Nakayama, H. and Nishimura, Y.
Osaka J. Math.
42 (2005), 243–256

THE ORIENTABILITY OF SMALL COVERS AND COLORING SIMPLE POLYTOPES

HISASHI NAKAYAMA and YASUZO NISHIMURA

(Received September 1, 2003)

Abstract

Small Cover is an n-dimensional manifold endowed with a $\mathbf{Z}_2^n$ action whose orbit space is a simple convex polytope P. It is known that a small cover over P is characterized by a coloring of P which satisfies a certain condition. In this paper we shall investigate the topology of small covers by the coloring theory in combinatorics. We shall first give an orientability condition for a small cover. In case $n = 3$, an orientable small cover corresponds to a four colored polytope. The four color theorem implies the existence of orientable small cover over every simple convex 3-polytope. Moreover we shall show the existence of non-orientable small cover over every simple convex 3-polytope, except the 3-simplex.

0. Introduction

"Small Cover" was introduced and studied by Davis and Januszkiewicz in [5]. It is a real version of "Quasitoric manifold," i.e., an n-dimensional manifold endowed with an action of the group $\mathbf{Z}_2^n$ whose orbit space is an n-dimensional simple convex polytope. A typical example is provided by the natural action of $\mathbf{Z}_2^n$ on the real projective space $\mathbf{R}P^n$ whose orbit space is an n-simplex. Let P be an n-dimensional simple convex polytope. Here P is *simple* if the number of codimension-one faces (which are called "*facets*") meeting at each vertex is n, equivalently, the dual K_P of its boundary complex $\partial(P)$ is an $(n-1)$-dimensional simplicial sphere. In this paper we shall handle a convex polytope in the category of combinatorics. We denote the set of facets of P (or the set of vertices of K_P) by $\mathcal{F}$. Associated to a small cover M over P, there exists a function $\lambda\colon \mathcal{F} \to \mathbf{Z}_2^n$ called a "characteristic function" of M. A basic result in [5] is that small covers over P are classified by their characteristic functions (cf. [5, Proposition 1.8]). The characteristic function is a (face-)coloring of P (or a vertex-coloring of K_P), which satisfies a "linearly independent condition" (see §1). Coloring convex n-polytopes have been studied actively in combinatorics. We shall investigate the topological properties of small covers through the coloring theory in combinatorics specially when $n = 3$.

The notion of small cover can be generalized to the case where the base space is

The second named author is supported by the 21 COE program "Constitution of wide-angle mathematical basis focused on knots"

more general than a simple convex polytope. An n-dimensional nice manifold P with corners such that the dual complex K_P is a simplicial decomposition of $\partial(P)$, is called a *simple polyhedral complex*. For a coloring of P which satisfies the linearly independent condition, we can construct an n-dimensional manifold with $\mathbf{Z}_2^n$-action over P in a similar way. We call it a small cover over a simple polyhedral complex P.

In this paper we give a criterion when a small cover over a simple convex polytope is orientable (Theorem 1.7). In case $n = 3$, this criterion implies that an orientable three-dimensional small cover corresponds to a 4-colored simple convex 3-polytope. Therefore the existence of an orientable small cover over every simple convex 3-polytope is equivalent to the four color theorem (Corollary 1.8). Next we shall discuss the colorability of a 3-polytope making allowance for the linearly independent condition, and prove existence of non-orientable small cover over every simple convex 3-polytope, except the 3-simplex (Theorem 2.3). The proof of Theorem 2.3 is given in a way similar to the proof of classical five color theorem by Kempe. Moreover we shall discuss the existence of non-orientable small cover over 4-colorable simple polyhedral handlebody with a positive genus (Theorem 3.1).

1. The orientability of small covers

At first we shall recall the definition and basic results of small covers in [5] or [3]. An n-dimensional convex polytope is *simple* if the number of codimension-one faces (which are called *facets*) meeting at each vertex is n. Equivalently, P is simple if the dual of its boundary complex is a simplicial decomposition of $(n-1)$-dimensional sphere. We denote the simplicial complex dual to P by K_P. In this paper we shall handle polytopes in the category of combinatorics and understand that two polytopes are identical if they are combinatorially equivalent. Therefore a considerable structure of a polytope P is only its face structure, i.e., the dual complex K_P. The natural action of $\mathbf{Z}_2^n$ on $\mathbf{R}^n$ is called the *standard representation* and its orbit space is $\mathbf{R}_+^n$.

DEFINITION 1.1. A manifold M endowed with the action of the group $\mathbf{Z}_2^n$ is a *small cover* over an n-dimensional simple convex polytope P if its orbit space is homeomorpic to P and M is locally isomorphic to the standard representation, i.e., there exists an automorphism θ of $\mathbf{Z}_2^n$ such that for any point $x \in M$, there exists stable neighborhoods $V \subset M$ of x and $W \subset \mathbf{R}^n$ and θ-equivariant homeomorphism $f\colon V \to W$ $(f(gv) = \theta(g)f(v))$.

Let $\pi_1\colon M_1 \to P$ and $\pi_2\colon M_2 \to P$ be two small covers over P. *An equivalence over P* is an automorphism θ of $\mathbf{Z}_2^n$, together with a θ-equivariant homeomorphism $f\colon M_1 \to M_2$, which covers the identity on P.

EXAMPLE 1.2. The group $\mathbf{Z}_2$ acts on S^1 by a reflection, and its orbit space is an interval I. Taking the n-fold product, we have a $\mathbf{Z}_2^n$ action on an n-dimensional torus $T^n = S^1 \times \cdots \times S^1$, which is a small cover over the n-cube I^n.

EXAMPLE 1.3. We have a usual $\mathbf{Z}_2^n$ action on the real projective space $\mathbf{R}P^n$ as follows:

$$(g_1, \ldots, g_n) \cdot [x_0, x_1, \ldots, x_n] = [x_0, g_1x_1, \ldots, g_nx_n].$$

This is a small cover over the n-simplex Δ^n.

Let $\pi : M \to P$ be a small cover over an n-dimensional simple convex polytope P. For a face F of P, the isotropy group at $x \in \pi^{-1}(\mathrm{int}\, F)$ is independent of the choice of x, denoted by G_F. In particular, if F is a facet, G_F is a rank-one subgroup, hence, it is determined by a generator $\lambda(F) \in \mathbf{Z}_2^n$. In this way we obtain a function $\lambda : \mathcal{F} \to \mathbf{Z}_2^n$ where $\mathcal{F}$ is the set of facets of P. This function is called *the characteristic function* of M. If F^{n-k} is a codimension-k face of P then $F = F_1 \cap \cdots \cap F_k$ where F_i's are the facets which contain F, and G_F is the rank-k subgroup generated by $\lambda(F_1), \ldots, \lambda(F_k)$. Therefore the characteristic function satisfies the following condition.

$(*)$ If $F_1, \ldots, F_n$ are the facets meeting at a vertex of P, then $\lambda(F_1), \ldots, \lambda(F_n)$ are linearly independent vectors of $\mathbf{Z}_2^n$.

In particular, the characteristic function $\lambda : \mathcal{F} \to \mathbf{Z}_2^n$ is a (face-)coloring of P (or a vertex-coloring of the dual graph K_P). We often call it *a linearly independent coloring* of P (or K_P). Conversely a coloring λ of P satisfying the linearly independent condition $(*)$ determines a small cover $M(P, \lambda)$ over P whose characteristic function is the given λ. The construction of $M(P, \lambda)$ is as follows. For each point $p \in P$, let $F(p)$ be the unique face of P which contains p in its relative interior. We define an equivalence relation on $P \times \mathbf{Z}_2^n$ as follows:

$$(1) \qquad (p, g) \sim (q, h) \Leftrightarrow p = q, \;\; g^{-1}h \in G_{F(p)}$$

where $G_{F(p)}$ is the subgroup generated by $\lambda(F_1), \ldots, \lambda(F_k)$ such that $F(p) = F_1 \cap \cdots \cap F_k$ $(F_i \in \mathcal{F})$. Then the quotient space $(P \times \mathbf{Z}_2^n)/\sim$ is $M(P, \lambda)$.

Theorem 1.4 ([5, Proposition 1.8]). *Let M be a small cover over P such that its characteristic function is $\lambda : \mathcal{F} \to \mathbf{Z}_2^n$. Then M is equivalent to $M(P, \lambda)$. In other words, the small cover is determined up to equivalence over P by its characteristic function.*

EXAMPLE 1.5. In case $n = 2$, P is a polygon and a characteristic function is a function $\lambda : \mathcal{F} \to \mathbf{Z}_2^2$ which satisfies the linearly independent condition $(*)$. Let $\{e_1, e_2\}$ be a basis of $\mathbf{Z}_2^2$. Since any pair of $\{e_1, e_2, e_1 + e_2\}$ is linearly independent, λ is just a 3-coloring of P. If (P, λ) is a k-gon colored by three colors (resp. two colors) then $M(P, \lambda)$ is the non-orientable surface $(k-2)\mathbf{R}P^2$ (resp. the orientable surface with genus $(k-2)/2$) endowed with a certain action of $\mathbf{Z}_2^2$ where $m\mathbf{R}P^2$ is the connected sum of m copies of $\mathbf{R}P^2$ (cf. [5, Example 1.20]).

REMARK 1.6. When an n-dimensional simple convex polytope P is s-colored ($s \geq n$), we understand that the image of the coloring function λ is a basis for $\mathbf{Z}_2^s$, and define the quotient space $Z(P, \lambda) = (P \times \mathbf{Z}_2^s)/\sim$ where the equivalence relation $\sim$ is given in a way similar to (1). It is called the "*manifold defined by the coloring* λ" in [7]. When $s = n$, $Z(P, \lambda)$ coincides with the small cover $M(P, \lambda)$, and is called the "*pullback from the linear model*" in [5]. In the special case $n = s = 3$, pullbacks from the linear model were studied by Izmestiev in details in [7].

Next we shall discuss the orientability condition of a small cover.

Theorem 1.7. *For a basis $\{e_1, \dots, e_n\}$ of $\mathbf{Z}_2^n$, a homomorphism $\epsilon\colon \mathbf{Z}_2^n \to \mathbf{Z}_2 = \{0, 1\}$ is defined by $\epsilon(e_i) = 1$ $(i = 1, \dots, n)$. A small cover $M(P, \lambda)$ is orientable if and only if there exists a basis $\{e_1, \dots, e_n\}$ of $\mathbf{Z}_2^n$ such that the image of $\epsilon\lambda$ is $\{1\}$.*

Proof. Let us calculate the n-dimensional integral homology group $H_n(M; \mathbf{Z})$ of a small cover $M = M(P, \lambda)$. The combinatorial structure of P defines a natural cellular decomposition of $M = (P \times \mathbf{Z}_2^n)/\sim$. We denote by (C_k, ∂_k) the chain complex associated with this cellular decomposition. In particular, C_n and C_{n-1} are the free abelian groups generated by $\{P\} \times \mathbf{Z}_2^n = \{(P, g) \mid g \in \mathbf{Z}_2^n\}$ and $(\mathcal{F} \times \mathbf{Z}_2^n)/\sim = \{[F, g] \mid F \in \mathcal{F},\ g \in \mathbf{Z}_2^n\}$, respectively, where the equivalence class of $\mathcal{F} \times \mathbf{Z}_2^n$ is defined by the equivalence relation $(F, g) \sim (F, \lambda(F) + g)$. We give an orientation on a facet F_i such that $\partial(P) = F_1 + \cdots + F_f$ where $f = \#\mathcal{F}$. Under these notations if $X = \sum_{g \in \mathbf{Z}_2^n} n_g(P, g) \in C_n$ $(n_g \in \mathbf{Z})$ is an n-cycle of M then

$$\partial_n(X) = \left[\sum_{g \in \mathbf{Z}_2^n} n_g \sum_{i=1}^{f} (F_i, g)\right] = \sum_{[F_i, g] \in (\mathcal{F} \times \mathbf{Z}_2^n)/\sim} (n_g + n_{\lambda(F_i)+g})[F_i, g] = 0.$$

Therefore $X \in \ker \partial_n$ if and only if $n_g = -n_{\lambda(F)+g}$ for any facet F and $g \in \mathbf{Z}_2^n$. The latter is equivalent to $n_g = (-1)^k n_{\lambda(F_{i_1})+\dots+\lambda(F_{i_k})+g}$ for any facets $F_{i_1}, \dots, F_{i_k}$ and $g \in \mathbf{Z}_2^n$. Suppose that there is no basis of $\mathbf{Z}_2^n$ such that $\epsilon\lambda \equiv 1$. It means that for each set of facets $F_{i_1}, \dots, F_{i_n}$ such that $e_j = \lambda(F_{i_j})$ $(1 \leq j \leq n)$ is a basis of $\mathbf{Z}_2^n$, there exists a facet F such that $\epsilon\lambda(F) = 0$, i.e., $\lambda(F) = e_{j_1} + \cdots + e_{j_k}$ where k is an even number. Then $n_g = (-1)^k n_{e_{j_1}+\dots+e_{j_k}+g} = n_{\lambda(F)+g} = -n_g$ that is $n_g = 0$ for any $g \in \mathbf{Z}_2^n$. Thus $H_n(M; \mathbf{Z}) = \ker \partial_n = 0$, and M is non-orientable. On the other hand, when there exists a basis of $\mathbf{Z}_2^n$ such that $\epsilon\lambda \equiv 1$, for any $g \in \mathbf{Z}_2^n$, the parity of $k = k(g)$ does not depend on the choice of F_{i_j}'s such that $g = \lambda(F_{i_1}) + \cdots + \lambda(F_{i_k})$. In fact, if $\lambda(F_{i_1}) + \cdots + \lambda(F_{i_k}) =$

$\lambda(F_{j_1}) + \cdots + \lambda(F_{j_l})$ then $\epsilon\lambda(F_{i_1}) + \cdots + \epsilon\lambda(F_{i_k}) = \epsilon\lambda(F_{j_1}) + \cdots + \epsilon\lambda(F_{j_l})$, therefore $k \equiv l \bmod 2$. Then $H_n(M; \mathbf{Z}) = \ker \partial_n \cong \mathbf{Z}$ is generated by $X = \sum_{g \in \mathbf{Z}_2^n} (-1)^{k(g)}(P, g)$, and M is orientable. $\square$

We call a linearly independent coloring which satisfies the orientability condition in Theorem 1.7 *an orientable coloring* of P (or K_P). In case $n = 2$, it is easy to see that an orientable coloring of a polygon is just a 2-coloring of P (see Example 1.5). In case $n = 3$, a three-dimensional small cover $M(P, \lambda)$ is orientable if and only if there exists a basis $\{\alpha, \beta, \gamma\}$ of $\mathbf{Z}_2^3$ such that the image of λ is contained in $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$. Since each triple of $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$ is linearly independent, the orientable coloring of a 3-polytope P is just a 4-coloring of P. By the four color theorem ([1]), we obtain the following corollary (in fact, the corollary below is equivalent to the four color rheorem).

Corollary 1.8. *There exists an orientable small cover over every simple convex* 3-*polytope.*

REMARK 1.9. Although there exists a small cover over every three-dimensional simple convex polytope, for each integer $n \geq 4$, there exists an n-dimensional simple convex polytope Q which admits no small cover (cf. [5, Nonexample 1.22]). In fact, a cyclic polytope C_k^n defined as the convex hull of $k \geq n + 1$ points on a curve $\gamma(t) = (t, t^2, \ldots, t^n)$ is a simplicial polytope such that the one-skeleton of C_k^n is a complete graph when $n \geq 4$ (see [2, §13]). Let Q_k^n be the simple polytope dual to C_k^n. Since the chromatic number of Q_k^n is k, the polytope Q_k^n admits no small cover whenever $k \geq 2^n$.

2. Existence of non-orientable small covers

We call a linearly independent coloring which does not satisfy the orientability condition in Theorem 1.7 *a non-orientable coloring*. In this section we shall discuss the existence of a non-orientable coloring over a simple polytope in case $n = 3$. We shall recall and use some notions of 3-polytopes and the graph theory. For further details see [2] or [6].

For a simple convex 3-polytope P, we set p_k the number of k-cornered facets of P. Then the numbers of facets, edges and vertices of P are $f_2 = \sum_{k \geq 3} p_k$, $f_1 = \sum_{k \geq 3} kp_k/2$ and $f_0 = \sum_{k \geq 3} kp_k/3$, respectively. Since the Euler number $f_0 - f_1 + f_2$ of $\partial(P)$ is two, we obtain immediately the following formula.

Lemma 2.1. *For any simple convex* 3-*polytope* P,

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k - 6)p_k. \tag{2}$$

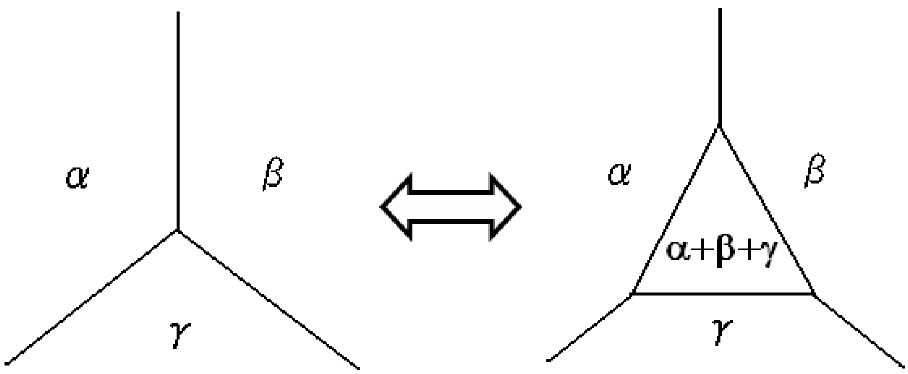

Fig. 1.

The lemma implies a well-known fact that each simple convex 3-polytope has a facet which has less than six edges.

We introduce an operation "*blow up*" for a vertex of a simple 3-polytope P (see Fig. 1). We cut around a vertex of P and create a new triangular facet there. The reverse operation of the blow up is called a "*blow down*." Notice that for any simple convex polytope except the 3-simplex, two triangular facets must not adjoin each other. Therefore except for the 3-simplex, the blow down for any triangular facet is possible. For the dual simplicial complex K_P, a blow up is operated for a 2-simplex of K_P and a blow down is operated for a vertex of K_P with degree three, respectively. The blow up can be done keeping the linearly independence. In fact, we can assign $\alpha + \beta + \gamma$ to the new triangle where α, β and γ are the colors assigned to three facets adjacent to the triangle. The blow up operation corresponds to *the equivariant connected sum* of $\mathbf{R}P^3$ at the fixed point of small covers corresponding to the vertex (see [5, 1.11]). When P is endowed with a linearly independent coloring, the blow down for a triangular facet can not be done keeping the linearly independence. From the above fact we obtain the following lemma immediately.

Lemma 2.2. *If a convex polytope P has a non-orientable coloring then each polytope obtained by blowing up P has also a non-orientable coloring.*

Theorem 2.3. *There exists a non-orientable small cover over every simple convex 3-polytope, except the 3-simplex.*

Proof. Let P be a simple convex polytope but not the 3-simplex. Operating the blow downs for triangular facets of P over again, P can be transformed to a polytope P' which does not have a triangular facet or is the triangular prism. In the latter case although the triangular prism can be transformed to the 3-simplex further by the blow down, we stop the operation because the 3-simplex is excepted from this the-

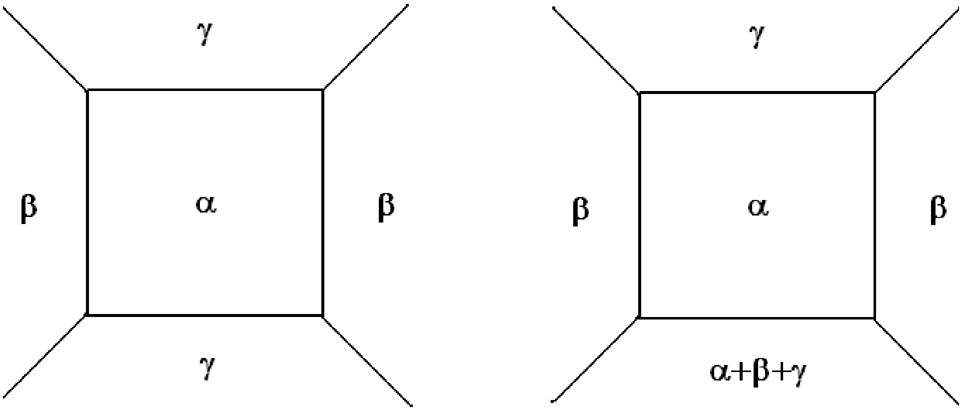

Fig. 2.

orem. By Lemma 2.2, a non-orientable coloring of P' leads to that of P. Therefore we can assume that P does not have a triangular facet or P is the triangular prism. We assume the four color theorem, and shall prove that some facets of 4-colored polytope can be repainted making allowance for the linearly independent condition and construct a non-orientable coloring. Here we assume that P is colored by four colors $\{\alpha, \beta, \gamma, \alpha+\beta+\gamma\}$ for some basis $\{\alpha, \beta, \gamma\}$ of $\mathbf{Z}_2^3$. (When P is colored by only three colors, we can repaint a facet and assume that P is 4-colored if necessary.) The case that P has a quadrilateral facet is immediate. In fact, the 4-coloring around a quadrilateral facet F must be the following situation: a center quadrilateral F is colored by α and two facets adjacent to F are colored by β and the rest two facets adjacent to F are colored by one color γ or two colors γ and $\alpha+\beta+\gamma$, respectively (see Fig. 2). In both cases we can repaint the center quadrilateral by $\alpha+\beta$ instead of α, and produce the non-orientable coloring. In particular, the triangular prism has a non-orientable coloring because it has a quadrilateral facet.

Suppose that P has no triangle and quadrilateral. By Lemma 2.1, P must have a pentagonal facet F. We can assume that the 4-coloring around F is in the following situation: the center pentagon F is colored by α and the adjacent five facets $F_1, \dots, F_5$ are 3-colored by β, γ, β, γ and $\alpha+\beta+\gamma$, respectively (see Fig. 3). Here we shall repaint some facets of P and construct a non-orientable coloring in a way similar to the proof of the classical Five Color Theorem by Kempe using the "Kempe chain" (cf. [8] or [6]). First we consider the $\{\alpha, \beta\}$-chain containing the pentagon F, i.e., the connected component of facets colored by α or β, which contains F. If the $\{\alpha, \beta\}$-chain has no elementary cycle containing F then we divide it by the edge $F \cap F_3$ into two chains, and the one side which contains F_3 can be repainted by $\alpha+\gamma$ and $\beta+\gamma$ instead of α and β, respectively. If the $\{\alpha, \beta\}$-chain has an elementary cycle containing F then F_2 and F_4 belong to a different component of $\{\gamma, \alpha+\beta+\gamma\}$-chain respectively, because of the Jordan curve theorem. Therefore the one side of them can

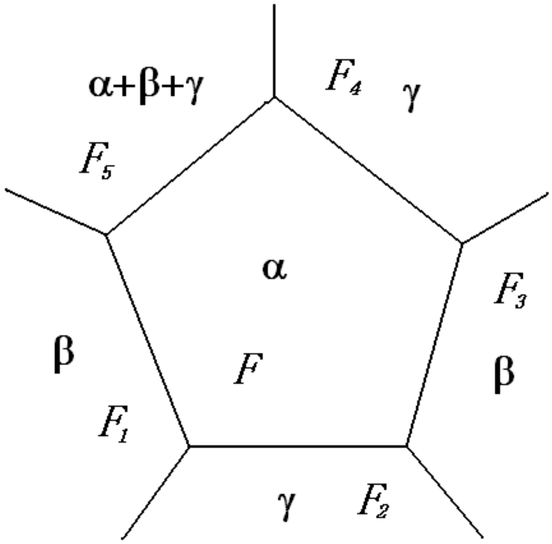

Fig. 3.

be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of γ and $\alpha + \beta + \gamma$, respectively. In both cases the repainted polytope is five or six-colored and the new coloring also satisfies the linearly independent condition. Therefore we obtain a non-orientable coloring of P. $\square$

3. Coloring simple polyhedral handlebodies

Let P be an n-dimensional nice manifold with corners. We say that P is a *simple polyhedral complex* if its dual complex K_P is a simplicial decomposition of $\partial(P)$. This condition implies that any intersection of two faces is a face of P. We can characterize a simple polyhedral complex by a pair of a manifold P and a simplicial decomposition K of $\partial(P)$. In fact a simplicial decomposition K of $\partial(P)$ determines the polyhedral structure of P as follows. For each simplex $\sigma \in K$, let F_σ denote the geometric realization of the poset $K_{\geq\sigma} = \{\tau \in K \mid \sigma \leq \tau\}$. We say that F_σ is a *codimension-k face* of P if σ is a $(k-1)$-simplex of K. Then its dual complex K_P is clearly same as K.

We call a facet-coloring of P (or a vertex-coloring of K_P) simply *a coloring* of P (or K_P). We denote by $\mathcal{F}$ the set of facets of P (or the set of vertices of K_P). A function $\lambda\colon \mathcal{F} \to \mathbf{Z}_2^n$ is called a linearly independent coloring of P (or K_P) if λ satisfies for P the condition $(*)$ in §1. We put $M(P, \lambda) = (P \times \mathbf{Z}_2^n)/\sim$, where the equivalence relation $\sim$ is defined as (1) in §1. We have a $\mathbf{Z}_2^n$-action on an n-dimensional manifold $M(P, \lambda)$ with the orbit space P. Conversely an n-dimensional manifold M endowed with a locally standard $\mathbf{Z}_2^n$-action whose orbit space is homeomorphic to P determines a characteristic function $\lambda\colon \mathcal{F} \to \mathbf{Z}_2^n$. Then M is equivalent to $M(P,\lambda)$ if the restriction on $\pi^{-1}(\operatorname{int} P)$ of the projection $\pi : M \to P$ is a trivial covering. We will say that $M(P,\lambda)$ is a *small cover* over P. (Warning: In [5], for each $(n-1)$-dimensional simplicial complex K, the simple polyhedral complex P_K is the cone on K. Then $M(P_K,\lambda)$ can be defined in a similar way, however, it is not always a manifold, and therefore it is called a "$\mathbf{Z}_2^n$*-space*" in [5].)

When P is an orientable simple polyhedral complex, the orientability condition of $M(P,\lambda)$ is same as the condition in Theorem 1.7. Therefore we may generalize the notion of (*non-*) *orientable coloring* of P (or K_P) to this case. Henceforth, we take P as a simple polyhedral handlebody with genus $g > 0$, i.e., a handlebody P together with a simplicial decomposition K_P of the orientable closed surface Σ_g with genus g. In this case, the formula (2) in Lemma 2.1 is generalized to

$$\sum_{k\geq 3}(k-6)p_k = 12(g-1) \tag{3}$$

where p_k is the number of k-cornered facets of P (or vertices of K_P whose degree is k). In the rest of this section we shall prove the following theorem.

Theorem 3.1. *Let P be a* 4*-colorable simple polyhedral handlebody with genus $g > 0$* (*equivalently, there exists an orientable small cover over P*). *If P has sufficiently many facets then there also exists a non-orientable small cover over P.*

Assume that P is colored by four colors $\{\alpha,\beta,\gamma,\alpha+\beta+\gamma\}$ for some basis $\{\alpha,\beta,\gamma\}$ of $\mathbf{Z}_2^3$. We shall repaint some facets of P and construct a non-orientable coloring. By the same reason as in the proof of Theorem 2.3, when P has a quadrilateral facet, the construction of non-orientable coloring is immediate.

Next we consider two operations "*blow down*" and "*blow up*" introduced in §2 (see Fig. 1). We can define these operations for a simple polyhedral handlebody P (or a simplicial decomposition of an orientable surface K_P) in a similar way. The blow up can be always done for any vertex of P together with a linearly independent coloring. Notice that for any simple polyhedral handlebody with a positive genus, two triangular facets must not adjoin each other. Therefore the blow down can be always done for any triangular facet of P. Operating the blow down for triangular facets of P one after another, we can reduce P to a simple polyhedral complex P' which has no triangular facet. As we have already seen in Lemma 2.2, a non-orientable coloring on P' can be extended on P. In the course of this process if a quadrilateral facet appears, we can also construct a non-orientable coloring of P. We assume that a quadrilateral facet does not appear during the reduction from P to P'. Generally if we operate blow up for a vertex of a triangular facet then a quadrilateral facet will be created. By the above assumption P must be obtained by blow up for original vertices of P' (but not for new vertices born by blow up). Therefore the number of facets of P is at most the sum of numbers of facets and vertices of P'. Consequently it is sufficient to prove Theorem 3.1 in the case that P does not have a quadrilateral or a triangle. In fact if the following proposition holds for any simple polyhedral handlebody which has more than N facets but not a quadrilateral and a triangle then Theorem 3.1 holds for any simple polyhedral handlebody which has more than $N + M$ facets where M is the maximum of numbers of vertices of simple polyhedral handlebodies which do not have more than N facets and a quadrilateral and a triangle.

Proposition 3.2. *Let P be a* 4*-colorable simple polyhedral handlebody with genus $g > 0$ such that P does not have a quadrilateral or a triangle* (*equivalently its dual K_P is a simpicial decomposition of an orientable surface Σ_g with genus $g > 0$ such that K_P does not have a vertex with degree three or four*)*. If P has sufficiently many facets then P has a non-orientable coloring.*

For a subset A of vertices of a simplicial complex K, we denote by Γ_A the subgraph of one-skeleton K^1 generated by A (which is called the *section subgraph*). We need the following lemma instead of the Jordan curve theorem.

Lemma 3.3. *Let K be a simplicial decomposition of the orientable closed surface Σ_g with genus $g > 0$, and (A, B) be a division of vertices of K $(A \coprod B = V(K))$ such that the section subgraphs Γ_A, Γ_B are both connected and have no cycle of length three. When $2g + 1$ edges are removed from $\Gamma_A \cup \Gamma_B$, either Γ_A or Γ_B is disconnected.*

Proof. Because Γ_A and Γ_B have no cycle of length three, each 2-simplex Δ of K intersects both of Γ_A and Γ_B, i.e., all vertices and only one edge of Δ belong to $\Gamma_A \cup \Gamma_B$. Therefore the number of 2-simplices of K and the number of edges of K which do not belong to $\Gamma_A \cup \Gamma_B$ coincide. Thus $\chi(\Gamma_A \cup \Gamma_B) = \chi(K) = 2 - 2g$ where $\chi(G)$ is the Euler number of G. Since Γ_A and Γ_B are both one-dimensional connected subcomplices of K, this means that the first Betti number of $\Gamma_A \cup \Gamma_B$ is $2g$, thus the lemma follows. $\square$

For a four-colored simple polyhedral handlebody P (or a simplicial decomposition K_P of a surface Σ_g), we consider a division of facets $\mathcal{F}$ into two Kempe chains in a way similar to the proof of Theorem 2.3, e.g., $\mathcal{F} = A \coprod B$ where A (resp. B) is a set of vertices which are colored by α or β (resp. γ or $\alpha + \beta + \gamma$). In this case $\{\alpha, \beta\}$-chain of P (resp. $\{\gamma, \alpha + \beta + \gamma\}$-chain) corresponds to the section subgraph Γ_A (resp. Γ_B) of K_P. We notice that there are three ways to divide $\mathcal{F}$ into two Kempe chains, i.e., $\{\alpha, \beta\} \coprod \{\gamma, \alpha+\beta+\gamma\}$, $\{\alpha, \gamma\} \coprod \{\beta, \alpha+\beta+\gamma\}$ and $\{\alpha, \alpha+\beta+\gamma\} \coprod \{\beta, \gamma\}$. If an $\{\alpha, \beta\}$-chain is disconnected then one of its connected component can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of α and β, respectively, and we obtain a non-orientable

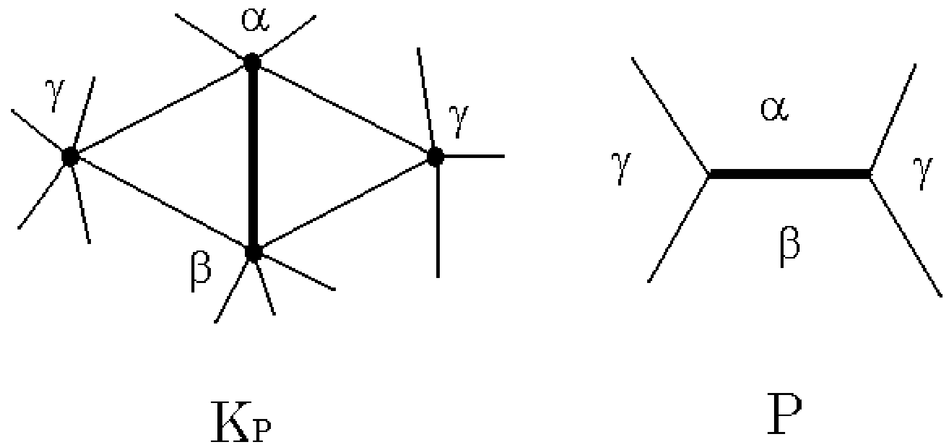

Fig. 4.

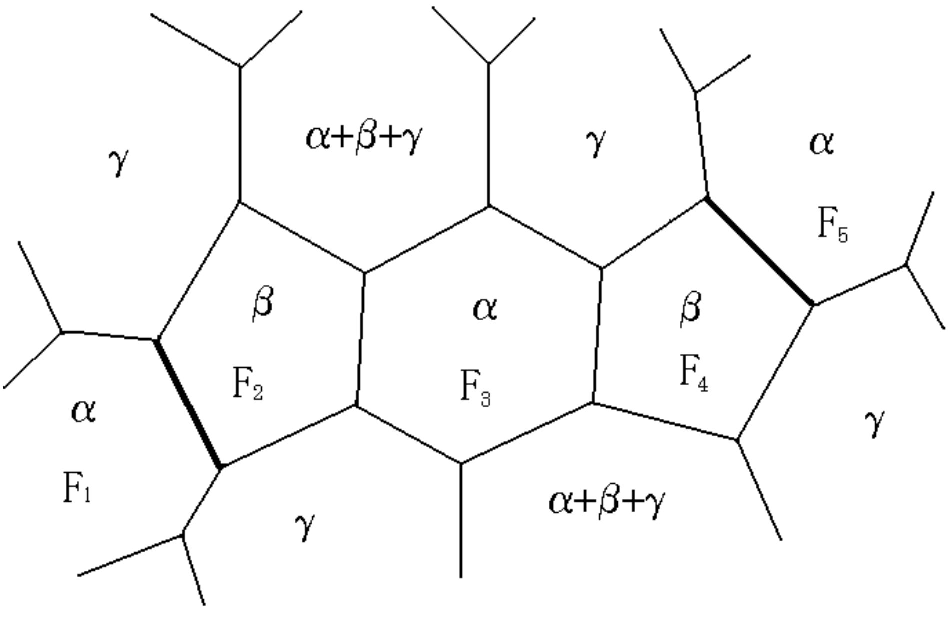

Fig. 5.

coloring of P. Assume that every chain is connected. Then each division of $\mathcal{F}$ into two chains satisfies the condition in Lemma 3.3.

In order to divide a connected Kempe-chain into two components we introduce a notion of a *cutable edge* of K_P (or P). An edge of K_P is called a *cutable edge* (*of type* $(\{\alpha, \beta\}, \gamma)$) when its star subcomplex of K_P (i.e., the subcomplex generated

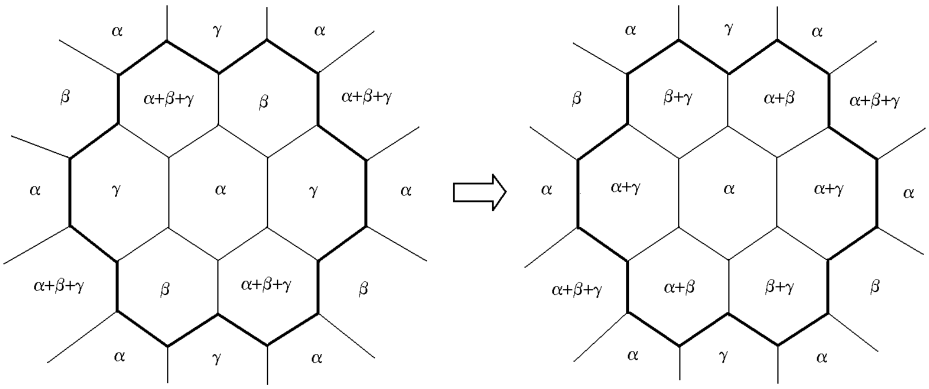

Fig. 6.

by simplices which contain the edge) is three-colored, i.e., the both end vertices of the edge are colored by $\{\alpha, \beta\}$ and others are colored by only one color γ. Similarly an edge of P is called a cutable edge when the dual edge is a cutable edge of K_P (see Fig. 4). A cutable edge of type $(\{\alpha, \beta\}, \gamma)$ is an edge of $\{\alpha, \beta\}$-chain. If there exist cutable edges of a same type $(\{\alpha, \beta\}, \gamma)$ such that an $\{\alpha, \beta\}$-chain becomes to be disconnected when they are removed, then one of its connected component can be repainted and we can construct a non-orientable coloring of P. For example, for a four-coloring of P shown in Fig. 5, $\{\alpha, \beta\}$-chain is the set of facets F_i's and $F_1 \cap F_2$ and $F_4 \cap F_5$ are cutable edges of the same type $(\{\alpha, \beta\}, \gamma)$. Here a component $F_2 \cup F_3 \cup F_4$ of $\{\alpha, \beta\}$-chain between two cutable edges can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of α and β, respectively, and we can construct a non-orientable coloring of P. We remark that the edge $F \cap F_3$ in Fig. 3 in the proof of Theorem 2.3 is a cutable edge which divides connected chain into two components. When P has more than $12g$ cutable edges, there exists a division $\mathcal{F} = A \coprod B$ into two chains (Γ_A, Γ_B) such that $\Gamma_A \cup \Gamma_B$ has more than $4g$ cutable edges because there are three ways to divide $\mathcal{F}$ into two Kempe chains. Here there are at most two types of cutable edges contained in Γ_A (or Γ_B), respectively. Then either of Γ_A or Γ_B becomes to be disconnected when cutable edges of a same type are removed because of Lemma 3.3. Therefore P can be repainted as a non-orientable coloring when P has more than $12g$ cutable edges.

Denote the number of facets of P with k-corners by p_k. By assumption $p_3 = p_4 = 0$. We notice that a facet with k-corners has at least two cutable edges if k is not a multiple of three. Therefore if

$$\sum_{k \not\equiv 0 \pmod 3} p_k > 12g \tag{4}$$

then there exist more than $12g$ cutable edges and P can be repainted as a non-orientable coloring. If there exists a hexagonal facet of P such that the six facets adjacent to it are all hexagonal, then the seven facets can be repainted as a non-orientable coloring as shown in Fig. 6 or they have at least two cutable edges. This is the case if $p_6 > \sum_{k\neq 6} kp_k$ because a k-cornered facet ($k \neq 6$) adjacent to at most k hexagonal facets. More generally, if $p_6 - \sum_{k\neq 6} kp_k > 7(t-1)$ then P can be repainted as a non-orientable coloring or there exist more than t hexagonal facets each of which has at least two cutable edges. Therefore, if the above inequality hold for $t = 12g - \sum_{k\not\equiv 0 \pmod 3} p_k$, i.e.

$$p_6 > \sum_{k\neq 6} kp_k + 7\left(12g - \sum_{k\not\equiv 0 \pmod 3} p_k - 1\right) \tag{5}$$

then P can be repainted as a non-orientable coloring.

Values of p_k's, which do not satisfy the above inequlities (4) and (5), are bounded. In fact, it follows from (3) and inequalities opposite to (4) and (5) that

$$\begin{aligned}
\sum_k p_k &\leq \sum_{k\neq 6}(k+1)p_k + 7\left(12g - \sum_{k\not\equiv 0 \pmod 3} p_k - 1\right) && \text{from } (5)^\dagger \\
&= -p_5 + \sum_{k>6}(k-6)p_k + 7\sum_{k\equiv 0 \pmod 3, k\geq 9} p_k + 84g - 7 \\
&= 7\sum_{k\equiv 0 \pmod 3, k\geq 9} p_k + 96g - 19 && \text{from } (3) \\
&\leq \frac{7}{3}\sum_{k\geq 7}(k-6)p_k + 96g - 19 \\
&= \frac{7}{3}(p_5 + 12g - 12) + 96g - 19 && \text{from } (3) \\
&= \frac{7}{3}p_5 + 124g - 47 \\
&\leq \frac{7}{3}\sum_{k\not\equiv 0 \pmod 3} p_k + 124g - 47 \\
&\leq 152g - 47 && \text{from } (4)^\dagger
\end{aligned}$$

where $(4)^\dagger$ and $(5)^\dagger$ are the inequalities opposite to (4) and (5), respectively. Therefore the proof of Theorem 3.1 is completed.

REMARK 3.4. The 4-colorability of graphs embedded into an orientable surface is an interesting problem. For example, we have the following conjecture (cf. [4, Conjecture 1.1]): *every simplicial decomposition of an orientable surface such that all vertices have even degree and all non-contractible cycles are sufficiently large* *is* 4-*colorable*. In case that $g = 1$ and a graph satisfies a special condition called "6-regular," the 4-colorability of toroidal 6-regular graph was studied in [4].

References

[1] K. Appel and W. Haken: *Every planner map is four colorable*, Illinois J. Math. **21** (1977), 429–490.
[2] A. Brøndsted: Introduction to Convex Polytopes, Springer, New York 1983.
[3] V.M. Buchstaber and T.E. Panov: *Torus actions and their applications in topology and combinatorics*, University Lecture Series, **24**, American Math. Soc. 2002.
[4] K.L. Collins and J.P. Hutchinson: *Four-coloring six-regular graphs on the torus*, CRM Proceedings and Lecture Notes, **23** (1999) 21–34.
[5] M. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), 417–451.
[6] R. Diestel: Graph Theory, Springer, New York, 2nd ed. 2000.
[7] I.V. Izmestiev: *Three-dimensional manifolds defined by coloring a simple polytope*, Math. Notes **69** (2001), 340–346.
[8] A.B. Kempe: *How to colour a map with four colours*, Nature **21** (26th February, 1880) 399–400.

Hisashi Nakayama
Research Institute of Systems
Planning, Inc.
23-23, Sakuragaoka-cho, Shibuya-ku
Tokyo 150-0031, Japan
e-mail: nakayama@isp.co.jp

Yasuzo Nishimura
Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558-8585, Japan
e-mail: nyasuzo@hkg.odn.ne.jp