'}^{\rho} f_\rho(z'))] \\
&= \log[-\mathrm{cr}(f_\rho(x), f_\rho(y), f_\rho(z), \Sigma_{gg'}^{\rho} f_\rho(z'))].
\end{aligned}$$

We remark that the slithering map $\Sigma_{gg'}^{\rho}$ is the extension of the horocyclic flow onto the ideal boundary. Indeed, the slithering map $\Sigma_{gg'}^{\rho}$ is constructed by the ordered product of $\Sigma_T^{\rho} \in \mathrm{PSL}_n\mathbb{R}$ as T ranges over all ideal triangles of $\widetilde{S} \setminus \widetilde{\lambda}$ separating g and g' ([**BD17**, Proposition 5.1]). Here the ideal triangles T is ordered from g to g'. All triangles T has two edges g_T and g'_T so that they separate g and g', and g_T (resp. g'_T) are near to g (resp. g'). The element Σ_T^{ρ} is defined by the parabolic element which sends g'_T to g_T, and this implies that $\Sigma_{gg'}^{\rho}$ is obtained by the horocyclic flow. Hence the last quantity is just equal to the value $\sigma^\rho(k)$ by the definition of the shearing cocycle. □

We construct an affine slice of $\mathcal{P}_\lambda$. Let $\mathcal{S}_\lambda$ be the slice of $\mathcal{P}_\lambda$ so that the first coordinate σ^ρ consists of the same entry, *i.e.* $\sigma_1^\rho = \cdots =$

$\sigma^{\rho}_{n-1} = \alpha$ where α is a $\mathbb{R}$-valued relative tangent cycle of λ, and the second coordinate is equal to 0. Let $x = (\sigma, 0)$ be a point of $\mathcal{S}_\lambda$, and let $\sigma = (\alpha, \cdots, \alpha)$. By the shearing cycle boundary condition for x, the boundary of the tangent cycle α is equal to zero since all $\tau_{pqr}(s^i_j)$-coordinates are 0. Then the quasi-additivity of α gives the additivity, so the entries α is just a transverse cocycle. Moreover, the positive intersection condition implies that, for any non-zero transverse measure μ on λ, the intersection number $\mu \cdot \alpha$ is positive. Hence α is a shearing cocycle, and there exists a Fuchsian representation which defines σ by the shearing parameterization. This argument shows the following conclusion.

THEOREM 4.4. *Let $\mathcal{S}_\lambda$ be the affine slice which is defined by the conditions that all $\tau_{pqr}(s^i_j)$-coordinates are equal to zero, and, for any oriented arc k tightly transverse to λ, the shearing class is of the form $\alpha(k) \cdot (1, \cdots, 1)^t$ where α is a transverse cocycle of λ. The restriction $\Phi_\lambda|_{F_n(S)} \colon F_n(S) \to \mathcal{S}_\lambda$ is surjective.*

3. The case of surfaces with boundary

3.1. The Hitchin component of surfaces with boundary. A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n\mathbb{R}$ is said to be *purely loxodromic respecting boundary* if the image of each boundary component via ρ is conjugate to an element with pairwise distinct, only real eigenvalues. We denote, by $R^{\mathrm{loxo}}_n(S)$, the space of representations which are purely loxodromic respecting boundary. In addition, we define $X^{\mathrm{loxo}}_n(S) = R^{\mathrm{loxo}}_n(S)/\mathrm{PSL}_n\mathbb{R}$, where the quotient is defined by the conjugate action.

Note that $\mathscr{T}(S)$ is contained in $X^{\mathrm{loxo}}_2(S)$, and $(\iota_n)_*(\mathscr{T}(S))$ is contained in $X^{\mathrm{loxo}}_n(S)$. The ($\mathrm{PSL}_n\mathbb{R}$-) Hitchin components $H_n(S)$ is the connected component of $X^{\mathrm{loxo}}_n(S)$ which contains the image $F_n(S) = (\iota_n)_*(\mathscr{T}(S))$.

3.2. The main result for surfaces with boundary. To define the Bonahon-Dreyer parameterization for surfaces with boundary, Bonahon and Dreyer used the result of Labourie and McShane.

THEOREM 4.5. *(Labourie-McShane* [**LaMc09**, Theorem 9.1.]*) Let S be a compact hyperbolic oriented surface with nonempty boundary, and $\rho\colon \pi(S) \to \mathrm{PSL}_n\mathbb{R}$ be a Hitchin representation. Then there exists* *a unique Hitchin representation* $\widehat{\rho}\colon \pi_1(\widehat{S}) \to \mathrm{PSL}_n\mathbb{R}$ *of the fundamental group of the double* $\widehat{S}$ *of* S *such that the restriction* $\widehat{\rho}$ *to* $\pi_1(S)$ *is equal to* ρ*.*

The extension $\widehat{\rho}$ of ρ is called the *Hitchin double*. For the flag curve $\widehat{\xi}_{\widehat{\rho}}\colon \partial\pi_1(\widehat{S}) \to \mathrm{Flag}(\mathbb{R}^n)$, we set $\xi_\rho = \widehat{\xi}_{\widehat{\rho}}|\partial\pi_1(S)$, the restriction to the boundary of $\pi_1(S)$. We call this restriction the *restricted flag curve*. In the parameterization of Hitchin representations in this case, we can use this restricted flag curves instead of the usual flag curves. (See [**BD14**, Section 7].) Then our results are extended to the case of surfaces with boundary. To check this, we focus on the doubling construction of $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representations. In the proof of the existence of Hitchin doubles ([**LaMc09**, Theorem 9.1]), we can see that the double of a $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation $\iota_n \circ \rho$ is again $\mathrm{PSL}_n\mathbb{R}$-Fuchsian. Especially, the Hitchin double $\widehat{\iota_n \circ \rho}$ is equal to the $\mathrm{PSL}_n\mathbb{R}$-Fuchsian representation $\iota_n \circ \widehat{\rho}$ induced by the hyperbolic double $\widehat{\rho}$ of the Fuchsian representation ρ. Thus the restricted flag curve of $\iota_n \circ \rho$ is the restriction of the Veronese flag curve of $\iota_n \circ \widehat{\rho}$, and our results are shown similarly in the case of compact surfaces with boundary.

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