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# SOME REMARKS ON THE CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATIONS

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Dedicated to the memory of Professor Hitoshi Kumano-go

(Received March 31, 1983)

## 0. Introduction

In the present paper we consider the Cauchy problem for the following equation

$$(0.1) \qquad Lu \equiv (i\partial_t + \tau\Delta + \sum_{j=1}^m b_j(x)\partial_{x_j} + c(x))u(x, t) = 0$$

with initial data $u_0(x)$ at $t=0$, where $\tau$ is a constant such that $0 \le \tau \le 1$, and $b_j(x), c(x)$ belong to $\mathscr{B}^\infty(R_x^m)$. $\mathscr{B}^\infty(R_x^m)$ denotes the set of $C^\infty$-functions whose derivatives of any order are all bounded. If $\tau$ is positive, the above equation (0.1) is the typical equation of non-kowalewskian type which is not parabolic. The study of the equation (0.1) is important for the study of equations of general non-kowalewskian type.

For real $s$ let $H_s$ be the Sobolev space with the usual norm $\|\cdot\|_s$ and let $H_\infty \equiv \bigcap_{s \in R} H_s$ be the Fréchet space with semi-norms $\|\cdot\|_s$, $s = 0, \pm 1, \pm 2, \cdots$. We say that the Cauchy problem for (0.1) is well posed for the future (resp. for the past) in the space $H_\infty$, if there exists a constant $T > 0$ (resp. $T < 0$) such that for any initial data $u_0(x) \in H_\infty$ a unique solution $u(x, t) \in \mathscr{E}_t^0([0, T]; H_\infty)$ of (0.1), which takes $u_0(x)$ at $t=0$, exists. Here, $f(x, t) \in \mathscr{E}_t^0([0, T]; H_\infty)$ means that the mapping: $[0, T] \ni t \to f(x, t) \in H_\infty$ is continuous in the topology of $H_\infty$.

Our purpose is to prove the following theorem corresponding to the so-called Lax-Mizohata theorem for equations of kowalewskian type (Lax [5], Mizohata [6]).

**Theorem.** *In order that equation* (0.1) *is well posed for the future or for the past in the space* $H_\infty$, *it is necessary that there exist constants $M$ and $N$ such that the inequality*

$$(0.2) \qquad \sup_{x \in R^m, \omega \in S^{m-1}} |\sum_{j=1}^m \int_0^\rho \mathrm{Re}\, b_j(x + 2\tau\theta\omega)\omega_j d\theta| \le M \log(1+\rho) + N$$

*holds for any* $\rho \ge 0$. $S^{m-1}$ *denotes the unit sphere in* $R^m$.

REMARK 1. J. Takeuchi in [8] first studied the Cauchy problem for equations of non-kowalewskian type in the frame of $L^2$ space.

REMARK 2. S. Mizohata in [7] proves the following. It is necessary for (0.1) to be well posed in the space $L^2$ that the inequality (0.2) with $M=0$ holds for any $\rho \geqq 0$. He proves it by constructing the asymptotic solution based on Birkhoff [1]. In the present paper we use the energy method.

REMARK 3. The author in [3] has given a sufficient condition for (0.1) to be well posed in the space $H_\infty$. In particular, from [3] and the above theorem we can see that in the case $m=1$ the condition (0.2) is necessary and sufficient for (0.1) to be well posed for the future and for the past in the space $H_\infty$.

When constant $\tau$ equals zero, equation (0.1) is kowalewskian. Then, we remark that our theorem gives the $H_\infty$ version of the Lax-Mizohata theorem.

Now, a solution $u(x, t)$ of the equation

$$(0.3) \qquad (i\partial_t + \Delta)u(x, t) = 0$$

with initial data $u_0(x)$ at $t=0$ is written by

$$(0.4) \qquad u(x, t) = C_0 \int e^{i|z|^2} u_0(x + 2\sqrt{t}z) dz$$
$$= (2\pi)^{-m} \int e^{-ix \cdot \xi - it|\xi|^2} \hat{u}_0(\xi) d\xi ,$$

where $C_0^{-1} = \int e^{i|z|^2} dz$ and $\hat{u}_0(\xi)$ is the Fourier transform for $u_0(x)$. (0.4) shows that equation (0.3) is not well posed in the space $\mathcal{E}$, but well posed in the space $H_\infty$. $\mathcal{E}$ is the space of infinitely differentiable functions with the customary topology. In fact, if (0.3) is well posed in the space $\mathcal{E}$, for any compact set $K$ in $R_x^m$ and any $T>0$ there exist a non-negative integer $l$ and a compact set $K'$ in $R_x^m$ such that

$$\sup_{K \times [0,T]} |u(\cdot, \cdot)| \leqq C_{K,K',T} \sum_{|\alpha| \leqq l} \sup_{K'} |\partial_x^\alpha u_0(\cdot)|$$

for a constant $C_{K,K',T}$. So, if the intersection of the support of $u_0(x)$ and $K'$ is empty, $u(x, T)$ equals zero for a point $x$ belonging to $K$. Hence, it follows from the first equality of (0.4) that for a point $x_0 \in K$

$$\int e^{i|z|^2} u_0(x_0 + 2\sqrt{T}z) dz = 0$$

is valid for any $u_0(x)$ whose support does not intersect $K'$. This is not true. On the other hand we have from the second equality of (0.4) $\|u(\cdot, t)\|_s = \|u_0(\cdot)\|_s$ ($s=0, \pm 1, \cdots$) for any $t$, which follows that (0.3) is well posed in the space $H_\infty$. Therefore, it is natural to consider the Cauchy problem for (0.1) in the frame of the space $H_\infty$ corresponding to the frame of the space $\mathcal{E}$ for the kowalewskian

type.

As is stated in Remark 2, we use the energy method. The technique used in the present paper is based on [6]. But, in particular, localizations in the present paper and [6] are quite different. Roughly speaking, in the present paper we localize the solution of (0.1) in phase space along the classical trajectory for the Hamiltonian $-\tau\Delta$. The symbol $w(x, t; \xi)$ of this localizing (pseudo-differential) operator is defined by the solution of "equation of motion for Hamilton function $-\tau|\xi|^2$"

$$(0.5) \qquad \partial_t w(x, t; \xi) = \{w(x, t; \xi), -\tau|\xi|^2\},$$

where for $C^1$-functions $f(x, \xi)$ and $g(x, \xi)$ $\{f, g\}(x, \xi)$ implies the Poisson bracket

$$\sum_{j=1}^{m}(\partial_{x_j}f\partial_{\xi_j}g - \partial_{\xi_j}f\partial_{x_j}g).$$

## 1. Notations and preliminaries

Let $x=(x_1, \cdots, x_m)$ denote a point of $R^m$ and let $\alpha=(\alpha_1, \cdots, \alpha_m)$ be a multi-index whose components $\alpha_j$ are non-negative integers. We use the usual notation.

$$|\alpha| = \alpha_1 + \cdots + \alpha_m, \ x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}, \ \alpha! = \alpha_1! \cdots \alpha_m!$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}, \ D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m}, \ \partial_{x_j} = \frac{\partial}{\partial x_j},$$

$$D_{x_j} = -i\frac{\partial}{\partial x_j}, \ \langle x \rangle = (1+|x|^2)^{1/2}.$$

Let $\mathcal{S}$ on $R^m$ denote the Schwartz space of rapidly decreasing functions. For $u(x)\in\mathcal{S}$ the Fourier transform $\hat{u}(\xi)$ is defined by

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx, \ x\cdot\xi = x_1\xi_1 + \cdots + x_m\xi_m.$$

For real $s$ we define the Sobolev space as the completion of $\mathcal{S}$ in the norm

$$||u||_s = \{\int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 d\xi\}^{1/2}, \ d\xi = (2\pi)^{-m}d\xi.$$

We first state the definitions and theorems with respect to pseudo-differential operators without proofs. Let $S_{0,0}^0$ be the set of $C^\infty$-functions such that for any $\alpha$, $\beta$ we have

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha,\beta},$$

where $p_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_x^\beta p(x, \xi)$ and $C_{\alpha,\beta} > 0$ are constants independent of $(x, \xi) \in R^{2m}$. $S_{0,0}^0$ is a Fréchet space provided with semi-norms $|p|_{l,l'}^{(0)} = \max_{|\alpha|\leq l, |\beta|\leq l'} \sup_{x,\xi}$ $|p_{(\beta)}^{(\alpha)}(x, \xi)| (l, l'=0, 1, \cdots)$. The pseudo-differential operator $P=p(x, D_x)$ with symbol $\sigma(P)(x, \xi)=p(x, \xi)$ is defined by

$$P\phi(x) = \int e^{ix\cdot\xi} p(x, \xi)\hat{\phi}(\xi)d\xi$$

for $\phi \in \mathcal{S}$. For $p_j(x, \xi) \in S_{0,0}^0$ $(j=1, 2)$ we define $q_\theta(x, \xi)$ $(0 \leq \theta \leq 1)$ by

(1.1)     $q_\theta(x, \xi)$

$$= O_s\text{-}\iint e^{-iy\cdot\eta} p_1(x, \xi+\theta\eta)p_2(x+y, \xi)dy\,d\eta$$

$$\equiv \lim_{\varepsilon\to 0} \iint e^{-iy\cdot\eta}\chi(\varepsilon y, \varepsilon\eta)p_1(x, \xi+\theta\eta)p_2(x+y, \xi)dy\,d\eta\,,$$

where $\chi(y, \eta)$ belongs to $\mathcal{S}(R^{2m})$ such that $\chi(0, 0)=1$. Then, it is known that $q_1(x, D_x)=p_1(x, D_x)\circ p_2(x, D_x)$, where "$\circ$" denotes the product of operators (see chap. 2 in [4]). We often write $q_1(x, \xi)=\sigma(P_1\circ P_2)(x, \xi)$.

**Theorem A** (Theorem 3.1 of chap. 2 in [4]). *Let define $q_1(x, \xi)$ by (1.1). Then, for any positive integer $\nu$ we get*

$$q_1(x, \xi) = \sum_{0 \leq |\gamma| \leq \nu-1} \frac{1}{\gamma!} p_1^{(\gamma)}(x, \xi)p_{2(\gamma)}(x, \xi) + \nu \sum_{|\gamma|=\nu} \int_0^1 \frac{(1-\theta)^{\nu-1}}{\gamma!} q_{\theta,\gamma}(x, \xi)d\theta\,,$$

*where*

$$q_{\theta,\gamma} = O_s\text{-}\iint e^{-iy\cdot\eta} p_1^{(\gamma)}(x, \xi+\theta\eta)p_{2(\gamma)}(x+y, \xi)dy\,d\eta\,.$$

**Theorem B** (Lemma 2.2 of chap. 7 in [4]). *For $q_\theta(x, \xi)$ defined by (1.1) we get*

$$|q_\theta|_{l,l}^{(0)} \leq C_l |p_1|_{l',l'}^{(0)} |p_2|_{l',l'}^{(0)}\,,$$

*where $l'=l+2[m/2+1]$ $(l=0, 1, 2, \cdots)$ and constants $C_l$ are independent of $\theta$ $(0 \leq \theta \leq 1)$, but depend on $l$. For real $r$ $[r]$ denotes the largest integer not greater than $r$.*

**Theorem C** (Calderón-Vaillancourt theorem, [2] or Theorem 1.6 of chap. 7 in [4]). *Let $p(x, \xi)$ belong to $S_{0,0}^0$. Then, we get*

$$\|p(x, D_x)\phi\| \leq C |p|_{l_0,l_0}^{(0)}\|\phi\|$$

*for $\phi \in \mathcal{S}$, where $\|\cdot\|=\|\cdot\|_0$, $l_0=2[m/2+1]$ and $C>0$ is a constant independent of $p(x, \xi)$ and $\phi$.*

Now, we shall prepare two lemmas. At first, we note that when $\tau$ is positive,

(1.2)     $$\sum_j \int_0^\rho \operatorname{Re} b_j(x+\tau\theta\omega)\omega_j d\theta = \frac{1}{\tau}\sum_j \int_{L_{x,x+\tau\rho\omega}} \operatorname{Re} b_j dx_j$$

holds. Here, integral $\int_{L_{x,x+\tau\rho\omega}}(\cdots)dx_j$ means curvilinear integral along the straight

line $L_{x,x+\tau\rho\omega}$ from a point $x \in R^m$ to a point $x+\tau\rho\omega \in R^m$.

**Lemma 1.1.** *The following* (i) *and* (ii) *are equivalent.*
(i) *The inequality* (0.2) *with constants* $M$ *and* $N$ *holds for any* $\rho \geqq 0$.
(ii) *The inequality*

$$(0.2)' \qquad \sup_{x \in R^m, \omega \in S^{m-1}} -\sum_{j=1}^{m} \int_0^\rho \mathrm{Re}\ b_j(x+2\tau\theta\omega)\omega_j d\theta \leqq M \log(1+\rho)+N$$

*holds for any* $\rho \geqq 0$.

Proof. We have only to show that (ii) yields (i). When $\tau$ equals zero, the proof is easy. We shall prove in the case $\tau > 0$. By (1.2) and (ii) we have

$$\sum_j \int_0^\rho \mathrm{Re}\ b_j(x+2\tau\theta\omega)\omega_j d\theta$$

$$= -\sum_j \int_0^\rho \mathrm{Re}\ b_j(x+2\tau\rho\omega+2\tau\theta(-\omega))\,(-\omega_j)d\theta$$

$$\leqq M \log(1+\rho)+N,$$

which completes the proof. \hfill Q.E.D.

We set

$$(1.3) \qquad b(x;\xi) = -\sum_{j=1}^{m} \mathrm{Re}\ b_j(x)\xi_j .$$

Then, we get

**Lemma 1.2.** *Assume that for any large constants* $M$ *and* $N$ *the inequality* (0.2) *does not hold. Then, for any large constant* $M$ *there exist sequences* $x^{(k)} \in R^m$, $\omega^{(k)} \in S^{m-1}$, $\rho_k \geqq 0$, $k=1, 2, \cdots$ *such that*

$$(1.4) \qquad \rho_k \to \infty \ \text{as} \ k \to \infty ,$$

$$(1.5) \qquad \int_0^{\rho_k} b(x^{(k)}+2\tau\theta\omega^{(k)};\ \omega^{(k)})d\theta \geqq M \log(1+\rho_k)+k$$

*and for any* $t \in [0, \rho_k]$

$$(1.6) \qquad \int_0^t b(x^{(k)}+2\tau\theta\omega^{(k)};\ \omega^{(k)})d\theta \geqq 0 .$$

Proof. Noting Lemma 1.1 and the assumption in this lemma, for any large constant $M$ we can find sequences $y^{(k)} \in R^m$, $\sigma^{(k)} \in S^{m-1}$, $\delta_k \geqq 0$, $k=1, 2, \cdots$ such that

$$(1.7) \qquad \int_0^{\delta_k} b(y^{(k)}+2\tau\theta\sigma^{(k)};\ \sigma^{(k)})d\theta \geqq M \log(1+\delta_k)+k .$$

Set $F_k(t) = \int_0^t b(y^{(k)}+2\tau\theta\sigma^{(k)};\ \sigma^{(k)})d\theta$ and let $t_k$ be the point at which $F_k(t)$

has the minimal value on $[0, \delta_k]$. Then, we shall prove that (1.4), (1.5) and (1.6) hold, if we determine $x^{(k)}$, $\omega^{(k)}$ and $\rho_k (k=1, 2, \cdots)$ by

(1.8)        $x^{(k)} = y^{(k)} + 2\tau t_k \sigma^{(k)}, \quad \omega^{(k)} = \sigma^{(k)}, \quad \rho_k = \delta_k - t_k$ .

We can see that for $t \in [0, \rho_k]$

(1.9)        $\displaystyle\int_0^t b(x^{(k)} + 2\tau\theta\omega^{(k)}; \omega^{(k)})d\theta$

$\displaystyle = \int_0^t b(y^{(k)} + 2\tau(t_k+\theta)\sigma^{(k)}; \sigma^{(k)})d\theta$

$\displaystyle = \int_{t_k}^{t+t_k} b(y^{(k)} + 2\tau\theta\sigma^{(k)}; \sigma^{(k)})d\theta$

$\displaystyle = F_k(t+t_k) - F_k(t_k)$ .

So, the choice of $t_k$ shows that (1.6) holds for $t \in [0, \rho_k]$. By (1.7)–(1.9) and $F_k(t_k) \leq F_k(0) = 0$, we have

(1.10)        $\displaystyle\int_0^{\rho_k} b(x^{(k)} + 2\tau\theta\omega^{(k)}; \omega^{(k)})d\theta$

$= F_k(\delta_k) - F_k(t_k)$

$\geq F_k(\delta_k)$

$\geq M \log(1+\delta_k) + k$

$\geq M \log(1+\rho_k) + k$ ,

which implies that (1.4) and (1.5) hold.                              Q.E.D.

## 2.  Localization in phase space and proof of Theorem

We prove our theorem by contradiction.  That is, we assume the following:

(A.1)   Equation (0.1) is well posed for the future or for the past in the space $H_\infty$.

(A.2)   Inequality (0.2) does not hold for any large constants $M$ and $N$.

Here, we may assume without loss of generality in place of (A.1)

(A.1)′   Equation (0.1) is well posed for the future in the space $H_\infty$.

Then, by the assumption (A.1)′ there exists a $T>0$ such that for any initial data $u_0(x) \in H_\infty$ a unique solution $u(x, t) \in \mathcal{E}_t^0([0, T]; H_\infty)$ of (0.1) exists.  Since the space $\mathcal{E}_t^0([0, T]; H_\infty)$ is a Fréchet space with semi-norms $\max_{0 \leq t \leq T} \|f(\cdot, t)\|_s$, $s=0, \pm1, \pm2, \cdots$, we see by the closed graph theorem that the mapping: $H_\infty \ni u_0(x) \to u(x, t) \in \mathcal{E}_t^0([0, T]; H_\infty)$ is continuous.  Consequently, there exist a non-negative integer $q$ and a constant $C(T)>0$ such that

(2.1)        $\|u(\cdot, t)\| \leq C(T)\|u_0(\cdot)\|_q$

holds for $t \in [0, T]$.

For the above $q$ we take a constant $M$ such that

(2.2)    $M > \dfrac{m}{2} + 2\left[\dfrac{m}{2} + 1\right] + 3q$

and fix it through sections 2 and 3. Then, since Lemma 1.2 holds from the assumption (A.2), for this $M$ we can take the sequences $x^{(k)} \in R^m$, $\omega^{(k)} \in S^{m-1}$, $\rho_k \geq 0$ $(k=1, 2, \cdots)$ satisfying (1.4), (1.5) and (1.6). Moreover, we take a positive constant $\delta$ such that

(2.3)    $M > \dfrac{m}{2} + 2\left[\dfrac{m}{2} + 1\right] + (3+\delta)q$ .

We can assume from (1.4) that

(2.4)    $\rho_k \geq 1$, $\rho_k^{-(2+\delta)} \leq T$

for any $k$. We also fix these sequences and $\delta$ hereafter.

Let $h(x)$ be the $C^\infty$-function such that

$$\begin{cases} h(x) = 1 & \text{on } \{x; \, |x| \leq 1/4\}, \\ \operatorname{supp} h(\cdot) \subset \{x; \, |x| \leq 1/2\}, \end{cases}$$

where $\operatorname{supp} h(\cdot)$ implies the support of the function $h(x)$. Let $w_{n,k}(x, t; \xi)$ be the solution of (0.5) with initial data $\rho_k^{m/2} h(\rho_k(x-x^{(k)}))h(\rho_k^2(\xi-n\omega^{(k)})/n)$ at $t=0$. Then, we can easily get

(2.5)    $w_{n,k}(x, t; \xi)$
$= \rho_k^{m/2} h(\rho_k(x-x^{(k)}-2\tau t\xi))h(\rho_k^2(\xi-n\omega^{(k)})/n)$ .

For the solution $u(x, t)$ of (0.1) we call $W_{n,k}u(x, t)=w_{n,k}(x, t; D_x)u(x, t)$ the localized solution (in phase space along the solution $(x^{(k)}+2n\tau t\omega^{(k)}, n\omega^{(k)}) \in R_{x,\xi}^{2n}$ of the canonical equation with initial value $(x^{(k)}, n\omega^{(k)})$ at $t=0$ for the Hamilton function $\tau|\xi|^2$) (see Lemma 2.3). We note

(2.6)    $\sigma([i\partial_t + \tau\Delta, W_{n,k}])(x, t; \xi)$
$= i\partial_t w_{n,k}(x, t; \xi) - i\{w_{n,k}, -\tau|\xi|^2\} + \tau\Delta w_{n,k}$
$= \tau(\Delta w_{n,k})(x, t; \xi)$ ,

where $[\cdot, \cdot]$ indicates the commutator of operators and $\Delta w_{n,k}(x, t; \xi)=\sum_j (\partial_{x_j}^2 w_{n,k})$ $(x, t; \xi)$. Equality (2.6) is essential for the proof of Theorem. For any multi-indices $\alpha$ and $\beta$ we set

(2.7)    $w_{n,k}^{\alpha,\beta}(x, t; \xi) = \rho_k^{m/2}(\partial_x^\alpha h)(x)(\partial_\xi^\beta h)(\xi)\bigg|_{\substack{x = \rho_k(x-x^{(k)}-2\tau t\xi). \\ \xi = \rho_k^2(\xi-n\omega^{(k)})/n}}$

We note that $w_{n,k}^{0,0}(x, t; \xi) = w_{n,k}(x, t; \xi)$.

Now, we define a series of solutions of (0.1) as in [6] by using $x^{(k)}$, $\omega^{(k)}$ and $\rho_k$ determined above. Namely, we define their initial values. We set hereafter throughout sections 2 and 3

$$(2.8) \qquad n = n(k) = \rho_k^{3+\delta}.$$

Let $\psi(x) \in \mathcal{S}$ be a function such that $\psi(0) = 2$ and

$$(2.9) \qquad \operatorname{supp} \hat{\psi}(\cdot) \subset \{\xi; h(\xi) = 1\},$$

and then, we define

$$(2.10) \qquad \hat{\psi}_k(\xi) = e^{-ix^{(k)} \cdot \xi} \hat{\psi}(\xi - n\omega^{(k)}) \qquad (n = \rho_k^{3+\delta}),$$

that is,

$$(2.10)' \qquad \psi_k(x) = e^{i(x-x^{(k)}) \cdot n\omega^{(k)}} \psi(x-x^{(k)}).$$

Let $u_k(x, t) \in \mathcal{E}_t^0([0, T]; H_\infty)$ be the solution of (0.1) with initial data $\psi_k(x)$ at $t=0$. Then, we can easily get by (2.1) and the definition of $\psi_k(x)$

$$(2.11) \qquad \|u_k(\cdot, t)\| \leq C_1(T) n^q$$

with a constant $C_1(T) > 0$ for $t \in [0, T]$. We set

$$(2.12) \qquad v_k^{\alpha,\beta}(x, t) = W_{n,k}^{\alpha,\beta} u_k(x, t),$$

where $W_{n,k}^{\alpha,\beta} = w_{n,k}^{\alpha,\beta}(x, t; D_x)$. We often write $v_k(x, t) = v_k^{0,0}(x, t)$. Since $\operatorname{supp} \hat{\psi}_k(\cdot) \subset \{\xi; h(\rho_k^2(\xi - n\omega^{(k)})/n) = 1\}$ is valid from (2.9), (2.10) and $\rho_k \geq 1$, we get

$$\|v_k(\cdot, 0)\|^2 = \|w_{n,k}(x, 0; D_x)u_k(\cdot, 0)\|^2$$
$$= \int \rho_k^m |h(\rho_k(x-x^{(k)}))\psi(x-x^{(k)})|^2 dx,$$

which follows from $\psi(0) = 2$ and (1.4) that for large $k$

$$(2.13) \qquad \|v_k(\cdot, 0)\| \geq \|h(\cdot)\| > 0.$$

Now, take a positive integer $s$ such that

$$(2.14) \qquad \delta\left[\frac{s+2}{2}\right] \geq \frac{m-2}{2} + 2\left[\frac{m}{2}+1\right] + (3+\delta)(q+1)$$

and set by the localized solution $v_k(x, t)$

$$(2.15) \qquad \sigma_k(t) = \sum_{0 \leq |\alpha+\beta| \leq s} (\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]} \|v_k^{\alpha,\beta}(\cdot, t)\|$$

for $t \in [0, T]$, where for real $r$ $[r]$ denotes the largest integer not greater than $r$. We remark that since $\rho_k/n = \rho_k^{-(2+\delta)}$ is not greater than $T$ for any $k$, $\sigma_k(t)$ has been

defined on the interval $[0, \rho_k/n]$. Then, we obtain

**Lemma 2.1.** *We have*

$$(2.16) \qquad \sigma_k(t) \leq C_0 \rho_k^{m/2 + 2[m/2+1] + (3+\delta)q}$$

*for any* $t \in [0, \rho_k/n]$ $(n = \rho_k^{3+\delta})$, *where* $C_0$ *is a constant independent of* $k$.

**Proposition 2.2.** *For large* $k$ *we get*

$$(2.17) \qquad \sigma_k(\rho_k/n) \geq C_1(1 + \rho_k)^M \qquad (n = \rho_k^{3+\delta})$$

*with a positive constant* $C_1$ *independent of* $k$.

Lemma 2.1 will be proved after the proof of Theorem and Proposition 2.2 will be proved in section 3.

Proof of Theorem. Since we have determined constant $\delta > 0$ so that (2.3) holds, (2.16) and (2.17) is not compatible for large $k$. Thus, we can prove Theorem. Q.E.D.

Proof of Lemma 2.1. By Theorem C we get

$$\|v_k(\cdot, t)\| \leq C \rho_k^{m/2} |h(\rho_k(x - x^{(k)} - 2\tau t\xi)) h(\rho_k^2(\xi - n\omega^{(k)})/n)|_{l_0, l_0}^{(0)} \|u_k(\cdot, t)\|$$
$$\leq C' \rho_k^{m/2 + l_0} \|u_k(\cdot, t)\|,$$

where $l_0 = 2[m/2 + 1]$. Here, we used $0 \leq \rho_k t \leq \rho_k^2/n = \rho_k^{-(1+\delta)}$ for $t \in [0, \rho_k/n]$. Consequently, we obtain from (2.11) for $t \in [0, \rho_k/n]$

$$\|v_k(\cdot, t)\| \leq C \rho_k^{m/2 + l_0} n^q$$

with another constant $C$ independent of $k$. In the same way we obtain for $t \in [0, \rho_k/n]$

$$(2.18) \qquad \|v_k^{\alpha \cdot \beta}(\cdot, t)\| \leq C_{\alpha, \beta} \rho_k^{m/2 + l_0} n^q$$

with constant $C_{\alpha, \beta}$ independent of $k$. Hence, we get (2.16) by $n = \rho_k^{3+\delta}$. Q.E.D.

**Lemma 2.3.** *If* $t \in [0, \rho_k/n]$ $(n = \rho_k^{3+\delta})$, *then we have*

$$(2.19) \qquad \text{supp } w_{n,k}^{\alpha \cdot \beta}(\cdot, t; \cdot)$$
$$\subset \{(x, \xi); |x - (x^{(k)} + 2n\tau t\omega^{(k)})| \leq 2/\rho_k, |\xi/n - \omega^{(k)}| \leq 1/(2\rho_k^2)\}.$$

Proof. If $(x, \xi) \in \text{supp } w_{n,k}^{\alpha \cdot \beta}(\cdot, t; \cdot)$, we have from the definition (2.7) of $w_{n,k}^{\alpha \cdot \beta}$

$$|x - (x^{(k)} + 2\tau t\xi)| \leq 1/(2\rho_k), \quad |\xi/n - \omega^{(k)}| \leq 1/(2\rho_k^2).$$

So, noting that $0 \leq \tau \leq 1$, it follows that

$$|x-(x^{(k)}+2n\tau t\omega^{(k)})|$$
$$\le |x-(x^{(k)}+2\tau t\xi)|+2n\tau t|\xi/n-\omega^{(k)}|$$
$$\le 2/\rho_k$$

for any $t\in[0, \rho_k/n]$.   This completes the proof.                    Q.E.D.

Now, if we use the equality (2.6), we can easily get for the localized solution $v_k(x, t)=W_{n,k}u_k(x, t)$

(2.20)  $Lv_k(x, t)$
$$= f_k(x, t)$$
$$\equiv \{[\sum_j b_j(x)\partial_{x_j}+c(x), W_{n,k}]+\tau(\Delta w_{n,k})(x, t; D_x)\}u_k.$$

Then, we obtain

**Lemma 2.4.** *Let $t\in[0, \rho_k/n]$ $(n=\rho_k^{3+\delta})$.  Then, for any $p=1, 2, \cdots$ we get*

(2.21)  $\|f_k(\cdot, t)\|$
$$\le \rho_k^2 \sum_{|\alpha+\beta|=2}\|v_k^{\alpha,\beta}(\cdot, t)\|+C_p n \sum_{1\le|\alpha+\beta|\le p+1}(\rho_k^2/n)^{|\alpha+\beta|}\|v_k^{\alpha,\beta}(\cdot, t)\|$$
$$+C_p n^{q+1}\rho_k^\lambda(\rho_k^2/n)^{p+1},$$

*where $\lambda=m/2+4[m/2+1]$ and $C_p$ is a positive constant independent of $k$.*

Proof.   We can easily see from (2.20)

(2.22)  $\|f_k(\cdot, t)\|$
$$\le \sum_j \|[b_j\partial_{x_j}, W_{n,k}]u_k(\cdot, t)\|+\|[c(x), W_{n,k}]u_k(\cdot, t)\|+\rho_k^2 \sum_{|\alpha+\beta|=2}\|v_k^{\alpha,\beta}(\cdot, t)\|.$$

We first consider the term $[b_j\partial_{x_j}, W_{n,k}]u_k(x, t)$.  If we use the notation (2.7), we can write

(2.23)  $[b_j\partial_{x_j}, W_{n,k}]u_k(x, t)$
$$= \rho_k b_j(x)W_{n,k}^{e_j,0}u_k(x, t)+[b_j, W_{n,k}]\partial_{x_j}u_k(x, t),$$

where $e_j$ is the multi-index whose $j$-th component is one and other components are all zero.  Then, for the first term of the right hand side of (2.23) its $L^2$ norm is estimated by the second term of the right hand side of (2.21).

We consider the second term in (2.23).  By Theorem A in section 1 we obtain

(2.24)  $\dfrac{1}{i}\sigma([b_j(x), W_{n,k}]\partial_{x_j})$
$$= -\{\sum_{1\le|\gamma|\le p}\frac{1}{\gamma!}D_x^\gamma b_j(x)\partial_\xi^\gamma w_{n,k}(x, t; \xi)\}\xi_j+r_{p,k}(x, t; \xi),$$

where $r_{p,k}(x, t; \xi)$ consists of the sum of

$$-(p+1)\frac{1}{\gamma!}\int_0^1 (1-\theta)^p d\theta \ O_s\text{-}\iint e^{-iy\cdot\eta}(D_x^\gamma b_j)\ (x+y)$$

$$(\partial_\xi^\gamma w_{n,k})\ (x, t;\ \xi+\theta\eta)\xi_j\ dyd\eta$$

over $\gamma$ such that $|\gamma|=p+1$. Using

$$O_s\text{-}\iint e^{-iy\cdot\eta}(D_x^\gamma\ b_j)\ (x+y)\ (\partial_\xi^\gamma w_{n,k})\ (x, t;\ \xi+\theta\eta)\xi_j\ dyd\eta$$

$$= O_s\text{-}\iint e^{-iy\cdot\eta}(D_x^\gamma\ b_j)\ (x+y)\ (\partial_\xi^\gamma w_{n,k})\ (x, t;\ \xi+\theta\eta)\ (\xi_j+\theta\eta_j)\ dyd\eta$$

$$-\theta\ O_s\text{-}\iint e^{-iy\cdot\eta}\ D_{y_j}(D_x^\gamma\ b_j)\ (x+y)\ (\partial_\xi^\gamma w_{n,k})\ (x, t;\ \xi+\theta\eta)\ dyd\eta$$

and then applying Theorem B, we get the estimates from (2.5) and Lemma 2.3

(2.25) $\quad |r_{p,k}(\cdot, t; \cdot)|_{0, l_0}^{(0)}$

$$\leqq C_{p,1}\ n\rho_k^\lambda \sum_{|\gamma_1+\gamma_2|=p+1}(\rho_k\ t)^{|\gamma_1|}(\rho_k^2/n)^{|\gamma_2|}$$

$$\leqq C_{p,2}\ n\rho_k^\lambda(\rho_k^2/n)^{p+1}$$

for $t\in[0,\ \rho_k/n]$, where $l_0=2[m/2+1]$ and $C_{p,1},\ C_{p,2}$ are positive constants depending only on $p$. Here, we used $\rho_k t\leqq\rho_k^2/n$ for $t\in[0,\ \rho_k/n]$. Consequently, applying Theorem C, we get

(2.26) $\quad \|r_{p,k}(x, t; D_x)u_k(\cdot, t)\|\leqq C_{p,3}\ n^{q+1}\rho_k^\lambda(\rho_k^2/n)^{p+1}$

by (2.11).

Next, we consider the first term in (2.24). We remark

(2.27) $\quad (\partial_\xi^\gamma w_{n,k})\ (x, t;\ \xi)\xi_j$

$$= \sum_{\alpha+\beta=\gamma}\frac{\gamma!}{\alpha!\,\beta!}\ (-2\tau t\rho_k)^{|\alpha|}(\rho_k^2/n)^{|\beta|}w_{n,k}^{\alpha;\beta}(x, t;\ \xi)\xi_j\ .$$

We can easily see

(2.28) $\quad \|W_{n,k}^{\alpha;\beta}\ D_{x_j}u_k(\cdot, t)\|$

$$\leqq \rho_k\|W_{n,k}^{\alpha+e_j;\beta}u_k(\cdot, t)\|+\|D_{x_j}\circ W_{n,k}^{\alpha;\beta}\ u_k(\cdot, t)\|\ ,$$

in which the second term is estimated by

$$K\ n\|W_{n,k}^{\alpha;\beta}\ u_k(\cdot, t)\|+C_{p,\alpha,\beta}\ n^{q+1}\rho_k^\lambda(\rho_k^2/n)^{p+1}\ ,$$

where $K=3\max\limits_{x\in R^m}|h(x)|$ and $C_{p,\alpha,\beta}$ are constants independent of $k$, but depend on $\alpha$ and $\beta$.

In fact, if we set

(2.29) $\quad \chi_{1,k}(\xi) = h(\rho_k(\xi - n\omega^{(k)})/3n)$,

we have

$$\|D_{x_j} \circ W_{n,k}^{\alpha;\beta} u_k(\cdot, t)\|$$
$$\leqq \|\chi_{1,k}(D_x) D_{x_j} \circ W_{n,k}^{\alpha;\beta} u_k(\cdot, t)\| + \rho_k \|(I - \chi_{1,k}(D_x)) \circ W_{n,k}^{\alpha+e_j,\beta} u_k(\cdot, t)\|$$
$$+ \|(I - \chi_{1,k}(D_x)) \circ W_{n,k}^{\alpha;\beta} D_{x_j} u_k(\cdot, t)\|.$$

Since supp $\chi_{1,k}(\cdot) \subset \{\xi; |\xi| \leqq 3n\}$ is valid, the term $\|\chi_{1,k}(D_x) D_{x_j} \circ W_{n,k}^{\alpha;\beta} u_k(\cdot, t)\|$ is estimated by $Kn \|W_{n,k}^{\alpha;\beta} u_k(\cdot, t)\|$. Apply Theorems A and B to the symbol $\sigma((I - \chi_{1,k}(D_x)) \circ W_{n,k}^{\alpha;\beta} D_{x_j})(x, t; \xi)$. Then, if we note from Lemma 2.3 that supp $(1 - \chi_{1,k}(\cdot)) \cap$ supp $w_{n,k}^{\alpha;\beta}(\cdot, t; \cdot) = \phi$ for $t \in [0, \rho_k/n]$, we can easily have

$$|\sigma((I - \chi_{1,k}(D_x)) \circ W_{n,k}^{\alpha;\beta} D_{x_j})(\cdot, t; \cdot)|_{l_0,l_0}^{(0)}$$
$$\leqq C'_{p,\alpha,\beta} n\rho_k^\lambda (\rho_k^2/n)^{p+1}$$

as in the proof of (2.25) for $t \in [0, \rho_k/n]$ with a constant $C'_{p,\alpha,\beta}$. So, we get

(2.30) $\quad \|(I - \chi_{1,k}(D_x)) \circ W_{n,k}^{\alpha;\beta} D_{x_j} u_k(\cdot, t)\|$
$$\leqq C'_{p,\alpha,\beta} n^{q+1} \rho_k^\lambda (\rho_k^2/n)^{p+1}$$

with another constant $C'_{p,\alpha,\beta}$. In the same way we can also estimate $\rho_k \|(I - \chi_{1,k}(D_x)) \circ W_{n,k}^{\alpha+e_j,\beta} u_k(\cdot, t)\|$.

Hence, noting that $\rho_k t \leqq \rho_k^2/n$ for $t \in [0, \rho_k/n]$, we obtain from (2.27)

(2.31) $\quad \|(\partial_\xi^\gamma w_{n,k})(x, t; D_x) D_{x_j} u_k(\cdot, t)\|$
$$\leqq C_\gamma (\rho_k^2/n)^{|\gamma|} \sum_{\alpha+\beta=\gamma} (\rho_k \|v_k^{\alpha+e_j,\beta}(\cdot, t)\| + n\|v_k^{\alpha,\beta}(\cdot, t)\|) + C_{p,\gamma} n^{q+1} \rho_k^\lambda (\rho_k^2/n)^{p+1}$$

for constants $C_\gamma$ and $C_{p,\gamma}$, which shows from (2.24) together with (2.26) that

(2.32) $\quad \|[b_j(x), W_{n,k}] \partial_{x_j} u_k(\cdot, t)\|$
$$\leqq C'_p n \sum_{1 \leqq |\alpha+\beta| \leqq p+1} (\rho_k^2/n)^{|\alpha+\beta|} \|v_k^{\alpha,\beta}(\cdot, t)\| + C'_p n^{q+1} \rho_k^\lambda (\rho_k^2/n)^{p+1}$$

for constants $C'_p$ independent of $k$. Since we can also estimate $\|[c(x), W_{n,k}] u_k(\cdot, t)\|$ in the same way, we can complete the proof. Q.E.D.

## 3. Proof of Proposition 2.2

We first prove for $v_k(x, t) = v_k^{0,0}(x, t)$ defined by (2.12)

**Lemma 3.1.** *Let* $t \in [0, \rho_k/n]$ $(n = \rho_k^{3+\delta})$. *Then, for any* $\nu = 1, 2, \cdots$

(3.1) $\quad \dfrac{1}{2} \dfrac{d}{dt} \|v_k(\cdot, t)\|^2$

$$\geqq \{b(x^{(k)}+2n\tau t\omega^{(k)};\, n\omega^{(k)})-A(1+\frac{n}{\rho_k})\}\|v_k(\cdot,\, t)\|^2$$

$$-\|f_k(\cdot,\, t)\|\times\|v_k(\cdot,\, t)\|-\tilde{C}_\nu\, n^{q+1}\rho_k^\lambda(\rho_k^2/n)^\nu\|v_k(\cdot,\, t)\|$$

*is valid, where* $\lambda$ *is the same constant in Lemma 2.4,* $A$ *is a constant independent of* $\nu$ *and* $k$, *and* $\tilde{C}_\nu$ *are constants independent of* $k$ *but depend on* $\nu$. *As set in section 1,* $b(x;\, \xi)$ *denotes* $-\sum_{j=1}^m \text{Re } b_j(x)\xi_j$.

Proof. From (2.20) we can see that

$$(3.2) \qquad \frac{d}{dt}\|v_k(\cdot,\, t)\|^2$$

$$= 2\text{Re }(\partial_t v_k(\cdot,\, t),\, v_k(\cdot,\, t))$$

$$= 2\text{Re } i((\tau\Delta+\sum_j b_j\partial_{x_j}+c)v_k(\cdot,\, t),\, v_k(\cdot,\, t))-2\text{Re } i(f_k(\cdot,\, t),\, v_k(\cdot,\, t))$$

$$\geqq -2\text{Re }(\sum_j (\text{Re } b_j)(x)D_{x_j}v_k(\cdot,\, t),\, v_k(\cdot,\, t))$$

$$-A_1\|v_k(\cdot,\, t)\|^2-2\|f_k(\cdot,\, t)\|\times\|v_k(\cdot,\, t)\|$$

for a constant $A_1$ independent of $k$. We shall estimate

$$-(\sum_j (\text{Re } b_j)(x)D_{x_j}v_k(\cdot,\, t),\, v_k(\cdot,\, t)) = (b(x;\, D_x)v_k(\cdot,\, t),\, v_k(\cdot,\, t))\,.$$

We write

$$(3.3) \qquad -(\text{Re } b_j)(x)D_{x_j}$$

$$= -(\text{Re } b_j)(x^{(k)}+2n\tau t\omega^{(k)})n\omega_j^{(k)}$$

$$+(\text{Re } b_j)(x^{(k)}+2n\tau t\omega^{(k)})\, (n\omega_j^{(k)}-D_{x_j})$$

$$+\{(\text{Re } b_j)(x^{(k)}+2n\tau t\omega^{(k)})-(\text{Re } b_j)(x)\}D_{x_j}$$

$$\equiv \sum_{j=1}^3 I_j\,.$$

We first estimate $I_2 v_k(x,\, t)$. Since supp $\mathcal{X}_{1,k}(\cdot)\subset\{\xi;\, |\xi-n\omega^{(k)}|\leqq 3n/(2\rho_k)\}$ holds for $\mathcal{X}_{1,k}(\xi)$ defined by (2.29), we see that

$$(3.4) \qquad \|\mathcal{X}_{1,k}(D_x)\circ(n\omega_j^{(k)}-D_{x_j})v_k(\cdot,\, t)\|$$

$$\leqq A_2\frac{n}{\rho_k}\|v_k(\cdot,\, t)\|$$

for a constant $A_2$ independent of $k$. Hereafter, in this proof, if there is no confusion, we do not indicate that constants are independent of $k$. Next, we write by $v_k(x,\, t)=W_{n,k}u_k(x,\, t)$

$$(3.5) \qquad J \equiv (I-\mathcal{X}_{1,k}(D_x))\circ(n\omega_j^{(k)}-D_{x_j})v_k$$

$$= (I-\mathcal{X}_{1,k}(D_x))\circ w_{n,k}(x,\, t;\, D_x)\, (n\omega_j^{(k)}-D_{x_j})u_k$$

$$- \frac{1}{i} \, \rho_k (I - \chi_{1,k}(D_x)) \circ w_{n,k}^{e,j_i,0}(x, t; D_x) u_k \, .$$

Apply Theorems A and B in section 1 to the term $p(x, t; \xi) = \sigma((I - \chi_{1,k}(D_x)) \circ w_{n,k}(x, t; D_x) \, (n\omega_j^{(k)} - D_{x_j})) \, (x, t; \xi)$. Then, we can show in the similar way to the proof of (2.30) that for any $\nu$

$$| p(\cdot, t; \cdot) |_{0,l_0}^{(0)} \leqq C_{\nu,1} \, n \rho_k^\lambda (\rho_k^2/n)^\nu$$

is valid for $t \in [0, \rho_k/n]$ and so we get

$$\| (I - \chi_{1,k}(D_x)) \circ w_{n,k}(x, t; D_x) \, (n\omega_j^{(k)} - D_{x_j}) u_k(\cdot, t) \|$$
$$\leqq C_{\nu,2} \, n^{q+1} \rho_k^\lambda (\rho_k^2/n)^\nu$$

for $t \in [0, \rho_k/n]$, where constants $C_{\nu,1}$ and $C_{\nu,2}$ depend only on $\nu$. In the same way we can also estimate $\rho_k \| (I - \chi_{1,k}(D_x)) \circ w_{n,k}^{e,j_i,0}(x, t; D_x) u_k(\cdot, t) \|$. Namely, we obtain

(3.6)    $\| J \|$
$$\leqq C_{\nu,3} \, n^{q+1} \rho_k^\lambda (\rho_k^2/n)^\nu \, ,$$

which shows together with (3.4) that

(3.7)    $\| I_2 v_k(\cdot, t) \|$
$$\leqq A_3 \frac{n}{\rho_k} \| v_k(\cdot, t) \| + C_{\nu,4} \, n^{q+1} \rho_k^\lambda (\rho_k^2/n)^\nu$$

for $t \in [0, \rho_k/n]$.

Next, we shall estimate $I_3 v_k(x, t)$. If we set

$$\chi_{2,k}(x) = h(\rho_k(x - x^{(k)} - 2n\tau t\omega^{(k)})/9) \, ,$$

supp $(1 - \chi_{2,k}(\cdot)) \cap$ supp $w_{n,k}^{\alpha,\beta}(\cdot, t; \cdot) = \phi$ holds for $t \in [0, \rho_k/n]$ from Lemma 2.3. So,

$$(I - \chi_{2,k}(x)) D_{x_j} v_k(x, t)$$
$$= (I - \chi_{2,k}(x)) \left\{ w_{n,k}(x, t; D_x) D_{x_j} u_k + \frac{1}{i} \, \rho_k w_{n,k}^{e,j_i,0}(x, t; D_x) u_k \right\}$$
$$= 0 \, .$$

That is,

(3.8)    $I_3 v_k(x, t)$
$$= \chi_{2,k}(x) \left\{ (\mathrm{Re} \, b_j) \, (x^{(k)} + 2n\tau t\omega^{(k)}) - (\mathrm{Re} \, b_j) \, (x) \right\} D_{x_j} v_k \, ,$$

which follows that

(3.9)    $\| I_3 v_k(\cdot, t) \| \leqq (A_4/\rho_k) \| D_{x_j} v_k(\cdot, t) \|$

for $t\in[0, \rho_k/n]$. Now, as in the proof of the estimate for the second term of the right hand side of (2.28) we get

$$\|D_{x_j}v_k(\cdot, t)\|$$
$$\leq A_5 n\|v_k(\cdot, t)\|+C_{\nu,5} n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{\nu},$$

which follows

(3.10)  $\|I_3 v_k(\cdot, t)\|$

$$\leq A_6 \frac{n}{\rho_k} \|v_k(\cdot, t)\|+C_{\nu,6} n^{q+1}\rho_k^{\lambda-1}(\rho_k^2/n)^{\nu}.$$

Using (3.2), (3.3), (3.7) and (3.10), we can complete the proof.    Q.E.D.

<u>Proof of Proposition 2.2.</u>   We can take a positive integer $p$ such that

(3.11)  $\displaystyle\sup_k n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{p+1}<\infty$,

noting $n=\rho_k^{3+\delta}$ and fix it.   Then, it is easily seen from Lemma 2.4 and Lemma 3.1 that

$$(3.12)\quad \frac{d}{dt}\|v_k(\cdot, t)\|\geq B(t; k)\|v_k(\cdot, t)\| - \text{const.} \frac{n}{\rho_k} \{(\rho_k^3/n)\sum_{|\alpha+\beta|=2}\|v_k^{\alpha,\beta}(\cdot, t)\|$$
$$+\sum_{1\leq|\alpha+\beta|\leq p+1}\rho_k(\rho_k^2/n)^{|\alpha+\beta|}\|v_k^{\alpha,\beta}(\cdot, t)\|\}- \text{const.},$$

where

$$(3.13)\quad B(t; k) = b(x^{(k)}+2n\tau t\omega^{(k)}; n\omega^{(k)})-A\left(1+\frac{n}{\rho_k}\right)$$

with the same constant $A$ in (3.1).   Since the inequality $\rho_k(\rho_k^2/n)^{|\gamma|}\leq(\rho_k^3/n)^{[(|\gamma|+1)/2]}$ $(|\gamma|\geq 1)$ is valid, we obtain from (3.12)

$$(3.14)\quad \frac{d}{dt}\|v_k(\cdot, t)\|$$

$$\geq B(t; k)\|v_k(\cdot, t)\|-\text{const.} \frac{n}{\rho_k}\sum_{1\leq|\alpha+\beta|\leq p+1}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}\|v_k^{\alpha,\beta}(\cdot, t)\|$$

$$- \text{const.}.$$

If we make the same process for $v_k^{\alpha,\beta}(x, t)=W_{n,k}^{\alpha,\beta}u(x, t)$ $((|\alpha+\beta|\geq 1)$ as for $v_k(x, t)=W_{n,k}u(x, t)$, corresponding to (3.14) we have

$$\frac{d}{dt}\|v_k^{\alpha,\beta}(\cdot, t)\|\geq B(t; k)\|v_k^{\alpha,\beta}(\cdot, t)\|$$

$$- C_{\alpha,\beta}\frac{n}{\rho_k}\sum_{1\leq|\tilde{\alpha}+\tilde{\beta}|\leq p+1}(\rho_k^3/n)^{[(|\tilde{\alpha}+\tilde{\beta}|+1)/2]}\|v_k^{\alpha+\tilde{\alpha},\beta+\tilde{\beta}}(\cdot, t)\|-C_{\alpha,\beta}$$

for constants $C_{\alpha,\beta}$ independent of $k$.   So, we obtain

(3.15)  $\dfrac{d}{dt}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}||v_k^{\alpha,\beta}(\cdot,\,t)||$

$\geqq B(t;\,k)\,(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}||v_k^{\alpha,\beta}(\cdot,\,t)||$

$-\,C_{\alpha,\beta}\dfrac{n}{\rho_k}\sum_{1\leqq|\tilde{\alpha}+\tilde{\beta}|\leqq p+1}(\rho_k^3/n)^{[(|\alpha+\tilde{\alpha}+\beta+\tilde{\beta}|+1)/2]}||v_k^{\alpha+\tilde{\alpha},\beta+\tilde{\beta}}(\cdot,\,t)||-C_{\alpha,\beta}\,.$

Here, we used

$$(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]+[(|\tilde{\alpha}+\tilde{\beta}|+1)/2]}$$

$$\leqq(\rho_k^3/n)^{[(|\alpha+\tilde{\alpha}+\beta+\tilde{\beta}|+1)/2]}$$

for $|\tilde{\alpha}+\tilde{\beta}|\geqq1$.

Now, we already determined $s$ so that (2.14) holds. Hence, if $|\alpha+\beta|\geqq s+1$, we have by (2.18)

$$\dfrac{n}{\rho_k}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}||v_k^{\alpha,\beta}(\cdot,\,t)||$$

$$\leqq C'_{\alpha,\beta}\dfrac{n}{\rho_k}(\rho_k^3/n)^{[(s+2)/2]}\rho_k^{m/2+2[m/2+1]}\,n^q$$

$$\leqq C''_{\alpha,\beta}<\infty$$

for any $k$ and for any $t\in[0,\,\rho_k/n]$. Therefore, for $\sigma_k(t)$ defined by (2.15) we obtain from (3.14) and (3.15)

(3.16)  $\dfrac{d}{dt}\sigma_k(t)\geqq(B(t;\,k)-C\dfrac{n}{\rho_k})\sigma_k(t)-O(1)$

for any $k$ and $t\in[0,\,\rho_k/n]$, where $C$ is a constant independent of $k$.

The integration of (3.16) gives

(3.17)  $\sigma_k(\rho_k/n)$

$\geqq(\exp\displaystyle\int_0^{\rho_k/n}B(\theta;\,k)-C\dfrac{n}{\rho_k}\,d\theta)$

$\times\{\sigma_k(0)-O(1)\displaystyle\int_0^{\rho_k/n}(\exp-\int_0^t B(\theta;\,k)-C\dfrac{n}{\rho_k}\,d\theta)\,dt\}\,.$

Here, we note from (3.13) that

$$B(\theta;\,k)=b(x^{(k)}+2n\tau\theta\omega^{(k)};\,n\omega^{(k)})-A(1+\dfrac{n}{\rho_k})\,.$$

Also, from the choice of $x^{(k)},\,\omega^{(k)},\,\rho_k$ we know that

$$\int_0^{\rho_k/n}b(x^{(k)}+2n\tau\theta\omega^{(k)};\,n\omega^{(k)})\,d\theta$$

$$=\int_0^{\rho_k}b(x^{(k)}+2\tau\theta\omega^{(k)};\,\omega^{(k)})\,d\theta$$

$$\geqq M\log(1+\rho_k)+k$$

and for $t \in [0, \rho_k/n]$

$$\int_0^t b(x^{(k)} + 2n\tau\theta\omega^{(k)}; n\omega^{(k)}) \, d\theta$$

$$= \int_0^{nt} b(x^{(k)} + 2\tau\theta\omega^{(k)}; \omega^{(k)}) \, d\theta$$

$$\geqq 0 .$$

Moreover, $\sigma_k(0) \geqq \|v_k(\cdot, 0)\| \geqq \|h(\cdot)\|$ holds for large $k$ by (2.13). Hence, if $k$ is large enough, we obtain from (3.17)

$$\sigma_k(\rho_k/n) \geqq C_1(1+\rho_k)^M$$

for a positive constant $C_1$, which shows Proposition 2.2.          Q.E.D.

REMARK 4.   In more detail we can see from the proof of Theorem the following is necessary in order that there exists a constant $T > 0$ such that for any initial data $u_0(x) \in H_\infty$ a unique solution $u(x, t) \in \mathcal{E}_t^0([0, T]; H_\infty)$ of (0.1) exists and the inequality (2.1) holds for some $q$.   For any $M$ greater than $m/2 + 2[m/2+1] + 3q$ there exists a constant $N$ such that the inequality (0.2) holds.

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## References

[1]   G.D. Birkhoff: *Quantum mechanics and asymptotic series,* Bull. Amer. Math. Soc. **39** (1939), 681–700.

[2]   A.P. Calderón and R. Vaillancourt: *On the boundedness of pseudo-differential operators,* J. Math. Soc. Japan **23** (1971), 374–378.

[3]   W. Ichinose: *Sufficient condition on $H_\infty$ well posedness for Schrödinger type equations,* Comm. Partial Differential Equations **9** (1984), 33–48.

[4]   H. Kumano-go: Pseudo-differential operators, M.I.T. Press, Cambridge, Massachusetts, and London, England, 1982.

[5]   P.D. Lax: *Asymptotic solutions of oscillatory initial value problem,* Duke Math. J. **24** (1957), 627–646.

[6]   S. Mizohata: *Some remarks on the Cauchy problem,* J. Math. Kyoto Univ. **1** (1961), 109–127.

[7]   S. Mizohata: *Sur quelques équations du type Schrödinger,* Journées "Equations aux dérivées partielles", Saint-Jean de Monts, 1981.

[8]   J. Takeuchi: *On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots,* J. Math. Kyoto Univ. **20** (1980), 105–124.

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