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THE ORIENTABILITY OF SMALL COVERS
AND COLORING SIMPLE POLYTOPES

HISASHI NAKAYAMA and YASUZO NISHIMURA

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Abstract
Small Cover is an \( n \)-dimensional manifold endowed with a \( \mathbb{Z}_2 \) action whose orbit space is a simple convex polytope \( P \). It is known that a small cover over \( P \) is characterized by a coloring of \( P \) which satisfies a certain condition. In this paper we shall investigate the topology of small covers by the coloring theory in combinatorics. We shall first give an orientability condition for a small cover. In case \( n = 3 \), an orientable small cover corresponds to a four colored polytope. The four color theorem implies the existence of orientable small cover over every simple convex 3-polytope. Moreover we shall show the existence of non-orientable small cover over every simple convex 3-polytope, except the 3-simplex.

0. Introduction

“Small Cover” was introduced and studied by Davis and Januszkiewicz in [5]. It is a real version of “Quasitoric manifold,” i.e., an \( n \)-dimensional manifold endowed with an action of the group \( \mathbb{Z}_2^n \) whose orbit space is an \( n \)-dimensional simple convex polytope. A typical example is provided by the natural action of \( \mathbb{Z}_2^n \) on the real projective space \( \mathbb{R}P^n \) whose orbit space is an \( n \)-simplex. Let \( P \) be an \( n \)-dimensional simple convex polytope. Here \( P \) is simple if the number of codimension-one faces (which are called “facets”) meeting at each vertex is \( n \), equivalently, the dual \( K_P \) of its boundary complex \( \partial(P) \) is an \( (n - 1) \)-dimensional simplicial sphere. In this paper we shall handle a convex polytope in the category of combinatorics. We denote the set of facets of \( P \) (or the set of vertices of \( K_P \)) by \( F \). Associated to a small cover \( M \) over \( P \), there exists a function \( \lambda: F \to \mathbb{Z}_2^2 \) called a “characteristic function” of \( M \). A basic result in [5] is that small covers over \( P \) are classified by their characteristic functions (cf. [5, Proposition 1.8]). The characteristic function is a (face-)coloring of \( P \) (or a vertex-coloring of \( K_P \)), which satisfies a “linearly independent condition” (see §1). Coloring convex \( n \)-polytopes have been studied actively in combinatorics. We shall investigate the topological properties of small covers through the coloring theory in combinatorics specially when \( n = 3 \).

The notion of small cover can be generalized to the case where the base space is

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more general than a simple convex polytope. An $n$-dimensional nice manifold $P$ with corners such that the dual complex $K_P$ is a simplicial decomposition of $\partial(P)$, is called a \textit{simple polyhedral complex}. For a coloring of $P$ which satisfies the linearly independent condition, we can construct an $n$-dimensional manifold with $\mathbb{Z}_2^n$-action over $P$ in a similar way. We call it a small cover over a simple polyhedral complex $P$.

In this paper we give a criterion when a small cover over a simple convex polytope is orientable (Theorem 1.7). In case $n = 3$, this criterion implies that an orientable three-dimensional small cover corresponds to a 4-colored simple convex 3-polytope. Therefore the existence of an orientable small cover over every simple convex 3-polytope is equivalent to the four color theorem (Corollary 1.8). Next we shall discuss the colorability of a 3-polytope making allowance for the linearly independent condition, and prove existence of non-orientable small cover over every simple convex 3-polytope, except the 3-simplex (Theorem 2.3). The proof of Theorem 2.3 is given in a way similar to the proof of classical five color theorem by Kempe. Moreover we shall discuss the existence of non-orientable small cover over 4-colorable simple polyhedral handlebody with a positive genus (Theorem 3.1).

1. \textbf{The orientability of small covers}

At first we shall recall the definition and basic results of small covers in [5] or [3]. An $n$-dimensional convex polytope is \textit{simple} if the number of codimension-one faces (which are called \textit{facets}) meeting at each vertex is $n$. Equivalently, $P$ is simple if the dual of its boundary complex is a simplicial decomposition of $(n - 1)$-dimensional sphere. We denote the simplicial complex dual to $P$ by $K_P$. In this paper we shall handle polytopes in the category of combinatorics and understand that two polytopes are identical if they are combinatorially equivalent. Therefore a considerable structure of a polytope $P$ is only its face structure, i.e., the dual complex $K_P$. The natural action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ is called the \textit{standard representation} and its orbit space is $\mathbb{R}^{n^*}$.

\textbf{Definition 1.1.} A manifold $M$ endowed with the action of the group $\mathbb{Z}_2^n$ is a \textit{small cover} over an $n$-dimensional simple convex polytope $P$ if its orbit space is homeomorphic to $P$ and $M$ is locally isomorphic to the standard representation, i.e., there exists an automorphism $\theta$ of $\mathbb{Z}_2^n$ such that for any point $x \in M$, there exists stable neighborhoods $V \subset M$ of $x$ and $W \subset \mathbb{R}^n$ and $\theta$-equivariant homeomorphism $f: V \rightarrow W$ ($f(gv) = \theta(g)f(v)$).

Let $\pi_1: M_1 \rightarrow P$ and $\pi_2: M_2 \rightarrow P$ be two small covers over $P$. An \textit{equivalence over $P$} is an automorphism $\theta$ of $\mathbb{Z}_2^n$, together with a $\theta$-equivariant homeomorphism $f: M_1 \rightarrow M_2$, which covers the identity on $P$.

\textbf{Example 1.2.} The group $\mathbb{Z}_2$ acts on $S^1$ by a reflection, and its orbit space is an interval $I$. Taking the $n$-fold product, we have a $\mathbb{Z}_2^n$ action on an $n$-dimensional
torus $T^n = S^1 \times \cdots \times S^1$, which is a small cover over the $n$-cube $I^n$.

Example 1.3. We have a usual $\mathbb{Z}_2^n$ action on the real projective space $\mathbb{R}P^n$ as follows:

$$(g_1, \ldots, g_n) \cdot [x_0, x_1, \ldots, x_n] = [x_0, g_1 x_1, \ldots, g_n x_n].$$

This is a small cover over the $n$-simplex $\Delta^n$.

Let $\pi: M \to P$ be a small cover over an $n$-dimensional simple convex polytope $P$. For a face $F$ of $P$, the isotropy group at $x \in \pi^{-1}(\text{int } F)$ is independent of the choice of $x$, denoted by $G_F$. In particular, if $F$ is a facet, $G_F$ is a rank-one subgroup, hence, it is determined by a generator $\lambda(F) \in \mathbb{Z}_2^n$. In this way we obtain a function $\lambda: \mathcal{F} \to \mathbb{Z}_2^n$ where $\mathcal{F}$ is the set of facets of $P$. This function is called the characteristic function of $M$. If $F^{n-k}$ is a codimension-$k$ face of $P$ then $F = F_1 \cap \cdots \cap F_k$ where $F_i$'s are the facets which contain $F$, and $G_F$ is the rank-$k$ subgroup generated by $\lambda(F_1), \ldots, \lambda(F_k)$. Therefore the characteristic function satisfies the following condition.

(*) If $F_1, \ldots, F_n$ are the facets meeting at a vertex of $P$, then $\lambda(F_1), \ldots, \lambda(F_n)$ are linearly independent vectors of $\mathbb{Z}_2^n$.

In particular, the characteristic function $\lambda: \mathcal{F} \to \mathbb{Z}_2^n$ is a (face-)coloring of $P$ (or a vertex-coloring of the dual graph $K_P$). We often call it a linearly independent coloring of $P$ (or $K_P$). Conversely a coloring $\lambda$ of $P$ satisfying the linearly independent condition (*) determines a small cover $M(P, \lambda)$ over $P$ whose characteristic function is the given $\lambda$. The construction of $M(P, \lambda)$ is as follows. For each point $p \in P$, let $F(p)$ be the unique face of $P$ which contains $p$ in its relative interior. We define an equivalence relation on $P \times \mathbb{Z}_2^n$ as follows:

$$(p, g) \sim (q, h) \iff p = q, \quad g^{-1}h \in G_{F(p)}$$

where $G_{F(p)}$ is the subgroup generated by $\lambda(F_1), \ldots, \lambda(F_k)$ such that $F(p) = F_1 \cap \cdots \cap F_k$ ($F_i \in \mathcal{F}$). Then the quotient space $(P \times \mathbb{Z}_2^n)/\sim$ is $M(P, \lambda)$.

Theorem 1.4 ([5, Proposition 1.8]). Let $M$ be a small cover over $P$ such that its characteristic function is $\lambda: \mathcal{F} \to \mathbb{Z}_2^n$. Then $M$ is equivalent to $M(P, \lambda)$. In other words, the small cover is determined up to equivalence over $P$ by its characteristic function.

Example 1.5. In case $n = 2$, $P$ is a polygon and a characteristic function is a function $\lambda: \mathcal{F} \to \mathbb{Z}_2^2$ which satisfies the linearly independent condition (*). Let $\{e_1, e_2\}$ be a basis of $\mathbb{Z}_2^2$. Since any pair of $\{e_1, e_2, e_1 + e_2\}$ is linearly independent, $\lambda$ is
just a 3-coloring of $P$. If $(P, \lambda)$ is a $k$-gon colored by three colors (resp. two colors) then $M(P, \lambda)$ is the non-orientable surface $(k-2)\mathbb{RP}^2$ (resp. the orientable surface with genus $(k-2)/2$) endowed with a certain action of $\mathbb{Z}_2^2$ where $m\mathbb{RP}^2$ is the connected sum of $m$ copies of $\mathbb{RP}^2$ (cf. [5, Example 1.20]).

Remark 1.6. When an $n$-dimensional simple convex polytope $P$ is $s$-colored ($s \geq n$), we understand that the image of the coloring function $\lambda$ is a basis for $\mathbb{Z}_2^n$, and define the quotient space $Z(P, \lambda) = (P \times \mathbb{Z}_2^n)/\sim$ where the equivalence relation $\sim$ is given in a way similar to (1). It is called the “manifold defined by the coloring $\lambda$” in [7]. When $s = n$, $Z(P, \lambda)$ coincides with the small cover $M(P, \lambda)$, and is called the “pullback from the linear model” in [5]. In the special case $n = s = 3$, pullbacks from the linear model were studied by Izmestiev in details in [7].

Next we shall discuss the orientability condition of a small cover.

Theorem 1.7. For a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}_2^n$, a homomorphism $\epsilon: \mathbb{Z}_2^n \to \mathbb{Z}_2 = \{0, 1\}$ is defined by $\epsilon(e_i) = 1$ ($i = 1, \ldots, n$). A small cover $M(P, \lambda)$ is orientable if and only if there exists a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}_2^n$ such that the image of $\epsilon\lambda$ is $\{1\}$. 

Proof. Let us calculate the $n$-dimensional integral homology group $H_n(M; \mathbb{Z})$ of a small cover $M = M(P, \lambda)$. The combinatorial structure of $P$ defines a natural cellular decomposition of $M = (P \times \mathbb{Z}_2^n)/\sim$. We denote by $(C_k, \partial_k)$ the chain complex associated with this cellular decomposition. In particular, $C_n$ and $C_{n-1}$ are the free abelian groups generated by $\{(P, g) \mid g \in \mathbb{Z}_2^n\}$ and $\{(F \times \mathbb{Z}_2^n)/\sim \mid \mathcal{F}, g \in \mathbb{Z}_2^n\}$, respectively, where the equivalence class of $F \times \mathbb{Z}_2^n$ is defined by the equivalence relation $(F, g) \sim (F, \lambda(F) + g)$. We give an orientation on a facet $F_i$ such that $\partial(P) = F_1 + \cdots + F_f$ where $f = \#F$. Under these notations if $X = \sum_{g \in \mathbb{Z}_2^n} n_g (P, g) \in C_n$ ($n_g \in \mathbb{Z}$) is an $n$-cycle of $M$ then

$$
\partial_n(X) = \sum_{g \in \mathbb{Z}_2^n} \sum_{i=1}^f (F_i, g) = \sum_{[F_i, g] \in \mathcal{F} \times \mathbb{Z}_2^n/\sim} (n_g + n_{\lambda(F_i)+g})[F_i, g] = 0.
$$

Therefore $X \in \ker \partial_n$ if and only if $n_g = -n_{\lambda(F)+g}$ for any facet $F$ and $g \in \mathbb{Z}_2^n$. The latter is equivalent to $n_g = (-1)^j n_{\lambda(F_i)+\cdots+\lambda(F_k)+g}$ for any facets $F_i, \ldots, F_k$ and $g \in \mathbb{Z}_2^n$. Suppose that there is no basis of $\mathbb{Z}_2^n$ such that $\epsilon\lambda \equiv 1$. It means that for each set of facets $F_i, \ldots, F_k$ such that $e_j = \lambda(F_j)$ ($1 \leq j \leq n$) is a basis of $\mathbb{Z}_2^n$, there exists a facet $F$ such that $\lambda(F) = 0$, i.e., $\lambda(F) = e_j + \cdots + e_k$ where $k$ is an even number. Then $n_g = (-1)^j n_{e_j+\cdots+e_k+g} = n_{\lambda(F)+g} = -n_g$ that is $n_g = 0$ for any $g \in \mathbb{Z}_2^n$. Thus $H_n(M; \mathbb{Z}) = \ker \partial_n = 0$, and $M$ is non-orientable. On the other hand, when there exists a basis of $\mathbb{Z}_2^n$ such that $\epsilon\lambda \equiv 1$, for any $g \in \mathbb{Z}_2^n$, the parity of $k = k(g)$ does not depend on the choice of $F_i$'s such that $g = \lambda(F_i)+\cdots+\lambda(F_k)$. In fact, if $\lambda(F_i)+\cdots+\lambda(F_k) = \lambda(F_k)+\cdots+\lambda(F_i)$.
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(1) \( + + \) then \( + + \) = \( + + \), therefore \( k \equiv 1 \) mod 2. Then \( H_\partial (M; \mathbb{Z}) = \ker \partial_n \cong \mathbb{Z} \) is generated by \( X = \sum_{g \in \mathbb{Z}^t} (-1)^{k(g)}(P, g) \), and \( M \) is orientable.

We call a linearly independent coloring which satisfies the orientability condition in Theorem 1.7 an orientable coloring of \( P \) (or \( K_P \)). In case \( n = 2 \), it is easy to see that an orientable coloring of a polygon is just a 2-coloring of \( P \) (see Example 1.5). In case \( n = 3 \), a three-dimensional small cover \( M(P, \lambda) \) is orientable if and only if there exists a basis \( \{ \alpha, \beta, \gamma \} \) of \( \mathbb{Z}_3 \) such that the image of \( \lambda \) is contained in \( \{ \alpha, \beta, \gamma, \alpha + \beta + \gamma \} \). Since each triple of \( \{ \alpha, \beta, \gamma, \alpha + \beta + \gamma \} \) is linearly independent, the orientable coloring of a 3-polytope \( P \) is just a 4-coloring of \( P \). By the four color theorem ([1]), we obtain the following corollary (in fact, the corollary below is equivalent to the four color theorem).

**Corollary 1.8.** There exists an orientable small cover over every simple convex 3-polytope.

**Remark 1.9.** Although there exists a small cover over every three-dimensional simple convex polytope, for each integer \( n \geq 4 \), there exists an \( n \)-dimensional simple convex polytope \( Q \) which admits no small cover (cf. [5, Nonexample 1.22]). In fact, a cyclic polytope \( C_k^n \) defined as the convex hull of \( k \geq n + 1 \) points on a curve \( \gamma(t) = (t, t^2, \ldots, t^n) \) is a simplicial polytope such that the one-skeleton of \( C_k^n \) is a complete graph when \( n \geq 4 \) (see [2, §13]). Let \( Q_k^n \) be the simple polytope dual to \( C_k^n \). Since the chromatic number of \( Q_k^n \) is \( k \), the polytope \( Q_k^n \) admits no small cover whenever \( k \geq 2^n \).

2. **Existence of non-orientable small covers**

We call a linearly independent coloring which does not satisfy the orientability condition in Theorem 1.7 a non-orientable coloring. In this section we shall discuss the existence of a non-orientable coloring over a simple polytope in case \( n = 3 \). We shall recall and use some notions of 3-polytopes and the graph theory. For further details see [2] or [6].

For a simple convex 3-polytope \( P \), we set \( p_k \) the number of \( k \)-cornered facets of \( P \). Then the numbers of facets, edges and vertices of \( P \) are \( f_2 = \sum_{k \geq 3} p_k \), \( f_1 = \sum_{k \geq 3} k p_k / 2 \) and \( f_0 = \sum_{k \geq 3} k p_k / 3 \), respectively. Since the Euler number \( f_0 - f_1 + f_2 \) of \( \partial(P) \) is two, we obtain immediately the following formula.

**Lemma 2.1.** For any simple convex 3-polytope \( P \),

\[
3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7}(k - 6)p_k.
\]
The lemma implies a well-known fact that each simple convex 3-polytope has a facet which has less than six edges.

We introduce an operation “blow up” for a vertex of a simple 3-polytope $P$ (see Fig. 1). We cut around a vertex of $P$ and create a new triangular facet there. The reverse operation of the blow up is called a “blow down.” Notice that for any simple convex polytope except the 3-simplex, two triangular facets must not adjoin each other. Therefore except for the 3-simplex, the blow down for any triangular facet is possible. For the dual simplicial complex $K_P$, a blow up is operated for a 2-simplex of $K_P$ and a blow down is operated for a vertex of $K_P$ with degree three, respectively. The blow up can be done keeping the linearly independence. In fact, we can assign $\alpha + \beta + \gamma$ to the new triangle where $\alpha$, $\beta$ and $\gamma$ are the colors assigned to three facets adjacent to the triangle. The blow up operation corresponds to the equivariant connected sum of $\mathbb{R}P^3$ at the fixed point of small covers corresponding to the vertex (see [5, 1.11]). When $P$ is endowed with a linearly independent coloring, the blow down for a triangular facet can not be done keeping the linearly independence. From the above fact we obtain the following lemma immediately.

**Lemma 2.2.** If a convex polytope $P$ has a non-orientable coloring then each polytope obtained by blowing up $P$ has also a non-orientable coloring.

**Theorem 2.3.** There exists a non-orientable small cover over every simple convex 3-polytope, except the 3-simplex.

Proof. Let $P$ be a simple convex polytope but not the 3-simplex. Operating the blow downs for triangular facets of $P$ over again, $P$ can be transformed to a polytope $P'$ which does not have a triangular facet or is the triangular prism. In the latter case although the triangular prism can be transformed to the 3-simplex further by the blow down, we stop the operation because the 3-simplex is excepted from this the-
orem. By Lemma 2.2, a non-orientable coloring of $P'$ leads to that of $P$. Therefore we can assume that $P$ does not have a triangular facet or $P$ is the triangular prism. We assume the four color theorem, and shall prove that some facets of 4-colored polytope can be repainted making allowance for the linearly independent condition and construct a non-orientable coloring. Here we assume that $P$ is colored by four colors \{\alpha, \beta, \gamma, \alpha + \beta + \gamma\} for some basis \{\alpha, \beta, \gamma\} of $\mathbb{Z}_2^3$. (When $P$ is colored by only three colors, we can repaint a facet and assume that $P$ is 4-colored if necessary.) The case that $P$ has a quadrilateral facet is immediate. In fact, the 4-coloring around a quadrilateral facet $F$ must be the following situation: a center quadrilateral $F$ is colored by $\alpha$ and two facets adjacent to $F$ are colored by $\beta$ and the rest two facets adjacent to $F$ are colored by one color $\gamma$ or two colors $\gamma$ and $\alpha + \beta + \gamma$, respectively (see Fig. 2). In both cases we can repaint the center quadrilateral by $\alpha + \beta$ instead of $\alpha$, and produce the non-orientable coloring. In particular, the triangular prism has a non-orientable coloring because it has a quadrilateral facet.

Suppose that $P$ has no triangle and quadrilateral. By Lemma 2.1, $P$ must have a pentagonal facet $F$. We can assume that the 4-coloring around $F$ is in the following situation: the center pentagon $F$ is colored by $\alpha$ and the adjacent five facets $F_1, \ldots, F_5$ are 3-colored by $\beta$, $\gamma$, $\beta$, $\gamma$ and $\alpha + \beta + \gamma$, respectively (see Fig. 3). Here we shall repaint some facets of $P$ and construct a non-orientable coloring in a way similar to the proof of the classical Five Color Theorem by Kempe using the “Kempe chain” (cf. [8] or [6]). First we consider the $\{\alpha, \beta\}$-chain containing the pentagon $F$, i.e., the connected component of facets colored by $\alpha$ or $\beta$, which contains $F$. If the $\{\alpha, \beta\}$-chain has no elementary cycle containing $F$ then we divide it by the edge $F \cap F_3$ into two chains, and the one side which contains $F_3$ can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of $\alpha$ and $\beta$, respectively. If the $\{\alpha, \beta\}$-chain has an elementary cycle containing $F$ then $F_2$ and $F_4$ belong to a different component of $\{\gamma, \alpha + \beta + \gamma\}$-chain respectively, because of the Jordan curve theorem. Therefore the one side of them can
be repainted by \( \alpha + \gamma \) and \( \beta + \gamma \) instead of \( \gamma \) and \( \alpha + \beta + \gamma \), respectively. In both cases the repainted polytope is five or six-colored and the new coloring also satisfies the linearly independent condition. Therefore we obtain a non-orientable coloring of \( P \).

\[\square\]

3. Coloring simple polyhedral handlebodies

Let \( P \) be an \( n \)-dimensional nice manifold with corners. We say that \( P \) is a simple polyhedral complex if its dual complex \( K_P \) is a simplicial decomposition of \( \partial(P) \). This condition implies that any intersection of two faces is a face of \( P \). We can characterize a simple polyhedral complex by a pair of a manifold \( P \) and a simplicial decomposition \( K \) of \( \partial(P) \). In fact a simplicial decomposition \( K \) of \( \partial(P) \) determines the polyhedral structure of \( P \) as follows. For each simplex \( \sigma \in K \), let \( F_\sigma \) denote the geometric realization of the poset \( K_{\geq \sigma} = \{ \tau \in K \mid \sigma \leq \tau \} \). We say that \( F_\sigma \) is a codimension-\( k \) face of \( P \) if \( \sigma \) is a \((k - 1)\)-simplex of \( K \). Then its dual complex \( K_P \) is clearly same as \( K \).

We call a facet-coloring of \( P \) (or a vertex-coloring of \( K_P \)) simply a coloring of \( P \) (or \( K_P \)). We denote by \( \mathcal{F} \) the set of facets of \( P \) (or the set of vertices of \( K_P \)). A function \( \lambda : \mathcal{F} \to \mathbb{Z}_2^p \) is called a linearly independent coloring of \( P \) (or \( K_P \)) if \( \lambda \) satisfies for \( P \) the condition (\( \star \)) in \( \S 1 \). We put \( M(P, \lambda) = (P \times \mathbb{Z}_2^p)/\sim \), where the equivalence relation \( \sim \) is defined as (1) in \( \S 1 \). We have a \( \mathbb{Z}_2^p \)-action on an \( n \)-dimensional manifold \( M(P, \lambda) \) with the orbit space \( P \). Conversely an \( n \)-dimensional manifold \( M \) endowed with a locally standard \( \mathbb{Z}_2^p \)-action whose orbit space is homeomorphic to \( P \) determines
a characteristic function \( \lambda : \mathcal{F} \to \mathbb{Z}_2 \). Then \( M \) is equivalent to \( M(P, \lambda) \) if the restriction on \( \pi^{-1}(\text{int} P) \) of the projection \( \pi : M \to P \) is a trivial covering. We will say that \( M(P, \lambda) \) is a small cover over \( P \). (Warning: In [5], for each \((n-1)\)-dimensional simplicial complex \( K \), the simple polyhedral complex \( P_K \) is the cone on \( K \). Then \( M(P_K, \lambda) \) can be defined in a similar way, however, it is not always a manifold, and therefore it is called a \( \mathbb{Z}_2^{n-1} \)-space" in [5].)

When \( P \) is an orientable simple polyhedral complex, the orientability condition of \( M(P, \lambda) \) is same as the condition in Theorem 1.7. Therefore we may generalize the notion of \((n-)\) orientable coloring of \( P \) (or \( K_P \)) to this case. Henceforth, we take \( P \) as a simple polyhedral handlebody with genus \( g > 0 \), i.e., a handlebody \( P \) together with a simplicial decomposition \( K_P \) of the orientable closed surface \( \Sigma_g \) with genus \( g \). In this case, the formula (2) in Lemma 2.1 is generalized to

\[
(3) \quad \sum_{k \geq 3} (k - 6)p_k = 12(g - 1)
\]

where \( p_k \) is the number of \( k \)-cornered facets of \( P \) (or vertices of \( K_P \) whose degree is \( k \)). In the rest of this section we shall prove the following theorem.

**Theorem 3.1.** Let \( P \) be a 4-colorable simple polyhedral handlebody with genus \( g > 0 \) (equivalently, there exists an orientable small cover over \( P \)). If \( P \) has sufficiently many facets then there also exists a non-orientable small cover over \( P \).

Assume that \( P \) is colored by four colors \( \{\alpha, \beta, \gamma, \alpha + \beta + \gamma\} \) for some basis \( \{\alpha, \beta, \gamma\} \) of \( \mathbb{Z}_2^3 \). We shall repaint some facets of \( P \) and construct a non-orientable coloring. By the same reason as in the proof of Theorem 2.3, when \( P \) has a quadrilateral facet, the construction of non-orientable coloring is immediate.

Next we consider two operations "blow down" and "blow up" introduced in §2 (see Fig. 1). We can define these operations for a simple polyhedral handlebody \( P \) (or a simplicial decomposition of an orientable surface \( K_P \)) in a similar way. The blow up can be always done for any vertex of \( P \) together with a linearly independent coloring. Notice that for any simple polyhedral handlebody with a positive genus, two triangular facets must not adjoin each other. Therefore the blow down can be always done for any triangular facet of \( P \). Operating the blow down for triangular facets of \( P \) one after another, we can reduce \( P \) to a simple polyhedral complex \( P' \) which has no triangular facet. As we have already seen in Lemma 2.2, a non-orientable coloring on \( P' \) can be extended on \( P \). In the course of this process if a quadrilateral facet appears, we can also construct a non-orientable coloring of \( P \). We assume that a quadrilateral facet does not appear during the reduction from \( P \) to \( P' \). Generally if we operate blow up for a vertex of a triangular facet then a quadrilateral facet will be created. By the above assumption \( P \) must be obtained by blow up for original vertices of \( P' \) (but not for new vertices born by blow up). Therefore the number of facets of \( P \) is
at most the sum of numbers of facets and vertices of $P'$. Consequently it is sufficient to prove Theorem 3.1 in the case that $P$ does not have a quadrilateral or a triangle. In fact if the following proposition holds for any simple polyhedral handlebody which has more than $N$ facets but not a quadrilateral and a triangle then Theorem 3.1 holds for any simple polyhedral handlebody which has more than $N + M$ facets where $M$ is the maximum of numbers of vertices of simple polyhedral handlebodies which do not have more than $N$ facets and a quadrilateral and a triangle.

**Proposition 3.2.** Let $P$ be a 4-colorable simple polyhedral handlebody with genus $g > 0$ such that $P$ does not have a quadrilateral or a triangle (equivalently its dual $K_P$ is a simplicial decomposition of an orientable surface $\Sigma_g$ with genus $g > 0$ such that $K_P$ does not have a vertex with degree three or four). If $P$ has sufficiently many facets then $P$ has a non-orientable coloring.

For a subset $A$ of vertices of a simplicial complex $K$, we denote by $\Gamma_A$ the subgraph of one-skeleton $K^1$ generated by $A$ (which is called the section subgraph). We need the following lemma instead of the Jordan curve theorem.

**Lemma 3.3.** Let $K$ be a simplicial decomposition of the orientable closed surface $\Sigma_g$ with genus $g > 0$, and $(A, B)$ be a division of vertices of $K$ ($A \coprod B = V(K)$) such that the section subgraphs $\Gamma_A$, $\Gamma_B$ are both connected and have no cycle of length three. When $2g + 1$ edges are removed from $\Gamma_A \cup \Gamma_B$, either $\Gamma_A$ or $\Gamma_B$ is disconnected.

Proof. Because $\Gamma_A$ and $\Gamma_B$ have no cycle of length three, each 2-simplex $\Delta$ of $K$ intersects both of $\Gamma_A$ and $\Gamma_B$, i.e., all vertices and only one edge of $\Delta$ belong to $\Gamma_A \cup \Gamma_B$. Therefore the number of 2-simplices of $K$ and the number of edges of $K$ which do not belong to $\Gamma_A \cup \Gamma_B$ coincide. Thus $\chi(\Gamma_A \cup \Gamma_B) = \chi(K) = 2 - 2g$ where $\chi(G)$ is the Euler number of $G$. Since $\Gamma_A$ and $\Gamma_B$ are both one-dimensional connected subcomplexes of $K$, this means that the first Betti number of $\Gamma_A \cup \Gamma_B$ is $2g$, thus the lemma follows. \[]

For a four-colored simple polyhedral handlebody $P$ (or a simplicial decomposition $K_P$ of a surface $\Sigma_g$), we consider a division of facets $\mathcal{F}$ into two Kempe chains in a way similar to the proof of Theorem 2.3, e.g., $\mathcal{F} = A \coprod B$ where $A$ (resp. $B$) is a set of vertices which are colored by $\alpha$ or $\beta$ (resp. $\gamma$ or $\alpha + \beta + \gamma$). In this case $\{\alpha, \beta\}$-chain of $P$ (resp. $\{\gamma, \alpha + \beta + \gamma\}$-chain) corresponds to the section subgraph $\Gamma_A$ (resp. $\Gamma_B$) of $K_P$. We notice that there are three ways to divide $\mathcal{F}$ into two Kempe chains, i.e., $\{\alpha, \beta\} \coprod \{\gamma, \alpha + \beta + \gamma\}$, $\{\alpha, \gamma\} \coprod \{\beta, \alpha + \beta + \gamma\}$ and $\{\alpha, \alpha + \beta + \gamma\} \coprod \{\beta, \gamma\}$. If an $\{\alpha, \beta\}$-chain is disconnected then one of its connected component can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of $\alpha$ and $\beta$, respectively, and we obtain a non-orientable
coloring of $P$. Assume that every chain is connected. Then each division of $\mathcal{F}$ into two chains satisfies the condition in Lemma 3.3.

In order to divide a connected Kempe-chain into two components we introduce a notion of a **cutable edge** of $K_P$ (or $P$). An edge of $K_P$ is called a **cutable edge** (of type $(\{\alpha, \beta\}, \gamma)$) when its star subcomplex of $K_P$ (i.e., the subcomplex generated
by simplices which contain the edge) is three-colored, i.e., the both end vertices of the edge are colored by \(\{\alpha, \beta\}\) and others are colored by only one color \(\gamma\). Similarly an edge of \(P\) is called a cutable edge when the dual edge is a cutable edge of \(K_P\) (see Fig. 4). A cutable edge of type \(\{\{\alpha, \beta\}, \gamma\}\) is an edge of \(\{\alpha, \beta\}\)-chain. If there exist cutable edges of a same type \(\{\{\alpha, \beta\}, \gamma\}\) such that an \(\{\alpha, \beta\}\)-chain becomes to be disconnected when they are removed, then one of its connected component can be repainted and we can construct a non-orientable coloring of \(P\). For example, for a four-coloring of \(P\) shown in Fig. 5, \(\{\alpha, \beta\}\)-chain is the set of facets \(F_1\)'s and \(F_1 \cap F_2\) and \(F_4 \cap F_3\) are cutable edges of the same type \(\{\{\alpha, \beta\}, \gamma\}\). Here a component \(F_2 \cup F_3 \cup F_4\) of \(\{\alpha, \beta\}\)-chain between two cutable edges can be repainted by \(\alpha + \gamma\) and \(\beta + \gamma\) instead of \(\alpha\) and \(\beta\), respectively, and we can construct a non-orientable coloring of \(P\).

We remark that the edge \(F \cap F_3\) in Fig. 3 in the proof of Theorem 2.3 is a cutable edge which divides connected chain into two components. When \(P\) has more than \(12g\) cutable edges, there exists a division \(\mathcal{F} = A \bigcup B\) into two chains \((\Gamma_A, \Gamma_B)\) such that \(\Gamma_A \cup \Gamma_B\) has more than \(4g\) cutable edges because there are three ways to divide \(\mathcal{F}\) into two Kempe chains. Here there are at most two types of cutable edges contained in \(\Gamma_A\) (or \(\Gamma_B\)), respectively. Then either of \(\Gamma_A\) or \(\Gamma_B\) becomes to be disconnected when cutable edges of a same type are removed because of Lemma 3.3. Therefore \(P\) can be repainted as a non-orientable coloring when \(P\) has more than \(12g\) cutable edges.

Denote the number of facets of \(P\) with \(k\)-corners by \(p_k\). By assumption \(p_3 = p_4 = 0\). We notice that a facet with \(k\)-corners has at least two cutable edges if \(k\) is not a multiple of three. Therefore if

\[
\sum_{k \neq 0 \pmod{3}} p_k > 12g
\]

then there exist more than \(12g\) cutable edges and \(P\) can be repainted as a non-
orientable coloring. If there exists a hexagonal facet of $P$ such that the six facets adjacent to it are all hexagonal, then the seven facets can be repainted as a non-orientable coloring as shown in Fig. 6 or they have at least two cutable edges. This is the case if $p_6 > \sum_{k \neq 6} kp_k$ because a $k$-cornered facet $(k \neq 6)$ adjacent to at most $k$ hexagonal facets. More generally, if $p_6 - \sum_{k \neq 6} kp_k > 7(t - 1)$ then $P$ can be repainted as a non-orientable coloring or there exist more than $t$ hexagonal facets each of which has at least two cutable edges. Therefore, if the above inequality hold for $t = 12g - \sum_{k \neq 0(\mod 3)} p_k$, i.e.

$$p_6 > \sum_{k \neq 6} kp_k + 7 \left( 12g - \sum_{k \neq 0(\mod 3)} p_k - 1 \right)$$

then $P$ can be repainted as a non-orientable coloring.

Values of $p_k$’s, which do not satisfy the above inequalities (4) and (5), are bounded. In fact, it follows from (3) and inequalities opposite to (4) and (5) that

$$\sum_k p_k \leq \sum_{k \neq 6} (k + 1)p_k + 7 \left( 12g - \sum_{k \neq 0(\mod 3)} p_k - 1 \right) \text{ from (5)$^\dagger$}$$

$$= \ -p_5 + \sum_{k > 6} (k - 6)p_k + 7 \sum_{k \equiv 0(\mod 3), k \geq 9} p_k + 84g - 7$$

$$= 7 \sum_{k \equiv 0(\mod 3), k \geq 9} p_k + 96g - 19 \text{ from (3)}$$

$$\leq \frac{7}{3} \sum_{k \geq 7} (k - 6)p_k + 96g - 19$$

$$= \frac{7}{3} (p_5 + 12g - 12) + 96g - 19 \text{ from (3)}$$

$$= \frac{7}{3} p_5 + 124g - 47$$

$$\leq \frac{7}{3} \sum_{k \equiv 0(\mod 3)} p_k + 124g - 47$$

$$\leq 152g - 47 \text{ from (4)$^\dagger$}$$

where (4)$^\dagger$ and (5)$^\dagger$ are the inequalities opposite to (4) and (5), respectively. Therefore the proof of Theorem 3.1 is completed.

Remark 3.4. The 4-colorability of graphs embedded into an orientable surface is an interesting problem. For example, we have the following conjecture (cf. [4, Conjecture 1.1]): every simplicial decomposition of an orientable surface such that all vertices have even degree and all non-contractible cycles are sufficiently large
is 4-colorable. In case that \( g = 1 \) and a graph satisfies a special condition called “6-regular,” the 4-colorability of toroidal 6-regular graph was studied in [4].

References


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