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Studies on the geometry of line bundles

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Contents

Introduction

A large part of the study of the geometry of projective varieties is concerned with the study of line bundles on them, which may be called the *geometry of line bundles* in short. Line bundles on varieties have a lot of information of the geometric properties of the varieties. For studying algebraic varieties, it is important to understand regular functions on them. In fact, the structure of an affine variety is completely determined by the ring of regular functions on the variety. On the other hand, any function defined on the whole of a projective variety is just a constant map. However, a global section of a line bundle on a variety gives a collection of nontrivial regular functions on open subsets which cover the variety, and hence we can study projective varieties by observing line bundles on them. A manifestation of this idea is the fact that, any morphism between projective varieties is given by a linear system associated to a line bundle on the source of the morphism. It is also quite important that the positivity of line bundles on an algebraic variety is deeply related to the global geometric structure of the variety. This thesis is a contribution to the geometry of line bundles in this sense. We discuss two topics in this direction separately in Chapter 1 and 2, respectively, as we explain below.

Chapter 1 is about a class of morphisms of algebraic varieties which we call *Mori dream morphisms* (MDM for short). We explain the motivation for studying this class of morphisms. The *minimal model program* (MMP), which is the central method of the birational classification of algebraic varieties, is the process of killing the loci along which the canonical line bundle is negative. Though the original MMP is about the canonical line bundle, there are many situation in which it is also meaningful to consider an MMP with respect to an arbitrary divisor D (we call it a D-MMP) on a variety, which kills the negative loci along the line bundle associated to D. We refer to [**KM98**] for more details of MMP. There is a class of varieties called *Mori dream spaces* (MDS), introduced by Hu and Keel in the seminal paper [**HK00**], on which a D-MMP works in the strongest sense; i.e., it exists and terminates either in a Mori fibre space or a good minimal model. On the other hand, it is quite important to consider the theory of MMP in the relative settings, where the varieties in each step of MMP admits a morphism to the fixed base variety that commutes with the contraction killing the negative loci with respect to the relative canonical line bundle. Hence it is natural to consider a class of morphisms $\pi\colon X \to U$ on which the relative D-MMP works for any divisor D on X. Furthermore, there have been many morphisms studied in various contexts (for example, toric morphisms, extremal contractions, and flat deformations of Fano varieties as studied in [**dFH12**] and [**dFH11**]) which have been vaguely recognized as "something like MDS over the base". We prove that some of them are actually MDMs in our sense in Section 1.6. In particular, by applying Theorem 0.0.2 below, we show that varieties relatively of Fano type over the bases are MDMs as in the non-relative case in Example 1.6.1.

The definition and some early examples of MDM, the relative version of MDS, are given in [**AW14**] (our definition of MDM is given in a slightly more general setting than that of the paper). It is a class of projective morphisms between normal quasi-projective varieties satisfying a few nice conditions on the relative cone of divisors (see Definition 1.2.1), which is the generalization of MDS in the sense that an MDM is an MDS if the target is a point. In the paper [**AW14**], they prove that a 4-dimensional local symplectic contraction is an MDM and study explicitly the structure of its relative movable cone for a concrete example. However, foundational properties of MDMs generalizing those of MDSs proved in [**HK00**], including the existence of the relative D-MMP for any divisor D, remained to be established. This is settled in this thesis.

We first show that the natural generalization of the results in [**HK00**, 1.11.PROPOSITION] holds. In particular, we have the following theorem.

Theorem 0.0.1 (= Theorem 1.4.6)**.** *Let $\pi\colon X \to U$ be an MDM and D a divisor on X. Then there exists an MMP for D over U and it terminates either in a Mori fibre space or a good minimal model (i.e., a model on which the strict transform of the divisor D is semiample over U).*

There is another important result on MDSs which should be generalized to MDMs. It is a characterization of MDS via Cox ring [**HK00**, 2.9. Proposition], which says that the $\mathbb{Q}$-factorial variety X such that $\mathrm{Pic}(X)_{\mathbb{Q}} \simeq \mathrm{N}^1(X)_{\mathbb{Q}}$ is an MDS if and only if its Cox ring is finitely generated, as an application of the theory of VGIT (Variation of Geometric Invariant Theory quotients). Many interesting examples of MDSs such as varieties of Fano type in characteristic 0, and K3 surfaces with only finitely many automorphisms are proved to be MDSs by applying the above characterization, hence it is worth to generalize it to the relative case.

We handle this problem in Section 1.5 and obtain the Theorem 0.0.2 below.

Theorem 0.0.2. *Let $\pi\colon X \to U$ be an algebraic fibre space between normal quasi-projective varieties. Assume that X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X/U)_{\mathbb{Q}} \simeq \mathrm{N}^1(X/U)_{\mathbb{Q}}$. Then X is an MDM if and only if its Cox sheaf is a finitely generated $\mathcal{O}_U$-algebra.*

The only if part, which is proved in Proposition 1.5.10, is an easy consequence of the Mori chamber decomposition of the relative effective cone (see Corollary 1.4.8) and the finite generation of the section algebras (see Corollary 1.4.7). To prove the if part, we study a version of VGIT for affine morphisms and show that some GIT quotients of affine morphisms are MDMs in Theorem 1.5.7, which is the generalization of [**HK00**, 2.3.THEOREM]. Moreover, we show in Proposition 1.5.11 that $\pi\colon X \to U$ is a GIT quotient of the relative spectrum of a Cox sheaf of X over U if the Cox sheaf is a finitely generated $\mathcal{O}_U$-algebra. Finally we obtain the if part of Theorem 0.0.2 by combing Theorem 1.5.7 and Corollary 1.5.13.

In Section 1.6, we give various examples of MDMs and study the functorial natures of MDMs, in particular with respect to compositions and base changes. In general the composition of two MDMs is not necessary an MDM. A typical example is the blowing up of $\mathbb{P}^2$ in nine general points (see Example 1.6.3). On the other hand we will show the following theorem, which says that if the composition of two algebraic fibre spaces is an MDM, then both of them are also MDMs.

Theorem 0.0.3. *Let $f\colon X \to Y$ be an algebraic fibre space between normal $\mathbb{Q}$-factorial quasi-projective varieties. Suppose that $\pi_1\colon X \to U$ and $\pi_2\colon Y \to U$ are algebraic fibre spaces to a quasi-projective normal variety U satisfying the following commutative* *diagram.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd} \tag{0.0.1}$$

If π_1 is an MDM, then both π_2 and f are MDMs.

Theorem 0.0.3 follows from Proposition 1.6.5 and Proposition 1.6.8, which is the generalization of [**Oka16**, Theorem 1.1].

In the final part of Chapter 1, we study base changes of MDMs. In general a base change of an MDM is not necessarily an MDM. In Example 1.6.12, we see this by considering a one parameter family of $K3$ surfaces such that the special fibre has infinitely many automorphisms. However, under a few reasonably strong conditions as in the following theorem, we can show that the base change of an MDM is also an MDM.

Theorem 0.0.4. *Let $f\colon X \to U$ be an MDM and $g\colon T \to U$ be a morphism between quasi-projective varieties. Let $W := X \times_U T$, and consider the following diagram.*

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & U \end{tikzcd} \tag{0.0.2}$$

p and q denote the natural projections. Assume the following three conditions:

(1) W is normal and $\mathbb{Q}$-factorial.

(2) The natural map $p^*\colon \mathrm{Pic}(X/U) \to \mathrm{Pic}(W/T)$ is surjective.

(3) $\mathrm{N}^1(W/T)_{\mathbb{Q}} \simeq \mathrm{Pic}(W/T)_{\mathbb{Q}}$ and the natural map $g^*f_*L \to q_*p^*L$ is surjective for any line bundle L on X.

Then q is an MDM.

The proof of this theorem is given as an application of Theorem 0.0.2. We see that the finite generation of the Cox sheaf of f implies the same for q under the conditions (2) and (3).

When the conditions (1) and (2) of Theorem 0.0.4 hold, we can check that the condition (3) holds if g is a flat proper morphism. Hence we have the following corollary.

Corollary 0.0.5. *Let $f\colon X \to U$ be an MDM and $g\colon T \to U$ be a morphism between quasi-projective varieties. Let $W := X \times_U T$ as in* (0.0.2)*. Assume the condition* (1) *and* (2) *in Theorem 0.0.4 and the following* (3′)*.*

(3') g is flat and proper.

Then q is an MDM.

We also give another proof of this corollary, where we directly investigate the geometry of W and confirm that q satisfies the conditions of Definition 1.2.1. The flatness of g is used to show that the small $\mathbb{Q}$-factorial modifications of X can be lifted to those of W.

The foundational results on MDMs obtained in this thesis should be useful for further investigations of MDM in nature, such as some families of moduli spaces over a base.

Chapter 2 is concerned with the Seshadri constants of polarized abelian varieties. The *Seshadri constant* is introduced in [**Dem92**] inspired by the Seshadri's characterization of ampleness, which measures the positivity of a line bundle at a point on a projective variety (see Section 2.1 for details). It is closely related to various geometric notions such as separation of jets, very ampleness of adjoint bundles, and the symplectic packing problem. We recommend [**Laz04**, Chapter 5] for details. For a polarized abelian variety (A, L), the Seshadri constant $\varepsilon(A, L; x)$ does not depend on the choice of a point $x \in A$, and hence we denote it by $\varepsilon(A, L)$. An important problem is to understand how $\varepsilon(A, L)$ reflects the global structure of (A, L).

For example, it is known that there exists a close relationship between the Seshadri constants and minimal period lengths (also called the Bauer-Sarnak invariants) of polarized complex tori. For details, see [**Laz96**], [**Bau98**] or [**Laz04**, Chapter 5]. Apart from this, Nakamaye [**Nak96**] proved the following very interesting result.

Theorem 0.0.6 (=[**Nak96**, Theorem 1.1])**.** *Let (A, L) be a polarized abelian variety of dimension n. Then $\varepsilon(A, L) \geq 1$. Moreover, $\varepsilon(A, L) = 1$ if and only if (A, L) is isomorphic to $(E, L_1) \times (B, L_2)$, where L_1 is a line bundle of degree 1 on an elliptic curve E and (B, L_2) is a polarized abelian variety of dimension $n - 1$.*

Another interesting result [**Nak96**, Lemma 3.3] from the same paper says that if there exists a curve C on A such that

$$\varepsilon_C(L) := \frac{C.L}{\mathrm{mult}_0(C)} < \frac{\sqrt[n]{L^n}}{n}, \tag{0.0.3}$$

then C is contained in a proper abelian subvariety of A.

In view of this, it is natural to ask wether there exists a proper abelian subvariety B which computes the Seshadri constant (i.e., $\varepsilon(A, L) = \varepsilon(B, L|_B)$) under the assumption of [**Nak96**, Lemma 3.3]. In general there exist infinitely many abelian subvarieties of a fixed abelian variety. Hence, for a sequence of curves $\{0 \in C_n\}_n$ such that $\{\varepsilon_{C_n}(L)\}_n$ converges to $\varepsilon(A, L)$, it is not clear whether we can take a subsequence such that all C_n are contained in the same proper abelian subvariety of A. Our first result gives an affirmative answer to this question.

Theorem 0.0.7 (=Theorem 2.2.3)**.** *Assume that*

$$\varepsilon(A, L) < \frac{\sqrt[n]{L^n}}{n}. \tag{0.0.4}$$

Then there exists a proper abelian subvariety B of A such that $\varepsilon(A, L) = \varepsilon(B, L|_B)$.

We prove Theorem 0.0.7 by combining [**Nak96**, Lemma 3.3] and some finiteness theorems for abelian varieties.

Applying Theorem 2.2.3 repeatedly, we obtain the following corollary.

Corollary 0.0.8. *For a polarized abelian variety (A,L), there exists an abelian subvariety $B \subset A$ of dimension k (possibly $A = B$) such that*

$$\varepsilon(A, L) = \varepsilon(B, L|_B) \geq \frac{\sqrt[k]{(L|_B)^k}}{k}. \tag{0.0.5}$$

We also prove the following theorem.

Theorem 0.0.9 (= Corollary 2.2.9)**.** *Let (A, L) be a polarized abelian variety of dimension n. Fix a positive real number a. Let D be an abelian divisor in A such that*

$$\sqrt[n]{L^n} \geq \sqrt[n-1]{a(L|_D)^{n-1}}. \tag{0.0.6}$$

If $\varepsilon(A, L) < a\sqrt[n]{L^n}/n$ holds, then $\varepsilon(A, L) = \varepsilon(D, L|_D)$. Moreover, if one can take $a > (\sqrt[n]{n})^{n-1}$, the upper bound $\varepsilon(A, L) < a\sqrt[n]{L^n}/n$ automatically holds.

We prove Theorem 0.0.7 and Theorem 0.0.9 in Section 2.2.

We discuss various applications of Theorem 0.0.7 and Theorem 0.0.9 in Section 2.3. By the argument in the proof of Theorem 0.0.9, we obtain an interesting relationship between the set of curves C with sufficiently small $\varepsilon_C(L)$ compered to L^n and the set of abelian divisors satisfying (0.0.6) (see Proposition 2.2.7 and (2.3.3) for details). For abelian surfaces, we obtain the following corollaries.

Corollary 0.0.10 (= Proposition 2.3.1)**.** *Let (S, L) be a polarized abelian surface. Assume that there exists a curve $C \ni 0$ such that*

$$\varepsilon_C(S, L) < \sqrt{\frac{L^2}{2}}. \tag{0.0.7}$$

Then C is elliptic, and it is the unique curve satisfying (0.0.7) *and containing* $0 \in S$*.*

Theorem 0.0.7 and Theorem 0.0.9 mentioned above mean that the computation of the Seshadri constant of an abelian variety can be reduced to that of its abelian subvariety in some cases. There exist many results which compute the Seshadri constants on concrete low-dimensional abelian varieties. [**Ste98**] and [**Kon03**] (respectively, [**BS01**] and [**Deb04**]) handle the case of the Theta divisors on Jacobian varieties of curves (resp. principally polarized abelian varieties). The Seshadri constants of abelian surfaces have been studied in further detail. For example, it is known that the Seshadri constants of abelian surfaces are rational (for more results, see Appendix of [**Bau98**], [**BS08**], [**BGS18**], etc). In the latter part of Section 2.3, keeping these previous works in mind, we give some applications of our theorems. In particular, we show Corollary 2.3.6 below. Note that this results is interesting in the sense that it gives an example such that the conditions for the Seshadri constant greatly affects the geometric structure of a polarized abelian variety.

Corollary 0.0.11. *Let (A, L) be a polarized abelian threefold. Assume $L^3 \leq 174$ and $\varepsilon(A, L) < \sqrt[3]{L^3}/3$. Then $\varepsilon(A, L) = 1$ or $4/3$. Moreover, if $\varepsilon(A, L) = 4/3$, A contains the Jacobian variety J of a genus two curve such that any curve satisfying $\varepsilon_C(L) < 21\sqrt[3]{L^3}/8$ is contained in J and $A \simeq J \times E$ for some elliptic curve E. If $L^3 \leq 60$, we obtain $\varepsilon(A, L) = 1$.*

The assumption $L^3 \leq 60$ is optimal for $\varepsilon(A, L) = 1$. In fact, for any $n \in 6\mathbb{Z}$ satisfying $n > 60$, we can construct examples of (A, L) satisfying $\varepsilon(A, L) < \sqrt[3]{L^3}/3$, $L^3 = n$, and $\varepsilon(A, L) \neq 1$ (see Example 2.3.8 for this).

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CHAPTER 1

On relative version of Mori dream spaces

This chapter devoted to the study of Mori dream morphisms, which is the generalization of the definition of Mori dream space to the relative setting in that a Mori dream space is precisely a Mori dream morphism whose target is a point.

Throughout this chapter, we assume that any varieties are normal and quasi-projective over the base field $\mathbb{C}$ unless otherwise stated.

1.1. Preliminaries

We use the following notations and definitions.

Definition 1.1.1. Let $\pi\colon X \to U$ be a projective morphism between varieties.

(1) $\mathrm{Pic}(X/U) \coloneqq \mathrm{Pic}(X)/\pi^*(\mathrm{Pic}(U))$ and $\mathrm{Pic}(X/U)_{\mathbb{Q}} \coloneqq \mathrm{Pic}(X/U) \otimes \mathbb{Q}$.

(2) $\mathrm{N}^1(X/U)_{\mathbb{Q}} \coloneqq \mathrm{Pic}(X/U)_{\mathbb{Q}}/\equiv$, where $D \equiv D'$ if $D.C = D'.C$ for any complete curve contracted by π.

(3) For a $\mathbb{Q}$-divisor D on X, we define the *stable base locus* of D over U as

$$\mathbf{B}(D/U) \coloneqq \bigcap_{0 \leq D' \sim_{\pi,\mathbb{Q}} D} \mathrm{Supp}(D'). \tag{1.1.1}$$

(4) We define the *augmented base loci* of D as

$$\mathbf{B}_+(D/U) \coloneqq \mathbf{B}(D - \epsilon A/U), \tag{1.1.2}$$

where A is π-ample divisor and ϵ is a sufficiently small positive rational number.

(5) For a Cartier divisor D on X, the rational map associated to the following natural map

$$\alpha_{D/U}\colon \pi^*\pi_*\mathcal{O}(D) \to \mathcal{O}(D) \tag{1.1.3}$$

is denoted by $\Phi_{D/U}\colon X \dashrightarrow \mathrm{Proj}_U(\mathrm{Sym}(\pi_*\mathcal{O}_X(D)))$. We often write α_D (respectively, Φ_D) instead of $\alpha_{D/U}$ (resp., $\Phi_{D/U}$) if no confusion is possible.

(6) We say that a $\mathbb{Q}$-divisor D is *π-semiample* (or relatively semiample over U) if there exists a morphism $X \to Y$ over U such that D on X is a pull-back of a $\mathbb{Q}$-divisor on Y ample over U. This is equivalent to the condition that α_{mD} is surjective for some positive integer m.

(7) We say that a $\mathbb{Q}$-divisor D is *π-movable* (or movable over U) if

$$\mathrm{codim}(\mathrm{Supp}(\mathrm{coker}(\alpha_{mD}))) \geq 2 \tag{1.1.4}$$

for some positive integer m. Since X is assumed to be normal this is equivalent to that there exists an open subset $V \subset X$ such that $\mathrm{codim}(X \backslash V) \geq 2$ and $D|_V$ is $\pi|_V$-semiample.

Remark 1.1.2. In [**Kaw88**, Section 2], π-movable means the condition

$$\mathrm{codim}(\mathrm{Supp}(\mathrm{coker}(\alpha_D))) \geq 2. \tag{1.1.5}$$

Our definition is slightly different but more convenient for our purposes.

Proposition 1.1.3. *Let $\pi\colon X \to U$ be as in Definition 1.1.1. Then a divisor D on X is π-semiample if and only if there exists an ample divisor A on U and a positive integer m such that $mD + \pi^*A$ is semiample over $\mathbb{C}$. Similarly, D is π-movable if and only if there exists an ample divisor A on U and a positive integer m such that $mD + \pi^*A$ is movable over $\mathbb{C}$.*

Proof. We prove the proposition only in the case which D is π-semiample since the other case obviously follows by the almost same argument. If $mD + \pi^*A$ is generated by global sections for some $m > 0$ and a divisor A on U, it is obvious that D is π-semiample. Hence we prove the other implication. Since D is π-semiample, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & U & \end{tikzcd} \tag{1.1.6}$$

and a π'-ample divisor A on Y such that $f^*(A) = D$. Since U is quasi-projective, there exists an ample divisor H on U such that $A + \pi'^*H$ is ample. Hence, for sufficiently large m, the divisor $mA + m\pi'^*H$ is very ample, and so it is generated by global sections. Then we obtain that

$$mD + \pi^*(mH) = f^*(mA + m\pi'^*H), \tag{1.1.7}$$

and it is also generated by global sections. $\square$

Corollary 1.1.4. *Under the same assumptions as in Proposition 1.1.3, D is π-movable if and only if* $\operatorname{codim}(\mathbf{B}(D/U)) \geq 2$.

Proof. If D is π-movable, then it follows that $\operatorname{codim}(\mathbf{B}(D/U)) \geq 2$ by Proposition 1.1.3. The other implication follows from [**BCHM10**, Proposition 3.5.4] $\square$

Lemma 1.1.5. *Let D be a π-effective divisor on X. Then we obtain an effective divisor F and a π-movable divisor M such that $D = M + F$ and $\pi_*\mathcal{O}_X(D) \simeq \pi_*\mathcal{O}_X(M)$.*

Proof. Let Z be a closed subscheme of X corresponding to the ideal sheaf on X defined by the image of the map $\alpha'_D \coloneqq \alpha_D \otimes \mathcal{O}_X(-D)$, whose support coincides with $\operatorname{Supp}(\operatorname{coker}(\alpha_D))$. For any prime divisor $F_i \subset Z$, we denote $k_i \coloneqq \operatorname{ord}_{F_i}(Z)$. Then let $F \coloneqq \Sigma_i k_i F_i$ and $M \coloneqq D - F$. We will check that M and F satisfy the desired condition locally over U. Consider an affine covering $U = \bigcup_{j=1}^r U_j$ of U. We denote $\pi^{-1}(U_j)$ by V_j. Let us consider

$$\alpha'_D|_{V_j}\colon H^0(V_j, \mathcal{O}_{V_j}(D)) \otimes \mathcal{O}_{V_j}(-D) \to \mathcal{O}_{V_j}. \tag{1.1.8}$$

Then obviously the closed subscheme defined by the image of α'_D is $Z \cap V_j$ and $\operatorname{ord}_{F_i \cap V_j}(Z \cap V_j) = k_i$ if $F_i \cap V_j \neq \emptyset$. This implies that any section in $H^0(V_j, \mathcal{O}_{V_j}(D))$ vanishes along the subscheme $\Sigma_{i=1}^{n_j} k_i(F_{l_i} \cap V_j)$, where $F_{l_1}, \ldots, F_{l_{n_j}}$ are all the components of F which have nonempty intersections with V_j.

Then the natural injective map

$$H^0(V_j, \mathcal{O}_{V_j}(D - F)) = H^0(V_j, \mathcal{O}_{V_j}(D - \Sigma_{i=1}^{n_j} k_{l_i} F_{l_i})) \hookrightarrow H^0(V_j, \mathcal{O}(D)), \tag{1.1.9}$$

is an isomorphism since the following map

$$H^0(V_j, \mathcal{O}_{V_j}(D)) \to H^0(\bigcup_{i=1}^{n_j} k_{l_i}F_{l_i}, \mathcal{O}_{\bigcup_{i=1}^{n_j} k_{l_i}F_{l_i},(D)}) \tag{1.1.10}$$

is the zero map. Hence we obtain $\pi_*\mathcal{O}_X(D) \simeq \pi_*\mathcal{O}_X(M)$. Moreover, for each $i = 1, \dots n_j$, we have a section

$$s_i \in H^0(V_j, \mathcal{O}_{V_j}(D - F)) \tag{1.1.11}$$

which does not vanish along F_{l_i}. This implies that the closed subscheme defined by the image of $\alpha'_{(D-F)}|_{V_j}$ has codimension at least two since it does not containing any F_{l_i} by the above argument. Since $\mathrm{Supp}(\mathrm{coker}(\alpha'_{D-F})) = \mathrm{Supp}(\mathrm{coker}(\alpha_{D-F}))$, it follows that $M = D - F$ is π-movable. $\square$

From now on we assume that $\pi\colon X \to U$ is an algebraic fibre space (i.e., a projective morphism satisfying $\mathcal{O}_U = f_*\mathcal{O}_X$) and X is $\mathbb{Q}$-factorial.

Lemma 1.1.6. *Let $f\colon X \to Y$ be an algebraic fibre space over U. Then f induces a natural injection*

$$f^*\colon \mathrm{Pic}(Y/U) \hookrightarrow \mathrm{Pic}(X/U). \tag{1.1.12}$$

Proof. Let L be a line bundle on Y. Note that $f_*(f^*L) \simeq L$ since f is an algebraic fibre space. This implies that if $[f^*L] = 0$ in $\mathrm{Pic}(X/U)$, then L comes from a line bundle on U. $\square$

1.1.1. **Rational maps.** Let us consider a dominant rational map $f\colon X \dashrightarrow Y$ over U.

Lemma 1.1.7. *f induces a natural map $f^*\colon \mathrm{Pic}(Y/U) \to \mathrm{Pic}(X/U)$. Moreover, If a divisor D on X satisfies $D \equiv_U 0$, then $f^*D \equiv_U 0$. The induced map $f^*\colon \mathrm{N}^1(Y/U)_{\mathbb{Q}} \to \mathrm{N}^1(X/U)_{\mathbb{Q}}$ is injective if f is birational.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
& \tilde{X} \arrow[dl, "\mu"'] \arrow[dr, "f'"] & \\
X \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\
& U &
\end{tikzcd} \tag{1.1.13}$$

where μ is a resolution of f and $\tilde{X}$ is nonsingular and projective over U. We can easily check that the morphism

$$f^*\colon \mathrm{Pic}(Y/U) \to \mathrm{Pic}(X/U); \quad [D] \mapsto [\mu_*(f'^*(D))] \tag{1.1.14}$$

does not depend on the choice of a resolution. Let us assume $D \equiv_U 0$. Consider the divisor $E := \mu^*(f^*(D)) - f'^*(D)$. Obviously, E is μ-exceptional, and $E \equiv_U \mu^*(f^*(D))$ since $f'^*(D) \equiv_U 0$. Take any curve C on $\tilde{X}$ such that $\mu(C) = \{pt\}$. We have

$$C.E = C.\mu^*(f^*(D)) - C.f'^*(D) = \mu_*(C).f^*(D) = 0. \tag{1.1.15}$$

By applying [**KM98**, Lemma 3.39] to μ, we obtain $E = 0$. Hence $\mu^*(f^*(D)) \equiv_U 0$, and the projection formula implies $f^*(D) \equiv_U 0$.

Let us prove the second assertion. Assume that f is birational and $f^*(D) \equiv_U 0$. Take a curve on C such that $f'(C) = \{pt\}$. Then we have

$$C.E = C.\mu^*(f^*(D)) - C.f'^*(D) = \mu_*(C).f^*(D) = 0, \tag{1.1.16}$$

where the last equality follows from that $f^*(D) \equiv_U 0$ and that $\mu(C)$ is contracted by π. By applying [**KM98**, Lemma 3.39] to the birational morphism f', we obtain that $E = 0$. Then for any curve C on Y such that $\pi'(C) = \{pt\}$, we obtain

$$D.C = f'^*(D).\tilde{C} = \mu^*(f^*(D)).\tilde{C} = f^*(D).\mu_*(\tilde{C}) = 0, \tag{1.1.17}$$

where $\tilde{C}$ is the strict transform of C in $\tilde{X}$. □

We recall the definition of rational contractions (see also [**HK00**]).

Definition 1.1.8. A rational map $f\colon X \dashrightarrow Y$ over U is a *rational contraction* if for some resolution (equivalently, for any resolution) $(p, q)\colon W \to X \times Y$ of f such that W is nonsingular and projective over U, every p-exceptional effective divisor E on W satisfies

$$q_*(\mathcal{O}_W(E)) = \mathcal{O}_Y. \tag{1.1.18}$$

An effective divisor F on X is *f-fixed* if any effective divisor D on W whose support is contained in the union of p-exceptional divisors and strict transform of F satisfies

$$q_*(\mathcal{O}_W(D)) = \mathcal{O}_Y. \tag{1.1.19}$$

Remark 1.1.9. Consider the following diagram.

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd} \tag{1.1.20}$$

If f is a rational contraction and π_1 is an algebraic fibre space, we can check that π_2 is also an algebraic fibre space by [**Laz04**, Example 2.1.12].

Lemma 1.1.10. *Let $\pi_i\colon X_i \to U$ $(i = 1, 2)$ be an algebraic fibre space and $f\colon X_1 \dashrightarrow X_2$ be a rational contraction over U. Then, for a Cartier divisor A on X_2 and an f-fixed divisor F on X_1, we obtain*

$$\pi_{1*}(f^*(A) + F) \simeq \pi_{2*}(A). \tag{1.1.21}$$

Proof. Consider the resolution $(p, q) : W \to X_1 \times X_2$ of f as in the Definition 1.1.8. Then $p^*(f^*(A)) - q^*A$ is p-exceptional, hence there exists effective p-exceptional divisors E_1 and E_2 such that

$$p^*(f^*(A)) + E_1 = q^*A + E_2. \tag{1.1.22}$$

Then we obtain

$$\pi_{1*}(f^*(A) + F) \tag{1.1.23}$$
$$\simeq \pi_{1*}p_*(p^*(f^*(A)) + p^*(F) + E_1) \tag{1.1.24}$$
$$= \pi_{2*}q_*(q^*A + E_2 + p^*(F)) \tag{1.1.25}$$
$$\simeq \pi_{2*}(A), \tag{1.1.26}$$

where the finial equality follows from the projection formula and the assumption that F is f-fixed. □

Definition 1.1.11. For a line bundle L on X, we define the section algebra over U as

$$\mathrm{R}_\pi(X, L) \coloneqq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \pi_*(L^m). \tag{1.1.27}$$

If $\mathrm{R}_\pi(X, L)$ is a finitely generated $\mathcal{O}_U$-algebra, we obtain the following rational map to the variety which is projective over U

$$\varphi_{L/U} \colon X \dashrightarrow \mathrm{Proj}_U(\mathrm{R}_\pi(X, L)). \tag{1.1.28}$$

Note that $\mathrm{Proj}_U(\mathrm{R}_\pi(X, L))$ is determined by the class of L in $\mathrm{Pic}(X/U)$ up to isomorphism over U (see [**Har77**, Chapter 2, lemma 7.9)]). We often denote $\varphi_{L/U}$ by φ_L if no confusion is possible.

Remark 1.1.12. φ_L has the following properties.

(1) For any $m > 0$ there exists a natural isomorphism

$$\mathrm{Proj}_U(\mathrm{R}_\pi(X, L)) \simeq \mathrm{Proj}_U(\mathrm{R}_\pi(X, L^m)), \tag{1.1.29}$$

which commutes with the rational maps φ_L and φ_{L^m}.

(2) For a sufficiently divisible $m > 0$, there exists the natural closed immersion

$$\mathrm{Proj}_U(\mathrm{R}_\pi(X, L^m)) \hookrightarrow \mathrm{Proj}_U(\mathrm{Sym}(\pi_*(L^m))), \tag{1.1.30}$$

which commutes with φ_{L^m} and Φ_{L^m}. This follows from [**HK00**, 1.5. LEMMA].

The following proposition is the relative version of [**HK00**, 1.6. LEMMA]

Proposition 1.1.13. *If $\mathrm{R}_\pi(X, L)$ is finitely generated, then $\varphi_L \colon X \dashrightarrow Y$ is a rational contraction and $L \sim_{\mathbb{Q},U} \varphi_L^*(A) + E$ for some relatively ample $\mathbb{Q}$-divisor A on Y over U and φ_L-fixed divisor E. Conversely, consider a rational contraction $f : X \dashrightarrow Y$ over U, and take a Cartier divisor A on Y which is relatively ample over U. Then*

(1) for any f-fixed divisor F, the map f is equal to $\varphi_{f^*(A)+F}$ up to isomorphisms over U, and

(2) f is regular if and only if $f^*(A)$ is π-semiample.

Proof. The arguments of the proof of [**HK00**, 1.6. LEMMA] can be easily generalized for the relative case by applying Lemma 1.1.5, Remark 1.1.12, and Lemma 1.1.10. □

Definition 1.1.14. Take $D_1, D_2 \in \mathrm{Pic}(X/U)_{\mathbb{Q}}$ with finitely generated section algebras over U. Then D_1 and D_2 are *Mori equivalent* if φ_{D_1} coincides with φ_{D_2} up to isomorphism.

Assume $\mathrm{Pic}(X/U)_{\mathbb{Q}} \simeq \mathrm{N}^1(X/U)_{\mathbb{Q}}$. A *Mori chamber* is the closure of a Mori equivalent class with non-empty interior in $\mathrm{Pic}(X/U)_{\mathbb{Q}}$.

Definition 1.1.15. A birational contraction $f \colon X \dashrightarrow Y$ over U is a *small $\mathbb{Q}$-factorial modification* (SQM for short) over U if f is isomorphic in codimension one and Y is $\mathbb{Q}$-factorial.

We can easily show the following properties of SQMs over U.

Remark 1.1.16. Let $g \colon X \dashrightarrow Y$ be an SQM over U. Then

(1) g induces an isomorphism $g^* \colon \mathrm{N}^1(Y/U)_{\mathbb{Q}} \to \mathrm{N}^1(X/U)_{\mathbb{Q}}$.
(2) $g^*(\mathrm{Eff}(Y/U)) = \mathrm{Eff}(X/U)$.
(3) $g^*(\mathrm{Mov}(Y/U)) = \mathrm{Mov}(X/U)$.
(4) Let $f \colon Y \dashrightarrow Z$ be a rational contraction, and $h \coloneqq f \circ g$. For any Cartier divisor D on Z such that f^*D is Cartier, it follows that $h^*(D) = g^*(f^*(D))$.

1.2. Definition of Mori dream morphisms

Let $\pi\colon X \to U$ be as in Definition 1.1.1 and assume that it is an algebraic fibre space.

Definition 1.2.1. π is said to be a *Mori dream morphism* (MDM) if it satisfies the following conditions.

(1) X is $\mathbb{Q}$-factorial.
(2) $\mathrm{Pic}(X/U)_{\mathbb{Q}} \simeq \mathrm{N}^1(X/U)_{\mathbb{Q}}$
(3) $\mathrm{Nef}(X/U)$ is a rational polyhedral cone generated by finitely many π-semiample divisors.
(4) There exists a finite collection of SQMs $g_i\colon X \dashrightarrow X_i$ $(i = 1 \dots r)$ over U such that every X_i satisfies (1), (2), (3), and

$$\mathrm{Mov}(X/U) = \bigcup_{i=1}^{r} g_i^*(\mathrm{Nef}(X_i/U)). \tag{1.2.1}$$

We can replace some of the conditions in the definition of MDM as follows.

Proposition 1.2.2. *In Definition 1.2.1, we can replace* (2) *and* (3) *with the following* (2′) *and* (3′).

(2′) *Every π-nef divisor is π-semiample.*
(3′) $\mathrm{Nef}(X/U)$ *is a rational polyhedral cone.*

Proof. Obviously, (2) and (3) imply (2′) and (3′), so that it is sufficient to show that (2′) implies (2). We prove that the natural surjection $\mathrm{Pic}(X/U)_{\mathbb{Q}} \to \mathrm{N}^1(X/U)_{\mathbb{Q}}$ is injective. Take $[D] \in \mathrm{Pic}(X/U)_{\mathbb{Q}}$ such that $D \equiv_U 0$. Since D is π-semiample, we obtain a morphism $\Phi\colon X \to Y$ over U such that $D = \Phi^*(A)$ for some relatively ample $\mathbb{Q}$-divisor A on Y over U. Then by the numericall condition of D, any curve contracted by π is also contracted by Φ. Hence, by the normality of U, there exists a morphism $g\colon U \to Y$ which satisfies the following diagram.

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & Y \\ & U \arrow[ur, "g"'] & \end{tikzcd} \tag{1.2.2}$$

Hence we obtain that $D = \Phi^*(A) = \pi^*(g^*(A))$, and so $[D] = 0 \in \mathrm{Pic}(X/U)$. □

Remark 1.2.3. Let $\pi\colon X \to U$ be an MDM. For an algebraic fibre space $f\colon X \to Y$ over U, the cone $f^*\,\mathrm{Nef}(Y/U)$ is a face of $\mathrm{Nef}(X/U)$. Indeed, we can check $f^*\,\mathrm{Nef}(Y/U) = \mathrm{Nef}(X/U) \cap \mathrm{NE}(f)^{\perp}$ by the argument as in the proof of Proposition 1.2.2, where $\mathrm{NE}(f) \subset \mathrm{N}_1(X/U)$ is the cone generated by the curves that are contracted by f.

1.3. Fan structure on $\mathrm{Mov}(X/U)$

Throughout this section, let $\pi\colon X \to U$ be an MDM and Y be a normal quasi-projective variety which is projective over U.

Proposition 1.3.1. *Let $f\colon X \dashrightarrow Y$ be a rational contraction over U. Then there exists an SQM $g_i\colon X \dashrightarrow X_i$ as in Definition 1.2.1 such that $h \coloneqq f \circ g_i^{-1}$ is an algebraic fibre* *space.*

$$\begin{tikzcd} X \arrow[r, dashed, "g_i"] \arrow[dr, dashed, "f"'] & X_i \arrow[d, "h"] \\ & Y \end{tikzcd} \tag{1.3.1}$$

Proof. We can prove this as in the last paragraph of the proof of [**HK00**, 1.11 Proposition] by applying Proposition 1.1.13. ☐

Corollary 1.3.2. *Under the assumptions in Proposition 1.3.1, the map*

$$f^*\colon \mathrm{N}^1(Y/U)_{\mathbb{Q}} \to \mathrm{N}^1(X/U)_{\mathbb{Q}} \tag{1.3.2}$$

is injective.

Proof. By Proposition 1.3.1 and Remark 1.1.16, we may assume that f is a morphism over U. Then $f^*(D) \equiv_U 0$ implies $f^*(D) \sim_{\mathbb{Q},U} 0$, and hence $D \sim_{\mathbb{Q},U} 0$ since f is an algebraic fibre space over U. ☐

The following corollary also immediately follows from Proposition 1.3.1.

Corollary 1.3.3. *There is no SQM of X over U other than $g_1, \ldots, g_r$ which appear in Definition 1.2.1.*

Let $\mathcal{M}_{X/U}$ be the set of all faces of the cones $g_i^*(\mathrm{Nef}(X_i/U))$ $(i = 1 \ldots r\ )$. Then the following theorem gives the generalization of [**HK00**, 1.11 Proposition (3)].

Theorem 1.3.4. *The above set $\mathcal{M}_{X/U}$ is a fan whose support is $\mathrm{Mov}(X/U)$. Moreover, there is a natural bijection α between the set of cones of $\mathcal{M}_{X/U}$ and the set of all rational contractions $f\colon X \dashrightarrow Y$ with Y projective over U, given by*

$$\alpha\colon f \mapsto f^*(\mathrm{Nef}(Y/U)). \tag{1.3.3}$$

Proof. We first prove that $\mathcal{M}_{X/U}$ is a fan. For this it is sufficient to show that $g_i^*(\mathrm{Nef}(X_i/U))$ and $g_j^*(\mathrm{Nef}(X_j/U))$ intersect along a common face for any $i, j = 1, \ldots r$ since this implies that $\sigma \cap \tau$ is a common face of σ and τ for any $\sigma, \tau \in \mathcal{M}_X$ by elementary arguments of convex geometry. We may assume that $X_i = X$ and $g_i = \mathrm{id}_X$ without loss of generality.

Take any $[D] \in \mathrm{Nef}(X/U) \cap g_j^*(\mathrm{Nef}(X_j/U))$. By replacing with a multiple, we may assume that D is an integral Cartier divisor and there exists a Cartier divisor D_j in $\mathrm{Nef}(X_j/U)$ such that $[D] = g_j^*[D_j]$. Then we obtain the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g_j"] \arrow[dr, "\varphi_D"'] & X_j \arrow[d, "\varphi_{D_j}"] \\ & Y_D \end{tikzcd} \tag{1.3.4}$$

such that $[D]$ and $[g_j^*(D_j)]$ are the pullbacks of some π-ample divisor on Y_D. This implies that $\varphi_D^*(\mathrm{Nef}(Y_D/U))$ is a common face of $\mathrm{Nef}(X/U)$ and $g_j^*(\mathrm{Nef}(X_j/U))$ containing D in its relative interior by Remark 1.2.3.

Let $A_1, A_2 \in \mathrm{Nef}(X/U)$ be $\mathbb{Q}$-divisors such that $A_1 + A_2 \in \mathrm{Nef}(X/U) \cap g_j^*(\mathrm{Nef}(X_j/U))$. Applying the above argument to $D \coloneqq A_1 + A_2$, we obtain $A_i \in \mathrm{Nef}(X/U)$ $(i = 1, 2)$, to conclude that $\mathrm{Nef}(X/U) \cap g_j^*(\mathrm{Nef}(X_j/U))$ is a face of $\mathrm{Nef}(X/U)$. Indeed, our assumption that $D \in \varphi_D^*(\mathrm{Nef}(Y_D/U))$ implies

$$A_1, A_2 \in \varphi_D^*(\mathrm{Nef}(Y_D/U)) \subset \mathrm{Nef}(X/U) \cap g_j^*(\mathrm{Nef}(X_j/U)) \tag{1.3.5}$$

since $\varphi_D^*(\mathrm{Nef}(Y_D/U))$ is a face of $\mathrm{Nef}(X/U)$. The same argument implies that $\mathrm{Nef}(X/U) \cap g_j^*(\mathrm{Nef}(X_j/U))$ is also a face of $\mathrm{Nef}(X_j/U)$.

For the second part, we define the inverse map β of α as follows. For any cone $\sigma \in \mathcal{M}_{X/U}$, take a line bundle L whose class is contained in the relative interior of σ. Then set $\beta(\sigma) := \varphi_L$. It follows that β does not depend on the choice of L, and it is the inverse map of α by the fact that the relative interiors of two different faces of a convex cone do not intersect each other and Proposition 1.1.13. □

Remark 1.3.5. Let $f\colon X \dashrightarrow Y$ be a rational contraction with Y projective over U (as in the following diagram) and $\sigma := f^*(\mathrm{Nef}(Y/U))$.

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & U & \end{tikzcd} \tag{1.3.6}$$

Then

(1) $\dim \sigma = \rho(Y/U)$.
(2) $f \circ (g_i)^{-1}$ is regular if and only if $\sigma \subset g_i^* \mathrm{Nef}(X_i/U)$.
(3) f is birational if and only if $\sigma \not\subseteq \partial(\mathrm{Eff}(X/U))$.

Proof. (1) and (2) follow from Corollary 1.3.2 and the proof of Proposition 1.3.1. For (3), note that f is birational if and only if the relative interior of $f^*(\sigma)$ is contained in the big cone of X over U. This implies (3). □

To prove Proposition 1.3.7 below, we need the following lemma.

Lemma 1.3.6. *Let $f\colon X \to Y$ be a birational morphism between projective varieties over U. For any π'-ample $\mathbb{Q}$-divisor D on Y, we obtain*

$$\mathbf{B}_+(f^*(D)/U) \subset \mathrm{Exc}(f) \tag{1.3.7}$$

Proof. We mimic the argument in the proof of [**BBP13**, Proposition 2.3].

Take $x \notin \mathrm{Exc}(f)$ and a π-ample divisor H on X such that $x \notin H$ (we can take such H because X is quasi-projective). Since f is an isomorphism around x, we obtain $f(x) \not\subseteq \mathbf{Z}(\mathcal{I})$ for the ideal sheaf $\mathcal{I} := f_*(\mathcal{O}_X(-H)) \subset \mathcal{O}_Y$. Take an ample divisor A on U such that $D' := D + \pi'^*A$ is ample on Y. Then for $k \gg 0$, $\mathcal{O}_Y(kD') \otimes \mathcal{I}$ is generated by global sections. Hence there exists a section

$$\tilde{s} \in H^0(Y, \mathcal{O}(kD') \otimes \mathcal{I}) \tag{1.3.8}$$

such that $\tilde{s}(f(x)) \neq 0$. Let $s \in \mathrm{H}^0(X, k(f^*D + \pi^*A) - H)$ be the section corresponding to $\tilde{s}$ via the following isomorphism

$$H^0(X, k(f^*D + \pi^*A) - H) \simeq H^0(Y, \mathcal{O}(kD') \otimes \mathcal{I}), \tag{1.3.9}$$

which is induced by the projection formula.

Then $s(x) \neq 0$ for $k \gg 0$, and hence $x \notin \mathbf{B}_+(f^*D/U)$. □

Proposition 1.3.7. *Under the same assumptions as in Remark 1.3.5, if f is birational and* $\mathrm{codim}\,\mathrm{Exc}(f) \geq 2$, *then $\sigma \not\subseteq \partial(\mathrm{Mov}(X/U))$.*

Proof. After replacing X to some SQM of X, we may assume that f is a birational morphism by Proposition 1.3.1. Take a divisor B whose class $[B]$ in $\mathrm{N}^1(X/U)$ is in the relative interior of σ. Then it follows that $B = f^*(D)$ for some π'-ample $\mathbb{Q}$-line bundle D on Y. For a π-ample divisor H and sufficiently small $\varepsilon \in \mathbb{Q}_{>0}$, we obtain

$$\mathbf{B}(B - \varepsilon H/U) \subset \mathrm{Exc}(f) \tag{1.3.10}$$

by Lemma 1.3.6. Then by the assumption and Corollary 1.1.4, we have $[B - \varepsilon H] \in \mathrm{Mov}(X/U)$. This implies $[B]$ is contained in the interior of $\mathrm{Mov}(X/U)$ since $[B] = [B - \varepsilon H] + [\varepsilon H]$ and $[H]$ is contained in the interior of $\mathrm{Mov}(X/U)$. □

1.4. Minimal model program

Throughout this section, let $\pi\colon X \to U$ be a Mori dream morphism unless otherwise stated.

Definition 1.4.1. Let $f\colon X \to Y$ be an algebraic fibre space over U.

(1) f is an *elementary contraction* if $\rho(X/U) - \rho(Y/U) = 1$.
(2) f is a *divisorial contraction* if it is a birational morphism which contracts divisors.
(3) f is *of fibre type* (or a *Mori fibre space*) if $\dim(X) > \dim(Y)$.
(4) Assume f is small and D is a divisor on X such that $-D$ is f-ample. Then an SQM $\varphi\colon X \dashrightarrow X'$ is a *D-flip* of f if $f' := f \circ \varphi^{-1}\colon X' \to Y$ is a morphism and $\varphi_*(D)$ on X' is f'-ample. If f is elementary, we sometimes simply call it the flip of f.

Remark 1.4.2. It is well-known that any elementary contraction is either divisorial, of fibre type, or small (for example, see [**KM98**, Proposition 2.5]). We remark the following facts.

(1) For an elementary divisorial contraction f, the exceptional locus $\mathrm{Exc}(f)$ is a prime divisor(for details, see [**KM98**, Proposition 2.5]).
(2) Assume that there exists a D-flip of f. Then $\mathrm{R}_f(X, D)$ is finitely generated and the D-flip X' is isomorphic to $\mathrm{Proj}_Y(\mathrm{R}_f(X, D))$. Furthermore, if f is an elementary contraction, the flip of f does not depend on the choice of a divisor D (for details, see [**KM98**, Corollary 6.4]).

Proposition 1.4.3. *Let $f\colon X \to Y$ be a small elementary contraction over U. Then there exists an SQM $g_i\colon X \dashrightarrow X_i$ as in Definition 1.2.1* (4) *which is a flip of f.*

Proof. Consider $\sigma := f^*(\mathrm{Nef}(Y/U))$, which is a facet of $\mathrm{Nef}(X/U)$ not contained in $\partial(\mathrm{Mov}(X/U))$ by Remark 1.3.5 and Proposition 1.3.7. Hence by the fan structure in Theorem 1.3.4, there exists an SQM $g_i\colon X \to X_i$ such that σ is a common facet of $\mathrm{Nef}(X/U)$ and $g_i^*(\mathrm{Nef}(X_i/U))$. Consider the composition $f' := f \circ {g_i}^{-1}\colon X_i \dashrightarrow Y$ and the hyperplane $H_\sigma := f^*(\mathrm{N}^1(Y/U)_{\mathbb{Q}})$ containing σ, which coincides with $\mathrm{NE}(f)^\perp = g_i^*(\mathrm{NE}(f')^\perp)$ by Remark 1.2.3. If a divisor D on X satisfies $D.\mathrm{NE}(f) < 0$, then D lies in the same side as ${g_i}^*(\mathrm{Nef}(X_i/U))$ of the hyperplane H_σ, and this implies ${g_i}_*(D).\mathrm{NE}(f') > 0$. Hence g_i is the flip of f. □

Proposition 1.4.4. *Consider an elementary divisorial contraction f over U as follows*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & U & \end{tikzcd} \tag{1.4.1}$$

Then π' is also an MDM.

Proof. By Remark 1.2.3, it follows that $f^*(\mathrm{N}^1(Y/U)_{\mathbb{Q}}) = \mathrm{NE}(f)^{\perp}$. Then we can prove that Y is $\mathbb{Q}$-factorial as in the proof of [**KM98**, Proposition 3.36].

If $L \in \mathrm{Nef}(Y/U)$, the pull back f^*L is also π-nef and π-semiample. Since f is an algebraic fibre space, we obtain $\pi^*\pi_*(f^*L) \simeq f^*(\pi'^*\pi'_*(L))$ by the projection formula, and $\alpha_{f^*(L)}\colon \pi^*\pi_*(f^*L) \to f^*L$ coincides with $f^*\alpha_L\colon f^*(\pi'^*\pi'_*(L)) \to f^*(L)$ via this isomorphism. Hence the π-semiampleness of f^*L implies the π'-semiampleness of L. Since $f^*(\mathrm{Nef}(Y/U))$ is a face of $\mathrm{Nef}(X/U)$, obviously $\mathrm{Nef}(Y/U)$ is a polyhedral cone. These arguments show that $\pi'\colon Y \to U$ satisfies the conditions ($2'$) and ($3'$) in Proposition 1.2.2.

Let E be the exceptional divisor of f. We will show that Y satisfies the condition (4) in Definition 1.2.1. Let $g_i\colon X \dashrightarrow X_i$ (for $i = 1, \ldots, r$) be the SQMs of X over U. Consider all the elementary divisorial contractions of X_i over U (for $i = 1, \ldots, r$) such that each exceptional divisor is the strict transform of E. We denote them as

$$f_j\colon X_{i_j} \to Y_j \tag{1.4.2}$$

for $j = 1, \ldots, m$. Note that each Y_j satisfies the conditions (1), (2), and (3) in Definition 1.2.1 as above. Let $\varphi_j \coloneqq f_j \circ g_{i_j} \circ f^{-1}$. Then we obtain the following diagram.

$$\begin{tikzcd} X \arrow[r, dashed, "g_{i_j}"] \arrow[d, "f"'] & X_{i_j} \arrow[d, "f_j"] \\ Y \arrow[r, dashed, "\varphi_j"] & Y_j \end{tikzcd} \tag{1.4.3}$$

Since each φ_j is an SQM of Y, it is sufficient to show that

$$\mathrm{Mov}(Y/U) = \bigcup_{j=1}^{m}(\varphi_j^*(\mathrm{Nef}(Y_j/U))). \tag{1.4.4}$$

Take $D \in \mathrm{Mov}(Y/U)$, then we can find $a \in \mathbb{Q}_{\geq 0}$ such that

$$M \coloneqq f^*D - aE \in \partial\,\mathrm{Mov}(X/U). \tag{1.4.5}$$

Indeed, if $f^*(D) \in \mathrm{Mov}(X/U)$, we can easily check $f^*(D) + \varepsilon E \notin \mathrm{Mov}(X/U)$ for any positive rational number ε, and so $f^*(D) \in \partial\,\mathrm{Mov}(X/U)$. Then assume that $f^*(D) \notin \mathrm{Mov}(X/U)$. Since D is π-movable, there exists a positive integer n such that $\mathrm{codim}(\mathrm{Supp}(\mathrm{coker}(\alpha_{nD}))) \geq 2$. Then we obtain $\mathrm{Supp}(\mathrm{coker}(\alpha_{f^*(nD)})) = E$ holds in codimension one since $\alpha_{f^*(nD)}$ coincides with $f^*\alpha_{nD}$ and it follows that

$$\mathrm{Supp}(\mathrm{coker}(f^*(\alpha_{nD/U}))) = f^{-1}(\mathrm{Supp}(\mathrm{coker}(\alpha_{nD/U}))) \tag{1.4.6}$$

by applying Nakayama's lemma to each stalk of the sheaves. By the proof of Lemma 1.1.5, there exists $a_0 > 0$ such that $f^*(nD) - a_0E$ is π-movable, and hence we can take $a > 0$ such that $f^*D - aE \in \partial\,\mathrm{Mov}(X/U)$.

Take $\tau \in \mathcal{M}_{X/U}$ such that $[M] \in \tau$, $\dim\tau = \rho(X/U) - 1$, and the hyperplane H_τ containing τ separates $\mathrm{Mov}(X/U)$ and $[E]$. Let $\varphi_\tau\colon X \dashrightarrow Z$ be a rational contraction corresponding to τ via Theorem 1.3.4. Then there exists an SQM $g\colon X \dashrightarrow X_k$ such that $h = \varphi_\tau \circ g^{-1}\colon X_k \to Z$ is an algebraic fibre space. Let E_k be the strict transform of E on X_k. Note that $E_k.\,\mathrm{NE}(h) < 0$ by the choice of τ, and hence h is birational and $\mathrm{Exc}(h) \subset E_k$. Since $\tau \subset \partial(\mathrm{Mov}(X/U))$, the morphism h is divisorial by Proposition 1.3.7. Therefore there exists some $j \in \{1, \ldots, m\}$ such that h coincide with $f_j\colon X_{i_j} \to Y_j$, where $i_j = k$. Moreover, there exists a relatively nef $\mathbb{Q}$-divisor M_j on Y_j over U such that $M \sim_{\mathbb{Q},U} g_{i_j}^*(h^*M_j) = g_{i_j}^*(f_j^*(M_j))$. This implies that $D \sim_{\mathbb{Q},U} \varphi_j^*(M_j)$, and so we obtain $[D] \in \varphi_j^*(\mathrm{Nef}(Y_j/U))$. □

Lemma 1.4.5. *Let D be a divisor on X which is not π-nef. Then there exists a D-negative elementary contraction $f\colon X \to Y$ over U.*

PROOF. Take a facet $\sigma \prec \mathrm{Nef}(X/U) \subset \mathrm{N}^1(X/U)_{\mathbb{Q}}$ such that $[D]$ lies on the other side of $\mathrm{Nef}(X/U)$ with respect to the hyperplane H_σ containing σ. Let $f\colon X \to Y$ be the morphism corresponding to σ. Then f is an elementary contraction and it follows that $(D.\,\mathrm{NE}(f)) < 0$ since $H_\sigma = \mathrm{NE}(f)^\perp$. □

Theorem 1.4.6. *Let D be a divisor on X. Then there exists an MMP for D over U and it terminates either in a Mori fibre space or a good minimal model (i.e., a model on which the strict transform of the divisor D is semiample over U).*

PROOF. We use an induction on $\rho(X/U)$. If $\rho(X/U) = 1$, then either D is π-nef or $-D$ is π-ample. When D is π-nef, it is π-semiample by the definition of MDM. If $-D$ is π-ample, then a D-negative contraction f contracts all the curves that is contracted by π. This implies that f coincides with π. Hence if π is a divisorial contraction, the divisor $f_*(D)$ is a $\mathbb{Q}$-Cartier divisor by Proposition 1.4.4, and it is nef and semiample over U since $f_*(D) \equiv_{\mathbb{Q},U} 0$. If π is a small contraction, we have a flip such that the strict transform of D is nef over U by Proposition 1.4.3. Finally, π is of fibre type if $\dim X > \dim U$.

Now assume that $\rho(X/U) \geq 2$ and D is not π-nef. Then consider a D-negative contraction $f\colon X \to Y$. If f is of fibre type, it is done. If f is a divisorial contraction, then the assumption of the induction and Proposition 1.4.4 implies the assertion. Assume that f is small. Then by Proposition 1.4.3, we only have to show that there does not exists an infinite sequence of D-negative flips. Since we know that there are only finitely many SQMs of X over U, it is sufficient to show that for any sequence of D-negative flips

$$X = X_0 \overset{\psi_0}{\dashrightarrow} X_1 \overset{\psi_1}{\dashrightarrow} \cdots \overset{\psi_{s-2}}{\dashrightarrow} X_{s-1} \overset{\psi_{s-1}}{\dashrightarrow} X_s, \tag{1.4.7}$$

$\psi_0 \circ \cdots \circ \psi_{s-1}$ is not an isomorphism. However, if we take a divisor E over X such that f is not isomorphic above the generic point of $\mathrm{center}_Y(E)$, it follows that $\mathrm{ord}_E(D_i) \leq \mathrm{ord}_E(D_{i+1})$ and $\mathrm{ord}_E(D) < \mathrm{ord}_E(D_1)$ as in the proof of [**KM98**, Lemma 3.38], where D_i is the strict transform of D in X_i and $\mathrm{ord}_E(D_i)$ is the coefficient of E in the divisor $\mu^*(D_i)$ for a resolution $\mu\colon \tilde{X}_i \to X_i$ such that E is a divisor on $\tilde{X}$. This shows that $\mathrm{ord}_E(D) < \mathrm{ord}_E(D_s)$, and hence $\psi_{s-1} \circ \cdots \circ \psi_0$ is not an isomorphism. □

Corollary 1.4.7. *For any divisor D on X, the $\mathcal{O}_U$-algebra $R_\pi(X, D)$ is finitely generated.*

PROOF. We may assume that $[D] \in \mathrm{Eff}(X/U)$. Then *Theorem* 1.4.6 implies that there exists a birational contraction $f\colon X \dashrightarrow Y$ over U and semiample divisor $D_Y := f_*(D)$ on Y since $[D] \in \mathrm{Eff}(X/U)$. By Lemma 1.1.10, we obtain $\mathrm{R}_\pi(X, D) \simeq \mathrm{R}_{\pi'}(Y, D_Y)$ since $D \simeq f^*(D_Y) + E$, where $\pi'\colon Y \to U$ is a natural map and E is an f-fixed divisor. □

Corollary 1.4.8. *There are finitely many birational contractions $h_i\colon X \dashrightarrow Y_i$ over U such that each $\pi_i\colon Y_i \to U$ $(i = 1, \ldots, k)$ is also an MDM, and there is a chamber decomposition of* $\mathrm{Eff}(X/U)$:

$$\mathrm{Eff}(X/U) = \bigcup_{i=1}^{r} (h_i^*(\mathrm{Nef}(Y_i/U)) \times \mathrm{Exc}(h_i)) \tag{1.4.8}$$

such that the interiors of the chambers do not intersect each other. Moreover, the cones $h_i^*(\mathrm{Nef}(Y_i/U)) \times \mathrm{Exc}(h_i)$ *in* $\mathrm{Eff}(X/U)$ *are precisely Mori chambers.*

Proof. For any divisor $D \in \mathrm{Eff}(X/U)$, we obtain a birational contraction $h\colon X \dashrightarrow Y'$ over U such that Y' is also an MDM and $D = h^*(D') + E$ for the relatively nef divisor $D' := h_*(D)$ on Y' over U and effective h-exceptional divisor h by running a D-MMP as in Theorem 1.4.6. By Theorem 1.3.4, there exists only finitely many rational contractions over U. Hence we obtain a finite collection of birational contractions $h_i\colon X \dashrightarrow Y_i$ satisfying (1.4.8).

The disjointness and the second assertion easily follow from Proposition 1.1.13 (1) and that each effective exceptional divisor of h_i is h_i-fixed. □

Corollary 1.4.9. $\mathrm{Eff}(X/U)$ *is a rational polyhedral cone.*

1.5. Characterization via Cox Sheaf

1.5.1. **VGIT for affine morphisms.** In this subsection we introduce the relative version of VGIT and prove that certain relative GIT quotients of affine morphisms are MDMs. First, we review VGIT for affine varieties. We refer to [**MFK94**] and Section 3 of [**AH09**] for details.

Let G be a connected reductive group over $\mathbb{C}$. $\chi(G)_{\mathbb{Q}} := \chi(G) \otimes \mathbb{Q}$ be the space of rational characters. Note that it is a finite dimensional vector space. Assume that G acts on a normal affine variety V. Let $\mathcal{O}_w$ be the trivial line bundle of V twisted by the character $w \in \chi(G)$. The *weight cone* of V will be denoted by

$$\Omega_G(V) := \mathrm{cone}\{w \in \chi(G) | H^0(V, \mathcal{O}_w)^G \neq 0\} \subset \chi(G)_{\mathbb{Q}}. \tag{1.5.1}$$

Note that, for each $w \in \chi(G)$, $w \in \Omega_G(V) \cap M$ holds if and only if the semi-stable locus V^{ss}_w with respect to $\mathcal{O}_w$ is non-empty.

Combing results from [**AH09**, Theorem 3.2], we obtain the following proposition.

Proposition 1.5.1. *There is a chamber decomposition*

$$\Omega_G(V) = \bigcup_{j=1}^{r} \sigma_j \tag{1.5.2}$$

satisfying the following properties.

1. *Each σ_j is a full dimensional rational polyhedral cone in $\chi(G)_{\mathbb{Q}}$.*
2. *$w, w' \in \mathrm{int}(\sigma_j) \cap \chi(G)$ holds for some j if and only if $w, w' \in \bigcup_{j=1}^{r} \mathrm{int}(\sigma_j)$ and $V^{ss}_w = V^{ss}_{w'}$.*
3. *If $w \in \sigma_j$, then $V^{ss}_{w'} \subset V^{ss}_w$ for any $w' \in \mathrm{int}(\sigma_j)$.*

We use the following lemma in the proof of Theorem 1.5.7 below.

Lemma 1.5.2. *The restriction map*

$$H^0(V, \mathcal{O}_w)^G \to H^0(V^{ss}_w, \mathcal{O}_w)^G \tag{1.5.3}$$

is an isomorphism.

Proof. Let $D_1 \dots, D_r$ be the codimension one components of $V \backslash V^{ss}_w$ and $V_k := V \backslash (\bigcup_{i=1}^{k} D_i)$. To see (1.5.3), we may assume that $V \backslash V^{ss}_w = \bigcup_{i=1}^{r} D_i$ since V is normal. Now it is sufficient to show that

$$H^0(V_{k-1}, \mathcal{O}_w)^G \simeq H^0(V_k, \mathcal{O}_w)^G \tag{1.5.4}$$

for all $k = 0, \ldots, r$, where we assume that $V_{-1} = V_0 = V$ for the sake of convenience. We prove this by induction on k. Obviously (1.5.4) holds for $k = 0$, and so assume that it holds for any $0 \leq l < k$. Note that the restriction map

$$H^0(V_{k-1}, \mathcal{O}_w)^G \hookrightarrow H^0(V_k, \mathcal{O}_w)^G \tag{1.5.5}$$

is injective for any k. Let

$$m_k := \min\{\mathrm{ord}_{D_k}(s) | s \in H^0(V_{k-1}, \mathcal{O}_w^{\otimes n})^G \text{ for some } n \in \mathbb{Z}_{>0}\}. \tag{1.5.6}$$

Then we have $m_k > 0$ since $s(D_k) = 0$ for any $s \in H^0(V_{k-1}, \mathcal{O}_w^{\otimes n})^G \simeq H^0(V, \mathcal{O}_w^{\otimes n})^G$, where the last isomorphism follows from the assumption of the induction. Fix $s \in H^0(V_{k-1}, \mathcal{O}_w)^G$ such that $\mathrm{ord}_{D_k}(s) = m_k$. Assume that (1.5.4) does not hold. Let us take any $t \in H^0(V_k, \mathcal{O}_w)^G \backslash H^0(V_{k-1}, \mathcal{O}_w)^G$. Then there exists $n > 0$ such that $t \cdot s^n \in H^0(V_{k-1}, \mathcal{O}_w^{\otimes n+1})^G$. Let n_0 be the minimal positive number satisfying this condition. By the choice of m_k, we obtain $\mathrm{ord}_{D_k}(t \cdot s^{n_0}) \geq m_k$. This implies that $\mathrm{ord}_{D_k}(t \cdot s^{n_0 - 1}) \geq 0$, and hence $t \cdot s^{n_0-1} \in H^0(V_{k-1}, \mathcal{O}_w^{\otimes n_0})^G$ since V is normal. This contradicts the choice of n_0, and so we obtain the desired equation (1.5.4).

□

Now let us consider the relative case. Let $\pi\colon V \to U$ be a G-invariant affine morphism of finite type, where we assume that U is a quasi-projective normal variety and V is also normal. For $\chi \in \chi(G)$, we define the semi-stable (respectively, stable) locus of V with respect to $\mathcal{O}_\chi$ over U as follows.

Definition 1.5.3. Let $U = \bigcup_i U_i$ be an affine covering of U. For a character χ, we define the semi-stable locus of $\mathcal{O}_\chi$ over U as

$$V^{ss}_{\chi/U} := \bigcup_i (V_i)^{ss}_\chi, \tag{1.5.7}$$

where $V_i := \pi^{-1}(U_i)$.

Similarly, we define the sable locus as

$$V^{s}_{\chi/U} := \bigcup_i (V_i)^{s}_\chi. \tag{1.5.8}$$

This definition does not depend on the choice of an affine covering of U by the lemma below. We say $\chi_1, \chi_2 \in M$ are *GIT equivalent* if they have the same semi-stable locus.

Lemma 1.5.4. *Suppose that U is an affine variety and $\chi \in \chi(G)$. For any affine open subset $U' \subset U$, let $V' := \pi^{-1}(U')$. Then we obtain $V'^{ss}_\chi = V^{ss}_\chi \cap V'$.*

Proof. It is obvious that $V'^{ss}_\chi \supset V^{ss}_\chi \cap V'$ by the definition of semi-stable locus of affine varieties. To prove the other inclusion, take any point $p \in V'^{ss}_\chi$ and a section $s \in \Gamma(V', \mathcal{O}_\chi^{\otimes m})^G$ such that $s(p) \neq 0$. Let $f \in H^0(U, \mathcal{O}_U)$ be a regular function on U such that $p \in D(f) \subset U'$. Since $\Gamma(V', \mathcal{O}_\chi^{\otimes m}) = \Gamma(U', \pi_* \mathcal{O}_\chi^{\otimes m})$ and the pull back of f is a G-invariant function on V, there is some positive number $N > 0$ such that $f^N s$ extends to a section $\tilde{s} \in \Gamma(V, \mathcal{O}_\chi^{\otimes m})^G$ by [**Har77**, Chapter 2, Lemma 5.3]. Obviously we obtain $\tilde{s}(p) \neq 0$ by the construction, so that $p \in V^{ss}_\chi \cap V'$. □

For each i in Definition 1.5.3, it is known that there exists the good quotient

$$q_{i_\chi}\colon (V_i)^{ss}_\chi \to Q_{i_\chi} := (V_i)^{ss}_\chi // G, \tag{1.5.9}$$

which admits natural morphism to $U_i \subset U$. The universality of categorical quotients and Lemma 1.5.4 imply that they can be glued, and we obtain the good quotient

$$q_\chi \colon V^{ss}_{\chi/U} \to Q_\chi := V^{ss}_{\chi/U}//G \tag{1.5.10}$$

and similarly the geometric quotient

$$q_\chi \colon V^{s}_{\chi/U} \to V^{s}_{\chi/U}/G, \tag{1.5.11}$$

both of which are morphisms over U. Moreover we obtain that

$$Q_\chi \simeq \operatorname{Proj}_U\Big(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} (\pi_*\mathcal{O}_\chi^{\otimes n})\Big)^G, \tag{1.5.12}$$

which can also be checked locally over U. Let $\pi_\chi \colon Q_\chi \to U$ denote the canonical morphism induced by the universality of the quotient.

Remark 1.5.5. Note that $\bigoplus_{n\geq 0}(\pi_*\mathcal{O}_\chi^{\otimes n})$ is finitely generated over U, and hence finitely generated over $\mathbb{C}$ since U is of finite type over $\mathbb{C}$. Therefore by applying the Nagata theorem (see, for example, [**Dol03**, Theorem 3.3]), $\bigoplus_{n\geq 0}(\pi_*\mathcal{O}_\chi^{\otimes n})^G$ is also finitely generated over U.

Let $\Omega_G(V/U)$ be the convex cone in $\chi(G)_\mathbb{Q}$ generated by $w \in \chi(G)$ such that $V^{ss}_{w/U} \neq \emptyset$. Then we can easily generalize Proposition 1.5.1 to the relative case.

Corollary 1.5.6. *There is a chamber decomposition*

$$\Omega_G(V/U) = \bigcup_{j=1}^{r} \sigma_j \tag{1.5.13}$$

satisfying the following properties.

(1) σ_j is a full dimensional rational polyhedral cone in $\chi(G)_\mathbb{Q}$.
(2) $w, w' \in \operatorname{int}(\sigma_j) \cap \chi(G)$ holds for some j if and only if $w, w' \in \bigcup_{j=1}^r \operatorname{int}(\sigma_j)$ and $V^{ss}_{w/U} = V^{ss}_{w'/U}$.
(3) If $w \in \sigma_j$, then $V^{ss}_{w'/U} \subset V^{ss}_{w/U}$ holds for any $w' \in \operatorname{int}(\sigma_j)$.

Proof. Let $U = \bigcup_i U_i$ be a finite affine covering. Note that $\Omega_G(V/U) = \Omega_G(V_i)$ for any i by Lemma 1.5.4. Then we obtain the proposition by taking intersections of the chambers appeared in Proposition 1.5.1 for all i whose interiors are non-empty. $\square$

We call each σ_j a *GIT chamber* of V over U.

Theorem 1.5.7. *Take $\chi \in \chi(G)$ such that Q_χ is a normal and $\pi_\chi \colon Q_\chi \to U$ is an algebraic fibre space satisfying $\operatorname{Pic}(Q_\chi/U)_\mathbb{Q} \simeq \mathrm{N}^1(Q_\chi/U)_\mathbb{Q}$. Assume the following properties.*

(1) $V^{ss}_{\chi/U} = V^{s}_{\chi/U}$, and $\operatorname{codim}_V(V \backslash V^{ss}_{\chi/U}) \geq 2$.
(2) Q_χ is $\mathbb{Q}$-factorial.
(3) Both of the following maps are isomorphisms.

$$\chi(G)_\mathbb{Q} \underset{\alpha}{\to} (\operatorname{Pic}(V)^G/\operatorname{Pic}(U))_\mathbb{Q} \overset{(1)}{\simeq} (\operatorname{Pic}(V^{ss}_{\chi/U})^G/\operatorname{Pic}(U))_\mathbb{Q} \underset{q_\chi^*}{\leftarrow} \operatorname{Pic}(Q_\chi/U)_\mathbb{Q}, \tag{1.5.14}$$

where we define $\alpha(\chi) := [\mathcal{O}_\chi] \in \operatorname{Pic}(V)^G/\operatorname{Pic}(U)$.

Then π_χ is an MDM.

Remark 1.5.8. Under the assumptions of the above theorem, we can check that there exists $L \in \operatorname{Pic}(Q_\chi)$ such that $q_\chi^* L \sim_\mathbb{Q} \mathcal{O}_y$ for any $y \in \chi(G)$. We write it as L_y. In the following proof V_y^{ss} denotes $V^{ss}_{y/U}$ for simplicity.

Proof of Theorem 1.5.7. Let $y \in \chi(G)$ be a character. Then for any sufficiently divisible $n > 0$ we obtain the canonical identification

$$\pi_*(\mathcal{O}_y{}^{\otimes n})^G = \pi_*(\mathcal{O}_y^{\otimes n}|_{V_\chi^{ss}})^G = \pi_{y*}(L_y^{\otimes n}), \tag{1.5.15}$$

where the first equality follows from that $\operatorname{codim}(V \backslash V_\chi^{ss}) \geq 2$ and the second equality follows from the descent. This implies that the isomorphism induced by (1.5.14)

$$\psi \colon \chi(G)_{\mathbb{Q}} \to \operatorname{Pic}(Q_\chi/U)_{\mathbb{Q}} \tag{1.5.16}$$

identifies $\Omega(V/U)$ with $\operatorname{Eff}(Q_\chi/U)_{\mathbb{Q}}$.

As in Remark 1.5.5, the $\mathcal{O}_U$-algebra $\bigoplus_{n\geq 0}(\pi_*\mathcal{O}_\chi^{\otimes n})^G$ is finitely generated. Then for any $y \in \chi(G)$, we obtain that $\mathrm{R}_{\pi_\chi}(Q_\chi, L_y)$ is finitely generated by (1.5.15). Let $f_y \colon Q_\chi \dashrightarrow Q_y$ be the canonical rational map between the quotients. Then we can check that f_y coincides with φ_{L_y} by (1.5.15). Moreover, by Proposition 1.1.13, there is a π_y-ample divisor A_y and a f_y-fixed divisor E_y such that

$$L_y \sim_{\mathbb{Q}} f_y^*(A_y) + E_y. \tag{1.5.17}$$

Now we compare the GIT chamber decomposition of $\Omega(V/U)$ and the Mori chamber decomposition of $\operatorname{Eff}(Q_\chi/U)_{\mathbb{Q}}$ via ψ. It is obvious that if two character y and y' are GIT equivalent, obviously L_y and $L_{y'}$ are Mori equivalent. This implies that GIT chamber decomposition is finer than Mori chamber decomposition. Hence there are only finitely many Mori chambers, and each of them is a rational polyhedral cone by Corollary 1.5.6.

Let us show that the Mori chambers coincide with the GIT chambers. Take $y, z \in \chi(G)$ such that L_y and L_z are in the interior of the same Mori chamber. Note that $f_z = f_y$ is a birational contraction since they are in the interior of $\operatorname{Eff}^1(Q_\chi/U)$. Considering the decomposition (1.5.17), E_y and E_z have the same support, which are the full divisorial exceptional locus of f_y, and the number of the component of E_y is $\rho(Q_\chi/U) - \rho(Q_y/U)$. This follows from Lemma 1.1.7, Proposition 1.1.13, and the assumption that L_z and L_y are in the interior of the Mori chamber. By considering the dimension, we obtain

$$\operatorname{Pic}(Q_\chi/U)_{\mathbb{Q}} = f_y^*(N^1(Q_y/U)) \times \langle \text{exceptional divisors of } f_y \rangle_{\mathbb{Q}}. \tag{1.5.18}$$

Let D be a prime divisor on Q_y and $\bar{D} \subset Q_\chi$ be its strict transform. We can write $\bar{D}$ as $\bar{D} = f^*A + E + \pi_\chi^*B$, where $A \in \operatorname{Pic}(Q_y)$, $B \in \operatorname{Pic}(U)$, and $f_{y*}E = 0$. Hence we have $\operatorname{Supp}(D) = \operatorname{Supp}(A + \pi_y^*B)$, and this implies that the Weil divisor D is linearly equivalent to the Cartier divisor $A + \pi_y^*B$ after multiplying both divisors by some positive integers. Therefore D is $\mathbb{Q}$-Cartier, and so Q_y is $\mathbb{Q}$-factorial.

Let us prove $V_y^{ss} = V_z^{ss}$. For this it is sufficient to show that $V_z^{ss} \subset V_y^{ss}$. Furthermore, we may assume that U is an affine variety since the problem is local over U. Let $p_z \in V_z^{ss}$ be a point. Note that there exists a very ample divisor A'_y on $Q_z = Q_y$ such that $\mathcal{O}_y^{\otimes n}|_{V_y^{ss}} = q_y^*A'_y$ by the GIT construction (this follows from (1.5.12) or [**MFK94**, Theorem 1.10]). There exists $\sigma' \in H^0(Q_y, A'_y)$ such that $\sigma'(q_z(p_z)) \neq 0$, and we obtain $\sigma \in H^0(V, \mathcal{O}_y^{\otimes n})^G$ satisfying $\sigma|_{V_y^{ss}} = q_y^*(\sigma')$ by Lemma 1.5.2. We claim that

$$\mathcal{O}_y^{\otimes n}|_{V_z^{ss}} = q_z^*A'_y \ \text{ and } \ \sigma|_{V_z^{ss}} = q_z^*\sigma'. \tag{1.5.19}$$

Consider the decomposition (1.5.17). We may assume that A_y is base point free over U by multiplying it by some positive integer. Then we obtain

$$H^0(Q_y, A_y) \simeq H^0(Q_\chi, f_y^*A_y) \xrightarrow[+E_y]{\sim} H^0(Q_\chi, f_y^*A_y + E_y) \tag{1.5.20}$$

$$= H^0(Q_\chi, L_y) \xrightarrow[q_\chi^*]{\sim} H^0(V_\chi^{ss}, \mathcal{O}_y^{\otimes n})^G = H^0(V, \mathcal{O}_y^{\otimes n})^G, \tag{1.5.21}$$

where the first isomorphism follows from Lemma 1.1.10. Since A_y is base point free, this implies that $V_\chi^{ss} \backslash (V_\chi^{ss} \cap V_y^{ss}) = q_\chi^{-1}(\mathrm{Supp}(E_y))$ up to codimension one. Similarly we can check that $V_\chi^{ss} \backslash (V_\chi^{ss} \cap V_z^{ss}) = q_\chi^{-1}(\mathrm{Supp}(E_z))$. However, since $\mathrm{Supp}(E_z) = \mathrm{Supp}(E_y)$ and $\mathrm{codim}(V \backslash V_\chi^{ss}) \geq 2$, we obtain that $V_z^{ss} = V_y^{ss}$ in codimension one, and so we have (1.5.19) by considering the restriction to $V_z^{ss} \cap V_y^{ss}$. Then by (1.5.19), it follows that $p_z \in V_y^{ss}$ since $\sigma(p_z) \neq 0$.

Now we know that Mori and GIT chambers coincide and each chamber is of the form $\overline{f_z^*(\mathrm{Ample}(Q_z/U))} \times \mathrm{excep}(f_z)$ such that f_z is birational and Q_z is $\mathbb{Q}$-factorial. In particular, $\mathrm{Mov}(Q_\chi/U)$ is the union of the chambers such that f_z is small. To conclude the proof, it is sufficient to show that $\mathrm{Nef}(Q_z/U)$ is generated by π_z-semiample divisors for such z. There exists the canonical identification $\mathrm{NE}^1(Q_z/U) = \mathrm{NE}^1(Q_\chi/U)$ since f_z is small. Let $C_z = {f_z}^* \mathrm{Nef}(Q_z/U)$ be a Mori chamber containing z. The same argument of the proof of (1.5.19) implies that $V_\chi^{ss} = V_z^{ss}$ in codimension one, and hence

$$(\pi_* \mathcal{O}_y|_{U_\chi})^G = (\pi_* \mathcal{O}_y)^G = (\pi_* \mathcal{O}_y|_{U_z})^G. \tag{1.5.22}$$

Take any $y \in \partial C_z$. We prove that $L_y \in \mathrm{Pic}(Q_z)$ is π_z-semiample. We can check this locally over U, and hence we assume that U is affine. Take any $p \in Q_z$. It is sufficient to show that there exists some $s \in H^0(Q_z, L_y^{\otimes n})$ such that $s(p) \neq 0$. To see this, take $p' \in V_z^{ss}$ such that $q_z(p') = p$. It follows that $p' \in V_z^{ss} \subset V_y^{ss}$ by Corollary 1.5.6 since $y \in \partial C_z$. Then there exists $s' \in H^0(V, \mathcal{O}_y^{\otimes n})^G$ such that $s'(p') \neq 0$ by the definition of the semi-stable locus. Note that we have the following diagram:

$$\begin{tikzcd}
H^0(V, \mathcal{O}_y^{\otimes n})^G \arrow[d, "\wr"] & \\
H^0(V_z^{ss}, \mathcal{O}_y^{\otimes n})^G \arrow[r, "\sim"] & H^0(V_\chi^{ss}, \mathcal{O}_y^{\otimes n})^G \\
H^0(Q_z, L_y^{\otimes n}) \arrow[r, "\sim"] \arrow[u, "q_z^*"] & H^0(Q_\chi, L_y^{\otimes n}), \arrow[u, "q_\chi^*", "\wr"']
\end{tikzcd} \tag{1.5.23}$$

where the horizontal isomorphisms follow from that $V_\chi^{ss} = V_z^{ss}$ and $Q_y = Q_\chi$ up to codimension one.

By the commutativity of the diagram, q_z^* is also an isomorphism, and hence there exists $s \in H^0(Q_z, L_y^{\otimes n})$ such that

$$s(p) = q_z^* s(p') = s'(p') \neq 0. \tag{1.5.24}$$

$\square$

1.5.2. **Cox sheaf and MDM.** Let $\pi_X \colon X \to U$ be an algebraic fibre space such that X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X/U)_\mathbb{Q} \simeq \mathrm{N}^1(X/U)_\mathbb{Q}$. For a collection of line bundles $\mathcal{L} \coloneqq (L_1, \ldots, L_r)$ on X, we define the following $\mathcal{O}_U$ algebra:

$$R(X/U, \mathcal{L}) \coloneqq \bigoplus_{m \in \mathbb{Z}^r} \pi_{X*}(\mathcal{L}^m), \tag{1.5.25}$$

where $\mathcal{L}^m \coloneqq \bigotimes_i L^{\otimes m_i}$ for $m = (m_1, \ldots, m_r)$. Similarly, we define

$$R(X/U, \mathcal{L})_+ \coloneqq \bigoplus_{m \in (\mathbb{Z}_{\geq 0})^r} \pi_{X*}(\mathcal{L}^m). \tag{1.5.26}$$

We call $R(X/U,\mathcal{L})$ a *Cox sheaf* of X/U if $([L_1]\dots,[L_r])$ is a basis of $\mathrm{Pic}(X/U)_{\mathbb{Q}}$. Note that the finite generation as an $\mathcal{O}_U$-algebra of a Cox sheaf does not depend on the choice of $\mathcal{L}$. We often take $\mathcal{L}$ such that $([L_1]\dots,[L_r])$ is a basis of $\mathrm{Pic}(X/U)_{\mathrm{free}}$.

Proposition 1.5.9. $\mathrm{Spec}_U(R(X/U,\mathcal{L}))$ *is normal.*

PROOF. Without loss generality, we may assume that U is an affine variety $\mathrm{Spec}(A)$, and hence it is sufficient to show that the A-algebra $R \coloneqq \bigoplus_{m\in\mathbb{Z}^r} H^0(X,\mathcal{L}^m)$ is normal. Consider the $\mathcal{O}_X$-algebra $\mathcal{R} \coloneqq \bigoplus_{m\in\mathbb{Z}^r} \mathcal{O}_X(\mathcal{L}^m)$ and the corresponding affine morphism $\varphi\colon P \coloneqq \mathrm{Spec}_X(R) \to X$. Then we obtain the natural identification

$$R = H^0(X,\mathcal{R}) = H^0(P,\mathcal{O}_P). \tag{1.5.27}$$

Take an open local trivialization $W \subset X$ of $L_1,\dots L_r$. Then $\varphi^{-1}(W) \simeq W \times (\mathbb{C}^*)^r$. Since W and $(\mathbb{C}^*)^r$ are normal, $\varphi^{-1}(W)$ is also normal, and hence P is. In particular, for each affine open subset $V_i \subset P$, we obtain that $\Gamma(V_i,\mathcal{O}_P)$ is normal. This implies that $H^0(P,\mathcal{O}_P)$ is also normal. $\square$

Proposition 1.5.10. *If π is an MDM, then a Cox sheaf $R(X/U,\mathcal{L})$ is finitely generated over $\mathcal{O}_U$.*

PROOF. Let $\{C_i\}$ be the set of all Mori chambers. Then by Corollary 1.4.8, $R(X/U,\mathcal{L}) = \bigcup R_{C_i}$. where

$$R_{C_i} \coloneqq \bigoplus_{\mathcal{L}^m\in C_i} \pi_*(\mathcal{L}^m). \tag{1.5.28}$$

is subring of $R(X/U,\mathcal{L})$. Then it is sufficient to show that each R_{C_i} is finitely generated. We will prove it for $i=1$. Take line bundles $J_1,\dots,J_d$ such that their classes in $\mathrm{Pic}(X/U)$ generate the cone C_1. Let $\mathcal{J} \coloneqq (J_1,\dots,J_d)$. Obviously, if $R_+(X/U,\mathcal{J})$ is finitely generated, then R_{C_1} is finitely generated. Hence it is sufficient to show that $R_+(X/U,\mathcal{J})$ is finitely generated. By Corollary 1.4.8, there exists a birational contraction $h_1\colon X \to Y_1$, a relatively semiample divisor A_j on Y over U, and a h_1-fixed divisor E_j such that $J_j = h_1^*A_j + E_j$ for each j. Let $\mathcal{A} \coloneqq (A_1,\dots,A_d)$ Hence $R_+(X/U,\mathcal{J})$ is finitely generated if and only if $R_+(X/U,\mathcal{A})$ by Lemma 1.1.10. However, $R_+(X/U,\mathcal{A})$ is finitely generated by the relative version of Zariski lemma (see [**HK00**, 2.8 Lemma]). $\square$

In the rest of this section, as an application of Theorem 1.5.7, we prove that the finite generation of a Cox sheaf implies that π is an MDM Consider the algebraic torus $T \coloneqq \mathrm{Hom}(\mathrm{Pic}(X/U)_{\mathrm{free}},\mathbb{C}^*)$ of dimension r. Then we have

$$\chi(T)_{\mathbb{Q}} \simeq \mathrm{Pic}(X/U)_{\mathbb{Q}} \simeq \bigoplus \mathbb{Q}L_i. \tag{1.5.29}$$

$L_{(y)}$ denotes the line bundle on X corresponding to a character $y \in \chi(T)$ via the above isomorphism.

Let $V \coloneqq \mathrm{Spec}_U(R(X/U,\mathcal{L}))$ and $\pi\colon V \to U$ be the natural morphism. We have the action of T on V which corresponds to the grading of $R(X/U,\mathcal{L})$ by $\mathbb{Z}^r$ as follows.

$$T \times \pi_{X*}(L_{(y)}) \to \pi_{X*}(L_{(y)})\colon (t,s) \mapsto y(t)s. \tag{1.5.30}$$

Note that π is G-invriant with respect to this action.

Proposition 1.5.11. *Under the above notation, assume that V is of finite type over U. If we take a character $\chi \in \chi(T)$ which corresponds to a π-ample line bundle, then $V^{ss}_{\chi/U} = V^{s}_{\chi/U}$, the quotient Q_χ is isomorphic to X, and* $\mathrm{codim}_V(V\backslash V^{ss}_{\chi/U}) \geq 2$.

Proof. Note that $(\mathcal{O}_{V/U,\chi})^T = \pi_{X*}(L_{(y)})$ for any character $y \in \chi(T)$ by definition. Then by (1.5.12) and Remark 1.5.5, we obtain that $\mathrm{R}_{\pi_X}(X, L_{(y)})$ is finitely generated over $\mathcal{O}_U$ and

$$Q_y = \mathrm{Proj}_U(\mathrm{R}_{\pi_X}(X, L_{(y)})). \tag{1.5.31}$$

In particular, if we take $\chi \in \chi(T)$ corresponding to a π-ample divisor $L_{(\chi)}$, then $Q_\chi = X$.

Let $\chi_1,\ \chi_2 \in \chi(T)$ be any two characters corresponding to π_X-ample line bundles. We prove that $V^{ss}_{\chi_1/U} = V^{ss}_{\chi_2/U}$. If we take sufficiently large number $N > 0$, then $A \coloneqq NL_{(\chi_1)} - L_{(\chi_2)}$ is π_X-ample. By the relative version of [**Laz04**, Example 1.2.22], the natural map

$$\pi_{X*}(nA) \otimes \pi_{X*}(nL_{(\chi_2)}) \to \pi_{X*}(nNL_{(\chi_1)}) \tag{1.5.32}$$

is surjective for any sufficiently large $n > 0$. This implies that

$$V^{ss}_{\chi_1/U} = V^{ss}_{\chi_2/U} \cap V^{ss}_{\chi_A}, \tag{1.5.33}$$

where χ_A is the character corresponding to A. In particular, $V^{ss}_{\chi_1/U} \subset V^{ss}_{\chi_2/U}$ and the other inclusion follows by the same argument. W denotes the semi-stable locus for a character corresponding to a π_X-ample divisor.

For $h \in V$, let $T_h \subset T$ be the isotropy group of h. We can easily check that if $t \in T_h$, then $y(t) = 1$ for any $y \in \chi(V)$ such that there exists an affine open subset $U' \subset U$ containing $\pi(h)$ and $s \in \Gamma(\pi_X^{-1}(U'), L_{(y)}) = \Gamma(U', \pi_*\mathcal{O}_y)^T$ satisfying $s(h) \neq 0$. Take characters $\chi_1, \ldots, \chi_r \in \chi(T)$ which generate $\chi(T)$ as a group, and assume that each χ_i corresponds to a π_X-ample divisor $L_{(\chi_i)}$. There exists a number $N_i > 0$ such that $\chi_i(t^{N_i}) = {\chi_i}^{N_i}(t) = 1$ when $t \in T_h$ for some $h \in V^{ss}_{\chi_i/U}$ by the above argument and the definition of the semi-stable locus. Since $\{\chi_i\}_i$ generate $\chi(T)$, we obtain that $y(g^N) = y^N(g) = 1$ for any $y \in \chi(T)$ by taking $N \coloneqq \prod_{i=1}^r N_i$. This implies that the isotropy group of any $h \in W$ is finite.

To prove W is the stable locus of $\chi \in \chi(T)$, it is sufficient to show that the orbit of each $h \in W$ is closed. However, this follows from the fact that the isotropy group of any $h' \in W$ is finite since if there exists a point $p \in \overline{o(h)}\backslash o(h)$, it follows that $\dim T_p > 0$. Finally we show that $\mathrm{codim}(V\backslash W) \geq 2$. Since the problem is local over U, we may assume that U is affine. Let L be a π_X-ample divisor on X. Take $\sigma, \tau \in H^0(X, L) = H^0(V, \mathcal{O}_{\chi_L})^T$ such that the zero divisors on X have no common components, where χ_L denotes the character corresponding to L via (1.5.29). Let $I \subset H^0(V, \mathcal{O}_V)$ be the ideal corresponding to the closed subset $V\backslash W$ with reduced structure. Then by the definition of the semi-stable locus, we obtain $\sigma, \tau \in I$. This implies that $\mathrm{codim}_V(V\backslash W) \geq 2$ by the same argument as in [**HK00**, Lemma 2.7]. □

Remark 1.5.12. In the proof of the above proposition, we proved that the isotropy group T_h has order N for any $h \in V^s_{\chi/U}$.

Corollary 1.5.13. *Under the same assumptions as in Proposition 1.5.11, $\pi\colon V \to U$ satisfies the conditions (1),(2),(3) in Theorem 1.5.7. Hence π is an MDM.*

Proof. The condition (1) and (2) follow from Proposition 1.5.11. Let us check (3) Note that there exists the natural isomorphism $\mathrm{Pic}(V^s_{\chi/U})^T_{\mathbb{Q}} \xleftarrow[q_\chi^*]{\sim} \mathrm{Pic}(X)_{\mathbb{Q}}$ by the Kempf descent Lemma [**DN89**, Theorem 2.3], Remark 1.5.12, and Proposition 1.5.11. Since q_χ is a morphism over U and $\mathrm{codim}(V\backslash V^s_{\chi/U}) \geq 2$, this induces the isomorphism q_χ^* in (1.5.14). Moreover, α is obviously injective and $\dim_{\mathbb{Q}} \chi(T)_{\mathbb{Q}} = \dim_{\mathbb{Q}}(\mathrm{Pic}(X/U)_{\mathbb{Q}})$ by the construction. Hence α is also an isomorphism. □

1.6. On conservation of Mori dreamness and some examples

In this section, we give various examples of MDMs and study the categorical properties of MDMs, in particular concerning compositions and base changes.

Example 1.6.1. Let $\pi\colon X \to U$ be an algebraic fibre space between normal quasi-projective varieties. Suppose that X is $\mathbb{Q}$-factorial and $K_X + \Delta$ is Kawamata log terminal for a $\mathbb{Q}$-divisor Δ on X. If $-(K_X + \Delta)$ is π-ample, then π is an MDM. Birkar, Cascini, Hacon, and Mckernan proved under these assumptions the finite generation of Cox sheaves (see [**BCHM10**, Corollary 1.1.9] and also [**BCHM10**, Corollary 1.3.2]). The isomorphism $\mathrm{N}^1(X/U)_{\mathbb{Q}} \simeq \mathrm{Pic}(X/U)_{\mathbb{Q}}$ follows from the assumption and the base point free theorem applied to π. Hence the morphism π is an MDM in the sense of this paper by Theorem 1.2.

Example 1.6.2. As in [**dFH12**, Section 4], let us consider a flat projective morphism $f\colon X \to T$ to a smooth curve T such that the fibre X_0 of $0 \in T$ is a Fano variety with $\mathbb{Q}$-factorial terminal singularities. After restricting T to a neighborhood of 0, the morphism f is a relative Fano variety over T with $\mathbb{Q}$-factorial terminal singularities by applying Corollary 3.2 and Proposition 3.5 in [**dFH11**]. It is an MDM by Example 1.6.1. In this case, as stated in [**dFH12**, Proposition 4.1], each fibre X_t of $t \in T$ is a Fano variety with $\mathbb{Q}$-factorial terminal singularities, which is an MDS by [**BCHM10**, Corollary 1.3.2].

Note that after taking a suitable étale base change of T, the following diagram

$$\begin{tikzcd} X_t \arrow[r, hook, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ \{t\} \arrow[r, hook, "g"] & T \end{tikzcd} \tag{1.6.1}$$

satisfies the assumptions of Theorem 1.6.9 below for any $t \in T$ (see Theorem 4.2, Theorem 4.3, and Corollary 4.7 in [**dFH12**] for details).

Below are examples of a pair of morphisms which are MDMs but their composition is not.

Example 1.6.3. Let U be a smooth variety and $\pi\colon X \to U$ be a blow-up at a point $u \in U$. Then we can easily see that $\mathrm{Pic}(X/U) = \mathbb{Q}[E] \simeq N^1(X/U)$ and $\mathrm{Nef}(X/U) = \mathrm{Mov}(X/U) = \mathbb{Q}_{\leq 0}[E]$, where E is the exceptional divisor. This implies that π is an MDM (we can also deduce this from Example 1.6.1). On the other hand, it is known that the blow-up of $\mathbb{P}^2$ in very general nine points is not a Mori dream space, although the blow-up in eight points is always an MDS.

Example 1.6.4. For any $\mathbb{Q}$-factorial variety X and locally free sheaf F on X, we can easily check that the projective bundle $\pi\colon \mathbb{P}_X(F) \to X$ is an MDM. However, [**GHPS12**, Theorem 1.1] implies that there exist a smooth toric variety X and a vector bundle F on X such that $\mathbb{P}_X(F)$ is not an MDS.

On the other hand, Proposition 1.6.5 and Proposition 1.6.8 below imply that if the composition of two algebraic fibre spaces is an MDM, both of them are also MDMs.

Proposition 1.6.5. *Let X be an MDM over U and f be an algebraic fibre space from X to a variety Y as in the following commutative diagram.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd} \tag{1.6.2}$$

Then f is an MDM.

Proof. Let $[D] \in \mathrm{Pic}(X/Y)$ be an f-nef divisor on X. Take a π_2-ample divisor A on Y. Then for sufficiently large $m \gg 0$, the divisor $D + mf^*A$ is π_1-nef since $\mathrm{NE}_1(X/U)$ is a finitely generated cone and we can take m such that the generators of $\mathrm{NE}_1(X/U)$ have non-negative intersection numbers with $D + mf^*A$. Hence $D + mf^*A$ is π_1-semiample, in particular it is f-semiample by Proposition 1.1.3. Note that the above argument also implies that any f-nef class in $\mathrm{N}^1(X/Y)_{\mathbb{Q}}$ comes from nef class in $\mathrm{N}^1(X/U)_{\mathbb{Q}}$, and hence $\mathrm{Nef}(X/Y)$ is a polyhedral cone since $\mathrm{Nef}(X/U)$ is. These argument shows that f satisfies $(2')$ and $(3')$ of Proposition 1.2.2.

We prove that f satisfies the condition (4) in Definition 1.2.1. Let $\sigma := f^*(\mathrm{Nef}(Y/U))$ be the cone corresponding to the map f and $\{g_i\}_{i=1}^{r'}$ be the subset of the all SQMs of X such that σ is a face of each $g_i^*(\mathrm{Nef}(X_i/U))$. For each $i = 1, \dots r'$, let f_i be the morphism corresponding to $\sigma \prec \mathrm{Nef}(X_i/U)$. There exists the following commutative diagram over U.

$$\begin{tikzcd} X \arrow[rr, dashed, "g_i"] \arrow[dr, "f"'] & & X_i \arrow[dl, "f_i"] \\ & Y & \end{tikzcd} \tag{1.6.3}$$

Then it is sufficient to prove

$$\mathrm{Mov}(X/Y) = \bigcup_{i=1}^{r'} g_i^*(\mathrm{Nef}(X_i/Y)). \tag{1.6.4}$$

For this, it is sufficient to show that the natural map

$$\bigcup_{i=1}^{r'} (\mathrm{Nef}(X_i/U)) \to \mathrm{Mov}(X/Y) \tag{1.6.5}$$

is surjective since we know that $\mathrm{Nef}(X_i/U) \to \mathrm{Nef}(X_i/Y)$ is surjective. Take any $[D] \in \mathrm{Mov}(X/Y)$. By applying Proposition 1.1.3, we may assume that $D \in \mathrm{Mov}(X/U)$. Take a π_2-ample divisor A on Y. Then $f^*(A)$ is in the relative interior of σ. Let $R := \mathbb{Q}_{\geq 0} f^*(A) \subset \mathrm{Mov}(X/U)$. Then we can take a cone V in $\mathrm{Mov}(X)$ which is open in $\mathrm{Mov}(X)$, contains R, and satisfies

$$V \subset \bigcup_{i=1}^{r'} g_i^*(\mathrm{Nef}(X_i/U)) \tag{1.6.6}$$

since $\{\mathrm{Nef}(X_i/U)\}_{i=1}^{r'}$ is the set of all the full dimensional cones in $\mathcal{M}_X$ containing σ. On the other hand, if we take sufficiently large $m \gg 0$, the divisor $D + mf^*(A)$ get closer to

R, and so $D + mf^*(A) \in V$. Hence (1.6.6) implies

$$D + mf^*(A) \in \bigcup_{i=1}^{r'} g_i^*(\mathrm{Nef}(X_i/U)), \tag{1.6.7}$$

and so the map (1.6.5) is surjective. □

Remark 1.6.6. From the above proof we deduce that the fan $\mathcal{M}_{X/Y}$ coincides with the fan $\mathrm{Star}(\sigma)$ associated to $\sigma = f^*(\mathrm{Nef}(Y/U)) \in \mathcal{M}_{X/U}$, where $\mathrm{Star}(\sigma)$ is the star of $\sigma \in \mathcal{M}_{X/Y}$ defined as in [**CLS11**, Section 3.2].

Corollary 1.6.7. *Let $\pi\colon X \to U$ be an MDM.*

(1) The map α in Theorem 1.3.4 (see (1.3.3)*) induces the bijection between the set of all faces of* $\mathrm{Nef}(X/U)$ *and the set of MDMs $f\colon X \to Y$ over U.*
(2) Let σ_i $(i = 1, 2)$ be two faces of $\mathrm{Nef}(X/U)$ *and MDM $f_i\colon X \to Y_i$ be the corresponding MDMs. Then $\sigma_2 \prec \sigma_1$ holds if and only if there exists an algebraic fibre space $h\colon Y_1 \to Y_2$ making the following diagram commutative.*

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "f_2"'] & Y_1 \arrow[d, "\pi_1"] \arrow[dl, "h"'] \\ Y_2 \arrow[r, "\pi_2"'] & U \end{tikzcd} \tag{1.6.8}$$

Proof. Theorem 1.3.4 and Proposition 1.6.5 immediately imply (1). Let us prove (2). If there is an algebraic fibre space h making the commutative diagram (1.6.8), we obtain $\sigma_2 = f_2^* \mathrm{Nef}(Y_2/U) \subset f_1^* \mathrm{Nef}(Y_1/U) = \sigma_1$. Since σ_1 and σ_2 are faces of the cone $\mathrm{Nef}(X/U)$, it follows that $\sigma_2 \prec \sigma_1$. To show the converse, assume that $\sigma_2 \prec \sigma_1$. It is sufficient to show that any curve C contracted by f_1 is also contracted by f_2. Take a π_2-ample divisor A on Y_2. Then there exists a π_1-nef divisor B on Y_1 such that $f_1^*(B) = f_2^*(A)$ in $\mathrm{N}^1(X/U)_{\mathbb{Q}}$ since $\sigma_2 \prec \sigma_1$. Then we see

$$f_{2*}(C).A = C.{f_2}^*A = C.{f_1}^*B = f_{1*}(C).B = 0. \tag{1.6.9}$$

Since A is π_2-ample, we obtain from (1.6.9) that $f_2(C) = \{pt\}$. □

The following Proposition 1.6.8 generalizes [**Oka16**, Theorem 1.1].

Proposition 1.6.8. *Let X be an MDM over U and Y be a $\mathbb{Q}$-factorial variety which is projective over U. Suppose that f is a surjective morphism from X to Y as in the following commutative diagram.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd} \tag{1.6.10}$$

Then π_2 is also an MDM.

Proof. We may assume that f is an algebraic fibre space or a finite morphism by the Stein factorization. In both cases, there is a natural injection $f^*\colon \mathrm{Pic}(Y)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}$ and $f^*\colon \mathrm{Pic}(Y/U)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}(X/U)_{\mathbb{Q}}$. First, we show that $\mathrm{Pic}(Y/U)_{\mathbb{Q}} \simeq \mathrm{N}^1(Y/U)_{\mathbb{Q}}$. For this, take a divisor D on Y such that $D \equiv_U 0$. Then there exists $A \in \mathrm{Pic}(U)_{\mathbb{Q}}$ such that $f^*D \sim_{\mathbb{Q}} \pi_1^*(A) = f^*(\pi_2^*(A))$ since we already know that $\mathrm{Pic}(X/U)_{\mathbb{Q}} \simeq \mathrm{N}^1(X/U)_{\mathbb{Q}}$ by the assumption that π_1 is an MDM. This implies that $\pi_2^*(A) \sim_{\mathbb{Q}} D$, and hence we have $\mathrm{Pic}(Y/U)_{\mathbb{Q}} \simeq \mathrm{N}^1(Y/U)_{\mathbb{Q}}$.

To conclude the proof, it is sufficient to show that a Cox sheaf of Y/U is finitely generated by Corollary 1.5.13. Let $L_1, L_2, \ldots, L_r \in \mathrm{Pic}(Y)$ such that $\{[L_i]\}_i$ is a $\mathbb{Z}$-basis of the free part of $\mathrm{Pic}(Y/U)$, and we write the free abelian group generated by L_i ($i = 1, \ldots, r$) as Γ_Y. Take line bundles $M_1 \ldots M_{r'}$ such that $\{[f^*L_1], \ldots, [f^*L_r], [M_1] \ldots [M_{r'}]\}$ is a basis of $\mathrm{Pic}(Y/U)_{\mathbb{Q}}$. Let M be the free abelian group generated by $\{M_j\}$ and $\Gamma_X \coloneqq f^*\Gamma_Y \oplus M$. Let T be a torus whose character group is isomorphic to M, and consider the natural action of T on $R \coloneqq \bigoplus_{\mathcal{L} \in \Gamma_X} \pi_{1*}\mathcal{L}$ which corresponds to the grading with respect to M. Then it follows that $R^T = \bigoplus_{\mathcal{L} \in f^*\Gamma_Y} \pi_{1*}\mathcal{L}$, and so it is finitely generated over U since R is finitely generated by the assumption that π_1 is an MDM (see Proposition 1.5.10).

If f is an algebraic fibre space it is obvious that

$$R^T \simeq \bigoplus_{\mathcal{L} \in \Gamma_Y} \pi_{2*}\mathcal{L}, \tag{1.6.11}$$

and this concludes the proof. In the case where f is finite, the finite generation of R^T is equivalent to that of $\bigoplus_{\mathcal{L} \in \Gamma_Y} \pi_{2*}\mathcal{L}$ by the same argument in Section 3.3 in [**Oka16**]. $\square$

Next, we consider base changes of MDMs.

Theorem 1.6.9. *Let $f\colon X \to U$ be an MDM and $g\colon T \to U$ be a morphism between normal quasi-projective varieties. Let $W \coloneqq X \times_U T$, and consider the following diagram.*

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & U \end{tikzcd} \tag{1.6.12}$$

p and q denote the natural projections. Assume the following three conditions:

(1) W is normal and $\mathbb{Q}$-factorial.

(2) The natural map $p^*\colon \mathrm{Pic}(X/U)_{\mathbb{Q}} \to \mathrm{Pic}(W/T)_{\mathbb{Q}}$ is surjective.

(3) $\mathrm{N}^1(W/T)_{\mathbb{Q}} \simeq \mathrm{Pic}(W/T)_{\mathbb{Q}}$ and the natural map $g^*f_*L \to q_*p^*L$ is surjective for any line bundle L on X.

Then q is an MDM.

Proof. Since f is an algebraic fibre space, we can easily check that each fibre of q is connected. Hence q is also an algebraic fibre space by considering the Stein factorization of q since T is normal and q is projective.

Let $\mathcal{L} \coloneqq (L_1, \ldots, L_r)$ be a collection of line bundles such that $([L_1], \ldots, [L_r])$ is a basis of $\mathrm{Pic}(X/U)_{\mathbb{Q}}$. Then a Cox sheaf of X/U is $\bigoplus_{m \in \mathbb{Z}^r} f_*(\mathcal{L}^m)$. By the assumption (3), we obtain the surjection

$$g^*(\bigoplus_{m \in \mathbb{Z}^r} f_*(\mathcal{L}^m)) \to \bigoplus_{m \in \mathbb{Z}^r} q_*(p^*\mathcal{L}^m). \tag{1.6.13}$$

Note that $(p^*L_1, \ldots, p^*L_r)$ generates $\mathrm{Pic}(W/T)_{\mathbb{Q}}$ since p^* is surjective, and so we may assume that $\mathcal{L}' \coloneqq (p^*L_1, \ldots, p^*L_{r'})$ is a basis of $\mathrm{Pic}(W/T)_{\mathbb{Q}}$ for some $1 \le r' \le r$. Let us consider the action of the torus $\mathcal{T} \coloneqq \mathrm{Hom}(\mathbb{Z}^{r-r'}, \mathbb{C}^*)$ on $\bigoplus_{m \in \mathbb{Z}^r} q_*(p^*\mathcal{L}^m)$ which corresponds the grading with respect to $(p^*L_{r'+1}, \ldots, p^*L_r)$. Then we obtain

$$(\bigoplus_{m \in \mathbb{Z}^r} (q_*p^*\mathcal{L}^m))^{\mathcal{T}} = \bigoplus_{m \in \mathbb{Z}^{r'}} q_*\mathcal{L}'^m. \tag{1.6.14}$$

By the assumption that f is an MDM and the surjection (1.6.13), $\bigoplus_{m\in\mathbb{Z}^r} q_*(p^*\mathcal{L}^m)$ is finitely generated, and its $\mathcal{T}$-invariant sheaf is also finitely generated. By (1.6.14), we obtain that $\bigoplus_{m\in\mathbb{Z}^{r'}} q_*\mathcal{L}'^m$ is finitely generated over T. Therefore q is an MDM by Corollary 1.5.13. $\square$

As an application of Theorem 1.6.9, we obtain the following corollary.

Corollary 1.6.10. *Let $f\colon X \to U$ be an MDM and $g\colon T \to U$ be a morphism between quasi-projective varieties. Let $W := X \times_U T$ as in* (1.6.12). *Assume the condition* (1) *and* (2) *in Theorem 1.6.9 and the following* (3′).

(3') g is flat and proper.

Then q is an MDM.

Proof. By the assumption, we have the following diagram.

$$\begin{tikzcd} \mathrm{Pic}(W/T)_{\mathbb{Q}} \arrow[d, "\varphi"'] & \mathrm{Pic}(X/U)_{\mathbb{Q}} \arrow[l, two heads, "p^*"'] \arrow[d, "\wr"] \\ \mathrm{N}^1(W/T)_{\mathbb{Q}} & \mathrm{N}^1(X/U)_{\mathbb{Q}} \arrow[l, two heads, "\overline{p^*}"'] \end{tikzcd} \tag{1.6.15}$$

The horizontal map $\overline{p^*}$ is the natural surjection induced by p^*. To see φ is an isomorphism, we claim that $\overline{p^*}$ is an isomorphism. For this, take $[L] \in \mathrm{N}^1(X/U)_{\mathbb{Q}}$ such that $\overline{p^*}([L]) = 0$. Let C be a curve on X such that $f(C) = u \in U$, and $t \in g^{-1}(u) \subset T$. Then the curve $C \times \{t\}$ is contained in W. We have $L.C = p_1^*(L).(C \times \{t\}) = 0$. This implies that $L \equiv_U 0$, and hence $\overline{p^*}$ is injective. Then by the diagram, φ is an isomorphism. The second condition of (3) easily follows from the flatness of g. In fact, the natural map $g^*f_*L \to q_*p^*L$ is an isomorphism for any line bundles L under the assumption. $\square$

We give another proof of Corollary 1.6.10. It is given by investigating the geometry of W, and we prove that q satisfies the conditions of Definition 1.2.1. In the second proof of Corollary 1.6.10 below, we show that the relative movable cone of W/T coincides with that of X/U under the isomorphism p^* in (1.6.15).

The second proof of Corollary 1.6.10. First we prove that all the maps in (1.6.15) are isomorphisms as in the first proof. Now we consider the nef cones and movable cones. We can easily see that a line bundle L on X is f-nef if and only if p^*L is q-nef as in the same argument of the proof of the injectivity of $\overline{p^*}$ in the first proof. Then $\mathrm{Nef}(W/T) = \mathrm{Nef}(X/U)$ via the isomorphism φ. Moreover, we show that L is f-semiample (respectively, f-movable) if and only if $p^*(L)$ is q-semiample (resp., q-movable) as follows. Consider the following map

$$\alpha_{p^*(L)/T}\colon q^*q_*(p^*(L)) \to p^*L. \tag{1.6.16}$$

By the flat base change theorem, we have the natural isomorphism $q^*q_*(p^*(L)) \simeq p^*(f^*f_*(L))$. This implies that $\alpha_{p^*(L)/T}$ coincides with the map $p^*(\alpha_{L/U})$, and hence

$$\mathrm{Supp}(\mathrm{coker}(\alpha_{p^*(L)/T})) = \mathrm{Supp}(\mathrm{coker}(p^*(\alpha_{L/U})) = p^{-1}(\mathrm{Supp}(\mathrm{coker}(\alpha_{L/U}))), \tag{1.6.17}$$

where we can check the second equality applying Nakayama's lemma to each stalk of the sheaves. Then we obtain $\mathrm{codim}_X(\mathrm{Supp}(\mathrm{coker}(\alpha_{L/U}))) = \mathrm{codim}_W(\mathrm{Supp}(\mathrm{coker}(\alpha_{p^*(L)/T})))$ by the flatness of p, and hence we conclude that L is f-semiample (resp., f-movable) if and only if $p^*(L)$ is q-semiample (resp., q-movable). Note that since $\mathrm{Nef}(X/U)$ is a polyhedral cone generated by finitely many f-semiample divisors, $\mathrm{Nef}(W/T) = \overline{p^*}(\mathrm{Nef}(X/U))$ is also generated by finitely many q-semiample divisors.

For an SQM $r\colon X \dashrightarrow X'$ over U, we show that $\tilde{r}\colon X \times_U T \dashrightarrow W' := X' \times_U T$ is an SQM over T. First, we prove that $\tilde{r}$ is small. Let $F \subset X$ be a closed subset of $\operatorname{codim}_X(F) \geq 2$ such that r is isomorphic in X/F. Then $\tilde{r}$ is an isomorphism outside the closed subset $\tilde{F} := F \times_U T \subset W$. Moreover, we obtain that $\operatorname{codim}_W(\tilde{F}) \geq 2$ by considering the dimension of the fibre of the natural map $\tilde{F} \to F$ since p is surjective and flat. Next we check that W' is $\mathbb{Q}$-factorial. Let D' be a Weil divisor on W'. Then $D := \tilde{r}^*(D')$ is a $\mathbb{Q}$-Cartier divisor since W is $\mathbb{Q}$-factorial. Then by the assumption (2), there exist $\mathbb{Q}$-divisors $D_X \in \operatorname{Pic}(X)_{\mathbb{Q}}$ and $D_T \in \operatorname{Pic}(T)_{\mathbb{Q}}$ such that

$$D + q^*(D_T) \sim_{\mathbb{Q}} p^*(D_X). \tag{1.6.18}$$

This implies that

$$D' \sim_{\mathbb{Q}} p'^*(r_*(D_X)) - q'^*(D_T), \tag{1.6.19}$$

where $p'\colon W' \to X'$ and $q'\colon W' \to T$ are the natural projections. Since the right side of (1.6.19) is $\mathbb{Q}$-Cartier, D' is also $\mathbb{Q}$-Cartier.

Let $r_i\colon X \dashrightarrow X_i$ $(i = 1 \dots k)$ be the all SQMs of X over U. Then $\tilde{r_i}\colon W \dashrightarrow W_i := X_i \times_U T$ are SQMs of W over T. Combining the above arguments, we have

$$\operatorname{Mov}(W/T) \simeq p^*(\operatorname{Mov}(X/U)) \tag{1.6.20}$$

$$= \bigcup_i p^*(r_i^*(\operatorname{Nef}(X_i/U))) \tag{1.6.21}$$

$$= \bigcup_i \tilde{r_i}^*(p_i{}^*(\operatorname{Nef}(X_i/U))) \tag{1.6.22}$$

$$\simeq \bigcup_i \tilde{r_i}^*(\operatorname{Nef}(W_i/T)). \tag{1.6.23}$$

Thus we conclude that q is an MDM. □

Remark 1.6.11. In Theorem 1.6.9 if $U = \operatorname{Spec}(\mathbb{C})$, so that X is an MDS, then the condition (2) is automatically satisfied by [**Har77**, III EXERCISE 12.6].

On the other hand, there exists an MDM whose special fibre is not an MDS as in the following example. Hence not an arbitrary base change of MDM is an MDM. The example violates (2) and the flatness of (3′) of Corollary 1.6.10.

Example 1.6.12. By [**MM64**, Theorem 4], there exists a smooth hypersurface $S_F \subset \mathbb{P}^3$ defined by a homogeneous quartic polynomial $F(x, y, z, w)$ such that $\#\operatorname{Aut}(S_F) = \infty$. Then S_F is not an MDS by [**AHL10**].

Take sufficiently general quartic polynomial $G(x, y, z, w)$ such that $S_F \cap S_G$ is a smooth curve. Let $\mu\colon X \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ along $S_F \cap S_G$. Then X is smooth and there is a morphism $f\colon X \to \mathbb{P}^1$ such that the fibre of a point $[a : b] \in \mathbb{P}^1$ is the hypersurface $S_{aF-bG} \subset \mathbb{P}^3$.

We show that f is an MDM. To see this, it is sufficient to show that X is an MDS because of Proposition 1.6.5. Let E be the μ-exceptional divisor. Then $\operatorname{Pic}(X) = \mu^*(\operatorname{Pic}(\mathbb{P}^3)) \oplus \mathbb{Z}E$, and we can check that $\operatorname{Nef}(X)$ is generated by $\mu^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $\mu^*(\mathcal{O}_{\mathbb{P}^3}(1)) - (1/4)E$. Note that some positive multiple of the second generator is the class of the fibre of f. In particular, $\operatorname{Nef}(X)$ is generated by semiample divisors. If $\varepsilon \in \mathbb{Q}_{>0}$, it is obvious that $\mu^*(\mathcal{O}_{\mathbb{P}^3}(1)) + \varepsilon E$ is not movable. On the other hand, $\mu^*(\mathcal{O}_{\mathbb{P}^3}(1)) - (1/4)E$ is not big since its self intersection number is zero, and so a divisor D in the outside of $\operatorname{Nef}(X)$ with respect to the ray generated by $\mu^*(\mathcal{O}_{\mathbb{P}^3}(1)) - (1/4)E$ is not peseudo-effective. Then we obtain $\operatorname{Nef}(X) = \operatorname{Mov}(X)$, and so X is an MDS.

Below is an example of a birational contraction which is not an MDM.

Example 1.6.13. Let X be the projective cone over a smooth plane cubic curve C defined by an equation $F(x, y, z) \in \mathbb{C}[x, y, z]$. By considering the blowing up at the vertex of the cone, we obtain the following birational contraction.

$$f\colon \widetilde{X} \coloneqq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(1)) \to X. \tag{1.6.24}$$

Note that $\mathrm{Pic}(\tilde{X})_{\mathbb{Q}}$ is not countable since $\mathrm{Pic}(\tilde{X}) \simeq \mathcal{O}_{\tilde{X}}(1) \oplus \pi^* \mathrm{Pic}(C)$ and C is an elliptic curve, where $\pi\colon \tilde{X} \to C$ is the canonical projection and $\mathcal{O}_{\tilde{X}}(1)$ is the tautological line bundle. On the other hand, $\mathrm{Pic}(X) \simeq \mathbb{Z}$. Indeed, we can see that $f^*(\mathrm{Pic}(X)) \simeq \mathrm{Ker}(s^*) \simeq \mathbb{Z}$ for the section $s\colon C \to \tilde{X}$ of π such that $s(C)$ is contracted by f and the induced map $s^* : \mathrm{Pic}(\tilde{X}) \to \mathrm{Pic}(C)$. Hence $\mathrm{Pic}(\tilde{X}/X)_{\mathbb{Q}}$ is not countable, and so it is not isomorphic to $\mathrm{N}^1(\tilde{X}/X)_{\mathbb{Q}} \simeq \mathbb{Q}$. Therefore f is not an MDM.

CHAPTER 2

On Seshadri constants of non-simple abelian varieties

In this chapter, we investigate the Seshadri constants of polarized abelian varieties. First we introduce some notations used in this chapter.

A *polarized abelian variety* is a pair (A, L) of an abelian variety A (i.e., smooth projective group scheme over $\mathbb{C}$) and an ample line bundle L on A. The identity of A is denoted by $0 \in A$. An irreducible closed subvariety $B \subset A$ is an *abelian subvariety* of A if B is a group subscheme of A by the inclusion. An abelian subvariety of A of codimension one is called an *abelian divisor* of A. A *curve* is a projective and integral scheme over $\mathbb{C}$ of dimension one. We say a curve in A generates an abelian subvariety B if B is the minimal abelian subvariety containing the curve.

2.1. Preliminaries

In this section, we recall some fundamental definitions and facts about the Seshadri constants. We recommend [**Laz04**, Chapter 5] and [**BDRH$^+$09**] for more details and overview of the Seshadri constants.

Let L be an ample line bundle on a smooth projective variety X of dimension n. For a point $x \in X$ and a curve C passing through x, we write

$$\varepsilon_{C,x}(L) \coloneqq \frac{L.C}{\operatorname{mult}_x C}, \tag{2.1.1}$$

where $\operatorname{mult}_x C$ is the multiplicity of C at x.

Definition 2.1.1. The Seshadri constant of L at x is defined by

$$\varepsilon(X, L; x) \coloneqq \inf_{x \in C}\{\varepsilon_{C,x}(L)\}, \tag{2.1.2}$$

where the infimum is taken over all curves passing through x.

Note that the definition immediately implies that $\varepsilon(X, L; x)$ is determined by the numerical class of L.

There is another equivalent definition of the Seshadri constant. Let μ be the blow-up of X at a point x and E be the exceptional divisor. Then

$$\varepsilon(X, L; x) = \sup\{t \in \mathbb{R}_{>0} \mid \mu^*L - tE \text{ is ample}\}. \tag{2.1.3}$$

This gives the well-known upper bound

$$\varepsilon(X, L; x) \leq \sqrt[n]{L^n}. \tag{2.1.4}$$

Definition 2.1.2. We say that a curve C is a *Seshadri curve* at x with respect to L if $\varepsilon_{C,x}(L) = \varepsilon(X, L; x)$.

Remark 2.1.3. In general Seshadri curves do not always exist. However, it is known that if X is a surface and $\varepsilon(X, L; x) < \sqrt{L^2}$, then there exists a Seshadri curve at x with respect to L.

If (A, L) is a polarized abelian variety, the Seshadri constant $\varepsilon(A, L; x)$ does not depend on the choice of a point $x \in A$, and hence we denote it by $\varepsilon(A, L)$. We also denote $\varepsilon_{C,0}(L)$ simply by $\varepsilon_C(L)$. Moreover, we say that a curve C is a Seshadri curve of (A, L) if C is a Seshadri curve at $0 \in A$ with respect to L.

2.2. Main results

The main results of this section are Theorem 2.2.3 and Corollary 2.2.9 below.

2.2.1. **Proof of Theorem 2.2.3.** The key ingredient of the proof is Lemma 2.2.2, which asserts the existence of the minimal element in the set of the Seshadri constants of polarized abelian subvarieties of (A, L) of bounded degree. We begin with some preparation.

Definition 2.2.1. Let A be an abelian variety of dimension n. Fix a positive real number r. We define the following sets.

(1) For a fixed abelian variety B of dimension k,

$$S_{B,r} \coloneqq \{[L'] \in \mathrm{NS}(B) \mid \ L' \text{ is an ample line bundle on B satisfying } {L'}^k < r \ \}/\sim, \tag{2.2.1}$$

where we define $[L_1] \sim [L_2]$ if there exists an automorphism f of B such that $[f^*L_1] = [L_2]$ in $\mathrm{NS}(B)$.

(2) For an ample line bundle L on A,

$$S^L_{k,r} \coloneqq \{B \mid B \text{ is an abelian subvariety of } A \text{ of dimension } k \text{ such that } (L|_B)^k < r \ \}. \tag{2.2.2}$$

Consider the following maps.

$$\varepsilon\colon S^L_{k,r} \to \mathbb{R}_{\geq 0};\ B \mapsto \varepsilon(B, L|_B). \tag{2.2.3}$$

Then we define the following sets.

$$E^L_{k,r} \coloneqq \mathrm{Im}(\varepsilon) = \{\varepsilon(B, L|_B) \in \mathbb{R}_{>0} \mid B \in S^L_{k,r}\}. \tag{2.2.4}$$

Lemma 2.2.2. *Let L be an ample line bundle on A. Then $E^L_{k,r}$ is a finite set for any $1 \leq k \leq n$ and r.*

Proof. Assume $S^L_{k,r} \neq \emptyset$. By [**LOZ96**, Theorem], there exist only finitely many isomorphism classes of abelian subvarieties of A of dimension k. Let $B_1, B_2 \ldots, B_t$ be representatives. Then we obtain the following map.

$$\alpha\colon S^L_{k,r} \hookrightarrow \bigcup_{i=1}^{t} S_{B_i,r};\quad B \mapsto [\varphi_B^*(L|_B)], \tag{2.2.5}$$

where $\varphi_B\colon B_i \to B$ is an isomorphism for some B_i.

Hence it is sufficient to show that $S_{B,r}$ is a finite set for a fixed k-dimensional abelian variety B since we can easily see that $\varepsilon(B, L|_B) = \varepsilon(B', L|_{B'})$ if $\alpha(B) = \alpha(B')$. However, this follows from the geometric finiteness theorem (for example, see [**Mil86**, Theorem 18.1]), which says that there exist only finitely many classes of ample line bundles of fixed degree in $\mathrm{NS}(A)$ up to the action of the group of automorphisms of A. □

Now we are ready to prove the theorem.

Theorem 2.2.3. *Assume that*

$$\varepsilon(A, L) < \frac{\sqrt[n]{L^n}}{n}. \tag{2.2.6}$$

Then there exists a proper abelian subvariety B of A such that $\varepsilon(A, L) = \varepsilon(B, L|_B)$.

Proof. Let k be the maximal dimension of the proper abelian subvarieties of A. Note that $1 \leq k < n$ by [**Nak96**, Lemma 3.3]. For each natural number $1 \leq i \leq k$, we write

$$r_i := \left(\frac{i \sqrt[n]{L^n}}{n} \right)^i. \tag{2.2.7}$$

Let

$$a_{k+1} := \frac{\sqrt[n]{L^n}}{n}. \tag{2.2.8}$$

For each $1 \leq i \leq k$, starting with $i = k$, we define a_i inductively as $a_i := \min\{\min(E^L_{i,r_i}), a_{i+1}\}$, where E^L_{k,r_i} is defined in (2.2.4). Obviously, the definition of a_i implies

$$\varepsilon(A, L) \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1} = \frac{\sqrt[n]{L^n}}{n}. \tag{2.2.9}$$

Now for the proof of the theorem, consider the following conditions for each i.

(1^i) $\varepsilon(A, L) < a_i$, and any curve C satisfying $\varepsilon(A, L) \leq \varepsilon_C(L) < a_i$ generates an abelian subvariety of dimension at most $(i - 1)$.

(2^i) $\varepsilon(A, L) = \varepsilon(B, L|_B)$ for some i-dimensional abelian subvariety B.

Note that our assumption (2.2.6) and [**Nak96**, Lemma 3.3] imply (1^{k+1}). To prove the theorem, it is sufficient to show that there exists some i, for which the condition (2^i) holds. However, this follows from the following Claim 2.2.4. $\square$

Claim 2.2.4. *Under the above notation,* (1^i) *implies either* (1^{i-1}) *or* (2^{i-1}) *for any* $2 \leq i \leq k + 1$.

Proof of the Claim. Assume that (1^i) holds. Note that it follows that $\varepsilon(A, L) \leq \min(E^L_{i-1,r_{i-1}})$ by the definition of the Seshadri constant. The equality implies the condition (2^{i-1}), so let us assume that the inequality is strict. In this case there exists a curve C satisfying

$$\varepsilon(A, L) \leq \varepsilon_C(L) < a_{i-1}. \tag{2.2.10}$$

If all the curves satisfying (2.2.10) generates abelian subvarieties of dimension at most $i - 2$, then this implies (1^{i-1}), so that the proof is done. Hence, for the contradiction, suppose that there exists a curve C which satisfies (2.2.10) and generates an abelian subvariety B of dimension $i - 1$ since we already know that $\dim B \leq i - 1$ by (1^i).

First assume that

$$\frac{\sqrt[n]{L^n}}{n} \leq \frac{\sqrt[i-1]{(L|_B)^{i-1}}}{i - 1}. \tag{2.2.11}$$

In this case, we obtain that

$$\varepsilon_C(L) < \frac{\sqrt[n]{L^n}}{n} \leq \frac{\sqrt[i-1]{(L|_B)^{i-1}}}{i - 1}, \tag{2.2.12}$$

where the first inequality follows from (2.2.9). Then [**Nak96**, Lemma 3.3] implies that C is contained in a proper abelian subvariety of B. However, this contradicts that C generates B.

Hence, let us assume that

$$\frac{\sqrt[n]{L^n}}{n} > \frac{\sqrt[i-1]{(L|_B)^{i-1}}}{i-1}. \tag{2.2.13}$$

Then it follows that $B \in S^L_{i-1,r_{i-1}}$ by the Definition 2.2.1. Then we obtain the inequality

$$\min(E^L_{i-1,r_{i-1}}) \leq \varepsilon_C(L). \tag{2.2.14}$$

However, this contradicts our assumption $\varepsilon_C(L) < a_{i-1} \leq \min(E^L_{i-1,r_{i-1}})$. Hence C can not generate an $i-1$ dimensional abelian subvariety and this concludes the proof. $\square$

2.2.2. **Proof of the Corollary 2.2.9.** For the proof, we define the nef threshold of a divisor D on an n-dimensional polarized abelian variety (A, L) as

$$\sigma(L, D) \coloneqq \sup\{t \in \mathbb{R} \mid L - tD \text{ is ample}\} \in \mathbb{R}_{>0} \cup \{\infty\}. \tag{2.2.15}$$

The following lemma is crucial for the proof of Corollary 2.2.9.

Lemma 2.2.5. *Let D be an abelian divisor of A. Then $\sigma(L, D)(L|_D)^{n-1} = L^n/n$.*

Proof. For any $x \in A \setminus D$, it follows that $(D + x) \cap (D) = \phi$. Hence we obtain $D^2 = 0$ in the Chow ring $A^2(A)$ since $(D + x)$ and D are in the same numerically class. By [**Bau08**, Proposition 1.1], $\sigma(L, D)$ is the multiplicative inverse of the maximal root of the polynomial

$$\chi(uL - M) = \frac{1}{n!}(uL - D)^n = \frac{1}{n!}(L^n u^n - n(L|_D)^{n-1}u^{n-1}) \in \mathbb{Q}[u]. \tag{2.2.16}$$

Then the straightforward computation implies the assertion. $\square$

Remark 2.2.6. We can also prove this lemma by applying the methods of the Okounkov body. For details, see the proof of [**Loz18**, Corollary 4.12].

Proposition 2.2.7. *Let D be an abelian divisor of A. Suppose that*

$$\sqrt[n]{L^n} > (\textit{respectively}, \geq) \sqrt[n-1]{a(L|_D)^{n-1}} \tag{2.2.17}$$

for a positive real number a. Then any curve C satisfying

$$\varepsilon_C(L) \leq (\textit{resp.} <) \frac{a\sqrt[n]{L^n}}{n} \tag{2.2.18}$$

is contained in D.

Proof. By Lemma 2.2.5 and the assumption (2.2.17), we obtain that

$$\sigma(L, D) > (\text{resp. } \geq) \frac{a\sqrt[n]{L^n}}{n}. \tag{2.2.19}$$

This implies that $L - \frac{(a\sqrt[n]{L^n})}{n} D$ is ample (resp. nef). Hence it follows that

$$L.C > (\text{resp. } \geq) \frac{a\sqrt[n]{L^n}(D.C)}{n}. \tag{2.2.20}$$

Then, by the assumption (2.2.18) and (2.2.20), we have

$$\frac{a\sqrt[n]{L^n}}{n} \geq (\text{resp. } >) \frac{L.C}{\operatorname{mult}_0(C)} > (\text{resp. } \geq) \frac{a\sqrt[n]{L^n}(D.C)}{n \operatorname{mult}_0(C)}, \tag{2.2.21}$$

so that

$$1 > \frac{D.C}{\operatorname{mult}_0 C}. \tag{2.2.22}$$

Now for a contradiction, we assume that C is not contained in D. Then we obtain

$$D.C \geq \operatorname{mult}_0(C)\operatorname{mult}_0(D) = \operatorname{mult}_0(C), \tag{2.2.23}$$

a contradiction. □

Lemma 2.2.8. *Let X be a smooth variety of dimension n, and D be a divisor containing a point $x \in X$. Assume*

$$\sqrt[n]{L^n} \geq \sqrt[n-1]{a(L|_D)^{n-1}} \tag{2.2.24}$$

for $a > \sqrt[n]{\frac{n^{n-1}}{\operatorname{mult}_x(D)}}$. Then we have the upper bound $\varepsilon(X, L; x) < a\sqrt[n]{L^n}/n$.

Proof. Suppose that $\varepsilon(X, L; x) \geq a\sqrt[n]{L^n}/n$.

$$\frac{a\sqrt[n-1]{a(L|_D)^{n-1}}}{n} \leq \frac{a\sqrt[n]{L^n}}{n} \leq \varepsilon(X, L; x) \leq \sqrt[n-1]{\frac{(L|_D)^{n-1}}{\operatorname{mult}_x(D)}}. \tag{2.2.25}$$

This contradicts to the assumption that $a > \sqrt[n]{\frac{n^{n-1}}{\operatorname{mult}_x(D)}}$. □

Combining Proposition 2.2.7 and Lemma 2.2.8, we obtain the following result.

Corollary 2.2.9. *Let (A, L) be a polarized abelian variety of dimension n. Fix a positive real number a. Let D be an abelian divisor in A such that*

$$\sqrt[n]{L^n} \geq \sqrt[n-1]{a(L|_D)^{n-1}}. \tag{2.2.26}$$

If $\varepsilon(A, L) < a\sqrt[n]{L^n}/n$ holds, then $\varepsilon(A, L) = \varepsilon(D, L|_D)$. Moreover, if one can take $a > (\sqrt[n]{n})^{n-1}$, the upper bound $\varepsilon(A, L) < a\sqrt[n]{L^n}/n$ automatically holds.

2.3. Applications

In this section, we give some applications of our theorems. First, we show some results about the uniqueness of Seshadri curves by applying Proposition 2.2.7.

Let (A, L) be a polarized abelian variety of dimension n. For any $a \in \mathbb{R}_{>0}$, we denote the set of all curves satisfying

$$\varepsilon_C(L) < \frac{a\sqrt[n]{L^n}}{n} \tag{2.3.1}$$

by $\mathcal{C}_a$. Moreover, we define

$$\mathcal{D}_a := \{D \subset A \mid D \text{ is an abelian divisor satisfying the following (2.3.2)}\}.$$

$$\sqrt[n]{L^n} \geq \sqrt[n-1]{a(L|_D)^{n-1}}. \tag{2.3.2}$$

Then, by Proposition 2.2.7, it follows that

$$\bigcup_{C \in \mathcal{C}_a} C \subset \bigcap_{D \in \mathcal{D}_a} D. \tag{2.3.3}$$

This observation implies the following Proposition 2.3.1 and Proposition 2.3.3.

Proposition 2.3.1. *Let (S, L) be a polarized abelian surface. Assume that there exists a curve $C \ni 0$ such that*

$$\varepsilon_C(L) < \sqrt{\frac{L^2}{2}}. \tag{2.3.4}$$

Then C is elliptic and it is the unique curve satisfying (2.3.4) *and containing $0 \in S$.*

Proof. The definition of the Seshadri constant implies that $\varepsilon(S, L) < \sqrt{\frac{L^2}{2}}$. Then, by Lemma 2.3.4 below, there exists an elliptic curve C_0 such that $\varepsilon_{C_0}(L) = \varepsilon(S, L)$. However, applying Proposition 2.2.7 as $a = \sqrt{2}$, we conclude that C_0 is the unique curve satisfies (2.3.4). $\square$

Remark 2.3.2. If $\sqrt{L^2}$ is irrational, it is already known that there are at most only a finitely many submaximal curves. In fact, by the proof of [**Bau98**, Theorem A.1.(a)], there exists an integer $k > 0$ and $D \in |kL|$ such that any curve satisfying $\varepsilon_C(L) < \sqrt{L^2}$ is an irreducible component of D by [**Bau99**, Lemma 5.2]. However, Proposition 2.3.1 implies that the Seshadri curve is unique and elliptic under the assumption (2.3.4).

Proposition 2.3.3. *Let (A, L) be a polarized abelian threefold. For any $a \in \mathbb{R}_{>0}$, if there exist at least two abelian divisors in $\mathcal{D}_a$, then there is at most only one curve in $\mathcal{C}_a$ and it is an elliptic curve. Moreover, if one can take $a > (\sqrt[3]{3})^2$, there exists exactly one curve in $\mathcal{C}_a$.*

Proof. Let D_1 and D_2 be different abelian divisors in $\mathcal{D}_a$. Then $D_1 \cap D_2$ with induced reduced structure is a reduced algebraic group of dimension one. Hence the identity component is an elliptic curve. Therefore we obtain the assertion since any curve in $\mathcal{C}_a$ is contained in the identity component of $D_1 \cap D_2$ by (2.3.3). For the latter part, it is sufficient to show $\mathcal{C}_a \neq \emptyset$. However, this follows from Lemma 2.2.8. $\square$

Next, we consider the polarized abelian threefolds such that $\varepsilon(A, L) < \sqrt[3]{L^3}$. We use the following fact from [**Bau98**, Theorem A.1.(b)] to show Corollary 2.3.5 below.

Lemma 2.3.4 (=[**Bau98**, Theorem A.1.(b)])**.** *Let (S, L) be a polarized abelian surface. Then one has a lower bound*

$$\varepsilon(S, L) \geq \min\left\{\varepsilon_0, \frac{\sqrt{14L^2}}{4}\right\}, \tag{2.3.5}$$

where ε_0 is the minimal degree of the elliptic curves in S with respect to L.

Then Theorem 2.2.3, Corollary 2.2.9 and Lemma 2.3.4 imply the following corollary.

Corollary 2.3.5. *Let (A, L) be a polarized abelian threefold. Assume that $\varepsilon(A, L) < \sqrt[3]{L^3}/3$.*

(1) If there exists an abelian surface S which satisfies

$$\sqrt[3]{L^3} > \frac{3\sqrt{14(L|_S)^2}}{4}, \tag{2.3.6}$$

then $\varepsilon(A, L) = \varepsilon(S, L|_S)$.

(2) Otherwise, $\varepsilon(A, L)$ is computed by an elliptic curve.

Proof. First, we prove (2). By Theorem 2.2.3, we may assume that there exists an abelian surface S' such that

$$\varepsilon(S', L|_{S'}) = \varepsilon(A, L) < \frac{\sqrt[3]{L^3}}{3} \leq \frac{\sqrt{14(L|_{S'})^2}}{4} \tag{2.3.7}$$

since if A does not contains an abelian surface, (2) obviously follows by [**Nak96**, Lemma 3.3].

Hence, applying Lemma 2.3.4, $\varepsilon(A, L)$ is the minimal degree of the elliptic curves in S' with respect to L.

On the other hand, the assumption of (1) implies

$$\sqrt[3]{L^3} > \frac{3\sqrt{14(L|_S)^2}}{4}. \tag{2.3.8}$$

Then we conclude the proof by Corollary 2.2.9. □

Corollary 2.3.6. *Let (A, L) be a polarized abelian threefold. Assume $L^3 \leq 174$ and $\varepsilon(A, L) < \sqrt[3]{L^3}/3$. Then $\varepsilon(A, L) = 1$ or $4/3$. Moreover if $\varepsilon(A, L) = 4/3$, A contains the Jacobian variety J of a genus two curve such that any curve satisfying $\varepsilon_C(L) < 21\sqrt[3]{L^3}/8$ is contained in J and $A \simeq J \times E$ for some elliptic curve E. If $L^3 \leq 60$, we obtain $\varepsilon(A, L) = 1$.*

Proof. Note that an ample line bundle on an abelian surface has the positive and even degree. Since $L^3 \leq 174$, if there exists an abelian surface S such that

$$\sqrt[3]{L^3} > \frac{3\sqrt{14(L|_S)^2}}{4}, \tag{2.3.9}$$

it must be $(L|_S)^2 = 2$. Hence, by Corollary 2.3.5, it follows that either

(1) $\varepsilon(A, L)$ is computed by an elliptic curve C, or
(2) A contains a principally polarized abelian surface $(S, L|_S)$ and $\varepsilon(A, L) = \varepsilon(S, L|_S)$.

If the first case occurs, we obtain $C.L = \varepsilon(A, L) = 1$ since $\varepsilon(A, L) < \sqrt[3]{L^3}/3 < 2$. Moreover, A is isomorphic to $C \times B$ for some abelian surface B by [**DH07**, Lemma 1]. In the second case, again by [**DH07**, Lemma 1], it follows that there exists an elliptic curve E such that $A \simeq S \times E$. Furthermore, $\varepsilon(A, L) = 1$ or $4/3$ since it is known that a principally polarized abelian surface is isomorphic to either the product of two elliptic curves with the product of line bundles of degree one or the Jacobian variety of a genus two curve with the Theta divisor, where its Seshadri constant is $4/3$ as proved in [**Ste98**, Proposition 2]. In the case $\varepsilon(A, L) = 4/3$, then S is the Jacobian variety of a genus two curve and any curve satisfying $\varepsilon_C(L) < 21\sqrt[3]{L^3}/8$ is contained in S by (2.3.9) and Proposition 2.2.7.

Finally, it is obvious that $\varepsilon(A, L) = 1$ if $L^3 \leq 60$ since $\sqrt[3]{60}/3 < 4/3$. □

Remark 2.3.7. It is known that any polarized abelian surface (S, L) satisfies $\varepsilon(S, L) \geq 4/3$ if it is not one (see [**Nak96**, Theorem 1.2]). Hence, in fact, the last assertion in Corollary 2.3.6 can be proven also directly from Nakamaye's [**Nak96**, Lemma 3.3] or Theorem 2.2.3. Indeed, if $L^3 \leq 60$, then $\varepsilon(A, L) < \sqrt[3]{L^3}/3$ implies $\varepsilon(A, L) < 4/3$. However, Theorem 2.2.3 implies that $\varepsilon(B, L|_B) = \varepsilon(A, L) < 4/3$ where B is an abelian surface or an elliptic curve. Hence $\varepsilon(A, L)$ must be one.

In Corollary 2.3.6, the assumption $L^3 \leq 60$ is optimal for $\varepsilon(A, L)$ to be one. In the following example, we construct polarized abelian threefolds (A, L) satisfying $\varepsilon(A, L) < \sqrt[3]{L^3}/3$, $L^3 = n$, and $\varepsilon(A, L) \neq 1$ for any $n \in 6\mathbb{Z}$ such that $n > 60$.

Example 2.3.8. Let (J, θ) be a pair of the Jacobian variety of a genus two curve and its Theta divisor. Then it follows that $\varepsilon(J, \theta) = 4/3$ by [**Ste98**, Proposition 2]. Assume (E, M_k) is a polarized elliptic curve with $\deg M_k = k > 0$. Consider $A := J \times E$ and the ample line bundle $L_k := pr_1^*\theta \otimes pr_2^*M_k$ on it. Then straightforward computation implies $L_k^3 = 6k$. Then we obtain

$$\varepsilon(A, L_k) = \min\{\varepsilon(J, \theta), \varepsilon(E, M_k)\} = \min\{4/3, k\}. \tag{2.3.10}$$

Hence, if $2 \leq k \leq 10$, we obtain $\varepsilon(A, L_k) = 4/3$ and $\varepsilon(A, L_k) > \sqrt[3]{L_k^3}/3$. If $11 \leq k$, we have $\varepsilon(A, L_k) = 4/3$, $L^3 = 6k > 60$ and $\varepsilon(A, L_k) < \sqrt[3]{L_k^3}/3$.

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