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Pathological phenomena in the wild McKay correspondence

Takahiro Yamamoto

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1 Introduction

Singularities are one of the most important themes in algebraic geometry. Quotient singularities, which appear when dividing a nonsingular variety by a finite group action, form a fundamental class of singularities. In characteristic zero, quotient singularities have good behavior. For example, they are log terminal singularities, which are a class of singularities used in minimal model program. Moreover, they have resolutions, in particular, Gorenstein quotient singularities of dimension ≤ 3 have crepant resolutions [4],[5],[6],[9],[10].

On the other hand, in positive characteristic, there are many pathological phenomena about quotient singularities. For instance, quotient singularities in positive characteristic are not log terminal in general (for instance see [16]). Whether quotient singularities have resolutions or not is not known, and there exist three-dimensional Gorenstein quotient singularities not having crepant resolution.

One of the natural questions is when quotient singularities are log terminal or belong to other classes of singularities appearing in the minimal model program. In characteristic zero, there is the Reid–Shepherd-Barron–Tai criterion [7, Theorem 3.21], which determines whether a quotient singularity is terminal/canonical by looking at elements of the finite group in question individually. But, in positive characteristic, this is no longer true, as was proved in the auther's master thesis. More precisely, there exist quotient singularities which are not log canonical such that the quotient by every cyclic subgroup of the total group is canonical. This illustrates how difficult it is to determine which classes given quotient singularities belong to. Our first main result below gives a partial answer to this problem.

Theorem 1.1 (Theorem 2.3,[21])**.** *Let k be an algebraic closed field of characteristic three. Let G be a group acting on $\mathbb{A}^3 = \mathbb{A}^3_k$ faithfull and linearly. Suppose that G has no pseudo-reflection. Then $\mathbb{A}^3/G$ is log terminal if and only if $\sharp G \notin 9\mathbb{Z}$. Moreover, if $\sharp G \in 9\mathbb{Z}$, then $\mathbb{A}^3/G$ is not log canonical.*

Note that from [16], wild quotient singularities by linear actions without pseudo-reflection in dimension three and characteristic $p > 3$ are not log terminal. Any three dimensional wild representation in characteristic two has pseudo-reflection. These are the reasons why we consider in dimension three and characteristic three.

Next we consider the question, when quotient singularities have crepant resolutions. As mentioned above, Gorenstein quotient singularities in dimension ≤ 3 have crepant resolutions in characteristic zero. When crepant resolution exists, we also ask whether the following theorem of Batyrev [1, Theorem 1.10] holds in positive characteristic:

Theorem 1.2. *Let $G \subset \mathrm{SL}_d(\mathbb{C})$ be a finite group. If the quotient variety $\mathbb{A}^d_{\mathbb{C}}/G$ has a crepant resolution $Y \to \mathbb{A}^d_{\mathbb{C}}/G$, then the topological Euler characteristic $e(Y)$ is equal to the number of the conjugacy classes of G:*

$$e(Y) = \sharp\mathrm{Conj}(G).$$

Let χ be the l-adic Euler characteristic, which is defined as the alternating sum of dimensions of l-adic cohomology. It coincides with the topological Euler characteristic in characteristic zero. We consider the equation of Batyrev's theorem using χ instead of e in positive characteristic.

In [8] and [2], it is shown that the quotient singularity of the canonical action of the symmetric group S_n on $\mathbb{A}^{2n}$ has a crepant resolution. In this case, the equation of Batyrev's theorem holds. In [16], there are some examples of quotient varieties associated to $\mathbb{Z}/p\mathbb{Z}$-representations which have crepant resolutions in characteristic p. Also in this case, the equation of Batyrev's theorem holds. However, in positive characteristic, there exist counterexamples of Batyrev's theorem.

Theorem 1.3 (Theorem 3.1)**.** *Let k be an algebraically closed field of characteristic three. Let $G \subset \mathrm{SL}_3(k)$ be a small finite subgroup. If $G \cong H \rtimes G'$ where*

H is a tame Abelian group and G' is $\mathbb{Z}/3\mathbb{Z}$ or $\mathfrak{S}_3$, then $\mathbb{A}^3/G$ has a crepant resolution $Y \to \mathbb{A}^3/G$ and

$$\chi(Y) = \begin{cases} \sharp\mathrm{Conj}(G) & (G' = \mathbb{Z}/3\mathbb{Z}) \\ \sharp\mathrm{Conj}(G) + 3 & (G' = \mathfrak{S}_3) \end{cases}$$

The case when H is trivial or the Klein four-group was treated in the author's master thesis [20]. In particular, the last theorem gives infinitely many examples of wild quotient singularities having crepant resolutions and infinitely many counterexamples to Batyrev's theorem in positive characteristic.

To prove Theorem 1.3, we explicitly construct crepant resolutions of quotient varieties. Since G has the normal subgroup H, we firstly construct a crepant resolution $\widetilde{Y} \to \mathbb{A}^3/H$ using the theory of toric varieties. Next, we construct a crepant resolution $\widetilde{X} \to \widetilde{Y}/G'$ which gives a crepant resolution of $\mathbb{A}^3/G$ by composing $\widetilde{Y}/G' \to \mathbb{A}^3/G$. We compute its Euler characteristic from the explicit structure of crepant resolutions.

We also compute, in an alternative way, the Euler characteristics of crepant resolutions in Theorem 1.3 using the wild McKay correspondence. The wild McKay correspondence was proved in [19]:

Theorem 1.4 ([19],Corollary 16.3)**.** *Let G be a finite group. Suppose that G acts on $V = \mathbb{A}^d$ linearly. Suppose that G has no pseudo-reflection. Then*

$$M_{st}(X) = \int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}_V}$$

for $X = V/G$.

Here $M_{st}(X)$ is the *stringy-motive* of X, G-Cov(D) is the moduli space of G-covers of D, $\mathbb{L} = [\mathbb{A}^1]$ in the Grothendieck ring, and $\boldsymbol{v}_V$ is a function on G-Cov(D) defined from the representation. The stringy-motive is an element of the Grothendieck ring of k-varieties localized and completed in some way. It has a lot of information of singularity of X. In particular, if there exists a crepant resolution $Y \to X$, then we have $M_{st}(X) = [Y]$. The stringy-motive is generalization of the stringy E-function in characteristic zero defined by Batyrev [1]. This theorem is generalization of the motivic McKay correspondence in characteristic zero proved by Batyrev [1] and Denef-Loeser [3]. In this paper, we use the stringy-point count, which is regarded as a realization of the wild McKay correspondence.

Theorem 1.5 ([17])**.** *Let G be a finite group. Let $K = \mathbb{F}_q((t))$ where q is a power of a prime number. Suppose that G acts on $V = \mathbb{A}^d_{\mathcal{O}_K}$ linealy and that G has no pseudo-reflection. Then*

$$\sharp_{st} V/G = \sum_{M \in G\text{-}\acute{E}t(K)} \frac{q^{d-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)}.$$

Here G-Ét(K) is the set of G-étale K-algebras, which are étale K-algebras M of $\dim_K(M) = \sharp G$ endowed with G-action such that $M^G = K$, H_M is the stabilizer subgroup of a connected component of Spec (M) when we write as $M = L^{\oplus n}$ by a Galois extension L/K, and $C_G(H_M)$ is the centralizer of H_M. We compute the right hand side of this equality directly in some of cases considered in Theorem 1.3.

Theorem 1.6 (Theorem 4.3,4.5,4.7)**.** *We follow the notation of Theorem 1.3 except that k now denote the finite field $\mathbb{F}_q$. Let $l \neq 3$ be a prime. We suppose that $q-1$ is divisible by l if $G' = \mathbb{Z}/3\mathbb{Z}$. We suppose that $q-1$ is divisible by $2l$ if $G' = \mathfrak{S}_3$. Then we have the following formulas:*

$$\sharp_{st}\mathbb{A}^3/G = \begin{cases} q^3 + \left(2 + \frac{l-1}{6}\right) q^2 + \frac{l-1}{6} q & (H = \mathbb{Z}/l\mathbb{Z}, G' = \mathbb{Z}/3\mathbb{Z}) \\ q^3 + \left(2 + \frac{(l-1)(l+4)}{6}\right) q^2 + \frac{(l-1)(l-2)}{6} q & (H = (\mathbb{Z}/l\mathbb{Z})^2, G' = \mathbb{Z}/3\mathbb{Z}) \\ q^3 + \frac{(l+5)(l+7)}{12} q^2 + \frac{(l+1)(l+5)}{12} q & (l \neq 2, H = (\mathbb{Z}/l\mathbb{Z})^2, G' = \mathfrak{S}_3) \\ q^3 + 6q^2 + q & (H = (\mathbb{Z}/2\mathbb{Z})^2, G' = \mathfrak{S}_3) \end{cases}.$$

In [17], the Serre-Bhargava mass formula is proved using the stringy-point count of the quotient variety. So the fomulas in Theorem 1.6 can be regarded as other versions of mass formula.

This theorem together with the Weil conjecture gives the Euler characteristics of crepant resolution.

Corollary 1.7. *In the situation of Theorem 1.6, if $Y \to \mathbb{A}^3/G$ is a crepant resolution, then*

$$\chi(Y) = \begin{cases} 3 + \frac{l-1}{3} & (H = \mathbb{Z}/l\mathbb{Z}, G' = \mathbb{Z}/3\mathbb{Z}) \\ 3 + \frac{l^2-1}{3} & (H = (\mathbb{Z}/l\mathbb{Z})^2, G' = \mathbb{Z}/3\mathbb{Z}) \\ \frac{(l-1)(l-2)}{6} + 2l + 4 & (H = (\mathbb{Z}/l\mathbb{Z})^2, G' = \mathfrak{S}_3) \end{cases}.$$

Corollary 1.7 coincides with the equality in Theorem 1.3.

To compute right hand side of equality in the wild McKay correspondence (Theorem 1.4), a key is computation of v-function. However, computing v-function is generally very difficult. Below are some examples of previously known computation:

- We consider the case when the n-th symmetric group $\mathfrak{S}_n$ acts on $\mathbb{A}^{2n} = (\mathbb{A}^2)^n$ canonically. Then the v-function is the same as the Artin conductor [13], which is used in the number theory.

- [12, Theorem 3.11] In characteristic $p > 0$, we consider the d-dimensional indecomposable representation V of $G = \mathbb{Z}/p^n\mathbb{Z}$ where $d \leq p^n$. For a Galois extension L/K whose Galois group is G with ramification jumps

  $$l_0 \leq l_1 \leq l_2 \leq \cdots \leq l_{n-1},$$

  we have

  $$\boldsymbol{v}_V(L) = \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-1}p^{n-1} < d, \\ 0 \leq i_0, \ldots, i_{n-1} < p}} \left\lceil \frac{i_0 l_0 p^{n-1} + i_1 l_1 p^{n-2} + \cdots + i_{n-1} l_{n-1}}{p^n} \right\rceil.$$

In all the known cases, the v-functions are determined from the ramification filtration of the extension L/K, as the last example indicates. We give an example where this is not the case:

Theorem 1.8. *There exist a finite group G and a G-representation whose v-fucntion is not determined by the ramification filtration of G.*

Therefore, for the computation of v-function, we need to use information of extension L/K more than the ramification filtration in general.

The outline of the paper is as follows. We prove Theorem 1.1 in section two. In section three, we construct counterexamples to Batyrev's theorem in positive characteristic. In section four, we give proofs of 1.6 and 1.7. Lastly, in section five, we illustrate the example that the v-function is not computed only from ramification filtration only.

2 A criterion for log terminal quotient singularities

Let k be an algebraic closed field. For a normal variety X with $\mathbb{Q}$-Cartier canonical divisor K_X, if there exists a proper birational map $\pi : Y \to X$ and hold the equation

$$K_Y = \pi^* K_X + \sum_{E:\text{prime}} a_E E,$$

we call E is an *exceptional divisor* of X and a_E is *discrepancy* of E. We call X is *terminal* (resp. *canonical*, *log terminal*, *log canonical*) if discrepancies > 0 (resp. $\geq 0,\ > -1,\ \geq -1$) for any prime exceptional divisors.

We already have the following result.

Theorem 2.1 ([20]). *Let k be an algebraic closed field of characteristic three. If the group $G = (\mathbb{Z}/3\mathbb{Z})^2$ acts on $\mathbb{A}^3_k$ linearly without pseudo-reflection, then the quotient variety $\mathbb{A}^3_k/G$ is not log canonical.*

The proof of this theorem is given by direct computation of a resolution. This theorem gives a counterexample of the equivalence of the first condition and the second one in the Reid–Shepherd-Barron–Tai criterion below. Let a finite group G acts on $\mathbb{C}^d$ linearly. For $g \in G$, we define the *age* $\text{age}(g)$ of g by

$$\text{age}(g) = \frac{1}{n}\sum_{i=1}^{d} a_i$$

if the representation matrix of g is diagonalized as

$$\begin{bmatrix} e^{\frac{2a_1\pi i}{n}} & & & \\ & e^{\frac{2a_2\pi i}{n}} & & \\ & & \ddots & \\ & & & e^{\frac{2a_d\pi i}{n}} \end{bmatrix}.$$

Proposition 2.2 (The Reid–Shepherd-Barron–Tai criterion,[7],Theorem 3.21)**.** *Suppose that a finite group G acts on $\mathbb{C}^d$ linearly without pseudo-reflection. Then the following three conditions are equivalent:*

- *the quotient variety $\mathbb{C}^d/G$ is canonical (resp. terminal),*
- *the quotient variety $\mathbb{C}^d/C$ is canonical (resp. terminal) for any cyclic subgroup C of G,*
- $\mathrm{age}(g) \geq 1$ *(resp.>1) for any $g \in G - \{1\}$.*

Let $G = (\mathbb{Z}/3\mathbb{Z})^2$. Any nontrivial cyclic subgroup C of G is $\mathbb{Z}/3\mathbb{Z}$. By [15], the quotient variety $\mathbb{A}^3/C$ is canonical. If the first and second condition of Reid–Shepherd-Barron–Tai criterion were equivalent in positive characteristic, $\mathbb{A}^3/G$ would be also canonical. But Theorem 2.1 says $\mathbb{A}^3/G$ is not log canonical. Hence these conditions are not equivalent in positive characteristic.

As an application of Theorem 2.1, we get the following criterion for quotients of $\mathbb{A}^3$ in characteristic three.

Theorem 2.3. *Let k be an algebraic field of characteristic three. Let G be a finite group. Suppose that G acts on $\mathbb{A}^3$ faithfully and linearly without pseudo-reflection. Then $\mathbb{A}^3/G$ is log terminal if and only if $\sharp G \notin 9\mathbb{Z}$. Moreover, if $\mathbb{A}^3/G$ is not log terminal, then it is not log canonical.*

We first prove auxiliary results which will be used in the proof of the theorem.

Lemma 2.4. *Let $\pi\colon X' \to X$ be a finite dominant morphism of varieties. Then, for any divisor E' over X', there exists a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[f]{} & X \end{array}$$

satisfying the following conditions:

- *Y and Y' are normal varieties.*
- *f' and f are birational.*
- *ρ is a morphism.*
- *The center of E' on Y' has codimension one.*
- *The closure of $\rho(\mathrm{cent}_{Y'}(E'))$ has codimension one.*

Proof. First, we take a birational morphism $\varphi\colon Y_0' \to X'$ from a normal variety Y_0' such that E' is a divisor on Y_0'. Let $\phi\colon X'' \to X'$ be the Galois closure of π. Thus the coordinate ring $\mathcal{O}_{X''}$ of X'' is the integral closure of the one $\mathcal{O}_{X'}$ of X'

in a Galois closure L of $K(X')/K(X)$. Let G be the Galois group of $L/K(X)$, and we put

$$G = \{g_1, g_2, \dots, g_d\}.$$

The group G acts on X'' naturally.

Let Y''_0 be the component of $Y'_0 \times_{X'} X''$ such that the morphism $\varphi' \colon Y''_0 \to X''$ is dominant. We denote a copy of Y''_0 endowed with the morphism

$$\sigma(g) \colon Y''_0 \xrightarrow{\varphi'} X'' \xrightarrow{g} X''$$

by $Y''_{0,g}$. We consider the variety Y'' given by the fiber product of all the $Y''_{0,g}$ over X'':

$$Y'' = (\cdots((Y''_{0,g_1} \times_{\sigma(g_1),X'',\sigma(g_2)} Y''_{0,g_2}) \times_{X'',\sigma(g_3)} Y''_{0,g_3}) \cdots) \times_{X'',\sigma(g_d)} Y''_{0,g_d}.$$

The G-action on Y'' is defined by the morphism $h : Y'' \to Y''$ induced from the morphisms $Y''_{0,g} \xrightarrow{id} Y''_{0,hg}$ for $h \in G$. Then the morphism $f'' \colon Y'' \to X''$ is G-equivariant because the diagram

$$\begin{array}{ccccc} Y''_{0,g} & \xrightarrow{\varphi'} & X'' & \xrightarrow{g} & X'' \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h} \\ Y''_{0,hg} & \xrightarrow[\varphi']{} & X'' & \xrightarrow[hg]{} & X'' \end{array}$$

is commutative. Let H be a subgroup of G such that $L^H = K(X')$. Let $Y' := Y''/H$ and $Y := Y''/G$. Since f'' is G-equivariant and since $X' = X''/H$ and $X = X''/G$, there exist natural morphisms $f' \colon Y' \to X'$ and $f \colon Y \to X$, which are birational. We also have natural morphisms $q \colon Y'' \to Y'$ and $\eta \colon Y' \to Y$.

$$\begin{tikzcd} Y'' \arrow[r] \arrow[rr, bend left, "f''"] \arrow[d, "q"'] & Y''_0 \arrow[r, "\varphi'"'] \arrow[d] & X'' \arrow[d, "\phi"] \\ Y' \arrow[rr, bend right, "f'"'] \arrow[d, "\eta"'] & Y'_0 \arrow[r, "\varphi"] & X' \arrow[d, "\pi"] \\ Y \arrow[rr, "f"'] & & X \end{tikzcd}$$

Let E'' be a prime divisor on Y'' contained in the pull-back of E' by $Y'' \to Y''_0 \to Y'_0$. Since q is finite, the push-forward q_*E' is a prime divisor on Y'. Let v'', v' be the valuations on $K(X''), K(X')$ corresponding to E'', E', respectively. From the construction of E'', we have $q(E'') = E'$. Moreover, since η is finite, η preserves the dimension. $\square$

Proposition 2.5. *Let $\pi\colon X' \to X$ be a finite dominant morphism of $\mathbb{Q}$-Gorenstein varieties. Assume that π is étale in codimension one. If X' is not log canonical (resp. not log terminal), then X is not log canonical (resp. not log terminal).*

Proof. We prove only the statement about log canonicity. The other is proved similarly.

Since X' is not log canonical, there is a prime divisor E' over X' with discrepancy smaller than -1. For this E', we apply Lemma 2.4, and write the resulting diagram as follows:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[f]{} & X \end{array}$$

We denote $\mathrm{cent}_{Y'}(E')$ again by E'. Let E be the closure of $\eta(E')$. The E' and E are prime divisors. We put

$$\begin{aligned} K_Y &= f^*K_X + aE + F, \\ K_{Y'} &= \eta^*K_Y + bE' + G', \\ \eta^*E &= tE' + H', \end{aligned}$$

where F, G', H' are divisors not containing E, E', and $a, b \in \mathbb{Q}$. Since π is étale in codimension one, $K_{X'} = \pi^*K_X$. We get

$$\begin{aligned} K_{Y'} &= \eta^*f^*K_X + (at+b)E' + \eta^*F + G' + aH' \\ &= (f')^*\pi^*K_X + (at+b)E' + \eta^*F + G' + aH' \\ &= (f')^*K_{X'} + (at+b)E' + \eta^*F + G' + aH' \end{aligned}$$

By assumption, $at + b < -1$. Hence $a < -\frac{b+1}{t}$. Since t is the ramification index of η along E', by [7, 2.41], $b \geq t-1$. Therefore we get $a < -1$, which shows that X is not log canonical. □

The following lemma is generalization of a well-known fact on Galois coverings to non-Galois ones (for instance, see [7]).

Lemma 2.6. *Let k be a field of positive characteristic p. Let $\pi\colon X' \to X$ is a (not necessarily Galois) finite dominant morphism of degree n between normal $\mathbb{Q}$-Gorenstein varieties over k. Suppose that π is étale in codimension one. If $n \notin p\mathbb{Z}$ and X' is log terminal, then X is log terminal.*

Proof. For a birational map $f\colon Y \to X$ from a normal variety Y, Let Y' be the normalization of the component of $X' \times_X Y$ dominating X'. Let $\rho\colon Y' \to Y$ and $f'\colon Y' \to X'$ be natural morphisms. We get the following diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[f]{} & X \end{array}$$

Fix an f-exceptional prime divisor E. When we write ρ^*E as $\sum_i r_i D_i$ with prime divisors D_i, we have $\sum_i r_i[D_i : E] = n$ where $[D_i : E]$ is the degree of $D_i \to E$. Since $n \notin p\mathbb{Z}$, one of the r_i is not divisible by p. Let E' be a prime divisor with such a coefficient r. We write

$$\begin{aligned} K_Y &= f^*K_X + aE + F, \\ K_{Y'} &= \rho^*K_Y + bE' + G', \\ \rho^*E &= rE' + H', \end{aligned}$$

where F, G', H' are divisors not containing E, E'. Since π is étale in codimension one, $K_{X'} = \pi^*K_X$. Hence we get

$$\begin{aligned} K_{Y'} &= \rho^*f^*K_X + (ar+b)E' + \rho^*F + G' + aH' \\ &= (f')^*\pi^*K_X + (ar+b)E' + \rho^*F + G' + aH' \\ &= (f')^*K_{X'} + (ar+b)E' + \rho^*F + G' + aH'. \end{aligned}$$

Since X' is log terminal, $ar + b > -1$. Since r is the ramification index of ρ and ρ is tame, by [7, 2.41], $b = r - 1$. Therefore,

$$a > \frac{-1-b}{r} = -1.$$

Hence X is log terminal. □

Proof of Theorem.2.3. We regard G as a subgroup of $\mathrm{GL}_n(k)$. We have only to consider that $\sharp G \in 3\mathbb{Z}$. From Sylow's theorem, G has a 3-Sylow group H. We have $\sharp H = 3^r$ from assumption.

Firstly, we consider the case that $\sharp G \in 3\mathbb{Z} - 9\mathbb{Z}$. Then H is a cyclic group. From [15, Corollary 6.25], the quotient variety $X' := \mathbb{A}^3/H$ is canonical. Let $\pi\colon X' \to X := \mathbb{A}^3/G$ be the canonical morphism. Then π is étale in codimension one. Therefore, X is log terminal by Lemma 2.6.

Next, we consider the case that $\sharp G \in 9\mathbb{Z}$. Since H is a 3-group, the center $Z(H)$ is not trivial. Take a $R \in Z(H)$ which order is three. Then we may assume that

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

since R is not a pseudo-reflection. We denote the centralizer of R in $\mathrm{SL}_n(k)$ by $C(R)$. For $X = [x_{ij}] \in C(R)$, we have

$$\begin{bmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{11} & x_{12} & x_{13} \end{bmatrix} = RX = XR = \begin{bmatrix} x_{13} & x_{11} & x_{12} \\ x_{23} & x_{21} & x_{22} \\ x_{33} & x_{31} & x_{32} \end{bmatrix},$$

it implies that

$$
\begin{aligned}
x_{11} &= x_{22} = x_{33}, \\
x_{12} &= x_{23} = x_{31}, \\
x_{13} &= x_{21} = x_{32}.
\end{aligned}
$$

Hence $C(R)$ is $(kI + kR + kR^2) \cap \mathrm{SL}_n(k)$. If $X = aI + bR + cR^2 \in C(R)$,

$$X^3 = (aI + bR + cR^2)^3 = (aI)^3 + (bR)^3 + (cR^2)^3 = (a^3 + b^3 + c^3)I.$$

Since $\det X = 1$, $a^3 + b^3 + c^3 = 1$. Therefore, all the elements of $C(R)$ are of order three. Now, since $H \subset C(R)$, H has a subgroup H' which is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$.

By Theorem 2.1, the quotient variety $X' := \mathbb{A}^3/H'$ is not log canonical. We consider the canonical morphism $\pi \colon X' \to X := \mathbb{A}^3/G$. Then π is étale in codimension one. By Proposition 2.5, X is not log canonical. □

3 Crepant resolution and Euler characteristic

In this section, we fix an algebraically field k of characteristic three. Fix l be a prime number except characteristic of k. We define *the l-adic Euler characteristic* $\chi(X)$ for k-variety X by the alternaving sum of the dimensions of the l-adic étale cohomology with compact support of X:

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}_l} H^i_{et,c}(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

Theorem 3.1. *Let G be a small finite subgroup of $\mathrm{SL}_3(k)$ which is written as $H \rtimes G'$ where H is a tame Abelian group and G' is the cyclic group $\mathbb{Z}/3\mathbb{Z}$ or the symmetric group $\mathfrak{S}_3$. If G acts on the affine space $\mathbb{A}^3 = \mathbb{A}^3_k$ canonically, then the quotient variety $X = \mathbb{A}^3/G$ have a crepant resolution $Y \to X$. Moreover, we have the following formulas:*

$$\chi(Y) = \begin{cases} \sharp\mathrm{Conj}(G) & (G_1 = \mathbb{Z}/3\mathbb{Z}) \\ \sharp\mathrm{Conj}(G) + 3 & (G_1 = \mathfrak{S}_3) \end{cases}$$

Note that $\chi(Y)$ does not depend on the choice of crepant resolution $Y \to X$.

We prove this theorem along the following strategy. We put $Y := \mathbb{A}^3/H$. We consider the following diagram:

$$
\begin{array}{ccccc}
 & & & & \widetilde{X} \\
 & & & & \downarrow \\
 & & \widetilde{Y} & \longrightarrow & X' \\
 & & \downarrow & & \downarrow \\
\mathbb{A}^3 & \longrightarrow & Y & \longrightarrow & X
\end{array}
$$

Since H is a tame Abelian group, we can construct Y as a toric variety. We take a toric crepant resolution $\widetilde{Y} \to Y$ such that the G'-action on Y lifts to $\widetilde{Y}$. Let X' be the quotient variety $\widetilde{Y}/G'$. Then we can take a crepant resolution $\widetilde{X} \to X'$. The composition $\widetilde{X} \to X' \to X$ is a crepant resolution of X. The Euler characteristic of $\widetilde{X}$ can be computed from this construction.

Firstly, we construct Y as follows. Since H is a tame Abelian subgroup of $\mathrm{SL}_3(k)$, we may assume that all the elements of H are diagonal matices. Let r be their maximal order. Any $h \in H - \{I\}$ has the form

$$\begin{bmatrix} \zeta_r^a & 0 & 0 \\ 0 & \zeta_r^b & 0 \\ 0 & 0 & \zeta_r^c \end{bmatrix}$$

where ζ_r is a fixed primitive root of unity in k and a, b, c are integers satifying $0 \le a, b, c \le r-1$ and $a+b+c = r$ or $2r$. We denote such $h \in H$ by $\frac{1}{r}[a, b, c]$. By this notation, we also regard elements H as points in $\mathbb{R}^3$. Let Γ be the lattice generated by the all elements of H and $\boldsymbol{e}_x = [1, 0, 0], \boldsymbol{e}_y = [0, 1, 0], \boldsymbol{e}_z = [0, 0, 1]$. Note that H is isomorphic to the quotient group $\Gamma/\mathbb{Z}^3$. We define Y to be the toric variety defined by the lattice Γ and the cone $\mathbb{R}^3_{\ge 0}$.

3.1 The case $G' \cong \mathbb{Z}/3\mathbb{Z}$

Now, we consider the case that $G' \cong \mathbb{Z}/3\mathbb{Z}$. We show that G' acts on Y by toric automorphisms.

Lemma 3.2. *The G'-action on Y gives a injection of G' into the set of the automorphisms of Y as a toric variety. Its image is generated by the morphism corresponding to the automorphism of Γ defined by*

$$\boldsymbol{e}_x \mapsto \boldsymbol{e}_y \mapsto \boldsymbol{e}_z \mapsto \boldsymbol{e}_x.$$

Proof. It is enough to show that G' is generated by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\sigma \in G'$ be a generator of G'. Take an element $\frac{1}{r}[a, b, c] \in H - \{I\}$. Since H is a normal subgroup of G, we can write

$$\sigma \frac{1}{r}[a, b, c] \sigma^{-1} = \frac{1}{r}[a', b', c'].$$

Put $\sigma = [s_{ij}]_{i,j=1}^3$. Then

$$\begin{bmatrix} \zeta_r^a s_{11} & \zeta_r^b s_{12} & \zeta_r^c s_{13} \\ \zeta_r^a s_{21} & \zeta_r^b s_{22} & \zeta_r^c s_{23} \\ \zeta_r^a s_{31} & \zeta_r^b s_{32} & \zeta_r^c s_{33} \end{bmatrix} = \begin{bmatrix} \zeta_r^{a'} s_{11} & \zeta_r^{a'} s_{12} & \zeta_r^{a'} s_{13} \\ \zeta_r^{b'} s_{21} & \zeta_r^{b'} s_{22} & \zeta_r^{b'} s_{23} \\ \zeta_r^{c'} s_{31} & \zeta_r^{c'} s_{32} & \zeta_r^{c'} s_{33} \end{bmatrix}.$$

If $a \notin \{a', b', c'\}$, the first column of σ is zero. It contradicts the fact that σ is invertible. Hence $a \in \{a', b', c'\}$. Similarly, we get $\{a, b, c\} = \{a', b', c'\}$. If $a = b = c$, since $a + b + c \in r\mathbb{Z}$, $3a \in r\mathbb{Z}$. Since $r \notin 3\mathbb{Z}$ and $0 \leq a \leq r - 1$, we get $a = 0$. That contradicts $\frac{1}{r}[a, b, c] \neq I$. Hence, permuting coordinates if necessary, we may assume that $a \neq b$ and $a \neq c$. If $a = a'$, then $s_{12} = s_{13} = s_{21} = s_{31} = 0$. Hence

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & \sigma' \end{bmatrix}.$$

where $\sigma' \in \mathrm{SL}_2(k)$ and it has order three. This implies that σ is a pseudo-reflection, which is impossible since G is small. Hence $a \neq a'$. Therefore, $a = b'$ or $a = c'$. Assume $a = b'$. We put

$$\frac{1}{r}[a'', b'', c''] = \sigma \frac{1}{r}[a', b', c']\sigma^{-1}.$$

We have $\{a', b', c'\} = \{a'', b'', c''\}$. Since $a = b'$, $b' \neq a' = b''$. If $b' = a''$, since $\sigma^3 = 1$, we have

$$a = b' = a'' = b.$$

It contradicts that $a \neq b$. Hence $b' = c''$, and we get

$$\sigma = \mathrm{diag}(s_{12}, s_{23}, s_{31}) \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since it is conjugate with

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

we may assume that

$$\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Similarly if $a = c'$, we may assume that

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

□

Let $\sigma \in G$ be

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We let G act on $\mathbb{R}^3$ by permutation of coordinates, i.e,

$$[x, y, z] = [z, x, y] \cdot \sigma.$$

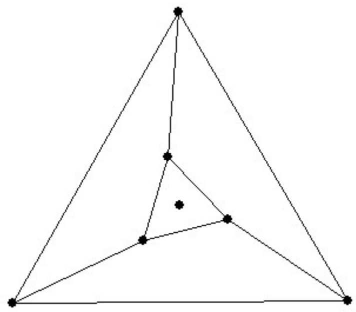

Figure 1:

For $\frac{1}{r}[a,b,c] \in H$, we have

$$\frac{1}{r}[a,b,c] \cdot \sigma = \sigma^{-1}\frac{1}{r}[a,b,c]\sigma = \frac{1}{r}[c,a,b].$$

Next we construct a toric crepant resolution $\widetilde{Y} \to Y$. Let Δ_i be the plane in $\mathbb{R}^3$ defined by

$$x + y + z = i$$

for $i = 1, 2$. Then $T := \mathbb{R}^3_{\geq 0} \cap \Delta_1$ is a triangle. Giving a subdivision of T is equivalent to giving a subdivision of the cone $\mathbb{R}^3_{\geq 0}$. We give a toric resolution $\widetilde{Y} \to Y$ by giving a subdivision of T. Firstly, we choose a point

$$\boldsymbol{a} = \frac{1}{r}[a,b,c] \in H \cap \Delta_1$$

such that the distance from the center $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ of T is minimal among all points in $H \cap \Delta_1$. Note that $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] \notin H$. We put $\boldsymbol{a}' = \boldsymbol{a}\cdot\sigma$ and $\boldsymbol{a}'' = \boldsymbol{a}\cdot\sigma^2$. We denote the triangle $\boldsymbol{a}\boldsymbol{a}'\boldsymbol{a}''$ by T_0. Obviously, T_0 is stable for the G'-action. We may assume that $a \leq b$ and $a \leq c$. We denote the squares $\boldsymbol{a}\boldsymbol{a}'\boldsymbol{e}_y\boldsymbol{e}_x, \boldsymbol{a}'\boldsymbol{a}''\boldsymbol{e}_z\boldsymbol{e}_y, \boldsymbol{a}''\boldsymbol{a}\boldsymbol{e}_x\boldsymbol{e}_z$ by S_1, S_2, S_3, respectively (Figure 1). We divide the square S_1 into triangles all whose vertices are exactly all the points in $H \cap S_1$. Since $S_1 \cdot \sigma = S_2, S_2 \cdot \sigma^2 = S_3$, the subdivision of S_1 gives the subdivisions of S_2 and S_3. Therefore, we have given the subdivision of T which is stable for G'-action. We denote the fun given by the above subdivision by Σ.

Lemma 3.3. *The fun Σ gives a crepant resolution of Y and is stable for the G'-action.*

Proof. Since the subdivision of T consists of triangles which containing no points of Γ except its vertices, the subdivision gives a toric resolution $\widetilde{Y} \to Y$ from Lemma 3.4. Since all the rays in Σ is generate a element in Δ_1, the resolution $\widetilde{Y} \to Y$ is crepant. □

Lemma 3.4. *Let $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$ be points in T. Suppose that no point of Γ is in the triangle $\boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3$ except its vertices. Then $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$ generate Γ.*

Proof. Let Γ_0 be the sublattice of Γ generated by $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$. To show $\Gamma = \Gamma_0$, we suppose that $\Gamma \neq \Gamma_0$ on the contrary. Then there exsits $\boldsymbol{x} = [x_1, x_2, x_3] \in \Gamma - \Gamma_0$. Since $\{\boldsymbol{t}_i\}_{i=1}^3$ is a basis of the vector space $\mathbb{R}^3$, we can write

$$\boldsymbol{x} = \sum_{i=1}^{3} c_i \boldsymbol{t}_i$$

by some $c_i \in \mathbb{R}$. Since

$$\boldsymbol{x} - \sum_{i=1}^{3} \lfloor c_i \rfloor \boldsymbol{t}_i = \sum_{i=1}^{3} (c_i - \lfloor c_i \rfloor) \boldsymbol{t}_i \in \Gamma - \Gamma_0,$$

we may assume that $0 \leq c_i \leq 1$ for $i = 1, 2, 3$. Since $\{\boldsymbol{t}_i\}_{i=1}^3 \subset \Delta_1$, we have

$$\sum_{i=1}^{3} c_i = \sum_{i=1}^{3} x_i = 1 \text{ or } 2.$$

Since

$$\sum_{i=1}^{3} \boldsymbol{t}_i - \boldsymbol{x} = \sum_{i=1}^{3} (1 - c_i) \boldsymbol{t}_i \in \Gamma - \Gamma_0,$$

we may assume that $\sum_{i=1}^{3} c_i = 1$. Thus $\boldsymbol{x}$ belongs to the inter section of Γ and the triangle $\boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3$. By asssumption, $\boldsymbol{x}$ is one of the vertices $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$. It contradicts that $\boldsymbol{x} \in \Gamma - \Gamma_0$. □

From Lemma 3.3, the toric variety $\widetilde{Y}$ defined by Σ is nonsingular and the toric morphism $\widetilde{Y} \to Y$ is a crepant resolution of Y. Since Σ is stable for the G'-action, the G'-action on Y lifts to $\widetilde{Y}$. Thus we can consider the quotient variety $\widetilde{Y}/G'$.

Proposition 3.5. *The quotient variety $\widetilde{Y}/G'$ has a crepant resolution $\widetilde{X} \to \widetilde{Y}/G'$.*

Proof. We denote the orbit of the torus action on $\widetilde{Y}$ corrresponding to a cone $s \in \Sigma$ by $O(s)$. Since the G'-invariant points in $\mathbb{R}^3$ form the line $\{[t, t, t] \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$, the singular locus in $\widetilde{Y}/G'$ is contained in $O(\boldsymbol{o}) \cup O(T_0)$, in particular, in the affine open subvariety defined by the cone $\mathrm{Cone}(T_0)$ corresponding to the triangle T_0. Hence the singularities of $\widetilde{Y}/G'$ are the same as the ones of the quotient variety $\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})$ for the small linear action. By [16, Corollary 6.25], $\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})$ has a crepant resolution. Therefore $\widetilde{Y}/G'$ has a crepant resolution $\widetilde{X} \to Y'/G'$. □

The composition $\widetilde{X} \to \widetilde{Y}/G' \to X$ is a crepant resolution of X. Hence we get the following theorem.

Theorem 3.6. *The quotient variety $X = \mathbb{A}^3/G$ has a crepant resolution.*

We will examine the crepant resolution $\widetilde{X} \to X$ given above in more details in order to compare the number of conjugacy classes of G and the Euler characteristic of $\widetilde{X}$.

Lemma 3.7. *We have*

$$\sharp\mathrm{Conj}(G) = \frac{\sharp H - 1}{3} + 3.$$

Proof. We denote the conjugacy class of $g \in G$ by $O_G(g)$. Since,

$$\sigma^{-1}\frac{1}{r}[a,b,c]\sigma = \frac{1}{r}[c,a,b]$$

and H is Abelian, we have

$$O_G\left(\frac{1}{r}[a,b,c]\right) = \left\{\frac{1}{r}[a,b,c], \frac{1}{r}[c,a,b], \frac{1}{r}[b,c,a]\right\}$$

for $\frac{1}{r}[a,b,c] \in H - \{I\}$. Since $r \notin 3\mathbb{Z}$ and $a+b+c = r$ or $2r$, we have $a+b+c \notin 3\mathbb{Z}$. Thus we have $a \neq b$ or $a \neq c$ or both. Hence $\frac{1}{r}[a,b,c] \neq \frac{1}{r}[c,a,b]$. Therefore, H contains $\frac{\sharp H - 1}{3} + 1$ conjugacy classes.

We show the conjugacy class $O_G(\sigma)$ containing σ is $H\sigma$. The inclusion $O_G(\sigma) \subset H\sigma$ follows from

$$\begin{aligned}
\left(\frac{1}{r}[\alpha,\beta,\gamma]\sigma^i\right)^{-1}\sigma\left(\frac{1}{r}[\alpha,\beta,\gamma]\sigma^i\right) &= \sigma^{-i}\frac{1}{r}[-\alpha,-\beta,-\gamma]\sigma\frac{1}{r}[\alpha,\beta,\gamma]\sigma^i \\
&= \sigma^{-i}\frac{1}{r}[-\alpha,-\beta,-\gamma]\frac{1}{r}[\beta,\gamma,\alpha]\sigma\sigma^i \\
&= \sigma^{-i}\left(\frac{1}{r}[\beta-\alpha,\gamma-\beta,\alpha-\gamma]\sigma\right)\sigma^i \\
&= \left(\sigma^{-i}\frac{1}{r}[\beta-\alpha,\gamma-\beta,\alpha-\gamma]\sigma^i\right)\sigma \in H\sigma.
\end{aligned}$$

To show the other inclusion, for $a,b,c \in \mathbb{Z}$ such that

$$a+b+c \equiv 0 \mod r,$$

we take $\alpha,\beta,\gamma \in \mathbb{Z}$ by

$$\begin{aligned}
\alpha &\equiv \frac{b+2c}{3} \mod r,\\
\beta &\equiv \frac{c+2a}{3} \mod r,\\
\gamma &\equiv \frac{a+2b}{3} \mod r.
\end{aligned}$$

Note that, since $r \notin 3\mathbb{Z}$, there exsits $d \in \mathbb{Z}$ such that $3d \equiv 1 \mod r$, and dividing by three in the above formulas means multiplying with d. Since $\frac{1}{r}[a,b,c] \in H$, the two matrices $\frac{1}{r}[b,c,a]$, $\frac{1}{r}[c,a,b]$ are also in H. Then

$$\frac{1}{r}[\alpha,\beta,\gamma] = \left(\frac{1}{r}[b,c,a]\frac{1}{r}[c,a,b]^2\right)^d \in H.$$

We get

$$\begin{aligned}\frac{1}{r}[\alpha,\beta,\gamma]^{-1}\sigma\frac{1}{r}[\alpha,\beta,\gamma] &= \frac{1}{r}[\beta-\alpha,\gamma-\beta,\alpha-\gamma]\sigma \\ &= \frac{1}{r}[d(2a-b-c),d(2b-c-a),d(2c-a-b)]\sigma \\ &= \frac{1}{r}[d(3a-r),d(3b-r),d(3c-r)]\sigma \\ &= \frac{1}{r}[a,b,c]\sigma.\end{aligned}$$

Thus $O_G(\sigma) \supset H\sigma$ and hence $O_G(\sigma) = H\sigma$.

Similary, we get $O_G(\sigma^2) = H\sigma^2$. Therefore,

$$\sharp\mathrm{Conj}(G) = \frac{\sharp H - 1}{3} + 3.$$

□

Theorem 3.8. *The Euler characteristic $\chi(\widetilde{X})$ of $\widetilde{X}$ is equal to the number of the conjugacy classes of G.*

Proof. Since H is tame and abelian, the Euler characteristic $\chi(\widetilde{Y})$ of $\widetilde{Y}$ is $\sharp H$. Using the decomposition of $\widetilde{Y}$ into torus orbits, we get

$$\left[\widetilde{Y}\right] = \sum_{s\in\Sigma}[O(s)]$$

in the Grothendiek ring of varieties. Since the Euler characteristic gives an additive map from the Grothendiek ring to $\mathbb{Z}$, we get

$$\sum_{s\in\Sigma}\chi(O(s)) = \sharp H.$$

If $\dim(s) \leq 2$, the Euler characteristic $\chi(O(s))$ is zero. Let Σ_3 be the set of cones of dimension three in Σ. Since a 3-dimensional cone corresponds to a torus orbit which is a point, we get

$$\sharp\Sigma_3 = \sum_{s\in\Sigma_3}\chi(O(s)) = \sharp H.$$

On the other hand, $\left[\widetilde{X}\right]$ is decomposed as

$$\left[\widetilde{X}\right] = \sum_{s\in(\Sigma-|\mathrm{Cone}(T_0)|)/G'}[O(s)] + [\widetilde{\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})}]$$

where $|\mathrm{Cone}(T_0)|$ is the set of the faces of $\mathrm{Cone}(T_0)$, $(\Sigma - |\mathrm{Cone}(T_0)|)/G'$ is the set of orbits for the G'-action on $\Sigma - |\mathrm{Cone}(T_0)|$, and $\widetilde{\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})}$ is a crepant resolution of $\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})$. Then, we have

$$\begin{aligned}\chi(\widetilde{X}) &= \sum_{s\in(\Sigma_3 - |\mathrm{Cone}(T_0)|)/G'} \chi(O(s)) + \chi(\widetilde{\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})}) \\ &= \sharp((\Sigma_3 - |\mathrm{Cone}(T_0)|)/G') + \chi(\widetilde{\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})})) \\ &= \frac{\sharp H - 1}{3} + \chi(\widetilde{\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})})).\end{aligned}$$

By [16, Corollary 6.21], $\chi(\widetilde{\mathbb{A}^3/(\mathbb{Z}/3\mathbb{Z})})) = 3$. Therefore,

$$\chi(\widetilde{X}) = \frac{\sharp H - 1}{3} + 3 = \sharp\mathrm{Conj}(G)$$

by Lemma 3.7. □

3.2 The case $G' \cong \mathfrak{S}_3$

Next we prove Theorem 3.1 in the case $G' \cong \mathfrak{S}_3$. The proof is similar to the one for the case $G' \cong \mathbb{Z}/3\mathbb{Z}$. We put $Y = \mathbb{A}^3/H$. The quotient variety Y is defined as a toric variety in the same way as in the previous case. We also assume as before that every matrix in H is diagonal. We give a toric crepant resolution of Y having a G'-action lifting the one on Y.

Lemma 3.9. *The G'-action on $\mathbb{A}^3$ is given by the subgroup of $\mathrm{SL}_3(k)$ generated by the matrices*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Proof. The group $\mathfrak{S}_3$ is generated by two elements σ, τ satisfying

$$\sigma^3 = \tau^2 = 1,\ \tau\sigma\tau = \sigma^2.$$

By Lemma 3.2, we may assume that the action of σ is defined by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We write $\tau = [\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3]$. Since $\tau\sigma\tau = \sigma^2$, we get

$$[\boldsymbol{t}_3, \boldsymbol{t}_1, \boldsymbol{t}_2] = \tau\sigma = \sigma^2\tau = [\sigma^2\boldsymbol{t}_1, \sigma^2\boldsymbol{t}_2, \sigma^2\boldsymbol{t}_3].$$

Hence $\boldsymbol{t}_2 = \sigma\boldsymbol{t}_1,\ \boldsymbol{t}_3 = \sigma^2\boldsymbol{t}_1$. Thus, we get

$$\tau = \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix}.$$

Take $\frac{1}{r}[a,b,c] \in H$ and put $\tau \frac{1}{r}[a,b,c]\tau = \frac{1}{r}[a',b',c']$. From the proof of Lemma 3.2, $a \in \{a',b',c'\}$. We may assume that $a \neq b$ and $a \neq c$. Hence one of x, y, z is not zero and the others are zero. Replacing τ by $\tau\sigma, \tau\sigma^2$ if necessary, we may assume that $z \neq 0, x = y = 0$. Since $\det \tau = -x^3 = 1$, we have $x = -1$ and hence

$$\tau = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

□

We use the symbols Γ, T in the same way as in the previous case. From the previous lemma, we may assume that

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ \tau = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

We define a G'-action on $\mathbb{R}^3$ by

$$[x,y,z] \cdot \sigma = [z,x,y],$$
$$[x,y,z] \cdot \tau = [z,y,x].$$

We denote the toric autmorphisms on $\mathbb{A}^3$ defined from the action of σ, τ by f_σ, f_τ, respectively. Since $\mathbb{A}^3$ is a toric variety, we denote $(-1,-1,-1)Q$ by $-Q$ for $Q \in \mathbb{A}^3$. Note that the G'-action on $\mathbb{A}^3$ is defined by

$$\sigma P = f_\sigma(P),\ \tau P = -f_\tau(P)$$

for $P \in \mathbb{A}^3$.

Let T_{xy} be the subset

$$\{\boldsymbol{x} = [x,y,z] \in T \cap \Gamma \mid x = y\}.$$

For $\boldsymbol{x} = [x_1, x_2, x_3] \in \mathbb{R}^3$, we define

$$d(\boldsymbol{x}) = \sum_{i=1}^{3} \left| \frac{1}{3} - x_i \right|.$$

Let $\boldsymbol{a} \in T_{xy}$ be a point satisfying the condition

$$d(\boldsymbol{a}) = \min \{d(\boldsymbol{x}) | \boldsymbol{x} \in T_{xy}\}.$$

Actually, $\boldsymbol{a}$ has the following property.

Lemma 3.10. *We have*

$$d(\boldsymbol{a}) = \min_{\boldsymbol{x} \in T \cap \Gamma} d(\boldsymbol{x}).$$

Proof. Suppose that there exists $\boldsymbol{x} \in T \cap \Gamma$ such that $d(\boldsymbol{x}) < d(\boldsymbol{a})$. We write

$$\boldsymbol{x} = \frac{1}{r}[x, y, z].$$

Since the G'-action on Γ preserves the value of d, we may assume that $x \leq y \leq z$. If two of them were equal, it would contradict the minimality of $d(\boldsymbol{a})$. Hence, we get $x < y < z$. Let $\boldsymbol{y} \in \Gamma$ be the point

$$\begin{aligned}
[1,1,1] - (\boldsymbol{x} + \boldsymbol{x} \cdot \tau) \cdot \sigma &= [1,1,1] - \frac{1}{r}[r-y, 2y, r-y] \cdot \sigma \\
&= [1,1,1] - \frac{1}{r}[r-y, r-y, 2y] \\
&= \frac{1}{r}[y, y, r-2y].
\end{aligned}$$

Since $\boldsymbol{x} \in T$, we have $x + y + z = r$. Hence, we get $x < \frac{r}{3}$, $y < \frac{r}{2}$, and $\frac{r}{3} < z$. Since $y < \frac{r}{2}$, $\boldsymbol{y} \in T_{xy}$. If $y < \frac{r}{3}$, we have $\left|\frac{r}{3} - x\right| > \left|\frac{r}{3} - y\right|$ and

$$\begin{aligned}
\left|\frac{r}{3} - z\right| - \left|\frac{r}{3} - (r-2y)\right| &= \left(z - \frac{r}{3}\right) - \left(\frac{2}{3}r - 2y\right) \\
&= z + 2y - r \\
&> z + y + x - r = 0.
\end{aligned}$$

Therefore, $d(\boldsymbol{y}) < d(\boldsymbol{x})$. On the other hand, if $y > \frac{r}{3}$, we have $\left|\frac{r}{3} - z\right| > \left|\frac{r}{3} - y\right|$ and

$$\begin{aligned}
\left|\frac{r}{3} - x\right| - \left|\frac{r}{3} - (r-2y)\right| &= \left(\frac{r}{3} - x\right) - \left(2y - \frac{2}{3}r\right) \\
&= r - (x + 2y) \\
&> r - (x + y + z) = 0.
\end{aligned}$$

Therefore, we get $d(\boldsymbol{y}) < d(\boldsymbol{x})$ again. Since $\boldsymbol{y} \in T_{xy}$, we have

$$d(\boldsymbol{a}) \leq d(\boldsymbol{y}) < d(\boldsymbol{x}) < d(\boldsymbol{a}).$$

This is a contradiction. □

To construct a subdivision of T giving a crepant toric resolution of Y, We show the following lemma.

Lemma 3.11. *We have*

$$T \cap \Gamma = \left\{ \frac{1}{r}[x, y, z] \,\middle|\, x, y, z \in \mathbb{Z}, 0 \leq x, y, z \leq r, x + y + z = r \right\}.$$

Proof. We put

$$\Gamma' = \left\{ \frac{1}{r}[x, y, z] \,\middle|\, x, y, z \in \mathbb{Z}, x + y + z \equiv 0 \mod r \right\}.$$

Note that the right hand of the desired equation is $T \cap \Gamma'$. We can write $\boldsymbol{a} = \frac{1}{r}[a, a, b]$. Let T_0 be the triangle whose vertices are $\boldsymbol{a}, \boldsymbol{a} \cdot \sigma, \boldsymbol{a} \cdot \sigma^2$. By Lemma 3.10, T_0 has only vertices as points of Γ. From Lemma 3.4, $\boldsymbol{a}, \boldsymbol{a} \cdot \sigma, \boldsymbol{a} \cdot \sigma^2$ form a $\mathbb{Z}$-basis of Γ, equivalently, $\boldsymbol{a}, \sigma^{-1}\boldsymbol{a}\sigma, \sigma\boldsymbol{a}\sigma^{-1}$ genarate H. Hence, r is the order of $\boldsymbol{a}$. Then

$$\gcd(a, b, r) = 1.$$

Let $d = \gcd(a, r)$. Since $2a + b = r$, we have $b = r - 2a$, in particular, b is divisible by d. Hence $\gcd(a, b, r) = d$. Therefore $\gcd(a, r) = 1$. Since

$$\det \begin{bmatrix} a & a \\ b & a \end{bmatrix} = a(a - b) = a(3a - r)$$

Since $\gcd(a, r) = 1$ and r is not divisible by three, this is invertible on $\mathbb{Z}/r\mathbb{Z}$. Hence $\boldsymbol{a}, \boldsymbol{a} \cdot \sigma, [0, 0, 1]$ generates Γ'. Since $[0, 0, 1] \in \Gamma$, Γ' is generated by $\boldsymbol{a}, \boldsymbol{a} \cdot \sigma, \boldsymbol{a} \cdot \sigma^2$. Therefore

$$T \cap \Gamma = \left\{ \frac{1}{r}[x, y, z] \middle| x, y, z \in \mathbb{Z}, 0 \leq x, y, z \leq r, x + y + z = r \right\}.$$

□

We give a subdivision of T by subdividing T with $3(r - 1)$ lines

$$x = \frac{i}{r}, y = \frac{i}{r}, z = \frac{i}{r} \quad (1 \leq i < r)$$

in Δ (Figure 2). Let Σ be a fan corresponding the above subdivision of T.

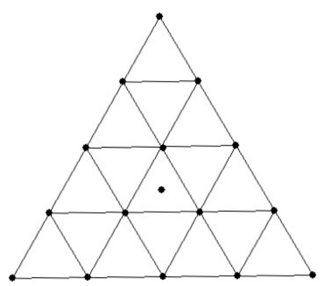

Figure 2:

Lemma 3.12. *The fan Σ gives a toric crepant resolution of Y with a G'-action lifting the one on Y.*

Proof. Let $\widetilde{Y}$ be the toric variety defined by Σ and let π be the canonical morphism $\widetilde{Y} \to Y$. Since any triangle of subdivision of T has no points of Γ except its vertices, the vertices of any triangle of subdivision are generate Γ from Lemma 3.4. Therefore, $\widetilde{Y}$ is nonsingular, and hence π is a resolution of Y. Since all the rays in Σ are generated by points of $\Gamma \cap \Delta_1$, π is crepant resolution. □

We denote the crepant resolution of Y given by Lemma 3.12 by $\pi : \widetilde{Y} \to Y$ as in the proof of Lemma 3.12.

Proposition 3.13. *The quotient variety $\widetilde{Y}/G'$ has a crepant resolution $\widetilde{X} \to \widetilde{Y}/G'$.*

Proof. Now, $G' = \langle\sigma\rangle \rtimes \langle\tau\rangle$. We have a crepant resolution $Y' \to \widetilde{Y}/\langle\sigma\rangle$ in Proposition 3.5. We can extends the action of τ to Y'. Let c be a 3-dimensional cone stable for the τ-action on Γ. On the open affine toric variety $U_c = \mathrm{Spec}\, k[s,t,u]$ of Y', τ acts by the morphism corresponding to

$$s \mapsto -u,\ t \mapsto -t,\ u \mapsto -s.$$

Since any cone in Σ which stable with respect to the τ-action is a face of a 3-dimensional cone, the fixed locus in Y' about $\langle\tau\rangle$-action is pure 1-dimensional. From [20, Theorem 5.5], $Y'/\langle\tau\rangle$ has a crepant resolution $\widetilde{x} \to Y'/\langle\tau\rangle$. Therefore, we can get the crepant resolution of $\widetilde{Y}/G'$ by the composition $\widetilde{X} \to Y'/\langle\tau\rangle \to \widetilde{Y}/G'$. □

Hence we get the following theorem.

Theorem 3.14. *The quotient singularity $\mathbb{A}^3/G$ has a crepant resolution.*

We compute the Euler characteritic of the crepant resolution $\widetilde{X}$ of $\mathbb{A}^3/G$.

Theorem 3.15. *We have*

$$\chi(\widetilde{X}) = \frac{(r-1)(r-2)}{6} + 2r + 4.$$

Proof. The G'-action on Γ gives the G'-action on $\mathbb{R}^3$. The subdivision given in Lemma 3.12 is G'-stable. We can separate the fan Σ defining $\widetilde{Y}$ into the G'-orbits with respect to the G'-action on $\mathbb{R}^3$. Let C_0 be the cone in Σ corresponding to the triangle T_0 given in the proof of Lemma 3.12. Let Σ_τ be the cones stable for the action of τ except C_0 and its faces. Let Σ' be the set of representatives of $(\Sigma - |C_0|)/G' - \Sigma_\tau$ where $|C_0|$ is the set of faces of C_0 and we choose the representatives of $(\Sigma - |C_0|)/G'$ containing Σ_τ. Then we have

$$\widetilde{Y}/G' = U_{C_0}/G' \sqcup \bigsqcup_{c\in\Sigma_\tau} O(c)/\langle\tau\rangle \sqcup \bigsqcup_{c\in\Sigma'} O(c).$$

Hence

$$\widetilde{X} = \pi^{-1}(U_{C_0}/G') \sqcup \bigsqcup_{c\in\Sigma_\tau} \pi^{-1}(O(c)/\langle\tau\rangle) \sqcup \bigsqcup_{c\in\Sigma'} \pi^{-1}(O(c)).$$

where $\pi : \widetilde{X} \to Y'/G'$ is the crepant resolution. By [20, Theorem 6.3], we have

$$\chi(\pi^{-1}(U_{C_0}/G')) = 6.$$

Since π is isomorphism on $O(c)$ for each $c \in \Sigma'$,

$$\chi\left(\bigsqcup_{c\in\Sigma'} \pi^{-1}(O(c))\right) = \sharp\{c \in \Sigma' \mid \dim(c) = 3\}.$$

We put

$$\Sigma_3 = \{c \in \Sigma \mid \dim(c) = 3\}.$$

From the construction of the subdivision of T, $\sharp\Sigma_3 = r^2$. Since an element of Σ_3 stable with respect to the σ-action is C_0 only, $\sharp((\Sigma_3 - C_0)/\langle\sigma\rangle) = \frac{r^2-1}{3}$. Since τ acts freely on $(\Sigma_3 - C_0)/\langle\sigma\rangle - \Sigma_\tau$, we get

$$\sharp\{c \in \Sigma' \mid \dim(c) = 3\} = \frac{1}{2}\left(\frac{r^2-1}{3} - (r-1)\right) = \frac{(r-1)(r-2)}{6}.$$

Lastly, we compute $\chi(\pi^{-1}(O(c)/\langle\tau\rangle))$ for $c \in \Sigma_\tau$. If $\dim(c) = 3$, $O(c)$ is a point on singular loci. Hence $\pi^{-1}(O(c)/\langle\tau\rangle)$ is $\mathbb{P}^1$ (see [20, Theorem 5.5]). Therefore,

$$\chi(\pi^{-1}(O(c)/\langle\tau\rangle)) = 2.$$

From construction of Σ, each $c \in \Sigma_\tau$ is a face of a 3-dimensional cone $c' \in \Sigma_\tau$. Then $O(c) \subset U_{c'}$. If c' is spaned by $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in T$, we may assume that

$$\boldsymbol{x}\cdot\tau = \boldsymbol{y},\ \ \boldsymbol{y}\cdot\tau = \boldsymbol{x},\ \ \boldsymbol{z}\cdot\tau = \boldsymbol{z}.$$

We denote the coordinate of $U_{c'}$ corresponding to $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ by x, y, z. The τ-action on $U_{c'}$ is defined by

$$(x, y, z) \to (-y, -x, -z).$$

If $\dim(c) = 2$, the orbit $O(c)$ is defined by $x = y = 0, z \neq 0$ in $U_{c'}$. Hence the τ-action is free. Hence

$$\chi(\pi^{-1}(O(c)/\langle\tau\rangle)) = \frac{\chi(O(c))}{2} = 0.$$

If $\dim(c) = 1$, the orbit $O(c)$ is defined by $z = 0, x \neq 0, y \neq 0$ on $U_{c'}$. Hence the fixed locus F is the line defined by $x = -y$ on $O(c)$. Each fiber of points in F is $\mathbb{P}^1$. Therefore,

$$\chi(\pi^{-1}(O(c)/\langle\tau\rangle)) = \chi(O(c) - F)/2 + \chi(F) \times \chi(\mathbb{P}^1) = 0$$

We get

$$\chi(\bigsqcup_{c\in\Sigma_\tau} \pi^{-1}(O(c)/\langle\tau\rangle)) = 2\sharp\{c \in \Sigma_\tau \mid \dim(c) = 3\} = 2(r-1).$$

Therefore

$$\chi(\widetilde{X}) = 6 + 2(r-1) + \frac{(r-1)(r-2)}{6} = \frac{(r-1)(r-2)}{6} + 2r + 4.$$

□

From the proof of Lemma 3.11,

$$\Gamma = \left\{ \frac{1}{r}[x, y, z] \middle| x, y, z \in \mathbb{Z}, x + y + z \equiv 0 \mod r \right\}.$$

Since $H = \Gamma/\mathbb{Z}$, H is

$$\left\{ \frac{1}{r}[x, y, z] \middle| 0 \leq x, y, z < r, x + y + z \equiv 0 \mod r \right\}.$$

From this, we get

Corollary 3.16. *We have*

$$\sharp\mathrm{Conj}(G) = \frac{(r-1)(r-2)}{6} + 2r + 1.$$

Hence the Euler characteristic computed in Theorem 3.15 is $\sharp\mathrm{Conj}(G) + 3$.

Proof. Since H is Abelian, if $h \in H$ and $ghg^{-1} \neq h$ then $g \notin H$. If this is the case, then ghg^{-1} is given as a permutation of coordinate of h regarding $h \in \Gamma$. Hence the normal subgroup H of G has the conjugacy classes represented by the identity element or

$$\frac{1}{r}[a, b, c] \ (0 \leq a < b < c < r, a + b + c \equiv 0 \mod r),$$

which is the case the coordinates are mutually distinct, or

$$\frac{1}{r}[a, (r-a)/2, (r-a)/2] \ (0 \leq a < r, a \equiv r \mod 2),$$

which is the case two of coordinates are equal and the sum of coordinates is one, or

$$\frac{1}{r}[2a, r-a, r-a] \ (0 < a < r/2),$$

which is the case two of coordinates are equal and the sum of coordinates is two.

The numbers of these elements with the last threee expressions are

$$\frac{(r-1)(r-2)}{6}, \ \left\lfloor \frac{r}{2} \right\rfloor, \ \left\lceil \frac{r}{2} \right\rceil - 1$$

respectively. Hence the number of conjugacy classes contained in H is

$$\frac{(r-1)(r-2)}{6} + r.$$

From the proof of Lemma 3.7, $H\sigma \subset O_G(\sigma)$. Since $\tau\sigma\tau = \sigma^2$, $\sigma^2 \in O_G(\sigma)$. From the proof of Lemma 3.7, $H\sigma^2 \subset O_G(\sigma)$. Hence the conjugacy class of σ contains $H\sigma \cup H\sigma^2$. Since $\tau H\sigma\tau = H\sigma^2$, $\tau H\sigma^2\tau = H\sigma$, we have $O_G(\sigma) = H\sigma \cup H\sigma^2$.

Since $\sigma^{-1}\tau\sigma = \sigma\tau$, for $h \in H$,

$$\begin{aligned} h\sigma\tau &= h\sigma^{-1}\tau\sigma \\ &= \sigma^{-1}(\sigma h\sigma^{-1}\tau)\sigma \end{aligned}$$

and

$$\begin{aligned} h\sigma^2\tau &= h\sigma\tau\sigma^{-1} \\ &= \sigma(\sigma^{-1}h\sigma\tau)\sigma^{-1}. \end{aligned}$$

Hence $h\sigma\tau, h\sigma^2\tau$ are conjugate to $h'\tau$ for some $h' \in H$. For $h = \frac{1}{r}[a,b,c]$,

$$h\tau h^{-1} = \frac{1}{r}[a,b,c]\left(\tau\frac{1}{r}[-a,-b,-c]\tau\right)\tau = \frac{1}{r}[a-c,0,c-a]\tau.$$

In particular, $h\tau h^{-1} = \frac{1}{r}[-1,0,1]\tau$ when $h = \frac{1}{r}[0,r-1,1] \in H$. Hence, the other conjugacy classes are represented by τ or

$$\frac{1}{r}[0,a,r-a]\tau \ (0 < a < r).$$

Therefore,

$$\sharp\mathrm{Conj}(G) = \frac{(r-1)(r-2)}{6} + 2r + 1.$$

□

4 Stringy point-counting and mass formulas

In this section, we compute the numbers of $\mathbb{F}_q$-points on some of crepant resolutions constructed above, where q is a power of three. We can compute those numbers from explicit description of resolutions. But now, we take alternative approach using stringy point-count. Let $k = \mathbb{F}_q$ and let $K = k((t))$. The field K has the valuation v_K. We denote the valuation ring of K by $\mathcal{O}_K$. For an $\mathcal{O}_K$-variety X, we denote its *stringy point-count* by $\sharp_{st}X$, which is defined as the volume of X with respect to a certain p-adic measure (for more detail, see [17]). If X has a crepant resolution $Y \to X$, we have

$$\sharp Y(k) = \sharp_{st}X.$$

Hence we can use this invariant for counting the k-points on a crepant resolution.

For a finite group G, a finite étale K-algebras M of degree $\sharp G$ endowed with a G-action and satisfying $M^G = \mathcal{O}_K$ is called a *G-étale K-algebra.* A *homomorphism* of G-étale K-algebras is a G-equivalent K-algebra homomorphism. We denote the set of the isomorphism classes of G-étale K-algebras by G-$\acute{\mathrm{E}}\mathrm{t}(K)$. Note that each $M \in G$-$\acute{\mathrm{E}}\mathrm{t}(K)$ is written as

$$M = L^{\oplus n}$$

for some H-extension L/K where H is a subgroup of G.

Let $V = \mathbb{A}^n_{\mathcal{O}_K} = \mathrm{Spec}\ \mathcal{O}_K[x_1, \ldots, x_n]$. For a linear G-action on V, we consider the quotient variety $X = V/G$. For $M \in G\text{-Ét}(K)$, we define its *tuning module* Ξ_M by

$$\Xi_M = \mathrm{Hom}^G_{\mathcal{O}_K}\left(\bigoplus_{i=1}^{n} \mathcal{O}_K x_i, \mathcal{O}_M\right).$$

This is a free $\mathcal{O}_K$-module of rank n. The *v-function* v_V is defined by

$$\boldsymbol{v}_V(M) = \frac{f_L}{\sharp G}\mathrm{length}_{\mathcal{O}_K} \frac{\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, \mathcal{O}_M)}{\mathcal{O}_M \cdot \Xi_M}$$

in the errata of [14], where L is a field such that $M = L^{\oplus n}$ and f_L is the inertia degree of L/K. When we have an $\mathcal{O}_M$-basis φ_i $(i = 1, 2, \ldots, n)$ of $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, \mathcal{O}_M)$ and an $\mathcal{O}_K$-basis $\psi_j = \sum_i c_{ij}\varphi_i$ $(j = 1, 2, \ldots, n)$ of Ξ_M,

$$\boldsymbol{v}_V(M) = v_K(\det(c_{ij})) \tag{1}$$

where v_K is the valuation of K. By [17, Corollary 7.5, Proposition 8.5], we have the following formula:

Theorem 4.1. *With the above notation,*

$$\sharp_{st} X = \sum_{M \in G\text{-}\acute{E}t(K)} \frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)}$$

where H_M is the stabilizer subgroup of a connected component of $\mathrm{Spec}(M)$*.*

We compute the right hand side of the equality of the theorem for some of the cases discussed in the last section.

4.1 The case $G \cong (\mathbb{Z}/l\mathbb{Z}) \rtimes (\mathbb{Z}/3\mathbb{Z})$

Let G be a group isomorphic to $H \rtimes G'$ where $H = \mathbb{Z}/l\mathbb{Z}$ with l a prime different from three and $G' = \mathbb{Z}/3\mathbb{Z}$. Then the set $G\text{-Ét}(K)$ is divided into three parts:

$$S_1 = \{K^{\oplus 3l}\},$$
$$S_2 = \{L^{\oplus l} \in G\text{-Ét}(K) \mid L/K : G'\text{-extension}\},$$
$$S_3 = \{L^{\oplus 3} \in G\text{-Ét}(K) \mid L/K : H\text{-extension}\}.$$

Case $M \in S_1$: When $M = K^{\oplus 3l}$, since M is unramified, we have $\boldsymbol{v}_V(M) = 0$.

Case $M \in S_2$: When $M = L^{\oplus l} \in S_2$, since $\boldsymbol{v}_V$ has convertibility from [13, Lemma 3.4], we have

$$\boldsymbol{v}_V(M) = \boldsymbol{v}_{V|_{G'}}(L)$$

where $V|_{G'}$ is the G'-representation obtained by restricting the G-representation V. The G'-extensions L/K are controlled by the Artin-Schreier theory. From this theory, L is defined by

$$L = K[x]/(x^3 - x - a)$$

with $a \in (k/\mathcal{P}(k)) \oplus \bigoplus_{3\nmid j>0} kt^{-j}$ where $\mathcal{P}$ is the Artin-Schreier map defined by $\mathcal{P}(x) = x^3 - x$. We put $j = -v_K(a)$. Since $V|_{G'}$ is a permutation representation, if $j > 0$, we get

$$\boldsymbol{v}_{V|_{G'}}(L) = \frac{d_{L/K}}{2} = \frac{(j+1)(3-1)}{2} = j+1$$

by [18, Lemma 11.1] and [11, Proposition 7, page 50]. Let

$$S_{2,m} = \{M \in S_2 \mid \boldsymbol{v}_V(M) = m\}$$

From the computation of $\boldsymbol{v}_{V|_{G'}}$, $S_{2,m}$ is not empty if and only if $m = 0$ or $m = j+1$ for $j \in \mathbb{Z}_{>0} - 3\mathbb{Z}$. If $m = 0$, then $v_K(a) = 0$. The number of such a is $\sharp(k/\mathcal{P}(k)) - 1 = 2$. If $m = j+1$, then $v_K(a) = -j$. Such a is written as

$$a = a_0 + \sum_{3\nmid i>0, i\leq j} a_i t^{-i}.$$

The number of choice of the coordinates a_i is $3(q-1)q^{j-1-\lfloor \frac{j}{3} \rfloor}$. Hence, we have

$$\sharp S_{2,m} = \begin{cases} 2 & (m=0) \\ 3(q-1)q^{m-2-\lfloor \frac{m-1}{3} \rfloor} & (m-1 \notin 3\mathbb{Z}, m>0) \\ 0 & (otherwise) \end{cases} . \qquad (2)$$

Case $M \in S_3$: Lastly we consider the case $M = L^{\oplus 3} \in S_3$. We have

$$\boldsymbol{v}_V(M) = \boldsymbol{v}_{V|_H}(L).$$

We now put an additional assumption $q-1$ divided by l, it means k has the l-th roots of unity.

Then the H-extensions L/K are controlled by the Kummer theory. Let $\mu \in k^\times$ be a generator of $k^\times$ and let $\zeta_l \in k$ be a primitive l-th root of unity. Then the multiplicative group $K^\times/(K^\times)^l$ is generated by μ and t. Hence L is generated by $\alpha \in L$ satisfying $\alpha^l = f$ where $f = \mu$ or $f = \mu^i t$ $(i = 0, 1, \dots, l-1)$. The H-action on L is defined by choosing $h \in H$ such that

$$\alpha \cdot h = \zeta_l \alpha.$$

We define the *age* of $h \in H$ by

$$\mathrm{age}(h) = \frac{1}{l}[a, b, c]$$

if $h = \frac{1}{l}[a, b, c]$.

Lemma 4.2. *For L as above, we have*

$$\boldsymbol{v}_{V|_H}(L) = \mathrm{age}(h) v_K(f).$$

Proof. We denote $h = \frac{1}{l}[a_1, a_2, a_3]$. Let $\varphi_1, \varphi_2, \varphi_3$ be the dual basis of x_1, x_2, x_3, i.e., φ_i are $\mathcal{O}_K$-linear maps from $\mathcal{O}_K[\boldsymbol{x}]_1$ to $\mathcal{O}_L$ defined by

$$\varphi_i(x_j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

Then

$$\begin{aligned} \sum_{i=1}^{3} c_i\varphi_i \in \Xi_L &\Leftrightarrow 1 \leq \forall j \leq 3, (\sum_{i=1}^{3} c_i\varphi_i)(x_j \cdot h) = (\sum_{i=1}^{3} c_i\varphi_i)(x_j) \cdot h \\ &\Leftrightarrow 1 \leq \forall j \leq 3, \zeta_l^{a_j} c_j = c_j \cdot h \\ &\Leftrightarrow 1 \leq \forall j \leq 3, \exists d_j \in \mathcal{O}_K, c_j = \alpha^{a_j} d_j. \end{aligned}$$

Therefore, the maps $\{\alpha^{a_i}\varphi_i\}$ are a basis of Ξ_L. Hence

$$\boldsymbol{v}_{V|_H}(L) = v_K(\det \operatorname{diag}(\alpha^{a_1}, \alpha^{a_2}, \alpha^{a_3})) = v_K(f^{\frac{a_1+a_2+a_3}{l}}) = \operatorname{age}(h) v_K(f).$$

□

When $M = L_1 \oplus L_2 \oplus L_3$, we may assume that $L_1\sigma = L_2$. Then, if the H-action on L_1 corresponds to h, the actions on L_2, L_3 correspond to $\sigma^{-1}h\sigma, \sigma h\sigma^{-1}$ respectively. Hence the G-action on M corresponds to the non-trivial G-conjugacy classes of elements of H. Let

$$S_{3,m} = \{M \in S_3 \mid \boldsymbol{v}_V(M) = m\}.$$

If $M \in S_{3,m}$, then the extension L/K correspoinding to M is determined by $f = \mu$. The number of G-action on M is $\sharp\mathrm{Conj}(H) - 1 = \frac{l-1}{3}$. Hence $S_{3,0} = \frac{l-1}{3}$. If $M \in S_{3,1}$, then the extension L/K is determined by $f = \mu^i t$. The number of G-action on M is equal to the number of the conjugacy classes containing age one elements. Since the inverses of age one elements have age two, the number of age one elements is $\frac{\sharp H - 1}{2}$. Hence $S_{3,1} = \frac{l(l-1)}{6}$. The number $\sharp S_{3,2}$ is computed similarly. We get

$$\sharp S_{3,m} = \begin{cases} \frac{l-1}{3} & (m = 0) \\ \frac{l(l-1)}{6} & (m = 1, 2) \\ 0 & (\mathit{otherwise}) \end{cases} . \tag{3}$$

Theorem 4.3. *Let l be a prime different from three and let q be a power of three such that $q-1$ is divisible by l. Let G be a small finite subgroup of $SL_3(k)$. Suppose that G has a normal group $H \cong \mathbb{Z}/l\mathbb{Z}$ and $G/H \cong \mathbb{Z}/3\mathbb{Z}$. Then we have*

$$\sum_{M \in G\text{-}\acute{E}t(K)} \frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} = q^3 + \left(2 + \frac{l-1}{6}\right) q^2 + \frac{l-1}{6} q.$$

Proof. We compute as

$$
\begin{aligned}
\sum_{M\in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} \frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} &= \sum_{M\in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} \\
&= \sum_{M\in S_1} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(1)} + \sum_{M\in S_2} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(G')} + \sum_{M\in S_3} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(H)} \\
&= \frac{q^3}{3p} + \sum_{m=0}^{\infty} \sum_{M\in S_{2,m}} \frac{q^{3-m}}{3} + \sum_{m=0}^{2} \sum_{M\in S_{3,m}} \frac{q^{3-m}}{l}. \qquad (4)
\end{aligned}
$$

From formula (2), we get

$$
\begin{aligned}
\sum_{m=0}^{\infty} \sum_{M\in S_{2,m}} \frac{q^{3-m}}{3} &= \sum_{m=0}^{\infty} \frac{q^{3-m}}{3} \sharp S_{2,m} \\
&= \frac{2}{3}q^3 + \sum_{p\nmid j>0} \frac{q^{2-j}}{3} \sharp S_{2,j+1} \\
&= \frac{2}{3}q^3 + \sum_{p\nmid j>0} \frac{q^{2-j}}{3} 3(q-1)q^{j-1-\lfloor \frac{j}{3} \rfloor} \\
&= \frac{2}{3}q^3 + q(q-1) \sum_{p\nmid j>0} q^{-\lfloor \frac{j}{3} \rfloor} \\
&= \frac{2}{3}q^3 + 2q(q-1) \sum_{l=0}^{\infty} q^{-l} \\
&= \frac{2}{3}q^3 + 2q(q-1) \frac{1}{1-q^{-1}} \\
&= \frac{2}{3}q^3 + 2q^2.
\end{aligned}
$$

For last sum in (4), we have

$$
\begin{aligned}
\sum_{m=0}^{2} \sum_{M\in S_{3,m}} \frac{q^{3-m}}{l} &= \sum_{m=0}^{2} \frac{q^{3-m}}{l} \sharp S_{3,m} \\
&= \frac{q^3}{l} \frac{l-1}{3} + \frac{q^2}{l} \frac{l(l-1)}{6} + \frac{q}{l} \frac{l(l-1)}{6} \\
&= \frac{l-1}{3l} q^3 + \frac{l-1}{6} q^2 + \frac{l-1}{6} q
\end{aligned}
$$

from (3). Therefore, we get

$$
\begin{aligned}
\sum_{M\in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} \frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} &= \frac{q^3}{3l} + \left(\frac{2}{3}q^3 + 2q^2\right) + \left(\frac{l-1}{3l}q^3 + \frac{l-1}{6}q^2 + \frac{l-1}{6}q\right) \\
&= q^3 + \left(2 + \frac{l-1}{6}\right) q^2 + \frac{l-1}{6} q.
\end{aligned}
$$

□

4.2 The case $G \cong (\mathbb{Z}/l\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$

Next we consider the case that the group G is isomorphic to $H \rtimes G'$ where $H = (\mathbb{Z}/l\mathbb{Z})^2$ with l a prime different from three and $G' = \mathbb{Z}/3\mathbb{Z}$. We assume that $q-1$ is divisible by l. Then the set G-Ét(K) is divided into four sets:

$$\begin{aligned}
S_1 &= \{K^{\oplus 3l^2}\},\\
S_2 &= \{L^{\oplus l^2} \in G\text{-Ét} \mid L/K : (\mathbb{Z}/3\mathbb{Z})\text{-extension}\},\\
S_3 &= \{L^{\oplus 3l} \in G\text{-Ét} \mid L/K : (\mathbb{Z}/l\mathbb{Z})\text{-extension}\},\\
S_4 &= \{L^{\oplus 3} \in G\text{-Ét} \mid L/K : (\mathbb{Z}/l\mathbb{Z})^2\text{-extension}\}.
\end{aligned}$$

Case $M \in S_1$: Since M is unramified, we have $\boldsymbol{v}_V(M) = 0$.

Case $M \in S_2$: We take an $M = L^{\oplus l^2} \in S_2$. Let L_0 be the first component of M. Since any element in G whose order is three is conjugate to σ or σ^2, we may assume that $\mathrm{Stab}_G(L_0) = G'$. Moreover, a normalizing group $N_G(G')$ is G'. Hence, a component L of M such that $\mathrm{Stab}_G(L) = G'$ is L_0 only. From the convertibility of $\boldsymbol{v}_V$, we have

$$\boldsymbol{v}_V(M) = \boldsymbol{v}_{V|_{G'}}(L_0).$$

The extension L_0/K is defined by

$$L_0 = K[x]/(x^3 - x - a)$$

by $a \in (k/(\mathcal{P}(k))) \oplus \bigoplus_{3 \nmid j > 0} kt^{-1}$. We put $j = -v_K(a)$. If $j = 0$, then L_0/K is unramified and we get $\boldsymbol{v}_{V|_{G'}}(L_0) = 0$. On the other hand, if $j > 0$, we get

$$\boldsymbol{v}_{V|_{G'}}(L_0) = \frac{d_{L_0/K}}{2} = j + 1.$$

Let

$$S_{2,m} = \{M \in S_2 \mid \boldsymbol{v}_V(M) = m\}.$$

By the same computation as the one for (2), we get

$$\sharp S_{2,m} = \begin{cases} 2 & (m = 0) \\ 3(q-1)q^{m-2-\lfloor \frac{m-1}{3} \rfloor} & (m - 1 \neq 3\mathbb{Z}, m > 0) \\ 0 & (otherwise) \end{cases}.$$

Case $M \in S_3$: For $M = L^{\oplus 3l} \in S_3$, we have

$$\boldsymbol{v}_V(M) = \boldsymbol{v}_{V|_{\mathbb{Z}/l\mathbb{Z}}}(L) = \mathrm{age}(h)v_K(f)$$

for $h \in H, f \in \mathcal{O}_K$ chosen as in Lemma 4.2. The number of elements of age two in H is

$$\begin{aligned}\sharp\{h \in H \mid \mathrm{age}(h) = 2\} &= \sharp\{h^{-1} \in H \mid \mathrm{age}(h) = 2\} \\ &= \sharp\left\{\frac{1}{l}[a,b,c] \in H \mid a+b+c = l, abc \neq 0\right\} \\ &= \binom{l-1}{2} = \frac{(l-1)(l-2)}{2}.\end{aligned}$$

We put

$$S_{3,m} = \{M \in S_3 \mid \boldsymbol{v}_V(M) = m\}.$$

Then we can compute $\sharp S_{3,m}$ similarly as (3). When $M \in S_{3,0}$, L_0 is determined from $f = \mu$ and the G-aciton on M corresponds to a nontrivial G-conjugacy class in H. Hence $\sharp S_{3,0} = \frac{l^2-1}{3}$. On the other hand, when $M \in S_{3,m}$ for $m = 1, 2$, L_0 is determined from $f = \mu^i t$ for $i = 0, 1, \ldots, l-1$ and the G-action on M corresponds to a conjugacy class of H whose age is m. Hence we get following formulas:

$$\sharp S_{3,m} = \begin{cases} \frac{l^2-1}{3} & (m = 0) \\ \frac{l(l-1)(l+4)}{6} & (m = 1) \\ \frac{l(l-1)(l-2)}{6} & (m = 2) \\ 0 & (\text{otherwise}) \end{cases}$$

Case $M \in S_4$: We consider $M = L^{\oplus 3} \in S_4$. Since L/K is a $(\mathbb{Z}/l\mathbb{Z})^2$-extension and $K^\times/(K^\times)^l = \{\mu^i t^j \mid 0 \leq i, j \leq l-1\}$, L has generators $\alpha, \beta \in L$ over K such that

$$\alpha^l = \mu, \beta^l = t.$$

The H-action on L corresponds to choosing the elements h_1, h_2 acting by

$$\begin{aligned} \alpha \cdot h_1 &= \zeta_l \alpha, & \beta \cdot h_1 &= \beta, \\ \alpha \cdot h_2 &= \alpha, & \beta \cdot h_2 &= \zeta_l \beta. \end{aligned}$$

Lemma 4.4. *For above L, we have*

$$\boldsymbol{v}_{V|_H}(L) = \mathrm{age}(h_2)$$

Proof. We define the $\mathcal{O}_K$-linear maps φ_i as in the proof of Lemma 4.2. We denote h_1, h_2 as $\frac{1}{l}[a_1, a_2, a_3], \frac{1}{l}[b_1, b_2, b_3]$. Then

$$\begin{aligned}\sum_{i=1}^{3} c_i \varphi_i \in \Xi_L &\Leftrightarrow 1 \leq \forall j \leq 3, \left(\sum_{i=1}^{3} c_i \varphi_i\right)(x_j \cdot h_m) = \left(\sum_{i=1}^{3} c_i \varphi_i\right)(x_j) \cdot h_m \text{ for } m = 1,2 \\ &\Leftrightarrow 1 \leq \forall j \leq 3, \zeta_l^{a_j} c_j = c_j \cdot h_1, \zeta_l^{b_j} c_j = c_j \cdot h_2 \\ &\Leftrightarrow 1 \leq \forall j \leq 3, \exists d_j \in \mathcal{O}_K, c_j = \alpha^{a_j} \beta^{b_j} d_j.\end{aligned}$$

Hence $(\alpha^{a_j}\beta^{b_j}\varphi_j)_{j=1}^3$ are $\mathcal{O}_K$-basis of Ξ_L. Therefore, we get

$$\boldsymbol{v}_{V|_H}(L) = v_K(\mu^{\frac{a_1+a_2+a_3}{l}} t^{\frac{b_1+b_2+b_3}{l}}) = \mathrm{age}(h_2).$$

□

The G-action on M is defined by the orbit of (h_1, h_2) in $(H-\{I\})\times(H-\{I\})$ about the G-action defined by taking componentwise conjugate. Note that this action is free. We put

$$S_{4,m} = \{M \in G\text{-Ét}(K) \mid \boldsymbol{v}_V(M) = m\}$$

for $m = 1, 2$. The number of pairs (h_1, h_2) with $\mathrm{age}(h_2) = 1$ is

$$\frac{(l-1)(l+4)}{2}(l^2 - l) = \frac{l(l-1)^2(l+4)}{2}$$

since h_1 is chosen from $H - \langle h_2 \rangle$. Since G-action on $(H - \{I\})^2$ is free, we have

$$\sharp S_{4,1} = \frac{l(l-1)^2(l+4)}{6}.$$

Similarly, the number of pairs (h_1, h_2) with $\mathrm{age}(h_2) = 2$ is

$$\frac{(l-1)(l-2)}{2}(l^2 - l) = \frac{l(l-1)^2(l-2)}{2},$$

and we have

$$\sharp S_{4,2} = \frac{l(l-1)^2(l-2)}{6}.$$

$$\sharp S_{4,m} = \left\{ \begin{array}{ll} \frac{1}{6}l(l-1)^2(l+4) & (m=1) \\ \frac{1}{6}l(l-1)^2(l-2) & (m=2) \end{array} \right. .$$

Theorem 4.5. *Let l be a prime different from three and let q be a power of three such that $q-1$ is divisible by l. Let G be a small finite subgroup of $SL_3(k)$. Suppose that G has a normal group $H \cong (\mathbb{Z}/l\mathbb{Z})^2$ and $G/H \cong \mathbb{Z}/3\mathbb{Z}$. Then we have*

$$\sum_{M \in G\text{-Ét}(K)} \frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} = q^3 + \left(2 + \frac{(l-1)(l+4)}{6}\right) q^2 + \frac{(l-1)(l-2)}{6} q$$

Proof. By the same computation as in the proof of Theorem 4.3, we have

$$\sum_{M \in S_2} \frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} = \frac{2}{3}q^3 + 2q^2.$$

From the above computations, we get

$$\sum_{M\in S_3}\frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} = \frac{1}{l^2}\left(q^3\sharp S_{3,0}+q^2\sharp S_{3,1}+q\sharp S_{3,2}\right)$$
$$= \frac{l^2-1}{3l^2}q^3+\frac{(l-1)(l+4)}{6l}q^2+\frac{(l-1)(l-2)}{6l}q,$$

and

$$\sum_{M\in S_4}\frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} = \frac{1}{l^2}\left(q^2\sharp S_{4,1}+q\sharp S_{4,2}\right)$$
$$= \frac{(l-1)^2(l+4)}{6l}q^2+\frac{(l-1)^2(l-2)}{6l}q.$$

Therefore,

$$\sum_{M\in G\text{-Ét}(K)}\frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)}$$
$$= \sum_{M\in S_1}\frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)}+\sum_{M\in S_2}\frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)}+\sum_{M\in S_3}\frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)}+\sum_{M\in S_4}\frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)}$$
$$= \frac{1}{3l^2}q^3+\left(\frac{2}{3}q^3+2q^2\right)+\left(\frac{l^2-1}{3l^2}q^3+\frac{(l-1)(l+4)}{6l}q^2+\frac{(l-1)(l-2)}{6l}q\right)$$
$$+\left(\frac{(l-1)^2(l+4)}{6l}q^2+\frac{(l-1)^2(l-2)}{6l}q\right)$$
$$= q^3+\left(2+\frac{(l-1)(l+4)}{6}\right)q^2+\frac{(l-1)(l-2)}{6}q.$$

□

4.3 The case $G \cong (\mathbb{Z}/l\mathbb{Z})^2 \rtimes \mathfrak{S}_3$

Lastly, we consider the case that $G \cong \mathfrak{S}_3$. We put $H \cong (\mathbb{Z}/l\mathbb{Z})^2$ with a prime $l \neq 3$ and $G' \cong \mathfrak{S}_3$. As a preparation, we describe some properties of G. We may assume that $\sigma, \tau \in G'$ be

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ \tau = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The subset $H - \{I\}$ is parted in two sets

$$H_f = \left\{\frac{1}{l}[a,b,c]\,\middle|\,\sharp\{a,b,c\}=3\right\},\quad H' = \left\{\frac{1}{l}[a,b,c]\,\middle|\,\sharp\{a,b,c\}=2\right\}.$$

Since any $h \in H'$ can be written as

$$h = \sigma^{-i}\frac{1}{l}[a,a,b]\sigma^{i} \quad (i = 0,1,2)$$

with $2a + b \in l\mathbb{Z}$, we have $\sharp H' = 3(l-1)$, and hence $\sharp H_f = (l-1)(l-2)$. From the proof of Corollary 3.16, the conjugacy classes of G is represented by $1, \sigma, \tau, \frac{1}{r}[c,d,c]\tau, \frac{1}{l}[c,d,c], \frac{1}{l}[a,b,c]$ where $\frac{1}{l}[e,d,e] \in H'$, $\frac{1}{l}[a,b,c] \in H_f$ and $a < b < c$. When we write $M \in G\text{-Ét}(K)$ as $M = L_1 \oplus L_2 \oplus \cdots \oplus L_r$ where L_i are copy of a Galois extension L/K, for any i, there exists $g \in G$ such that $g(L_1) = L_i$. Then the stable subgroups H_M, H'_M of L_1, L_i are conjugate by g. Therefore, we may assume that, if $H_M \not\subset H$, then $H_M \cap G' \neq \emptyset$.

Suppose that $q \equiv 1 \mod 2l$ and $q > l+1$. The set $G\text{-Ét}(K)$ of G-étale K-algebras is parted into following seven sets.

$$\begin{aligned}
S_1 &= \{M \in G\text{-Ét}(K) \mid H_M \text{ is trivial}\},\\
S_2 &= \{M \in G\text{-Ét}(K) \mid H_M = \langle\tau\rangle\},\\
S_3 &= \{M \in G\text{-Ét}(K) \mid H_M = \langle\sigma\rangle\},\\
S_4 &= \{M \in G\text{-Ét}(K) \mid H_M \cong \mathbb{Z}/l\mathbb{Z} \text{ and } H_M \subset H\},\\
S_5 &= \{M \in G\text{-Ét}(K) \mid H_M = G'\},\\
S_6 &= \{M \in G\text{-Ét}(K) \mid H_M = H\},\\
S_7 &= \{M \in G\text{-Ét}(K) \mid H_M \cap H \cong \mathbb{Z}/l\mathbb{Z} \text{ and } H_M \cap G' = \langle\tau\rangle\}.
\end{aligned}$$

We computed $\boldsymbol{v}_V(M)$ in the previous cases except $M \in S_5 \cup S_7$. Let

$$S_{i,m} = \{M \in S_i \mid \boldsymbol{v}_V(M) = m\}.$$

Case $M \in S_1$: Since M is unramified, we have $\boldsymbol{v}_V(M) = 0$.

Case $M \in S_2$: Since $H_M \cong \mathbb{Z}/2\mathbb{Z}$, the extension L/K is determined by $f \in \{\mu, t, \mu t\}$. From Lemma 4.2, we have

$$\boldsymbol{v}_V(M) = v_K(f).$$

Note that τ is diagonalized as $\mathrm{diag}(1,-1,-1)$ and hence $\mathrm{age}(\tau) = 1$. Thus

$$\sharp S_{2,m} = \begin{cases} 1 & (m = 0) \\ 2 & (m = 1) \end{cases}$$

Since $C_G(H_M) = \left\langle \frac{1}{l}[1, 2l-2, 1], \tau \right\rangle$, we have

$$\sum_{M \in S_2} \frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} = \frac{1}{2l}\left(q^3 + 2q\right) \tag{5}$$

Case $M \in S_3$: We take an $M = L^{\oplus l^2} \in S_2$. From the convertibility of $\boldsymbol{v}_V$, we have

$$\boldsymbol{v}_V(M) = \boldsymbol{v}_{V|_{G'}}(L).$$

The extension L_0/K is defined by

$$L_0 = K[x]/(x^3 - x - a)$$

by $a \in (k/(\mathcal{P}(k))) \oplus \bigoplus_{3\nmid j>0} kt^{-1}$. We put $j = -v_K(a)$. If $j = 0$, then L_0/K is unramified and we get $\boldsymbol{v}_{V|_{G'}}(L_0) = 0$. On the other hand, if $j > 0$, we get

$$\boldsymbol{v}_{V|_{G'}}(L_0) = \frac{d_{L_0/K}}{2} = j + 1.$$

Since the normalizer $N_G(H_M)$ is G', $\sharp[N_G(H_M) : H_M] = 2$. From (2), we get

$$\sharp S_{3,m} = \begin{cases} 1 & (m = 0) \\ \frac{3(q-1)q^{m-2-\lfloor \frac{m-1}{3} \rfloor}}{2} & (m - 1 \neq 3\mathbb{Z}, m > 0) \\ 0 & (otherwise) \end{cases}.$$

Therefore

$$\sum_{M \in S_3} \frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} = \frac{1}{3}q^3 + q. \tag{6}$$

Case $M \in S_4$: For $M = L^{\oplus 6l} \in S_4$, we have

$$\boldsymbol{v}_V(M) = \boldsymbol{v}_{V|_{\mathbb{Z}/l\mathbb{Z}}}(L) = \mathrm{age}(h) v_K(f)$$

for $h \in H, f \in \mathcal{O}_K$ chosen as in Lemma 4.2. We put

$$S_{4,m,f} = \{M \in S_{4,m} \mid h \in H_f\},\ S'_{4,m} = \{M \in S_{4,m} \mid h \in H'\}.$$

Then we can compute $\sharp S_{4,m}$ similarly as (3). Note that the G-aciton on M corresponds to a nontrivial G-conjugacy class in H. Since

$$H' = \left\langle \frac{1}{l}[1, 2l-2, 1] \right\rangle \cup \left\langle \frac{1}{l}[2l-2, 1, 1] \right\rangle \cup \left\langle \frac{1}{l}[1, 1, 2l-2] \right\rangle,$$

we have

$$\{h \in H' \mid \mathrm{age}(h) = 1\} = \{\frac{1}{l}[a, l-2a, a], \frac{1}{l}[l-2a, a, a], \frac{1}{l}[a, a, l-2a] \mid 0 < a \leq \frac{l}{2}\},$$
$$\{h \in H' \mid \mathrm{age}(h) = 2\} = \{\frac{1}{l}[a, 2l-2a, a], \frac{1}{l}[2l-2a, a], \frac{1}{l}[a, a, 2l-2a] \mid \frac{l}{2} < a \leq l-1\}.$$

Since $O_G(h) = 3$ for $h \in H'$, we have

$$\sharp S'_{4,m} = \begin{cases} l - 1 & (m = 0) \\ l \left\lfloor \frac{l}{2} \right\rfloor & (m = 1) \\ l \left\lceil \frac{l}{2} \right\rceil - 1 & (m = 2) \end{cases}$$

On the other hand, since

$$\sharp\{h \in H \mid \mathrm{age}(h) = i\} = \begin{cases} \frac{(l-1)(l+4)}{2} & (i = 1) \\ \frac{(l-1)(l-2)}{2} & (i = 2) \end{cases}$$

we have

$$\sharp\{h \in H' \mid \mathrm{age}(h) = i\} = \begin{cases} \frac{(l-1)(l+4)}{2} - 3\left\lfloor \frac{l}{2} \right\rfloor & (i = 1) \\ \frac{(l-1)(l-2)}{2} - 3\left(\left\lceil \frac{l}{2} \right\rceil - 1\right) & (i = 2) \end{cases}$$

and hence

$$\sharp S_{4,m,f} = \begin{cases} \frac{l^2-1-3(l-1)}{6} & (m = 0) \\ \frac{l}{6}\left(\frac{(l-1)(l+4)}{2} - 3\left\lfloor \frac{l}{2} \right\rfloor\right) & (m = 1) \\ \frac{l}{6}\left(\frac{(l-1)(l-2)}{2} - 3\left(\left\lceil \frac{l}{2} \right\rceil - 1\right)\right) & (m = 2) \end{cases}$$

Since

$$\sharp C_G(H_M) = \begin{cases} l^2 & (h \in H_f) \\ 2l^2 & (h \in H') \end{cases}$$

we get following formulas:

$$\begin{aligned} \sum_{M \in S_4} \frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} &= q^3 \left(\frac{1}{l^2}\sharp S_{4,0,f} + \frac{1}{2l^2}\sharp S'_{4,0}\right) \\ &\quad + q^2 l \left(\frac{1}{l^2}\sharp S_{4,1,f} + \frac{1}{2l^2}\sharp S'_{4,1}\right) \\ &\quad + ql \left(\frac{1}{l^2}\sharp S_{4,2,f} + \frac{1}{2l^2}\sharp S'_{4,2}\right) \\ &= \frac{l^2-1}{6l^2}q^3 + \frac{(l-1)(l+4)}{12l}q^2 + \frac{(l-1)(l-2)}{12l}q \end{aligned} \tag{7}$$

Case $M \in S_5$: In this case, L/K is an $\mathfrak{S}_3$-extension. Put $Q = L^\sigma$. Then L/Q is a $\mathbb{Z}/3\mathbb{Z}$-extension and Q/K is a $\mathbb{Z}/2\mathbb{Z}$-extension. By the Kummer theory and the Artin-Schreier theory, Q is generated by $\alpha \in Q$ over K such that $\alpha^2 \in \{\mu, t, \mu t\}$ and L is generated by $\beta \in L$ over Q such that

$$\beta^3 - \beta \in \left(\mathbb{F}_3\lambda \oplus \bigoplus_{3 \nmid j > 0} k_Q \pi_Q^{-j}\right) =: RP_Q$$

where k_Q is the residue field of Q, $\lambda \in k_Q - \mathcal{P}(k_Q)$, and π_Q is a uniformizer of Q. Note that we can choose π_Q satisfying $\pi_Q \cdot \tau = \pm\pi_Q$. Hence the set RP_Q is invariant under the action of τ. We may assume that

$$\begin{aligned} \alpha \cdot \sigma &= \alpha, \\ \alpha \cdot \tau &= -\alpha, \\ \beta \cdot \sigma &= \beta + 1. \end{aligned}$$

We put $\beta \cdot \tau = \sum_{i=0}^{2} c_i \beta^i$ with $c_i \in Q$. Since $\tau\sigma = \sigma^2\tau$, we have

$$\beta \cdot (\tau\sigma) = \beta \cdot (\sigma^2\tau)$$

and

$$(c_0 + c_1 + c_2) + (c_1 - c_2)\beta + c_2\beta^2 = (c_0 - 1) + c_1\beta + c_2\beta^2.$$

Then, $c_2 = 0, c_1 = -1$, in other words, $\beta \cdot \tau = c_0 - \beta$. Since $\tau^2 = 1$,

$$\beta = (\beta \cdot \tau) \cdot \tau = (c_0 \cdot \tau - c_0) + \beta.$$

Hence $c_0 \cdot \tau = c_0$ and $c_0 \in K$. Let $b = \beta^3 - \beta$. Then

$$b \cdot \tau = (\beta^3 - \beta) \cdot \tau = (c_0^3 - c\_0) - b$$

We get $b \cdot \tau + b = c_0^3 - c_0$. Since the set RP_Q is closed under the τ-action and addition, $b \cdot \tau + b \in RP_Q$. Now $RP_Q \cap \mathcal{P}(K) = \{0\}$. Therefore $b \cdot \tau = -b$ and $c_0 \in \mathbb{F}_3$. By replacing β with $\beta \pm 1$ if necessary, we may assume that $\beta \cdot \tau = -\beta$.

If $\alpha^2 = \mu$, any element of k_Q is written as $s\alpha + u$ by $s, u \in \mathbb{F}_q$. We have

$$\begin{aligned} \mathcal{P}(s\alpha + u) &= \mathcal{P}(s\alpha) + \mathcal{P}(u) \\ &= s^3\alpha^3 - s\alpha + u^3 - u \\ &= (s^3\mu - s)\alpha + u^3 - u. \end{aligned}$$

Hence $\mathcal{P}(k_Q) \cap \mathbb{F}_q = \mathcal{P}(\mathbb{F}_q)$. Then we can choose λ from $\mathbb{F}_q$ and $\pi_Q = t$. We put $b = a_0\lambda + \sum_{3\nmid j>0} a_j t^{-j}$ where $a_0 \in \mathbb{F}_3,\ a_j \in k_Q$. Then

$$b \cdot \tau = a_0\lambda + \sum_{3\nmid j>0} (a_j \cdot \tau) t^{-j}$$

Since $b \cdot \tau = -b$, $a_j \in \alpha\mathbb{F}_q$ for any j. Hence

$$b \in \bigoplus_{3\nmid j>0} \alpha\mathbb{F}_q t^{-j}.$$

If $\alpha \neq \mu$, we can choose α as uniformaizer of Q. Then b is written as $a_0\lambda + \sum_{3\nmid j>0} a_j \alpha^{-j}$ where $a_0 \in \mathbb{F}_3,\ a_j \in k_Q = \mathbb{F}_q$. Since $b \cdot \tau = -b$ and

$$b \cdot \tau = a_0\lambda + \sum_{3\nmid j>0} (-1)^j a_j \alpha^{-j},$$

we get $a_j = 0$ for any $j \in 2\mathbb{Z}$, and hence

$$b \in \bigoplus_{j>0, \gcd(j,6)=1} \mathbb{F}_q \alpha^{-j}$$

. Let $\varphi_1, \varphi_2, \varphi_3$ be the dual basis of x_1, x_2, x_3. Then

$$
\begin{aligned}
(\sum_{i=1}^{3} c_i\varphi_i) \in \Xi_L \Leftrightarrow 1 \leq \forall j \leq 3,\ & \begin{cases} (\sum_{i=1}^{3} c_i\varphi_i)(x_j \cdot \sigma) = ((\sum_{i=1}^{3} c_i\varphi_i)(x_j)) \cdot \sigma \\ (\sum_{i=1}^{3} c_i\varphi_i)(x_j \cdot \tau) = ((\sum_{i=1}^{3} c_i\varphi_i)(x_j)) \cdot \tau \end{cases} \\
&\Leftrightarrow \begin{cases} c_1 = c_2 \cdot \sigma = c_3 \cdot \sigma^2 \\ c_1 = -c_3 \cdot \tau, c_2 = -c_2 \cdot \tau \end{cases} \\
&\Leftrightarrow c_1 = c_2 \cdot \sigma^2, c_3 = c_2 \cdot \sigma, c_2 \cdot \tau = -c_2 \\
&\Leftrightarrow c_1 = c_2 \cdot \sigma^2, c_3 = c_2 \cdot \sigma, c_2 \in (K\alpha \oplus K\beta \oplus K\alpha\beta^2) \cap \mathcal{O}_K
\end{aligned}
$$

Since $v_L(\alpha) = 3v_Q(\alpha) \in 3\mathbb{Z}$ and $v_L(\beta) = v_Q(b) \notin 3\mathbb{Z}$, we have

$$
v_L(\alpha) \not\equiv v_L(\beta) \not\equiv v_L(\alpha\beta^2) \not\equiv v_L(\alpha) \mod 3.
$$

Hence, for $a_1, a_2, a_3 \in K$,

$$
v_L(a_1\alpha + a_2\beta + a_3\alpha\beta^2) = \min(v_L(a_1\alpha), v_L(a_2\beta), v_L(a_3\alpha\beta^2)).
$$

Thus,

$$
(K\alpha \oplus K\beta \oplus K\alpha\beta^2) \cap \mathcal{O}_K = \alpha\mathcal{O}_K + t^{n_1}\beta\mathcal{O}_K + t^{n_2}\alpha\beta^2\mathcal{O}_K
$$

where

$$
n_1 = \left\lceil \frac{-v_L(\beta)}{e_{L/K}} \right\rceil, n_2 = \left\lceil \frac{-2v_L(\beta) - v_L(\alpha)}{e_{L/K}} \right\rceil
$$

and $e_{L/K}$ is the ramification index of L/K. Note that

$$
e_{L/K} = \begin{cases} 3 & (\alpha^2 = \mu) \\ 6 & (otherwise) \end{cases}
$$

Hence, we can take a basis of Ξ_L as the morphisms ψ_1, ψ_2, ψ_3 corresponding to $c_2 = \alpha,\ t^{n_1}\beta,\ t^{n_2}\alpha\beta^2$ respectively:

$$
\psi_1 = \alpha \sum_{i=1}^{3} \varphi,\ \psi_2 = t^{n_1} \sum_{i=1}^{3} \beta \cdot \sigma^{i+1}\varphi_i,\ \alpha\psi_3 = t^{n_2} \sum_{i=1}^{3} \beta \cdot \sigma^{i+1}\varphi_i
$$

Therefore

$$
\begin{aligned}
\boldsymbol{v}_V(M) &= v_K\left(\det \begin{bmatrix} \alpha & \alpha & \alpha \\ \beta - 1 & \beta & \beta + 1 \\ \alpha(\beta-1)^2 & \alpha\beta^2 & \alpha(\beta+1)^2 \end{bmatrix}\right) + n_1 + n_2 \\
&= v_K(\alpha^2) + n_1 + n_2.
\end{aligned}
$$

by the formula (1). We put

$$
S_{5,m,j} = \{M \in S_5 \mid v_K(\alpha^2) = m,\ v_{L^\sigma}(b) = -j\}
$$

for $m \in \{0,1\}$ and $j > 0$. Then, we get

$$\boldsymbol{v}_V(M) = \begin{cases} \lceil \frac{j}{3} \rceil + \lceil \frac{2j}{3} \rceil & (m = 0, j \in \mathbb{Z} - 3\mathbb{Z}) \\ 1 + \lceil \frac{j}{6} \rceil + \lceil \frac{2j-3}{6} \rceil & (m = 1, j \in \mathbb{Z}, \gcd(j,6) = 1) \end{cases}$$

for $M \in S_{5,m,j}$. We have

$$\sharp \left\{ b \in \bigoplus_{3\nmid j>0} \alpha \mathbb{F}_q t^{-j} \middle| v_Q(b) = -j \right\} = (q-1)q^{j - \lfloor \frac{j}{3} \rfloor - 1}$$

when $\alpha^2 = \mu$ and

$$\sharp \left\{ b \in \bigoplus_{j>0, \gcd(j,6)=1} \mathbb{F}_q \alpha^{-j} \middle| v_Q(b) = -j \right\} = (q-1)q^{j - 2\lfloor \frac{j+1}{3} \rfloor - 1}$$

when $\alpha^2 = t, \mu t$. Since the G'-action on L corresponding to $-b$ is conjugate with the one corresponding to b, we have

$$\sharp S_{5,m,j} = \begin{cases} \frac{1}{2}(q-1)q^{j - \lfloor \frac{j}{3} \rfloor - 1} & (m = 0, j \in \mathbb{Z} - 3\mathbb{Z}) \\ (q-1)q^{j - 2\lfloor \frac{j+1}{3} \rfloor - 1} & (m = 1, j \in \mathbb{Z}, \gcd(j,6) = 1) \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{M \in S_5} \frac{q^{3 - \boldsymbol{v}_V(M)}}{C_G(H_M)} &= \sum_{m=0}^{1} \sum_{j=1}^{\infty} \sum_{M \in S_{5,m,j}} q^{3 - \boldsymbol{v}_V(M)} \\ &= \sum_{j \in \mathbb{Z}_{>0} - 3\mathbb{Z}} \sum_{M \in S_{5,0,j}} q^{3 - \lceil \frac{j}{3} \rceil - \lceil \frac{2j}{3} \rceil} \\ &\quad + \sum_{j \in \mathbb{Z}_{>0}, \gcd(j,6)=1} \sum_{M \in S_{5,1,j}} q^{2 - \lceil \frac{j}{6} \rceil - \lceil \frac{2j-3}{6} \rceil} \\ &= \sum_{l=0}^{\infty} \sum_{M \in S_{5,0,3l+1}} q^{1-3l} + \sum_{l=0}^{\infty} \sum_{M \in S_{5,0,3l+2}} q^{-3l} \\ &\quad + \sum_{l=0}^{\infty} \sum_{M \in S_{5,0,6l+1}} q^{1-3l} + \sum_{l=0}^{\infty} \sum_{M \in S_{5,0,6l+5}} q^{-3l-1} \\ &= \frac{1}{2} \sum_{l=0}^{\infty} (q-1)q^{1-l} + \sum_{l=0}^{\infty} (q-1)q^{1-l} \\ &\quad + \sum_{l=0}^{\infty} (q-1)q^{1-l} + \sum_{l=0}^{\infty} (q-1)q^{-l-1} \\ &= 2q^2 + q \end{aligned} \tag{8}$$

Case $M \in S_6$: We consider $M = L^{\oplus 6} \in S_4$. Since L/K is a $(\mathbb{Z}/l\mathbb{Z})^2$-extension and $K^\times/(K^\times)^l = \{\mu^i t^j \mid 0 \le i, j \le l-1\}$, L has generators $\alpha, \beta \in L$ over K such that

$$\alpha^l = \mu, \beta^l = t.$$

The H-action on L corresponds to choosing the elements h_1, h_2 acting by

$$\begin{aligned} \alpha \cdot h_1 &= \zeta_l \alpha, & \beta \cdot h_1 &= \beta, \\ \alpha \cdot h_2 &= \alpha, & \beta \cdot h_2 &= \zeta_l \beta. \end{aligned}$$

From Lemma 4.4,

$$\boldsymbol{v}_V(M) = \mathrm{age}(h_2).$$

The G-action on M is defined by the orbit of (h_1, h_2) in $(H - \{I\}) \times (H - \{I\})$ about the G-action defined by taking componentwise conjugate. Note that this action is free. The number of pairs (h_1, h_2) with $\mathrm{age}(h_2) = 1$ is

$$\frac{(l-1)(l+4)}{2}(l^2 - l) = \frac{l(l-1)^2(l+4)}{2}$$

since h_1 is chosen from $H - \langle h_2 \rangle$. Since G-action on $(H - \{I\})^2$ is free, we have

$$\sharp S_{6,1} = \frac{l(l-1)^2(l+4)}{12}.$$

Similarly, the number of pairs (h_1, h_2) with $\mathrm{age}(h_2) = 2$ is

$$\frac{(l-1)(l-2)}{2}(l^2 - l) = \frac{l(l-1)^2(l-2)}{2},$$

and we have

$$\sharp S_{6,2} = \frac{l(l-1)^2(l-2)}{12}.$$

$$\sharp S_{6,m} = \begin{cases} \frac{l(l-1)^2(l+4)}{12} & (m = 1) \\ \frac{l(l-1)^2(l-2)}{12} & (m = 2) \end{cases}.$$

Threrefore, we have

$$\sum_{M \in S_6} \frac{q^{3 - \boldsymbol{v}_V(M)}}{C_G(H_M)} = \frac{1}{l^2}\left(\frac{l(l-1)^2(l-4)}{12}q^2 + \frac{l(l-1)^2(l-2)}{12}q\right) \tag{9}$$

Case $M \in S_7$: Since $H_M \cap G' = \langle \tau \rangle$ is acts on $H_M \cap H$ by conjugate, $H_M \cap H = \left\langle \frac{1}{l}[1, 2l-2, 1] \right\rangle$ or $H_M \cap H = \left\langle \frac{1}{l}[1, 0, l-1] \right\rangle$. If $H_M \cap H$ is the former, then $H_M = \langle h\tau \rangle \cong \mathbb{Z}/2l\mathbb{Z}$ for some $h \in H$. In the latter case, $H_M = \langle \frac{1}{l}[1, 0, l-1], \tau \rangle$ which is isomorphic to the dihedral group. Since the tame part of absolute Galois group of K is abelian, the latter appears only if $l = 2$.

When $H_M \cong \mathbb{Z}/2l\mathbb{Z}$, H_M is generated by $h\tau$ for some $h = \frac{1}{l}[a,b,c] \in H$. Since $h\tau$ is conjugate with

$$\frac{1}{l}[0, l-a, a]h\tau\frac{1}{l}[0, a-l, -a] = h\frac{1}{l}[-a, 0, a]\tau = \frac{1}{l}[0, b, l-b]\tau,$$

we may assume that $H_M = \left\langle \frac{1}{l}[0, 1, l-1]\tau \right\rangle$. From the Kummar theory, there exists a generator $\alpha \in L$ over K such that

$$\alpha^{2l} \in \{\mu, \mu t^l, \mu^2 t^l\} \cup \{\mu^i t \mid 0 \le i \le 2l-1\} \cup \left\{\mu^{2j-1}t^2 \mid 1 \le j \le l\right\}$$

and the generator $h \in H_M$ such that $\alpha \cdot h\tau = \zeta_{2l}\alpha$ where ζ_{2l} is a $2l$-th primitive root of unity. Note that if $l = 2$, $\mu^2 t^2$ is in $(K^\times)^2$, and hence

$$\alpha^4 \in \{\mu, \mu t^2, t, \mu t, \mu^2 t, \mu^3 t\}$$

Since $H_M = \left\langle \frac{1}{l}[0, 1, l-1]\tau \right\rangle$, h is written as

$$\left(\frac{1}{l}[0, 1, l-1]\tau\right)^{2i-1} = \frac{1}{l}[(i-1)(l-1), i, i(l-1)] \ (1 \le i \le l, 2i-1 \ne l).$$

Lemma 4.6. *For L as above, we have*

$$\boldsymbol{v}_{V|_{H_M}}(L) = \boldsymbol{v}_V(M) = \begin{cases} 0 & (\alpha^{2l} = \mu) \\ 1 & (v_K(\alpha^{2l}) \le 2, i \ge \frac{l}{2}) \\ 2 & (v_K(\alpha^{2l}) \le 2, i < \frac{l}{2}) \\ 1 & (v_K(\alpha^{2l}) = l) \end{cases}$$

Proof. We write α^2 as $\mu^m t^n$ and h as $\frac{1}{l}[a,b,c]$. Note that $a+b+c = \text{age}(h)l$. If $n = 0$, $\boldsymbol{v}_{V|_{H_M}}(L) = 0$ since L/K is unramified. Let $\varphi_1, \varphi_2, \varphi_3$ be the dual basis of x_1, x_2, x_3. Then

$$\begin{aligned}
(\sum_{i=1}^{3} c_i\varphi_i) \in \Xi_L &\Leftrightarrow 1 \le \forall j \le 3,\ (\sum_{i=1}^{3} c_i\varphi_i)(x_j \cdot h\tau) = (\sum_{i=1}^{3} c_i\varphi_i)(x_j) \cdot h\tau \\
&\Leftrightarrow \begin{cases} -\zeta_l^a c_3 = c_1 \cdot h\tau, \\ -\zeta_l^b c_2 = c_2 \cdot h\tau, \\ -\zeta_l^c c_1 = c_3 \cdot h\tau \end{cases} \\
&\Leftrightarrow c_1 \cdot (h\tau)^2 = \zeta_l^{a+c} c_1,\ c_2 \in \alpha^{2b+l}K \cap \mathcal{O}_K,\ c_3 = -\zeta_l^{-a} c_1 \cdot h\tau \\
&\Leftrightarrow c_1 \cdot h\tau = \pm\zeta_{2l}^{a+c} c_1,\ c_2 \in \alpha^{2b+l}K \cap \mathcal{O}_K,\ c_3 = -\zeta_l^{-a} c_1 \cdot h\tau \\
&\Leftrightarrow c_1 \in (\alpha^{a+c}K + \alpha^{a+c+l}K) \cap \mathcal{O}_K,\ c_2 \in \alpha^{2b+l}K \cap \mathcal{O}_K,\ c_3 = -\zeta_l^{-a} c_1 \cdot h\tau
\end{aligned}$$

Now we have

$$(\alpha^{a+c}K + \alpha^{a+c+l}K) \cap \mathcal{O}_K = \alpha^{a+c}t^{n_1}\mathcal{O}_K + \alpha^{a+c+l}t^{n_2}\mathcal{O}_K$$

where

$$n_1 = -\left\lfloor \frac{(a+c)n}{2l} \right\rfloor,\ n_2 = -\left\lfloor \frac{(a+c+l)n}{2l} \right\rfloor,$$

and

$$\alpha^{2b+l} K \cap \mathcal{O}_K = \alpha^{2b+l} t^{n_3} \mathcal{O}_K$$

where

$$n_3 = -\left\lfloor \frac{(2b+l)n}{2l} \right\rfloor.$$

Hence we get an $\mathcal{O}_K$-basis

$$t^{n_1}(\alpha^{a+c}\varphi_1 - \zeta_{2l}^{c-a}\alpha^{a+c}\varphi_3),\ t^{n_3}\alpha^{2b+l}\varphi_2,\ t^{n_2}(\alpha^{a+c+l}\varphi_1 + \zeta_{2l}^{c-a}\alpha^{a+c+l}\varphi_3)$$

of Ξ_L. By the formula (1),

$$\begin{aligned} \boldsymbol{v}_{V|_{H_M}}(L) &= v_K\left(\det \begin{bmatrix} \alpha^{a+c} & 0 & \alpha^{a+c+l} \\ 0 & \alpha^{2b} & 0 \\ -\zeta_{2l}^{c-a}\alpha^{a+c} & 0 & \zeta_{2l}^{c-a}\alpha^{a+c+l} \end{bmatrix}\right) + n_1 + n_2 + n_3 \\ &= v_K(\alpha^{2(a+b+c+l)}) + n_1 + n_2 + n_3 \\ &= (\mathrm{age}(h)+1)n + n_1 + n_2 + n_3 \end{aligned}$$

Since $a + c = l\mathrm{age}(h) - b$,

$$n_1 = -\left\lfloor \frac{(l\mathrm{age}(h) - b)n}{2l} \right\rfloor,\ n_2 = -\left\lfloor \frac{(l\mathrm{age}(h) + l - b)n}{2l} \right\rfloor.$$

Hence

$$(\mathrm{age}(h)+1)n + n_1 + n_2 + n_3 = \begin{cases} 2n - \left\lfloor \frac{(l-b)n}{2l} \right\rfloor - \left\lfloor \frac{(2l-b)n}{2l} \right\rfloor - \left\lfloor \frac{(2b+l)n}{2l} \right\rfloor & (\mathrm{age}(h) = 1) \\ 3n - \left\lfloor \frac{(2l-b)n}{2l} \right\rfloor - \left\lfloor \frac{(3l-b)n}{2l} \right\rfloor - \left\lfloor \frac{(2b+l)n}{2l} \right\rfloor & (\mathrm{age}(h) = 2) \end{cases}$$

Since $\left\lfloor \frac{(3l-b)n}{2l} \right\rfloor = 1 + \left\lfloor \frac{(l-b)n}{2l} \right\rfloor$, we have

$$\begin{aligned} \boldsymbol{v}_{V|_{H_M}}(L) &= (\mathrm{age}(h)+1)n + n_1 + n_2 + n_3 \\ &= 2n - \left\lfloor \frac{(l-b)n}{2l} \right\rfloor - \left\lfloor \frac{(2l-b)n}{2l} \right\rfloor - \left\lfloor \frac{(2b+l)n}{2l} \right\rfloor \\ &= \begin{cases} 2 - \left\lfloor \frac{l-b}{2l} \right\rfloor - \left\lfloor \frac{2l-b}{2l} \right\rfloor - \left\lfloor \frac{2b+l}{2l} \right\rfloor & (n=1) \\ 4 - \left\lfloor \frac{l-b}{l} \right\rfloor - \left\lfloor \frac{2l-b}{l} \right\rfloor - \left\lfloor \frac{2b+l}{l} \right\rfloor & (n=2) \\ 2l - \left\lfloor \frac{l-b}{2} \right\rfloor - \left\lfloor \frac{2l-b}{2} \right\rfloor - \left\lfloor \frac{2b+l}{2} \right\rfloor & (n=l) \end{cases} \\ &= \begin{cases} 2 - \left\lfloor \frac{2b+l}{2l} \right\rfloor & (n=1) \\ 2 - \left\lfloor \frac{2b}{l} \right\rfloor & (n=2) \\ 2l - b - \left\lfloor \frac{l-b}{2} \right\rfloor - \left\lfloor \frac{2l-b}{2} \right\rfloor - \left\lfloor \frac{l}{2} \right\rfloor & (n=l) \end{cases} \end{aligned}$$

If $n = 1, 2$, then we have

$$\boldsymbol{v}_{V|_{H_M}}(L) = \begin{cases} 2 & (b < \frac{l}{2}) \\ 1 & (b \geq \frac{l}{2}) \end{cases}$$

If $n = l$, then we have

$$\boldsymbol{v}_{V|_{H_M}}(L) = \begin{cases} 2l - b - \frac{3l-2b-1}{2} - \lfloor \frac{l}{2} \rfloor & (l \neq 2) \\ 1 & (l = 2) \end{cases} = 1$$

□

Therefore, if $l \neq 2$, then

$$\sharp S_{7,m} = \begin{cases} l - 1 & (m = 0) \\ \frac{3l(l-1)}{2} + 2(l-1) & (m = 1) \\ \frac{3l(l-1)}{2} & (m = 2) \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{M \in S_7} \frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} &= \frac{1}{2l} \left(q^3 \sharp S_{7,0} + q^2 \sharp S_{7,1} + q \sharp S_{7,2} \right) \\ &= \frac{l-1}{2l} q^3 + \frac{(l-1)(3l+4)}{4l} q^2 + \frac{3(l-1)}{4} q. \end{aligned} \tag{10}$$

If $l = 2$, since $\frac{1}{l}[0, 1, 1]\tau = \tau(\frac{1}{l}[1, 1, 0]\tau)\tau$, we have

$$\sharp\{M \in S_{7,m} \mid H_M \cong \mathbb{Z}/4\mathbb{Z}\} = \begin{cases} 1 & (m = 0) \\ 5 & (m = 1) \end{cases}$$

When H_M isomorphic to dihedral group, $l = 2$ and $H_M \cong (\mathbb{Z}/2\mathbb{Z})^2$. From Lemma 4.4 and any element of H_M is age one, $\boldsymbol{v}_V(M) = 1$. The choices of a ordered pair of generators of H_M are $(h, \tau), (\tau, h), (\tau, h\tau)$ up to conjugate where $h = \frac{1}{l}[1, 0, 1]$. Therefore, we have

$$\sharp\{M \in S_{7,1} \mid H_M \cong (\mathbb{Z}/2\mathbb{Z})^2\} = 3$$

and hence

$$\sharp S_{7,m} = \begin{cases} 1 & (m = 0) \\ 8 & (m = 1) \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{M \in S_7} \frac{q^{3-\boldsymbol{v}_V(M)}}{C_G(H_M)} &= \frac{1}{4} \left(q \sharp S_{7,1} + q^2 \sharp S_{7,2} \right) \\ &= \frac{1}{4} q^3 + 2q^2. \end{aligned} \tag{11}$$

Therefore, we get

Theorem 4.7. *Let l be a prime different from three and let q be a power of three such that $q-1$ is divisible by l. Let G be a small finite subgroup of $SL_3(k)$. Suppose that G has a normal group $H \cong (\mathbb{Z}/l\mathbb{Z})^2$ and $G/H \cong \mathfrak{S}_3$. Then we have*

$$\sum_{M \in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} \frac{q^{n-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} = \begin{cases} q^3 + \frac{(l+5)(l+7)}{12} q^2 + \frac{(l+1)(l+5)}{12} q & (l \neq 2) \\ q^3 + 6q^2 + q & (l = 2) \end{cases}$$

Proof. If $l \neq 2$, then

$$\begin{aligned} \sum_{M \in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} &= \sum_{i=1}^{7} \sum_{M \in S_i} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} \\ &= q^3 + \frac{(l+5)(l+7)}{12} q^2 + \frac{(l+1)(l+5)}{q} \end{aligned}$$

from the equations (5), (6), (7), (8), (9), and (10). If $l = 2$, then

$$\begin{aligned} \sum_{M \in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} &= \sum_{i=1}^{7} \sum_{M \in S_i} \frac{q^{3-\boldsymbol{v}_V(M)}}{\sharp C_G(H_M)} \\ &= q^3 + 6q^2 + q \end{aligned}$$

from the equations (5), (6), (7), (8), (9), and (11). □

4.4 Computing Euler characteristic

From Theorems 4.3, 4.5, and 4.7, we can get the Euler characteristic of a crepant resolution of the associated quotient variety. We get the following formula from the Weil conjecture.

Proposition 4.8. *For a smooth variety Y over $\mathbb{F}_q$. Suppose that*

$$\sharp Y(\mathbb{F}_{q^m}) = \sum_{i=1}^{n} a_i q^{im}$$

where $a_i \in \mathbb{Z}$. Then

$$\chi(Y) = \sum_{i=1}^{n} a_i.$$

Proof. Let $Z(t) \in \mathbb{Q}[[t]]$ be the zeta function of Y, which is defined as

$$Z(t) = \exp\left(\sum_{m=1}^{\infty} \sharp Y(\mathbb{F}_{q^m}) \frac{t^m}{m} \right),$$

or equivalently

$$\frac{d}{dt} \log Z(t) = \sum_{m=1}^{\infty} \sharp Y(\mathbb{F}_{q^m}) t^{m-1}.$$

From the assumption, we have

$$\sum_{m=1}^{\infty} \sharp Y(\mathbb{F}_{q^m}) t^{m-1} = \sum_{i=1}^{n} \frac{a_i q^i}{1 - q^i t}.$$

Hence

$$Z(t) = \prod_{i=1}^{n} \frac{1}{(1 - q^i t)^{a_i}}.$$

From the Weil conjecture, we get

$$\chi(Y) = \sum_{i=1}^{n} a_i.$$

□

Corollary 4.9. *Let G be a finite group considered in Theorem 4.3, 4.5, or 4.7. Suppose that $\mathbb{A}^3/G$ have a crepant resolution $Y \to \mathbb{A}^3/G$. Then*

$$\chi(Y) = \begin{cases} 3 + \frac{l-1}{3} & (H = \mathbb{Z}/l\mathbb{Z}, G' = \mathbb{Z}/3\mathbb{Z}) \\ 3 + \frac{l^2-1}{3} & (H = (\mathbb{Z}/l\mathbb{Z})^2, G' = \mathbb{Z}/3\mathbb{Z}) \\ \frac{(l-1)(l-2)}{6} + 2l + 4 & (H = (\mathbb{Z}/l\mathbb{Z})^2, G' = \mathfrak{S}_3) \end{cases}.$$

Proof. By the property of the stringy-point count, we have

$$\sharp Y(\mathbb{F}_q) = \sharp_{st} \mathbb{A}^3/G.$$

Hence the assertion follows from the previous proposition and Theorem 4.3, 4.5, or 4.7. □

5 An example where the v-function is not determined by the ramification filtration

Let k be an algebraic closed field of characteristic $p > 0$. Let K be the field $k((t))$ of Laurent power series and let $\mathcal{O}_K$ be the valuation ring of K. We consider the 2-dimensional $\mathcal{O}_K$-representation $V = \mathbb{A}^2_{\mathcal{O}_K}$ of $G = (\mathbb{Z}/p\mathbb{Z})^2$ defined by

$$\sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tau = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

for some $a \in k - \mathbb{F}_p$ where σ, τ are generators of G.

We define a subset $J \subset K$ by

$$J = \bigoplus_{p \nmid j, j > 0} k t^{-j},$$

which is an additive group of representatives of $K/\mathcal{P}(K)$. Let F/K be a G-extension. Since F^σ/K and F^τ/K are $(\mathbb{Z}/p\mathbb{Z})$-extensions, there exist $\alpha, \beta \in F$

uniquely such that $F^\sigma = K[\alpha], F^\tau = K[\beta], \alpha^p - \alpha = g_1 \in J, \beta^p - \beta = g_2 \in J$, and

$$\alpha \cdot \tau = \alpha + 1, \qquad\qquad \beta \cdot \sigma = \beta + 1.$$

by the Artin-Schreier theory. In this case, we denote F by F_{g_1,g_2} where the pair (g_1, g_2) is chosen from

$$J^{(2)} = \{(g_1, g_2) \in J^2 \mid g_1 \neq 0,\ g_2 \notin \mathbb{F}_p g_1\}.$$

We denote $\boldsymbol{v}_V(F_{g_1,g_2})$ by $\boldsymbol{v}_V(g_1, g_2)$ for simplicity.

Let $\mathcal{O}_K[x_1, x_2]$ be the coordinate ring of V.

Proposition 5.1. *The tuning module Ξ_F of $F \in G\text{-}\acute{E}t(K)$ is isomorphic to*

$$\Theta_F := \{m \in \mathcal{O}_F \mid m(\sigma-1)^2 = 0, m(\tau-1) = am(\sigma-1)\}$$

as an $\mathcal{O}_K$-module by the following maps

$$\Xi_F \ni \varphi \mapsto \varphi(x_2) \in \Theta_F,$$
$$\Theta_F \ni m \mapsto m\varphi_2 + m(\sigma-1)m\varphi_1 \in \Xi_F.$$

Proof. Let $\varphi \in \Xi_F$ and let $m = \varphi(x_2)$. Since $\varphi \in \Xi_F$, we get

$$m(\sigma-1)^2 = \varphi(x_2(\sigma-1)^2) = \varphi(0) = 0$$

and

$$m(\tau-1) = \varphi(x_2(\tau-1)) = \varphi(ax_1) = a\varphi(x_2(\sigma-1)) = am(\sigma-1).$$

Thus $m \in \Theta_F$.

On the other hand, for $m \in \Theta_F$, we define

$$\varphi = (m(\sigma-1))\varphi_1 + m\varphi_2$$

where $\varphi_i \colon \mathcal{O}_K[x_1, x_2]_1 \to \mathcal{O}_F$ are $\mathcal{O}_K$-linear maps defined by $\varphi_j(x_i) = \delta_{ij}$. Then

$$\varphi(x_1)\sigma = m(\sigma-1)\sigma = m(\sigma-1) = \varphi(x_1) = \varphi(x_1\sigma),$$
$$\varphi(x_1)\tau = m(\sigma-1)\tau = (am(\sigma-1) + m)(\sigma-1) = m(\sigma-1) = \varphi(x_1) = \varphi(x_1\tau),$$
$$\varphi(x_2)\sigma = m\sigma = m(\sigma-1) + m = \varphi(x_1 + x_2) = \varphi(x_2\sigma),$$
$$\varphi(x_2)\tau = m\tau = am(\sigma-1) + m = \varphi(ax_1 + x_2) = \varphi(x_2\tau).$$

These equalities show that φ is G-equivariant and belongs to Ξ_F.

Therefore, we have two $\mathcal{O}_K$-homorphisms

$$\Xi_F \ni \varphi \mapsto \varphi(x_2) \in \Theta_F,$$
$$\Theta_F \ni m \mapsto m\varphi_2 + m(\sigma-1)\varphi_1 \in \Xi_F$$

These maps are inverse of each other. ∎

We get the following formula for v-function.

Theorem 5.2. *For $(g_1, g_2) \in J^{(2)}$, we put $f = a^p g_1 + g_2$. Then*

$$\boldsymbol{v}_V(g_1, g_2) = \left\lceil -\frac{\min\{v_K(g_1), pv_K(f)\}}{p^2} \right\rceil$$

Proof. The field F^σ endowed with the action of $\langle\tau\rangle$ can be regarded as a cyclic representation of $\mathbb{Z}/p\mathbb{Z}$ over K. Since

$$\alpha^i(\tau-1) = (\alpha+1)^i - \alpha^i = i\alpha^{i-1} + \sum_{j=0}^{i-2} \binom{i}{j} \alpha^j$$

for $i \geq i$, $\alpha^i(t-1)^j$ is a polynomial of α degree $i-j$. In particular, $\alpha^{p-1}(t-1)^{p-1} \neq 0$. Hence the matrix of τ on this representation has only one Jordan block. The same is true for F^τ endowed with the action of $\langle\sigma\rangle$. Hence we can choose K-bases $(A_i)_{i=0}^{p-1} \subset F^\sigma$ and $(B_j)_{j=0}^{p-1} \subset F^\tau$ of F^σ and F^τ respectively which satisfy

$$\begin{aligned} A_i \cdot (\tau-1) &= A_{i-1} \quad (i = 1, 2, \dots, p-1), \\ B_j \cdot (\sigma-1) &= B_{j-1} \quad (j = 1, 2, \dots, p-1), \end{aligned}$$

and $A_0 = B_0 = 1, A_1 = \alpha, B_1 = \beta$. Then $(A_i B_j)_{i,j=0}^{p-1}$ is a K-basis of F.

Let $m \in \Theta_F$. Then we can write $m = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} c_{ij} A_i B_j$ with $c_{ij} \in K$. We can rephrase the two equations defining Θ_F the terms of c_j as follows:

$$\begin{aligned} m(\sigma-1)^2 = 0 \;&\Leftrightarrow\; 0 \leq i \leq p-1,\ 2 \leq {}^\forall j \leq p-1,\ c_{ij} = 0, \\ m(\tau-1) = am(\sigma-1) \;&\Leftrightarrow\; \sum_{i=0}^{p-2}\sum_{j=0}^{p-1} c_{i+1,j} A_i B_j = \sum_{i=0}^{p-1}\sum_{j=0}^{p-2} c_{i,j+1} A_i B_j \\ &\Leftrightarrow\; \left\{ \begin{array}{c} 1 \leq {}^\forall i \leq p-1, c_{i1} = 0 \\ c_{10} = ac_{01} \\ 2 \leq {}^\forall i \leq p-1,\ c_{i0} = 0 \end{array} \right. . \end{aligned}$$

Therefore, we can write $m = c_0 + c_1(a\alpha+\beta)$ with $c_0, c_1 \in K$. Let s be $-v_F(a\alpha+\beta)$. Then,

$$m \in \mathcal{O}_F \;\Leftrightarrow c_0 \in \mathcal{O}_K \text{ and } c_1 \in t^{\left\lceil \frac{s}{p^2} \right\rceil} \mathcal{O}_K.$$

We get an $\mathcal{O}_K$-basis $1, t^{\left\lfloor \frac{s}{p^2} \right\rfloor}(a\alpha+\beta)$ of Θ_F. Via the isomorphism in Proposition 5.1, this corresponds to the $\mathcal{O}_K$-basis

$$\varphi_2, \quad t^{\left\lfloor \frac{s}{p^2} \right\rfloor}(\varphi_1 + (a\alpha+\beta)\varphi_2)$$

of Ξ_F.

Since the formula (1),

$$\boldsymbol{v}_V(F) = v_K\left(t^{\left\lfloor \frac{s}{p^2} \right\rfloor} \det\left(\begin{bmatrix} 0 & 1 \\ 1 & a\alpha+\beta \end{bmatrix}\right)\right) = \left\lfloor \frac{s}{p^2} \right\rfloor.$$

On the other hand, $\gamma := a\alpha + \beta$ satisfies

$$\begin{aligned} N_{F/F^\tau}(\gamma) &= \gamma^p - \gamma \\ &= (a^p\alpha^p + \beta^p) - (a\alpha + \beta) \\ &= a^p(\alpha + g_1) + (\beta + g_2) - a\alpha - \beta \\ &= (a^p - a)\alpha + a^p g_1 + g_2. \end{aligned}$$

We get

$$s = -v_{F^\tau}((a^p - a)\alpha + a^p g_1 + g_2).$$

Now

$$v_{F^\tau}(\alpha) = v_K(N_{F^\tau/K}(\alpha)) = v_K(g_1),$$

which is not divided by p. Since

$$v_{F^\tau}(a^p g_1 + g_2) = p v_K(a^p g_1 + g_2) \in p\mathbb{Z},$$

we get $v_{F^\tau}(\alpha) \neq v_{F^\tau}(a^p g_1 + g_2)$ and

$$\begin{aligned} s &= -v_{F^\tau}((a^p - a)\alpha + a^p g_1 + g_2) \\ &= -\min\{v_{F^\tau}(\alpha), v_{F^\tau}(a^p g_1 + g_2)\} \\ &= -\min\{v_K(g_1), p v_K(a^p g_1 + g_2)\} \end{aligned}$$

Therefore, we get the value of v-function as

$$\boldsymbol{v}_V(g_1, g_2) = \left\lceil -\frac{\min\{v_K(g_1), p v_K(f)\}}{p^2} \right\rceil.$$

□

Corollary 5.3. *We keep the notation of this section. Then the value of $\boldsymbol{v}_V$ at a G-extension F/K is not determined by the ramification filtration of G associated to F/K.*

Proof. In the situation of Theorem 5.2, $g_1 = t^{-(p^2-1)}$ and $g_2 = ct^{-(p^2-1)} + t^{-1}$ where $c \in k - \mathbb{F}_p$. Then

$$\boldsymbol{v}_V(g_1, g_2) = \begin{cases} p & (c \neq -a^2) \\ 1 & (c = -a^2) \end{cases}.$$

In particular, $\boldsymbol{v}_V(g_1, g_2)$ depends on the value of c. We will show that the ramification filtration is independent of c, which proves the corollary.

The upper ramification filtration is compatible with passing to a quotient group, and hence the upper ramification filtration is determined from the ones of all the intermediate fields of F/K. An intermediate field is determined from a subgroup of $\mathbb{F}_p g_1 + \mathbb{F}_p g_2$. A ramification filtration of an intermediate field is determined from valuations of all elements of the corresponding subgroup. Then the upper ramification filtration of F is determined from the valuations of all the elements $\mathbb{F}_p g_1 + \mathbb{F}_p g_2$. Since $c \notin \mathbb{F}_p$, each nonzero element of $\mathbb{F}_p g_1 + \mathbb{F}_p g_2$ has valuation -3. Therefore, the ramification filtration of F_{g_1, g_2} does not depend on c. □

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References

[1] V.V. Batyrev, (1999). Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *J. Eur. Math. Soc.* 1, 5-33

[2] M. Brion, S. Kumar, (2005). Frobenius Splitting Methods in Geometry and Representation Theory. *Progress in Mathematics* 231. Boston, MA:Birkhauser Boston Inc.

[3] J. Denef, F. Loeser, (2002). Motivic integration, quotient singularities and the McKay correspondence. *Compositio Math.*, 131(3):267-290.

[4] Y. Ito, (1994). Crepant resolutions of trihedral singularities. *Proc. Japan Acad.* 70, 131-136

[5] Y. Ito, (1995). Crepant resolution of trihedral singularities and the orbifold Euler characterisitc, *Internat. J. Math.* 6:1, 33-43

[6] Y. Ito, (1995). Gorenstein quotient singularities of monomial type in dimension three, *J. Math. Sci. Univ. Tokyo* 2, 419-440

[7] J. Kollár, (2013). Singularities of the minimal model program. *Cambridge Tarcts in Mathematics,* Vol. 200. Cambridg University Press.

[8] S. Kumar, J.F. Thomsen,(2001). Frobenious splitting of Hilbert schemes of points on surfaces. *Mathematische Annalen* 319, no.4, 797-808

[9] D. Makushevich, (1993). Resolution of $\mathbb{C}^3/H_{168}$, *Technion prepreint*

[10] S-S. Roan, (1996). Minimal resolutions of Gorenstein orbifolds in dimension three, *Topology* 35, no.2, 489-508

[11] J.-P. Serre, Local fields, *Graduate Texts in Mathematics 67, Springer*

[12] M. Tanno,T. Yasuda, The wild McKay correspondence for cyclic groups of prime power order. *arXiv:2006.12048.*

[13] M.M. Wood, T. Yasuda, (2015). Mass formulas for local Galois representations and quotient singularities I : a comparison of counting gunctions, *Int. Math. Res. Not.* IMRN 2015(3)

[14] M.M. Wood, T. Yasuda, (2017). Mass formulas for local Galois representations and quotient singularities II : dualities and resolution of singularities. *Algebra Number Theory*, 11(4):817-840.

[15] T. Yasuda,(2019). Discrepancies of p-cyclic quotient varieties. *J. Math. Sci. Univ. Tokyo* 26 1-14.

[16] T. Yasuda, (2014). The p-cyclic McKay correspondence via motivic integration. *Comp. Math.* 150, 1125-1168

[17] T. Yasuda, (2017). The wild McKay correspondence and p-adic measures. *J. Eur. Math. Soc*, 19, 3709-3734.

[18] T. Yasuda, (2016). Wilder McKay correspondences. *Nagoya Math. J.* 221(1), 111-164

[19] T. Yasuda, (2019). Motivic integration over wild Deligne-Mumford stacks. *arXiv:1908.02932.*

[20] T. Yamamoto, (2018). A counterexmaple to the McKay correspondence in positive characteristic., Master thesis, Osaka University

[21] T. Yamamoto, (2019). Pathological quotient singularities in characteristic three which are not log canonical, to appear in Michigan Mathematical Jurnal.