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DRINFELD MODULAR FUNCTIONS

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0. Introduction

In 1973, Drinfeld introduced the notion of elliptic modules, which is now known as Drinfeld modules. After that the analogies between number fields and function fields have many interesting new aspects. Drinfeld modular function theory is one of these.

Drinfeld modular functions were studied by several mathematicians and known to have many properties analogous to those of classical elliptic functions, such as the generators of modular function fields, Galois groups between them. ([5],[8],[9]).

In the first part of this note, we establish some more properties of Drinfeld modular functions in analogy with those obtained by Shimura. In [12], Shimura proved his exact-sequence and his reciprocity law. Lang proved the exact-sequence another way in [11] using the isogeny theory. Shimura's proof of the reciprocity law is not easy. For example he used the parametrizations of the models of a modular function field over $\boldsymbol{Q}$. In [11], Lang avoided Shimura's method using the decomposition gorups which is well-known in algebraic number theory. In this article, we will follow Shimura's method to prove the exact-sequence, and Lang's method to prove the reciprocity law in the function field case.

In the second part, we go on to study two variable Drinfeld modular functions in analogy with two variable elliptic functions studied by Berndt. ([1]). In [2], he also gerneralized Shimura's exact-sequence and his reciprocity law corresponding to this extended modular function fields. We discuss the analogies of these in the Drinfeld setting.

1. Definitions and basic facts

Let $A=\boldsymbol{F}_q[T]$, $k=\boldsymbol{F}_q(T)$, k_∞ be the completion of k at $\infty=(\frac{1}{T})$, and C the completion of the algebraic closure of k_∞. Then C has an absolute value extending that of k_∞. By an A-lattice in C, we mean a projective A-submodule Λ of C which is discrete in the topology of C. A meromorphic function f on C is said to be *even* if $f(\mu z)=f(z)$ for every $\mu\in\boldsymbol{F}_q^*$. A meromorphic function f on C is

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called a *lattice function* for Λ if $f(z+\lambda)=f(z)$ for every $\lambda\in\Lambda$. For an A-lattice Λ, we define the lattice function

$$e_\Lambda(z)=z\prod_{\lambda\in\Lambda}{}'(1-z/\lambda).$$

The basic properties of the function $e_\Lambda: C\to C$ are ([8], I.2)

(i) e_Λ is entire, i.e. it converges uniformly on bounded sets.
(ii) e_Λ has simple zeros at the points of Λ and no further zeros.
(iii) e_Λ is unique up to constant multiple with properties (i) and (ii).
(iv) e_Λ is $\boldsymbol{F}_q$-linear and surjective.
(v) For $c\in C$, $e_{c\Lambda}(cz)=ce_\Lambda(z)$.

Lemma 1.1.
a) *If a meromorphic function f on C is even and has a zero of order m at u, then f has a zero of the same order at μu for every $\mu\in\boldsymbol{F}_q^*$.*
b) *If, moreover, f is a lattice function for Λ, then $(q-1)$ divides $ord_u(f)$ for $u\in\Lambda$.*

Proof. a) Let

$$f(z)=a(z-u)^m+\text{higher terms}$$

be the expansion of f around u. Then

$$\begin{aligned}f(z)=f(\mu^{-1}z)&=a(\mu^{-1}z-u)^m+\text{higher terms}\\&=\mu^{-m}a(z-\mu u)^m+\text{higher terms}.\end{aligned}$$

Hence we get the result.
b) We may assume $u=0$. Write

$$f(z)=az^m+\text{higher terms}.$$

Then

$$f(\mu z)=\mu^m az^m+\text{higher terms}.$$

Thus $\mu^m=1$, so that $(q-1)\mid m$.

Proposition 1.2. *The field of even lattice functions for Λ is generated by $e_\Lambda^{q-1}(z)$ over C.*

Proof. Exactly the same proof as in the classical case replacing $\wp(z)$ by $e_\Lambda^{q-1}(z)$ and using Lemma 1.1 would give the result.

By a *morphism* of lattices Λ and Λ' we mean a number $c \in C$ with $c\Lambda \subset \Lambda'$. Two lattices Λ and Λ' are said to be *similar* if $\Lambda' = c\Lambda$ for some nonzero $c \in C$. For each lattice Λ, it is well-known that we associate a Drinfeld module ϕ^Λ so that $e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$ and the association $\Lambda \mapsto \phi^\Lambda$ defines an equivalence of the categories of rank r A-lattices in C to the category of Drinfeld modules of rank r over C, which maps a similarity class to an isomorphism class.

From now on we only consider Drinfeld modules of rank 2 over C, so that we do not need to distinguish them from rank 2 A-lattices. $GL_2(k_\infty)$ acts on the rank 2 lattices in the usual manner. Then $\Lambda = \gamma\Lambda$ for $\gamma \in GL_2(k_\infty)$ if and only if $\gamma \in GL_2(A)$. The similarity classes of rank 2-lattices can be represented by $\Omega = C - k_\infty$ if we identify $z \in \Omega$ to the lattice $\Lambda_z = [z, 1]$ generated by z and 1. Therefore the set of the isomorphism classes of Drinfeld modules of rank 2 is parametrized by $GL_2(A)\backslash\Omega$.

For a Drinfeld module ϕ of rank 2 over C and $N \in A$, let $D(\phi, N)$ be the A-submodule $\operatorname{Ker}\phi_N$. We call $D(\phi, N)$ the set of *N-division points.*

By a *level N-structure* we mean an isomorphism

$$\alpha : (N^{-1}/A)^2 \to D(\phi, N).$$

An isomorphism between two Drinfeld modules with level N-structures is defined as in the classical case. Let $\Gamma(N) = \{\gamma \in \Gamma\colon\ \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$. Then the isomorphism classes of Drinfeld modules of rank 2 with level N-structure are parameterized by $\Gamma(N)\backslash\Omega$. Let ρ be the Carlitz module which is defined by $\rho_T = TX + X^q$. Then ρ corresponds to the lattice $\bar{\pi}A$ for some $\bar{\pi} \in C$. Let $N \in A$ and $\Lambda_N = \operatorname{Ker}\rho_N$. Let k_N be the field extension of k generated by Λ_N, which we call the *N-th cyclotomic function field.* Then we have

Theorem 1.3 ([10]) a). *k_N is a Galois extension of k with Galois group isomorphic to $(A/N)^*$. The action of $a \in (A/N)^*$ on Λ_N is that of ρ_a*

b) *k_N is ramified only on the divisors of N and ∞. The inertia group at ∞ is*

$$F_q^* \subset (A/N)^*.$$

c) *Let k_N^+ be the fixed field of F_q^*. Then k_N^+ is generated by λ^{q-1}, $\lambda \in \Lambda_N$, over k.*

Theorem 1.4 ([8]). *$\Gamma(N)\backslash\Omega$ can be given a structure of an affine curve $Y(N)$ over C. If we add some cusps to $Y(N)$, we get a projective curve $X(N)$. $X(N)$ can be defined over k_N^+.*

DEFINITION 1.5. Let Γ be a congruence subgroup of $GL_2(A)$. A function $f : \Omega \to C$ is a *modular form* of weight k, if the following conditions are satisfied;

(i) For $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$ and $z\in\Omega$, we have

$$f(\gamma z)=(cz+d)^k f(z)$$

(ii) f is meromorphic on Ω in the rigid analytic sense

(iii) f is meromorphic at the cusps of Γ, that is, f has a Laurent series expansion in $t_N=e^{-1}(\frac{\bar{\pi}z}{N})$, where $\Gamma(N)\subset\Gamma$.

In (ii) and (iii) if we replace meromorphic by holomorphic, we call such a modular form a *holomorphic* modular form.

We say that a holomorphic modular form f is a *cusp form* if f vanishes at all the cusps. A modular form of weight 0 is called a *modular function.*

Let ϕ be the Drinfeld module of rank 2 associated to the lattice $\Lambda_z=[z,1]$. For each $a\in A$ with $\deg a=d$

$$\phi_a=\sum_{i=0}^{2d} l_i(a,z)X^{q^i}.$$

Then $l_i(a,z)$ is a holomorphic modular form of weight q^i-1. In particular, let $a=T$. Then

$$\phi_T=TX+g(z)X^q+\Delta(z)X^{q^2}.$$

The functions $g(z)$ and $\Delta(z)$ are the most important modular forms, and $\Delta(z)$ is called the *discriminant* function. In fact, $\Delta(z)$ is a cusp form. $j(z)=g(z)^{q+1}/\Delta(z)$ is a modular function.

Let M_k be the set of all holomorphic modular forms of weight k for $GL_2(A)$, and $M=\oplus_k M_k$. Then as in the classical case, we have;

Theorem 1.6 ([9]). $M=C[g,\Delta]$.

The function j gives a bijection

$$GL_2(A)\backslash\Omega \underset{j}{\overset{\sim}{\rightarrow}} C.$$

Therefore $X(1)$ is the projective j-line and $C(X(1))=C(j)$. Since $X(N)$ is the projective model of the affine curve $\Gamma(N)\backslash\Omega$, the group of ramified covering for $X(N)$ over $X(1)$ is $\Gamma(1)/\Gamma(N)$, which is isomorphic to $\widetilde{GL_2}(A/N)/Z(\boldsymbol{F}_q)$ where

$$\widetilde{GL_2}(A/N)=\{\gamma\in GL_2(A/N):\det\gamma\in\boldsymbol{F}_q^*\}.$$

For $a=(a_1,a_2)\in(N^{-1}A/A)^2$, define

$$e_a(z)=e_{\Lambda_z}(a_1z+a_2)$$

where $\Lambda_z=[z,1]$. Then e_a is a modular form of weight -1 for $\Gamma(N)$. Define the *Fricke function*

$$h_a(z)=g(z)e_a(z)^{q-1}.$$

Then $\gamma\in GL_2(A)$ acts on h_a via $h_a^\gamma(z)=h_{a\cdot\gamma}(z)$. It is easy to see that $h_{a\cdot\gamma}(z)=h_a(\gamma z)$ and that $h_a^\gamma=h_a$ for $\gamma\in\Gamma(N)$.

Theorem 1.7 ([5]). *$C(X(N))$ is a Galois extension of $C(X(1))=C(j)$ generated by the Fricke functions h_a, $a\in(N^{-1}A/A)^2$. The Galois group is $\widetilde{GL_2}(A/N)/Z(\boldsymbol{F}_q)$ and its action on h_a is that given above.*

We now consider the field

$$F_N=k(j,h_a:a\in(N^{-1}A/A)^2)$$

Then we have

Theorem 1.8 ([8]).
(i) *The algebraic closure of k in F_N is k_N^+.*
(ii) $\mathrm{Gal}(F_N/k(j))=GL_2(A/N)/Z(\boldsymbol{F}_q)$ *with its action on h_a given by $h_a^\gamma=h_{a\gamma}$.*
(iii) *The elements in F_N have their coefficients of t_N-expansions in k_N.*
(iv) *The subgroup $\{\left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right):a\in(A/N)^*\}\simeq(A/N)^*$ of $GL(2,A/N)$ acts on the coefficients as the action of $(A/N)^*$ on k_N.*

If one follows the methods in [11], pp66–67, one gets

Corollary 1.9. *The action of $\gamma\in GL(2,A/N)$* on *k_N^+ is given by*

$$(\lambda^{q-1})^\gamma=[\rho_{\det\gamma}(\lambda)]^{q-1}$$

for $\lambda\in\Lambda(N)$.

2. Shimura exact sequence

Let $F=\cup_N F_N$. We discuss the structure of automorphism group $\mathrm{Aut}_k(F)$ of F over k. For each finite place v of k, let $G_v=GL_2(k_v)$. Define

$$U_f=\prod_{\substack{v\\ \text{finite}}} GL_2(A_v)$$

and

$$G(A_f)=\prod_{\substack{v\\ \text{finite}}}{}' GL_2(k_v)$$

where Π' means the restricted product with respect to U_f. Let $\tilde{k}^{ab}$ be the maximal abelian extension of k where ∞ splits completely. Then we have a natural map

$$\sigma: G(A_f) \to \mathrm{Gal}(\tilde{k}^{ab}/k)$$

given by

$$\sigma(x)=\left(\frac{(\det x)^{-1}, \tilde{k}^{ab}}{k}\right).$$

REMARK. We use $\tilde{k}^{ab}$, instead of the maximal abelian extension of k, because the algebraic closure of k in F_N is k_N^+.

For $u \in U_f$, define $\tau(u) \in \mathrm{Gal}(F/F_1)$ by $h_a^{\tau(u)}=h_{au}$ for every $a \in k^2/A^2$. Then we get

Proposition 2.1. (i) *The sequence*

$$1 \to F_q^* \to U_f \to \mathrm{Gal}(F/F_1) \to 1$$

is exact.

(ii) $\tau(u)=\sigma(u)$ *on* $\tilde{k}^{ab}$.

(iii) $h^{\tau(\gamma)}=h \circ \gamma$ *for every* $h \in F$ *and* $\gamma \in GL_2(A)$.

Now it is easy to see that $GL_2(A_f)=GL_2(k)\cdot U_f$. Hence we have to define the action of $GL_2(k)$ on F. For $\gamma \in GL_2(k)$ and $h \in F$, define

$$h^{\tau(\gamma)}=h \circ \gamma.$$

Then on $GL_2(k) \cap U_f = GL_2(A)$, the two definitions coincide from the fact that $h_{a\cdot\gamma}(z)=h_a(\gamma z)$ and $j(\gamma z)=j(z)$. To show that τ is a well-defined homomorphism, we need the following proposition.

Proposition 2.2. (i) *For every* $\gamma \in GL_2(k)$ *and for every* $h \in F$, *the function* $h \circ \gamma$ *belongs to* F.

(ii) *If* $\gamma_1, \gamma_2 \in GL_2(k)$, $u_1, u_2 \in U$ *and* $\gamma_1 u_1 = u_2 \gamma_2$, *then* $(j \circ \gamma_1)^{\tau(u_1)}=j \circ \gamma_2$, *and* $(h_a \circ \gamma_1)^{\tau(u_1)}=h_{a\cdot u_2} \circ \gamma_2$ *for every* $a \in k^2/A^2$, $\neq 0$.

Proof. Let $F'=k(h \circ \gamma\colon h \in F,\ \gamma \in GL_2(k))$. Choose a point $z_0 \in \Omega$ such that the specialization map $f \to f(z_0)$ defines an isomorphism of F' onto $F_0'=\{f(z_0)\colon f \in F'\}$. Taking suitable scalar multiples of γ_1 and γ_2 instead of γ_1 and γ_2, we may assume that γ_1^{-1} and γ_2^{-1} belong to $M_2(A)$. Define a Drinfeld module ϕ of rank 2 such that

$$\phi_T = TX + X^q + \frac{1}{j(z_0)} X^{q^2}.$$

Let $\Lambda_{z_0} = [z_0, 1]$ and Λ be the corresponding lattice for ϕ. Then $\Lambda = c\Lambda_{z_0}$ for some $c \in C$. Define

$$\xi : C \to C$$

by $\xi(x) = c \cdot e_{\Lambda_{z_0}}(x)$. Since c is isomorphism of Λ_{z_0} to Λ, it is easy to see that $g(z_0) = c^{q-1}$. Let $z_i = \gamma_i z_0$ for $i = 1,2$. Replacing z_0 by z_i, we can define ϕ^i, c_i and η_i corresponding to ϕ, c and ξ. Then we have

$$\eta_i \circ a = \phi^i_a \circ \eta_i \quad \text{for all} \quad a \in A.$$

We also have that $g(z_i) = c_i^{q-1}$. If $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(k)$, define $\mu_\alpha = cz_0 + d$. Let $\mu_i = \mu_{\gamma_i}$ for $i = 1,2$. Then we have

$$\mu_i^{-1} \begin{pmatrix} z_0 \\ 1 \end{pmatrix} = \gamma_i^{-1} \begin{pmatrix} z_i \\ 1 \end{pmatrix}.$$

It follows that the multiplication by $c^{-1}\mu_i^{-1}c_i$ defines an isogeny

$$c\Lambda_{z_0} \to c_i\Lambda_{z_i},$$

hence it induces an isogeny

$$\lambda_i : \phi \to \phi^i.$$

Then we have the following commutative diagram.

$$\begin{array}{ccccc} \Lambda_{z_0} & \to & C & \xrightarrow{\xi} & C \\ {\scriptstyle \mu_i^{-1}} \downarrow & & {\scriptstyle \mu_i^{-1}} \downarrow & & {\scriptstyle \lambda_i} \downarrow \\ \Lambda_{z_i} & \to & C & \xrightarrow{\eta_i} & C \end{array}$$

Let σ be the automorphism of $k(h(z_0) : h \in F)$ over $k(j(z_0))$ such that $h_a(z_0)^\sigma = h_{au_1}(z_0)$ for all $a \in k^2/A^2$, $\neq 0$. Extend it to an automorphism of C, and denote it again by σ. Then we see that

$$\xi\left(a\begin{pmatrix} z_0 \\ 1 \end{pmatrix}\right)^\sigma = \varepsilon\xi\left(au_1\begin{pmatrix} z_0 \\ 1 \end{pmatrix}\right),$$

for some $\varepsilon \in F_q^*$, because

$$\begin{aligned}\left(\left(\xi\left(a\begin{pmatrix}z_0\\1\end{pmatrix}\right)\right)^{\sigma}\right)^{q-1} &= \left(\left(ce_{\Lambda_{z_0}}\left(a\begin{pmatrix}z_0\\1\end{pmatrix}\right)\right)^{q-1}\right)^{\sigma}\\ &= \left(g(z_0)e_{\Lambda_{z_0}}\left(a\begin{pmatrix}z_0\\1\end{pmatrix}\right)^{q-1}\right)^{\sigma}\\ &= h_a(z_0)^{\sigma}\\ &= h_{au_1}(z_0)\\ &= \left(ce_{\lambda_{z_0}}\left(au_1\begin{pmatrix}z_0\\1\end{pmatrix}\right)\right)^{q-1}\\ &= \xi\left(au_1\begin{pmatrix}z_0\\1\end{pmatrix}\right)^{q-1}.\end{aligned}$$

Then following the method of Proposition 6.22 of [12], our proposition follows. The only thing to note in (i) is that any field extension is separable in the classical case, but we need to check it in our case. So we need to show that F' is separable over F. Then it reduces to showing that $j(\gamma z)$ and $h_a(\gamma z)$ are separable over F. Let $z'=\gamma z$. Since the coefficients of the polynomial

$$\prod_{a\in(N^{-1}A/A)^2}(X-h_a(z'))$$

are invariant under $GL_2(A)$ and holomorphic on Ω, they lie in $k(\Lambda_N)[j(z')]$. In fact, they lie in $k[j(z')]$, because they are fixed by

$$\left\{\begin{pmatrix}1&0\\0&d\end{pmatrix}: d\in(A/NA)^*\right\}.$$

(e.g. Theorem 1.8 (ii), (iv)). Then it reduces to showing that $j(\gamma z)$ is separable over F since $h_a(z')$'s are distinct. Following the method in [11, pp54–55], we have $j(\gamma z)$ is integral over $A[j]$ and separable over $k(j)$. This completes the proof.

Theorem 2.3. *The sequence*

$$1\to k^*\to G(A_f)\xrightarrow{\tau}\mathrm{Aut}_k(F)\to 1$$

is exact.

Proof. We claim that

$$k^* \cdot U_N = \{x \in G(A_f) : \tau(x) = id \quad \text{on} \quad F_N\},$$

where

$$U_N = \{u = (u_v) \in U_f : u_v \equiv 1 \mod N \cdot M_2(A_v)\}.$$

Let $x \in G(A_f)$ be in the kernel of τ. Write $x = u\gamma$ with $u \in U_f$, $\gamma \in GL_2(k)$. Since $j^{\tau(x)} = j \circ \gamma = j$, it is easy to see that $\gamma = y \cdot \gamma'$ for some $y \in k^*$, $\gamma' \in GL_2(A)$. Then we have

$$h_a^{\tau(x)} = h_{au} \circ \gamma = h_{au} \circ \gamma' = h_{au\gamma'}.$$

It follows that

$$h_a = h_{au\gamma'} \quad \text{for all} \quad a \in (N^{-1}/A)^2.$$

We regard $u\gamma'$ as an element in $GL_2(A/N)$. Putting a equal to $(\frac{1}{N}, 0)$, $(0, \frac{1}{N})$ and $(\frac{1}{N}, \frac{1}{N})$ respectively, it is easy to see that

$$u\gamma' \equiv \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \mod NA,$$

for some $\varepsilon \in F_q^*$. Thus our claim follwos. Then it is easy to see that

$$\ker \tau = \bigcap_N k^* U_N = k^*.$$

Exactly the same proof as in the classical case in [12] gives that τ is surjective.

3. Shimura's reciprocity law for F

Let L be an imaginary quadratic field, that is, L is a quadratic extension of k where ∞ does not split completely. Let $I = I_L$ be the group of the ideles in L without ∞-component, and let L^{ab} the maximal abelian extension of L where the infinite place splits completely. Let

$$(\ ,L) : I \to \mathrm{Gal}(L^{ab}/L)$$

be the Artin-homomorphism in class field theory. Then we have

Theorem 3.1 [6,(4.5)]. *For any $\omega_0 \in L \cap \Omega$, we have*

(i) *$j(\omega_0)$ lies in L^{ab} and $j(\omega_0)^{(s^{-1},L)} = j(s[\omega_0, 1])$ for $s \in L$.*

(ii) *L^{ab} is generated by $\{j(\omega_0), g(\omega_0)e_a(\omega_0)^{q-1} : a \in (k/A)^2\}$ over L if $j(\omega_0) \neq 0$ and generated by $\{\Delta(\omega_0)e_a(\omega_0)^{q^2-1} : a \in (k/A)^2\}$ over L if $j(\omega_0) = 0$.*

Fix a point $\omega_0 \in L \cap \Omega$. We set the following notations;

$$R=k[j]$$

$$S=\text{the integral closure of the ring } R \text{ in } F$$

$$\mathfrak{m}=\{f\in R: f(\omega_0)=0\}$$

$$\mathfrak{M}=\{f\in S: f(\omega_0)=0\}.$$

Then $\mathfrak{M}$ is a maximal ideal of S lying above $\mathfrak{m}$. If η is an automorphism of S which maps $\mathfrak{M}$ onto $\mathfrak{M}$ then η induces an sutomorphism on the residue class field, denoted by

$$\bar{\eta}: \bar{S} \to \bar{S}.$$

We identify $\bar{S}$ with the set of all elements $\bar{f}=f(\omega_0)$, $f\in S$. Let $G_{\mathfrak{M}}$ be the decomposition group of $\mathrm{Gal}(F/F_1)$. If $\sigma\in G_{\mathfrak{M}}$, then we will denote by $\bar{\sigma}$ its image in the Galois group of $S/\mathfrak{M}$ over $R/\mathfrak{m}$.

Lemma 3.2. *Suupose $j(\omega_0)\neq 0$. If $\tau(u)\in \mathrm{Gal}(F/F_1)$ satisfies*

$$h_a^{\tau(u)}(\omega_0)=h_a(\omega_0)$$

for all $a\in(k/A)^2$, then $\bar{f}^{\overline{\tau(u)}}=\bar{f}$ for all $f\in S$.

Proof. It suffices to show that $\bar{f}^{\tau(u)}=f$ for $f\in S\cap F_N$ for any given N. Then we can view u as an element in $GL_2(A/NA)$. Since $h_a^{\tau(u)}(\omega_0)=h_a(\omega_0)$ for $a=(\frac{1}{N},0)$, $(0,\frac{1}{N})$ and $(\frac{1}{N},\frac{1}{N})$, it is easy to see that

$$u=\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}\in GL_2(A/NA)$$

for some $\varepsilon\in \boldsymbol{F}_q^*$. Thus $\overline{\tau(u)}=1$, hence our Lemma follows.

We will show that Lemma 3.2 still holds in the case that $j(\omega_0)=0$ if we change h_a slightly. Define

$$h_a'(z)=\Delta(z)e_{\Lambda_z}\left(a\begin{pmatrix}z\\1\end{pmatrix}\right)^{q^2-1}$$

Then $\tau(u)$ acts on h_a' via $h_a'^{\tau(u)}=h_{au}'$.

Lemma 3.3. *Suppose $j(\omega_0)=0$. If $\tau(u)\in \mathrm{Gal}(F/F_1)$ satisfies*

$$h_a'^{\tau(u)}(\omega_0)=h_a'(\omega_0)$$

for all $a\in(k/A)^2$, then $\bar{f}^{\overline{\tau(u)}}=\bar{f}$ for all $f\in S$.

Proof. Since ω_0 is $GL_2(A)$-equivalent to an element in $F_{q^2}-F_q$, we assume that $\omega_0 \in F_{q^2}-F_q$. Then we have $\Lambda_{\omega_0}=F_{q^2}[T]$. We may assume that $f \in F_N$ for some N. Write

$$u=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A/NA).$$

Putting a equal to $(\frac{1}{N},0)$, $(0,\frac{1}{N})$ and $(\frac{T}{N},\frac{1}{N})$ respectively to the equation

$$h_a'^{\tau(u)}(\omega_0)=h_a'(\omega_0),$$

it is easy to see that $u \in GL_2(F_q)$ and

$$a\omega_0+b=\varepsilon\omega_0$$

$$c\omega_0+d=\varepsilon$$

for some $\varepsilon \in F_{q^2}^*$, so that $\tau(u)$ lies in the inertia group at ω_0.

Define an embedding

$$q: L^* \to GL_2(k)$$

$$s \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

satisfying $s\omega_0=a\omega_0+b$ and $s=c\omega_0+d$. By the continuity, we can extend q to an embedding of the idele group A_L^* of L, again denoted by

$$q: A_L^* \to G(A_f).$$

Then we have the following Shimura's reciprocity law.

Theorem 3.4. *Suppose $f \in F$ is defined at ω_0. If $L^{ab}(f(\omega_0))$ is a separable extension of L^{ab} then $f(\omega_0)$ lies in L^{ab} and $f^{\tau(q(s))}(\omega_0)=f(\omega_0)^{(s^{-1},L)}$. If $L^{ab}(f(\omega_0))$ is not separable over L^{ab}, then it is a purely inseparable extension of L^{ab}. In this case, we can extend (s^{-1},L) uniquely to an embedding of $L^{ab}(f(\omega_0))$ over L. We denote it again by (s^{-1},L). Then $f^{\tau(q(s))}(\omega_0)=f(\omega_0)^{(s^{-1},L)}$.*

Proof. Write $q(s)=u\cdot\alpha$, $u \in U_f$, $\alpha \in GL_2(k)$. First we will show that our Theorem holds in the case that $j(\omega_0)\neq 0$. Let ϕ^{ω_0} be the Drinfeld module satisfying

$$\phi_T^{\omega_0}=TX+X^q+\frac{1}{j(\omega_0)}X^{q^2}.$$

Then we can find c such that $c\Lambda_{\omega_0}$ is the corresponding lattice to ϕ^{ω_0}. Then

$c^{q-1}=g(\omega_0)$. Replacing ω_0 by $\alpha(\omega_0)$, we can define $\phi^{\alpha(\omega_0)}$ and find d corresponding to c satisfying $d^{q-1}=g(\alpha(\omega_0))$. Let $\mu=\mu_\alpha$ be as in the proof of Proposition 2.2. Following the classical method in [11, pp150–151], we have the following commutative diagram

$$(*)\qquad \begin{array}{ccc} (k/A)^2 & \to & k\Lambda_{\omega_0}/\Lambda_{\omega_0} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle s\mu^{-1}} \\ (k/A)^2 & \to & k\Lambda_{\alpha(\omega_0)}/\Lambda_{\alpha(\omega_0)} \end{array}$$

with $s\Lambda_{\omega_0}=\mu\Lambda_{\alpha(\omega_0)}$. Then we have

$$\begin{aligned} j(\omega_0)^{(s^{-1},L)} &= j(s[\omega_0,1]) \text{ by (3.1)} \\ &= j^{\tau(q(s))}(\omega_0). \end{aligned}$$

Let σ be any automorphism of C such that $\sigma=(s^{-1},L)$ on L^{ab}. Since d is well-defined up to the multiplication by an element in F_q^*, the following commutative diagram follows from [6,(1.12)].

$$(**)\qquad \begin{array}{ccc} k\Lambda_{\omega_0}/\Lambda_{\omega_0} & \xrightarrow{ce_{\Lambda_{\omega_0}}} & Tor(\phi^{\omega_0}) \\ {\scriptstyle s\mu^{-1}}\downarrow & & \downarrow{\scriptstyle \sigma} \\ k\Lambda_{\alpha(\omega_0)}/\Lambda_{\alpha(\omega_0)} & \xrightarrow[de_{\Lambda_{\alpha(\omega_0)}}]{} & Tor(\phi^{\alpha\omega_0}). \end{array}$$

By $(*)$ and $(**)$, we have

$$de_{\Lambda_{\alpha(\omega_0)}}\left(au\begin{pmatrix}\alpha\omega_0\\1\end{pmatrix}\right)=\left(ce_{\Lambda_{\omega_0}}\left(a\begin{pmatrix}\omega_0\\1\end{pmatrix}\right)\right)^{\sigma}$$

for $a\in(k/A)^2$. Taking the $(q-1)$-st power, we have

$$h_a^{\tau(q(s))}(\omega_0)=h_a(\omega_0)^{(s^{-1},L)}.$$

Thus our theorem holds for $f=j, h_a$.

Next we prove that the relation of the theorem is true for all elements in F. Define

$$\begin{aligned} R' &= L[j] \\ S' &= \text{the integral closure of the ring } R' \text{ in } L\cdot F \\ \mathfrak{m}' &= \{f\in R' : f(\omega_0)=0\} \\ \mathfrak{M}' &= \{f\in S' : f(\omega_0)=0\} \end{aligned}$$

$$G_{\mathfrak{M}'} = \text{the decomposition group of } \mathrm{Gal}(L\cdot F/L(j)).$$

where $L\cdot F$ is the compositum of L and F. Since the infinite place does not split in L, it is easy to see that

$$\mathrm{Gal}(L\cdot F/L(j)) = \mathrm{Gal}(F/k(j)).$$

Thus we will view $\tau(u)$ as an element in $\mathrm{Gal}(L\cdot F/L(j))$.

We claim that there exists an element $\rho\in \mathrm{Aut}_L(L\cdot F)$ which maps S' and $\mathfrak{M}'$ onto S' and $\mathfrak{M}'$ respectively satisfying $\rho=\tau(q(s))$ on $k(j)$ and $\bar{\rho}=\sigma$ on $\overline{S'}$. Extend $\tau(\alpha)$ to the automorphism of $L\cdot F$ such that $f^{\tau(\alpha)}=f\circ\alpha$ for $f\in L\cdot F$, and denote it again by $\tau(\alpha)$. Since $j\circ\alpha$ is integral over $L[j]$, $\tau(\alpha)$ induces an automorphism of S'. The formula $j(\alpha\omega_0)=j(\omega_0)^{(s^{-1},L)}$ shows that $\mathfrak{m}'^{\tau(\alpha)}\subset\mathfrak{M}'$ because $\tau(\alpha)$ leaves the constants fixed. Consequently $\mathfrak{m}'\subset\mathfrak{M}'^{\tau(\alpha^{-1})}$. Since $\mathfrak{M}'^{\tau(\alpha^{-1})}$ and $\mathfrak{M}'$ are prime ideals lying above $\mathfrak{m}'$, there exists an element $\pi\in\mathrm{Gal}(L\cdot F/L(j))$ such that $\mathfrak{M}'^{\pi}=\mathfrak{M}'^{\tau(\alpha^{-1})}$, whence we obtain $\mathfrak{M}'^{\pi\circ\tau(\alpha)}=\mathfrak{M}'$. Clearly $\pi\circ\tau(\alpha)=\tau(q(s))$ on $k(j)$. Then $\sigma|_{\overline{S'}}\circ(\overline{\pi\circ\tau(\alpha)})^{-1}$ lies in Galois group of $\overline{S'}$ over $\overline{R'}$. By the surjectivity of [11, pp364], there exists an automorphim $\lambda\in G_{\mathfrak{M}'}$ satisfying $\overline{\lambda\circ\pi\circ\tau(\alpha)}=\sigma$ on $\overline{S'}$. Put $\rho=\lambda\circ\pi\circ\tau(\alpha)$. Then ρ satisfies all the requirements of our claim.

Then $\tau(q(s))\rho^{-1}\in\mathrm{Gal}(L\cdot F/L(j))=\mathrm{Gal}(F/k(j))$ satisfies the requirements of Lemma 3.2, whence

$$\bar{f}^{\overline{\tau(q(s))}}=\bar{f}^{\bar{\rho}}=\bar{f}^{\sigma}$$

for all $f\in S$. Since $\bar{S}$ is a normal extension of $\bar{R}$ containing L_{ab} and the equation $\bar{f}^{\overline{\tau(q(s))}}=\bar{f}^{\sigma}$ holds for any extension σ of (s^{-1},L), our theorem follows.

The case that $j(\omega_0)=0$ follows similarly by taking Lemma 3.3 into account.

4. Two variable modular functions

Define

$$X=C\times\Omega.$$

Let

$$G=\left\{g=(r_1,r_2;\alpha)=\begin{pmatrix}1 & r_1 & r_2\\ 0 & a & b\\ 0 & c & d\end{pmatrix}:r_1,r_2\in k_\infty,\ ad-bc\neq 0\right\}$$
$$=k_\infty^2\cdot GL(2,k_\infty).$$

We fix the following notations;

$$G(1)=A^2\cdot GL(2,A)$$

$$G^0(N)=A^2\cdot\Gamma(N)$$
$$G(N)=(NA)^2\cdot\Gamma(N)$$

We let $g=(r_1,r_2,\alpha)$ in G act on X via

$$g\cdot(v,\omega)=(\frac{v+r_1\omega+r_2}{c\omega+d},\frac{a\omega+b}{c\omega+d}).$$

Let

$$t=t(\omega)=e_{\bar{\pi}A}^{-1}(\bar{\pi}\omega)\quad\text{and}\quad\varepsilon(v)=e_{\bar{\pi}A}^{-1}(\bar{\pi}v).$$

Put

$$e(v,\omega)=e_{[\omega,1]}(v):=e_\omega(v).$$

Then from [7]

$$e(v,\omega)=\bar{\pi}^{-1}\varepsilon^{-1}\prod_{a\in A-\{0\}}\frac{f_a(t)-\varepsilon^{-1}t^{|a|}}{f_a(t)},$$

where $f_a(t)=\rho_a(t^{-1})t^{|a|}$, $|a|=q^{\deg a}$. Thus

$$\bar{\pi}e(v,\omega)\in A(\varepsilon)((t)),$$

because $f_a(t)\in A[t]$ with the constant term in F_q^*. We know from [9] that $\bar{\pi}^{1-q}g(\omega)\in A[[t]]$, $\bar{\pi}^{1-q^2}\Delta(\omega)\in A[[t]]$, and

$$j(\omega)=\frac{1}{t^{q-1}}+h(t^{q-1}),\quad\text{for some}\quad h(t)\in A[[t]].$$

Define the *first Weber function* $z(v,\omega)$ by

$$z(v,\omega)=g(\omega)e(v,\omega)^{q-1}.$$

Then $z(v,\omega)$ has an expansion in $k(\varepsilon)((t))$.

Let K be a subfield of C containing k. Let $\mathfrak{R}_N(K)$ (resp. $\mathfrak{R}_N^0(K)$) be the field of functions f on X such that

(1) f is meromorphic on X in the rigid analytic sense.

(2) f is $G(N)$ (resp. $G^0(N)$)-invariant, that is,

a) $f(v+r_1\omega+r_2,\omega)=f(v,\omega)$ if $r_1,r_2\in NA$ (resp. $r_1,r_2\in A$)

b) $f(\frac{v}{c\omega+d},\alpha(\omega))=f(v,\omega)$.

(3) f has a meromorphic $t_N=t(\frac{\omega}{N})$-expansion with coefficients in K, that is,

there exist $R>0$, $\mathfrak{n}>0$ and $L\in N$ such that for $0<|\varepsilon|<R$, $0<|t|<\mathfrak{n}|\varepsilon|^L$, and all $g\in G(1)$, we have

$$f(g(v,\omega))=\sum_{\nu>-\infty} b_\nu^g(f)t_N^\nu$$

with $b_\nu^g\in K(\varepsilon_N)$, $\varepsilon_N=\varepsilon(\frac{v}{N})$.

Put

$$\mathfrak{R}_N(C)=\hat{\mathfrak{R}}_N \quad\text{and}\quad \mathfrak{R}_N(k_N)=\mathfrak{R}_N.$$

Then imitating the proof of Theorem 1 of [1] and using Proposition 1.2 we get;

Theorem 4.1. $\mathfrak{R}_1(K)=K(j(\omega),z(v,\omega))$.
For $(r,s)\in A^2-\{(0,0)\}$, put

$$e_{r,s}(\omega)=e(\frac{r\omega+s}{N},\omega)$$

$$z_{r,s}(\omega)=z(\frac{r\omega+s}{N},\omega)$$

and

$$w_{r,s}(v,\omega)=\frac{e(v,\omega)}{e(\frac{r\omega+s}{N},\omega)}.$$

Then it is not hard to see that $z_{r,s}(\omega)$, $w_{r,s}(v,\omega)\in\mathfrak{R}_N^0$.

Theorem 4.2. *Assume that K contains k_N. Then we have*

a) $\mathfrak{R}_N^0(K)=K(j,(z_{r,s})_{r,s},z,(w_{r,s})_{r,s})$

b) $\mathrm{Gal}(\mathfrak{R}_N^0(K)/\mathfrak{R}_1(K))=\widetilde{GL}(2,A/N)$

c) $\mathrm{Gal}(\mathfrak{R}_N^0/\mathfrak{R}_1)=GL(2,A/N)$.

Proof. Note that $\mathfrak{G}^0=G(1)/G^0(N)=GL(2,A)/\Gamma(N)=\widetilde{GL}(2,A/N)$, and that $\{\sigma_d=\begin{pmatrix}1&0\\0&d\end{pmatrix}:d\in(A/N)^*\}$ acts on the N-*th* root λ_N of ρ by $\sigma_d(\lambda_N)=\rho_d(\lambda_N)$. Then exactly the same proof as in the classical case ([1], Theorem 2) works replacing $\{\pm1\}$ by F_q^* and $SL(2,Z/N)$ by $\widetilde{GL}(2,A/N)$.

Now let, for $(r,s)\in A^2$,

$$Z_{r,s}(v,\omega)=z(\frac{v+r\omega+s}{N},\omega).$$

Then we easily get

(1) $Z_{r,s}(v+r_1\omega+s,\omega)=Z_{r+r_1,s+r_2}(v,\omega)$
(2) $Z_{r,s}(v+r_1\omega+r_2,\omega)=Z_{r,s}(\omega)$ if $(r_1,r_2)\equiv(0,0) \bmod N$.
(3) $Z_{r,s}(\frac{v}{c\omega+d},\alpha(\omega))=Z_{(r,s)\alpha}(v,\omega)\equiv Z_{r,s}(v,\omega)$ if $\alpha\equiv 1 \bmod N$.

It is straightforward to see that $Z_{r,s}(g(v,\omega))$ lies in $k_N(\varepsilon_N)((t_N))$, for any $g\in G(1)$. Following the methods in [1] we get;

Theorem 4.3. *Assume that K contains k_N.*

a) $\mathfrak{R}_N(K)=K(j,(z_{r,s})_{r,s},z,(w_{r,s})_{r,s},(Z_{r,s})_{r,s})$

b) $\mathrm{Gal}(\mathfrak{R}_N(K)/\mathfrak{R}_1(K))=(A/N)^2\cdot GL(2,A/N)$

c) $\mathrm{Gal}(\mathfrak{R}_N/\mathfrak{R}_1)=(A/N)^2\cdot \widetilde{GL}(2,A/N)$.

Let

$$\phi_T^{\omega}=T+g(\omega)\tau+\Delta(\omega)\tau^2$$

be the Drinfeld module associated to the lattice $[\omega,1]$. Then $\Delta(\omega)=\frac{g^{q+1}(\omega)}{j(\omega)}$. For $M\in A$ we have

$$\begin{aligned} z(Mv,\omega)&=g(\omega)e_\omega(Mv)^{q-1}\\ &=g(\omega)(\phi_M^{\omega}(e_\omega(v)))^{q-1}\\ &=g(\omega)(\sum h_i g^{\frac{q^i-1}{q-1}}e_\omega(v)^{q^i})^{q-1} \quad h_i\in k(j)\\ &=h_M(g(\omega)e_\omega(v)^{q-1}), \end{aligned}$$

for some homogeneous polynomial $h_M\in k(j)[X]$. For $r,s\in A$, we have

$$\begin{aligned} e_\omega(\frac{r\omega}{N})&=\sum_{i=0}^{2\deg r} h_i g(\omega)^{\frac{q^i-1}{q-1}}e_\omega(\frac{\omega}{N})^{q^i}\\ &=e_\omega(\frac{\omega}{N})\cdot \text{polynomial in } z_{1,0} \text{ with coefficients in } k(j), \end{aligned}$$

and

$$\begin{aligned} e_\omega(\frac{s}{N})&=\sum_{i=0}^{2\deg s} l_i g(\omega)^{\frac{q^i-1}{q-1}}e_\omega(\frac{1}{N})^{q^i}\\ &=e^{\omega}(\frac{1}{N})\cdot \text{polynomial in } z_{0,1} \text{ with coefficients in } k(j). \end{aligned}$$

Then

$$z_{r,s}(\omega)=g(\omega)(e_\omega(\frac{r\omega}{N})+e_\omega(\frac{s}{N}))^{q-1}$$

$$=g(\omega)(\sum_{v=0}^{q-1} a_v e_\omega(\frac{r\omega}{N})^v e_\omega(\frac{s}{N})^{q-1-v})$$

$$=\sum_{v=o}^{q-1}(\frac{e_{1,0}}{e_{0,1}})^v\cdot z_{0,1}\cdot\text{polynomial in } z_{1,0} \text{ and } z_{0,1}.$$

Therefore we have

$$z_{r,s},\ \frac{e_{r,s}}{e_{0,1}}\in k(j,z_{1,0},z_{0,1},\frac{e_{1,0}}{e_{0,1}})$$

Note that

$$w_{r,s}=\frac{e_{0,1}}{e_{r,s}}\cdot w_{0,1}.$$

Therefore we have

$$\mathfrak{R}_N^0(K)=K(j,z_{1,0},z_{0,1},\frac{e_{1,0}}{e_{0,1}},z,w_{0,1}).$$

Also

$$\begin{aligned}
&Z_{r,s}(v,\omega)\\
&=z(\frac{v+r\omega+s}{N},\omega)\\
&=g(\omega)(e_\omega(\frac{v}{N})+e_{r,s}(\omega))^{q-1}\\
&=g(\omega)\sum_{v=0}^{q-1} a_v e_\omega(\frac{v}{N})^{q-1-v} e_{r,s}(\omega)^v\\
&=\sum a_v Z_{0,0}\cdot\left(\frac{e_{r,s}(\omega)}{e_\omega(\frac{v}{N})}\right)^v\\
&=\sum a_v Z_{0,0}\left(\frac{e_{1,0}(\omega)}{e_\omega(\frac{v}{N})}\cdot\text{polynomials in } z_{1,0}+\frac{e_{0,1}(\omega)}{e_\omega(\frac{v}{N})}\cdot\text{polynomial in } z_{0,1}\right)^v.
\end{aligned}$$

Therefore

$$\mathfrak{R}_N(K)=K(j,z_{1,0},z_{0,1},\frac{e_{1,0}}{e_{0,1}},z,w_{0,1},\frac{e_{1,0}}{e_\omega(\frac{v}{N})},Z_{0,0}).$$

5. The Automorphism of $\mathfrak{R}$ over k

Let $\mathfrak{R}=\cup\mathfrak{R}_N$. For any L, define $G_L=L^2\cdot GL_2(L)$, as G in section 4. By Theorem 4.3 (c), we have

$$\begin{aligned}\mathrm{Gal}(\mathfrak{R}/\mathfrak{R}_1)&=\varprojlim(A/NA)^2\cdot GL(A/NA)\\&=\prod_{v:finite}(A_v)^2\cdot GL_2(A_v)\\&=:\bar{U}.\end{aligned}$$

Since the level is not fixed in $\mathfrak{R}$, write $z_{r,s}=z_a$, $w_{r,s}=w_a$, $Z_{r,s}=Z_a$ if $a=(\frac{r}{N},\frac{s}{N})$. For any element $\bar{u}=(m,u)\in\bar{U}$, let $\tau(\bar{u})$ be the corresponding element in $\mathrm{Gal}(\mathfrak{R}/\mathfrak{R}_1)$. Then $\tau(\bar{u})$ acts on $\mathfrak{R}_N$ via

$$\begin{gathered}j,z\mapsto j,z\\z_a,w_a\mapsto z_{au},w_{au}\\Z_a\mapsto Z_{au+\frac{m}{N}}.\end{gathered}$$

We need some explanation about the notation $au+\frac{m}{N}$. There exists a canonical isomorphism

$$\varphi:(k/A)^2\to\coprod_{v:finite}(k_v/A_v)^2.$$

Then $\varphi^{-1}(au+\frac{m}{N})\in(N^{-1}A/A)^2$. Denote it by $au+\frac{m}{N}$. Define $f^{\tau(g)}:=f\circ g$ for $g\in G_k$ and $f\in\mathfrak{R}$. Following the method in [2], $\tau(g)\in\mathrm{Aut}_k(\mathfrak{R})$ for all $g\in G_k$, and two difinitions of τ are the same on the intersection of their defined domains, $G_k\cap\bar{U}=G(1)$.

Define

$$G(A_f)=:\prod{}'k_v^2\cdot GL_2(k_v),$$

where $\prod'$ means the restricted product with respect to $\bar{U}$. Then we have

$$G(A_f)=G_k\cdot\bar{U}=\bar{U}\cdot G_k.$$

Under this decomposition, we can define the action of $G(A_f)$ on $\mathfrak{R}$ via τ. To show that

$$\tau:G(A_f)\to\mathrm{Aut}_k(\mathfrak{R})$$

is a well-defined homomorphism, we need the following proposition.

Proposition 5.1. *If $g\bar{u}=\bar{u}'g'$ for $g,g'\in G_k$ and $\bar{u},\bar{u}'\in\bar{U}$, then $\tau(g)\tau(\bar{u})=\tau(\bar{u}')\tau(g')$.*

Proof. Write $g=(l,\alpha)$, $g'=(l',\alpha')$, $\bar{u}=(m,u)$ and $\bar{u}'=(m',u')$. Since $g\bar{u}=\bar{u}'g'$, we have $\alpha u=u'\alpha'$ and $m+lu=l'+m'\alpha'$. It suffices to show that two actions are the same on the generators of $\mathfrak{R}_N$. By Proposition 2.2, two actions are the same on j, z_b. If the level N is 1, Z_b is equal to z. Thus it suffices to show that these actions are the same on Z_b and w_b. Choose a point $z_0=(v_0,\omega_0)\in C\times\Omega$ such that the specialization $f\mapsto f(z_0)$ defines an isomorphism of $\mathfrak{R}$ to $\mathfrak{R}_0=\{f(z_0)\,|\,f\in\mathfrak{R}\}$. Under this specialization map, it suffices to show our assertions for $Z_b(z_0)$, $w_b(z_0)$. We will view $\tau(\bar{u})$ as an automorphism of $\mathfrak{R}_0$. Extend it to an automorphism of C, and denote it by σ. Throughout this proof, we will denote v_0,ω_0 simply by v,ω. Let $a=b\alpha+\frac{l}{N}$. By definition, we have

$$Z_b^{\tau(g)\tau(\bar{u})}=z\left(\frac{1}{\mu}\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right),\alpha(\omega)\right)^{\tau(\bar{u})}.$$

On the other hand,

$$\begin{aligned}Z_b^{\tau(\bar{u}')\tau(g')}&=z\left(\frac{v+l'\binom{\omega}{1}}{\mu' N}+\left(bu'+\frac{m'}{N}\right)\begin{pmatrix}\alpha'(\omega)\\1\end{pmatrix},\alpha'(\omega)\right)\\&=z\left(\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\begin{pmatrix}\omega\\1\end{pmatrix}\right)\frac{1}{\mu'},\alpha'(\omega)\right)\end{aligned}$$

Thus we are going to show that two last terms of above equations are equal. First we assume that α^{-1}, $\alpha'^{-1}\in M_2(A)$. Let ϕ^ω be the Drinfeld module satisfying

$$\phi_T^\omega=TX+X^q+\frac{1}{j(\omega)}X^{q^2}.$$

In the proof of Proposition 2.2, we can find $c(\omega)$ such that $c(\omega)\Lambda_\omega$ is the corresponding lattice to ϕ^ω satisfying $g(\omega)=c(\omega)^{q-1}$. Let $c=c(\omega)$, $d=c(\alpha(\omega))$ and $d'=c(\alpha'(\omega))$. Then the multiplication by $c^{-1}\mu^{-1}d$ defines an isogeny

$$C/c\Lambda_\omega\to C/d\Lambda_{\alpha(\omega)},$$

and it induces the isogeny

$$\lambda:\phi^\omega\to\phi^{\alpha(\omega)}.$$

Similarly we can have tthe isogeny

$$\lambda':\phi^\omega\to\phi^{\alpha'(\omega)}.$$

Following the proof of Proposition 2.2, we can find $\varepsilon\in \boldsymbol{F}_q^*$ such that

$$\lambda^\sigma=\varepsilon\lambda'. \tag{1}$$

$$de_{\Lambda_{\alpha(\omega)}}\circ\frac{1}{\mu}=\lambda\circ ce_{\Lambda_\omega}. \tag{2}$$

$$d'e_{\Lambda_{\alpha'(\omega)}}\circ\frac{1}{\mu'}=\lambda'\circ ce_{\Lambda_\omega}. \tag{3}$$

We claim that there exists $\varepsilon'\in \boldsymbol{F}_q^*$ such that

$$\left(ce_{\Lambda_\omega}\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right)^\sigma=\varepsilon' ce_{\Lambda_\omega}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\begin{pmatrix}\omega\\1\end{pmatrix}\right). \tag{4}$$

Taking $(q-1)$-th power of left hand side of (4), we have

$$\left[g(\omega)e_{\Lambda_\omega}\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)^{q-1}\right]^{\tau(\bar{u})}.$$

By definition of $\tau(\bar{u})$, this is equal to

$$\begin{aligned}&g(\omega)e_{\Lambda_\omega}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\begin{pmatrix}\omega\\1\end{pmatrix}\right)^{q-1}\\ &=\left[ce_{\Lambda_\omega}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right]^{q-1}.\end{aligned}$$

This proves our claim. Now we have

$$\begin{aligned}&z\left(\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)\frac{1}{\mu},\alpha(\omega)\right)^\sigma\\
&=\left\{g(\alpha(\omega))e_{\Lambda_{\alpha(\omega)}}\left(\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)\frac{1}{\mu}\right)^{q-1}\right\}^\sigma\\
&=\left\{\left(de_{\Lambda_{\alpha(\omega)}}\left(\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)\frac{1}{\mu}\right)\right)^{q-1}\right\}^\sigma\\
&=\left\{\left(\lambda\circ ce_{\Lambda_\omega}\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right)^{q-1}\right\}^\sigma \quad \text{by (2)}\\
&=\left\{\lambda^\sigma\circ\left(ce_{\Lambda_\omega}\left(\frac{v}{N}+a\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right)^\sigma\right\}^{q-1}\\
&=\left\{\varepsilon\varepsilon'\lambda'\circ\left(ce_{\Lambda_\omega}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right)\right\}^{q-1} \quad \text{by (1), (4)}\end{aligned}$$

$$=\left\{d'e_{\Lambda_{\alpha'(\omega)}}\left(\frac{1}{\mu'}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\binom{\omega}{1}\right)\right)\right\}^{q-1} \quad \text{by (3)}$$

$$=g(\alpha'(\omega))e_{\Lambda_{\alpha'(\omega)}}\left(\frac{1}{\mu'}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\binom{\omega}{1}\right)\right)^{q-1}$$

$$=z\left(\frac{1}{\mu'}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\binom{\omega}{1}\right),\alpha'(\omega)\right).$$

This proves our assertion for Z_b in the case of α^{-1}, $\alpha'^{-1}\in M_2(A)$. In general case, write $\alpha=r\beta$ and $\alpha'=r\beta'$, with $r\in A$ and β^{-1}, $\beta'^{-1}\in M_2(A)$. Then the multiplication by $c^{-1}r\mu^{-1}d$ defines an isogeny

$$C/c\Lambda_\omega\to C/d\Lambda_{\alpha(\omega)},$$

and it induces the isogeny

$$\lambda:\phi^\omega\to\phi^{\alpha(\omega)}.$$

Then equation (1) is unchanged whereas equations (2) and (4) are changed as follows.

$$de_{\Lambda_{\alpha(\omega)}}\circ\frac{1}{\mu}=\lambda\circ ce_{\Lambda_\omega}\circ r^{-1}. \tag{2'}$$

$$\left(ce_{\Lambda_\omega}\left(\frac{1}{r}\left(\frac{v}{N}+a\binom{\omega}{1}\right)\right)\right)^\sigma=\varepsilon' ce_{\Lambda_\omega}\left(\frac{1}{r}\left(\frac{v}{N}+\left(au+\frac{m}{N}\right)\binom{\omega}{1}\right)\right). \tag{4'}$$

Similarly we can show that two actions are same on Z_b. Now it remains to show that two actions are same on w_b. We will only prove it under the assumption that $\alpha^{-1},\alpha'^{-1}\in M_2(A)$. The general case can be shown similarly. By definition, we have

$$w_a(v,\omega)^{\tau(\bar{u}')\tau(g')}=\frac{e_{\Lambda_{\alpha'(\omega)}}(\frac{1}{\mu'}(v+l'\binom{\omega}{1}))}{e_{\Lambda_{\alpha'(\omega)}}(au'\binom{\alpha'(\omega)}{1})}.$$

On the other hand,

$$w_a(v,\omega)^{\tau(g)\tau(\bar{u})}=\left\{\frac{e_{\Lambda_{\alpha(\omega)}}(\frac{1}{\mu}(v+l\binom{\omega}{1}))}{e_{\Lambda_{\alpha(\omega)}}(a\binom{\alpha(\omega)}{1})}\right\}^\sigma$$

$$=\frac{\{de_{\Lambda_{\alpha(\omega)}}(\frac{1}{\mu}(v+l\binom{\omega}{1}))\}^\sigma}{de_{\Lambda_{\alpha(\omega)}}(a\binom{\alpha(\omega)}{1})\}^\sigma}$$

$$=\frac{\{\lambda\circ ce_{\Lambda_\omega}(v+l\binom{\omega}{1})\}^\sigma}{\{\lambda\circ ce_{\Lambda_\omega}(a\alpha\binom{\omega}{1})\}^\sigma} \quad \text{by (2).}$$

Taking the $(q-1)$-th power and using the difinition of $\tau(\bar{u})$, we can prove followings as we did in (4).

$$\left\{ce_{\Lambda_\omega}\left(v+l\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right\}^\sigma=\varepsilon_1\left(ce_{\Lambda_\omega}\left(v+(lu+m)\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right)$$

and

$$\left\{ce_{\Lambda_\omega}\left(a\alpha\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right\}^\sigma=\varepsilon_2\left(ce_{\Lambda_\omega}\left(a\alpha u\begin{pmatrix}\omega\\1\end{pmatrix}\right)\right),$$

for some $\varepsilon_1,\varepsilon_2\in F_q^*$. By definition of $\tau(\bar{u})$, the following equation holds, which implies $\varepsilon_1=\varepsilon_2$.

$$\left(\frac{e_{\Lambda_\omega}(v+l(\begin{smallmatrix}\omega\\1\end{smallmatrix}))}{e_{\Lambda_\omega}(a\alpha(\begin{smallmatrix}\omega\\1\end{smallmatrix}))}\right)^\sigma=\left(\frac{e_{\Lambda_\omega}(v+(lu+m)(\begin{smallmatrix}\omega\\1\end{smallmatrix}))}{e_{\Lambda_\omega}(a\alpha u(\begin{smallmatrix}\omega\\1\end{smallmatrix}))}\right).$$

Together with these facts and by (1), we have

$$\begin{aligned}
w_a(v,\omega)^{\tau(g)\tau(\bar{u})}&=\frac{\varepsilon\varepsilon_1\lambda'(ce_{\Lambda_\omega}(v+(lu+m)(\begin{smallmatrix}\omega\\1\end{smallmatrix})))}{\varepsilon\varepsilon_2\lambda'(ce_{\Lambda_\omega}(a\alpha u(\begin{smallmatrix}\omega\\1\end{smallmatrix})))}\\
&=\frac{d'e_{\Lambda_{\alpha'(\omega)}}(\frac{1}{\mu'}(v+(lu+m)(\begin{smallmatrix}\omega\\1\end{smallmatrix})))}{d'e_{\Lambda_{\alpha'(\omega)}}(\frac{1}{\mu'}(a\alpha u(\begin{smallmatrix}\omega\\1\end{smallmatrix}))}\quad\text{by (3)}\\
&=\frac{e_{\Lambda_{\alpha'(\omega)}}(\frac{1}{\mu'}(v+(l'+m'\alpha')(\begin{smallmatrix}\omega\\1\end{smallmatrix})))}{e_{\Lambda_{\alpha'(\omega)}}(\frac{1}{\mu'}(au'\alpha'(\begin{smallmatrix}\omega\\1\end{smallmatrix}))}\\
&=\frac{e_{\Lambda_{\alpha'(\omega)}}(\frac{1}{\mu'}(v+l'(\begin{smallmatrix}\omega\\1\end{smallmatrix})))}{e_{\Lambda_{\alpha'(\omega)}}(au'(\begin{smallmatrix}\alpha'(\omega)\\1\end{smallmatrix}))}.
\end{aligned}$$

This proves our assertion for w_a, hence our proposition follows.

Proposition 5.2. $\tau: G(A_f)\to \mathrm{Aut}_k(\mathfrak{R})$ *is injective.*

Proof. Let $x=(n,\alpha)\in G(A_f)$ be a kernel of τ. By Theorem 2.3, α equals to $\begin{pmatrix}y&0\\0&y\end{pmatrix}$ for some $y\in k^*$. Write $n=l+m\alpha$, $l\in k^2$, $m\in\Pi(A_v)^2$. Then $x=\bar{u}\cdot g$ with $\bar{u}=(m,1)\in\bar{U},g=(l,\alpha)\in G_k$. Since $z^{\tau(x)}=z$, it's easy to see that $y\in F_q^*$ and $l\in A^2$. Since $\omega_a^{\tau(x)}=\omega_a$, it follows that $y=1$. Thus $x=(n,1)\in\bar{U}$. Finally, $n=0$ because $Z_a^{\tau(x)}=Z_a$.

6. Shimura's reciprocity law for $\mathfrak{R}$

Let L, L^{ab}, I and $(\ ,L)$ be as in section 3. Fix a point $z_0=(v_0,\omega_0)\in(L\times L)\cap X$.

We set the following notations;

$$R=k[j,z]$$

$$S=\text{the integral closure of the ring } R \text{ in } \mathfrak{R}$$

$$\mathfrak{m}=\{f\in R:f(z_0)=0\}$$

$$\mathfrak{M}=\{f\in S:f(z_0)=0\}$$

$$G_{\mathfrak{M}}=\text{the decomposition group of } \mathrm{Gal}(\mathfrak{R}/\mathfrak{R}_1).$$

Under this fixed point z_0, we can define $\bar{S}$, $\bar{\sigma}$ and $\bar{f}$ for $\sigma\in G_{\mathfrak{M}}$, $f\in S$ as we did in section 3.

Lemma 6.1. *Suppose $j(\omega_0)\neq 0$. If $\tau(\bar{u})\in\mathrm{Gal}(\mathfrak{R}/\mathfrak{R}_1)$ satisfies*

$$z_a^{\tau(\bar{u})}(z_0)=z_a(z_0)$$

$$Z_a^{\tau(\bar{u})}(z_0)=Z_a(z_0)$$

for all $a\in(k/A)^2$, then $\bar{f}^{\overline{\tau(\bar{u})}}=\bar{f}$ for all $f\in S$.

Proof. It suffices to show that $\bar{f}^{\overline{\tau(\bar{u})}}=\bar{f}$ for all $f\in S\cap\mathfrak{R}_N$, where $\deg N$ is suffiiciently large. Then $\bar{u}=(m,u)$ can be viewed as an element in $(A/NA)^2\cdot GL_2(A/NA)$. Since $z_a^{\tau(\bar{u})}(z_0)=z_a(z_0)$ for $a=(\frac{1}{N},0)$, $(0,\frac{1}{N})$ and $(\frac{1}{N},\frac{1}{N})$, it is easy to see that

$$u=\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}\in GL_2(A/NA)$$

for some $\varepsilon\in \boldsymbol{F}_q^*$. We claim that

$$v_0+m\begin{pmatrix}\omega_0\\ 1\end{pmatrix}\equiv\varepsilon v_0 \mod N\Lambda_{\omega_0}$$

Putting a equal to $(\frac{1}{N},0)$, $(0,\frac{1}{N})$ and $(\frac{T}{N},0)$, respectively, to the equation $Z_a^{\tau(\bar{u})}(z_0)=Z_a(z_0)$, we have

$$v_0+\varepsilon\omega_0+m\begin{pmatrix}\omega_0\\ 1\end{pmatrix}\equiv\varepsilon_1(v_0+\omega_0) \mod N\Lambda_{\omega_0} \tag{1}$$

$$v_0+\varepsilon+m\begin{pmatrix}\omega_0\\ 1\end{pmatrix}\equiv\varepsilon_2(v_0+1) \mod N\Lambda_{\omega_0} \tag{2}$$

$$v_0+\varepsilon T\omega_0+m\begin{pmatrix}\omega_0\\ 1\end{pmatrix}\equiv\varepsilon_3(v_0+T\omega_0) \mod N\Lambda_{\omega_0} \tag{3}$$

for some ε_1, ε_2 and $\varepsilon_3 \in \boldsymbol{F}_q^*$. Write $v_0 = x\omega_0 + y$ with $x, y \in k$. Then (1)–(2) gives that

$$(\varepsilon_1 - \varepsilon_2)x + (\varepsilon_1 - \varepsilon) \equiv 0 \mod NA \tag{4}$$

$$(\varepsilon_1 - \varepsilon_2)y + (\varepsilon - \varepsilon_2) \equiv 0 \mod NA. \tag{5}$$

If $x \notin \boldsymbol{F}_q$ (respectively $y \notin \boldsymbol{F}_q$), then (4) (respectively (5)) gives that $\varepsilon_1 = \varepsilon_2 = \varepsilon$ because $\deg N$ is large. Then our claim follows from (1). If both x and y lie in $\boldsymbol{F}_q$ then the equation (1)–(3) gives that $\varepsilon_3 = \varepsilon$, whence our claim follows from (3). Thus we can write

$$\frac{v_0 + m\binom{\omega_0}{1}}{\varepsilon} = v_0 + N_1\omega_0 + N_2. \tag{6}$$

for some $N_1, N_2 \in NA$. It follows that

$$\begin{aligned} \overline{f^{\tau(\bar{u})}} &= f(v_0 + N_1\omega_0 + N_2, \omega_0) \\ &= \overline{f(v + N_1\omega + N_2, \omega)} \\ &= \overline{f(v, \omega)} \\ &= \bar{f} \end{aligned}$$

When $j(\omega_0) = 0$, we change z_a and Z_a slightly by

$$z'_a = \Delta(\omega) e\left(a\begin{pmatrix}\omega \\ 1\end{pmatrix}, \omega\right)^{q^2-1}$$

$$Z'_a = \Delta(\omega) e\left(\frac{v}{N} + a\begin{pmatrix}\omega \\ 1\end{pmatrix}, \omega\right)^{q^2-1}.$$

Then we have

Lemma 6.2. *Suppose $j(\omega_0) = 0$. If $\tau(\bar{u}) \in \mathrm{Gal}(\mathfrak{R}/\mathfrak{R}_1)$ satisfies*

$$z_a'^{\tau(\bar{u})}(z_0) = z'_a(z_0)$$

$$Z_a'^{\tau(\bar{u})}(z_0) = Z'_a(z_0)$$

for all $a \in (k/A)^2$, then $\overline{f^{\tau(\bar{u})}} = \bar{f}$ for all $f \in S$.

Define an embedding

$$\bar{q} : L^* \to k^2 \cdot GL_2(k)$$

$$s \to (l(s), q(s))$$

with the properties;

$$s\begin{pmatrix}\omega_0\\1\end{pmatrix}=q(s)\begin{pmatrix}\omega_0\\1\end{pmatrix} \tag{7}$$

$$sv_0=v_0+l(s)\begin{pmatrix}\omega_0\\1\end{pmatrix}. \tag{8}$$

By the continuity, we can extend $\bar{q}$ to an embedding of the idele groups A_L^* of L, and denote it again by

$$\bar{q}:A_L^*\to G(A_f).$$

Then Shimura's reciprocity law in our case is given by

Theorem 6.3. *Suuppose $f\in\mathfrak{R}$ is defined at z_0. If $L^{ab}(f(z_0))$ is a separable extension of L^{ab} then $f(z_0)$ lies in L^{ab} and $f^{\tau(\bar{q}(s))}(z_0)=f(z_0)^{(s^{-1},L)}$. If $L^{ab}(f(z_0))$ is not separable over L^{ab}, then it is a purely inseparable extension of L^{ab}. In this case, we can extend (s^{-1},L) uniquely to an embedding of $L^{ab}(f(z_0))$ over L. We denote it again by (s^{-1},L). Then $f^{\tau(\bar{q}(s))}(z_0)=f(z_0)^{(s^{-1},L)}$.*

Proof. Write $\bar{q}(s)=\bar{u}g$ with $\bar{u}=(m,u)\in\bar{U}$, $g=(l,\alpha)\in G_k$. Then we have

$$q(s)=u\alpha \tag{9}$$

$$l(s)=l+m\alpha. \tag{10}$$

We will show that our Theorem holds for $j(\omega_0)\neq 0$. By Theorem 3.4, our theorem holds for $f=j,z_a$. For $f=Z_a$, we have

$$\begin{aligned}
&Z_a(z_0)^\sigma\\
&=\left(\left(ce_{\Lambda_{\omega_0}}\left(\frac{v_0}{N}+a\begin{pmatrix}\omega_0\\1\end{pmatrix}\right)\right)^{q-1}\right)^\sigma\\
&=\left(\left(ce_{\Lambda_{\omega_0}}\left(s^{-1}\left(\frac{v_0+l(s)\binom{\omega_0}{1}}{N}+au\mu\begin{pmatrix}\alpha\omega_0\\1\end{pmatrix}\right)\right)\right)^\sigma\right)^{q-1} && \text{by (7), (8), (9)}\\
&=\left(de_{\Lambda_{\alpha\omega_0}}\left(\frac{v_0+l(s)\binom{\omega_0}{1}}{N\mu}+au\begin{pmatrix}\alpha\omega_0\\1\end{pmatrix}\right)\right)^{q-1} && \text{by } (**)\\
&=g(\alpha\omega_0)e_{\Lambda_{\alpha\omega_0}}\left(\frac{v_0+l\binom{\omega_0}{1}}{N\mu}+\left(au+\frac{m}{N}\right)\begin{pmatrix}\alpha\omega_0\\1\end{pmatrix}\right)^{q-1} && \text{by (10)}\\
&=Z_a^{\tau(\bar{q}(s))}(z_0),
\end{aligned}$$

where σ is any automorphism of C such that $\sigma=(s^{-1},L)$ on L^{ab}. Since $z=Z_a$ if the level N is equal to 1, we have proved our theorem for the special functions

$f=j,z,z_a,Z_a$. Define

$$R'=L[j,z],$$
$$\mathfrak{m}'=\{f\in R':f(z_0)=0\}.$$

Note that L is contained in $\mathfrak{R}$, so we need not define S', $\mathfrak{M}'$ as in the proof of Theorem 3.4 because they are the same with S, $\mathfrak{M}$, in this case. It is not hard to see that $j\circ g$ and $z\circ g$ are integral over $k[j,z]$. Then the proof of the following claim is mostly the same as that of the claim in the proof of Theorem 3.4.

Claim: There exists an element $\rho\in\mathrm{Aut}_k(\mathfrak{R})$ which maps S and $\mathfrak{M}$ onto S and $\mathfrak{M}$ respectively satisfying $\rho=\tau(\bar{q}(s))$ on $k(j,z)$ and $\bar{\rho}=\sigma$ on $\bar{S}$.

The rest are mostly the same as the proof of Theorem 3.4 by taking Lemma 6.1 into account. In the case that $j(\omega_0)=0$, we can argue similarly by taking Lemma 6.2 into account and by replacing z, z_a and Z_a by $z'=\Delta(\omega)e(v,\omega)^{q^2-1}$, z'_a and Z'_a respectively.

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