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Representation of Riemann Surfaces

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The present paper is a continuation of the previous paper "On the ideal boundary of abstract Riemann surfaces[1]" and its purpose is to investigate the covering properties of Riemann surfaces of some classes.

Let R be a Riemann surface and let $\{R_n\}$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$ $(n=1, 2, \cdots)$.

Class 0_{HAB} and 0_{HAD}. Let R' be a Riemann surface ($\subset R$) with compact relative boundary $\partial R'$. If there exists no non-constant harmonic function $U(z)$ in R' such that $U(z)=0$ on $\partial R'$, $\sup|U(z)|<\infty$ $(D(U(z))<\infty)$ and the conjugate harmonic function of $U(z)$ has vanishing periods along every dividing cut, we say $R'\subset 0_{HAB}(\subset 0_{HAD})$.

Class 0^0_{AB}, 0^0_{AD}, 0^0_{ASD}. If any non compact domain G of R with compact or non compact relative boundary ∂G tolerates no non-constant bounded, Dirichlet bounded or spherical area bounded analytic function with vanishing real part on ∂G, we say $R\in 0^0_{AB}$, 0^0_{AD} or 0^0_{ASD} respectively.

Theorem 1. *The properties* $R'\in 0_{HAB}$, 0_{HAD} *and* $R\in 0^0_{AB}$, 0^0_{AD}, 0^0_{ASD} *are ones depending only on the ideal boundary.*

Proof. Our assertion for $R\in 0^0_{AD}$, 0^0_{AB} and 0^0_{ASD} is evident. We shall prove for the other classes. Suppose $R'\notin 0_{HAB}$ or 0_{HAD}. Then there exists a harmonic function in R' such that $U(z)=0$ on $\partial R'$, every period of its conjugate function along a dividing cut is zero and $\sup|U(z)|<\infty$ or $D(U(z))<\infty$. Let $R''(\subset R')$ be a Riemann surface with compact relative boundary such that $R''-R'$ is compact and $\partial R'\cap\partial R''=0$, where R'' may consist of a finite number of components. Let $V_n(z)$ be a harmonic function in $R''\cap R_n$ such that $V_n(z)=U(z)$ on $\partial R''$, $\frac{\partial V_n(z)}{\partial n}=0$ on $\partial R^n\cap R''$. Then $V_n(z)$ converges to a function $V(z)$ in mean. It is clear that $V(z)$ has the conjugate harmonic function with vanishing periods along every dividing cut. $V(z)$ has M.D.I. (minimal Dirichlet integral) which is equal

1) Z. Kuramochi: On the ideal boundary of abstract Riemann surfaces: Osaka Math. 10, 1958.

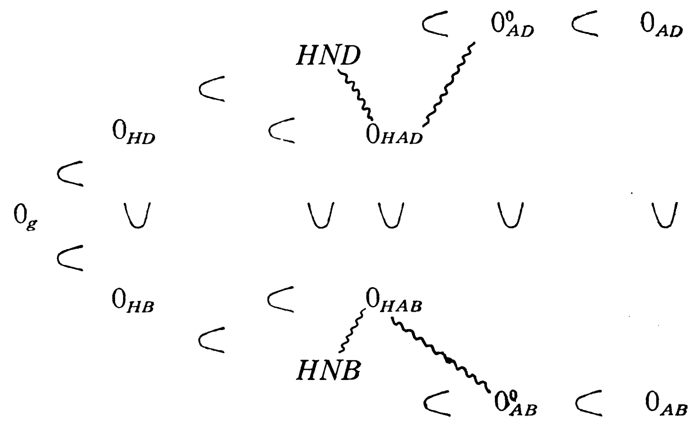
to $\int_{\partial R''} V(z)\frac{\partial V(z)}{\partial n}ds$ and $\sup|V(z)|<\infty$. We show that $U(z)-V(z)$ is a non-constant and satisfies the above conditions. On the contrary, suppose $U(z)\equiv V(z)$. Let $R'''(\subset R'')$ be a Riemann surface with compact relative boundary $\partial R'''$ such that $\partial R''\cap\partial R'''=0$ and $R''-R'''$ is compact. Then $\max_{z\in\partial R''} U(z)<\max_{z\in\partial R'''} U(z)$. On the other hand, by considering $U(z)(\equiv V(z))$ in R'', $\max_{z\in\partial R''} U(z)>\max_{z\in\partial R'''} U(z)$. This is a contradiction. Hence $U(z)-V(z)$ is non-constant and it is clear that $\sup|U(z)-V(z)|<\infty$, if $\sup|U(z)|<\infty$ and $D(U(z)-V(z))<\infty$, if $D(U(z))<\infty$ and further the conjugate function of $U(z)-V(z)$ has vanishing periods along every dividing cut. Next, let $R''\notin 0_{HAD}$ or 0_{HAB} and let $U(z)$ a non-constant harmonic function satisfying the above conditions. Then since both dist $(\partial R'', \partial R')$ and dist $(\partial R'', \partial R''')$ are positive, we can construct by Neumann's alternierendes Verfahren a harmonic function $U^*(z)$ in R' such that $U^*(z)$ is harmonic in $R'-R'''$, $U^*(z)=0$ on $\partial R'$ and $U(z)-U^*(z)$ has M.D.I. over R'', whence $U^*(z)$ has the conjugate harmonic function with vanishing periods along every dividing cut and $\sup|U^*(z)|<\infty$ for $\sup|U(z)|<\infty$ and $D(U^*(z))<\infty$ for $D(U(z))<\infty$ respectively. We can prove that $U^*(z)$ is non-constant as above. Hence $R\notin 0_{HAD}$ or 0_{HAB} respectively.

The classes 0_{HAB} and 0_{HAD} are generalizations of 0_{AB} and 0_{AD} of Riemann surfaces of finite genus, in this case evidently $0^0_{AB}\subset 0_{AB}(=0_{HAB})$ and $0^0_{AD}\subset 0_{AD}(=0_{HAD})$ respectively. But in general cases, there exists a Riemann surface with positive boundary belonging to 0^0_{ASD} and not belonging to 0_{HAD}. For exmple, let $R-R_0$ be a Riemann surface with positive ideal boundary and with one ideal boundary component $\mathfrak{p}$ which has two different bounded minimal functions $N(z, p_1)$ and $N(z, p_2)$ (p_1 and p_2 lie on $\mathfrak{p}$) and let $R\subset HND$[2] $(N=2)$. Then $U(z)=N(z, p_1)-N(z, p_2)=0$ on ∂R_0, $U(z)$ is harmonic in $R-R_0$, $D(U(z))<4\pi\max(\sup N(z, p_1), \sup N(z, p_2))$ and $U(z)$ has the conjugate harmonic function with vanishing periods along every dividing cut. Hence $R-R_0\notin 0_{HAD}$. On the other hand, it is clear $R-R_0\in 0^0_{AD}$[3]. Similar facts occur for 0_{HAB} and 0^0_{HAB}.

2) HNB (HND) means a class of Riemann surfaces on which at most N number of linearly independent bounded (Dirichlet bounded) harmonic functions exist.

3) Let 0_g be a class of Riemann surface with null-boundary. Let R be a Riemann surface $\in(H2D-0_g)$ and $\notin 0^0_{AD}$. Then there exists a non compact domain G in R such that a non constant Dirichlet bounded analytic function with vanishing real part on ∂G exists in G. Clearly there exists a non-constant Dirichlet bounded harmonic function vanishing on ∂G exists in G. Then in at least one of $G_1=G\cap v(p_1)$ and $G_2=G\cap v(p_2)$ there exists a Dirichlet bounded harmonic function vanishing on $\partial G_i(i=1.2.)$. Then by Theorem 10 (On the ideal boundary of Riemann surfaces) there exists no Dirichlet bounded analytic function, where $v(p_1)$ and $v(p_2)$ are neighbourhoods of p_1 and p_2 with respect to Martin's topology. This is a contradiction. Hencc $R\in 0^0_{AD}$.

Hence we have the following



∿∿ means that there is no inclusion relation.

From the above example, we see that the properties $R \in 0_{HAB}$, 0_{HAD}, 0^0_{AB} and 0^0_{AD}, 0^0_{ASD} depend not only on the *size of the ideal boundary but also on the complexity of the ideal boundary.* On the other hand, the properties $R \in 0_{AB}$[4] or 0_{AD} sometimes depend only on *geometrical structure of R,* for instance, the location of genus and branch points.

Exceptional set. $\mathfrak{E}_0$ (=*set of capacity zero*), $\mathfrak{E}_{AB}$, $\mathfrak{E}_{AD}$, $\mathfrak{E}_2$ (=*set of areal measure zero*). Let F be a closed set in the w-plane. If in the complementary domain of F, there exists no non-constant bounded (Dirichlet bounded) analytic function, we say $F \subset \mathfrak{E}_{AB}$ ($\mathfrak{E}_{AD}$). Clearly $\mathfrak{E}_0 \subset \mathfrak{E}_{AB} \subset \mathfrak{E}_{AD} \subset \mathfrak{E}_2$.

In the following, we suppose that an analytic function $f(z)$ is defined in R or $R-R_0$ or non compact domain G of R, whose values fall on the w-plane.

1. Properties of connected pieces.

Let K; $|w-w_0|<r$ be a circle and let ψ be a connected piece over K. Suppose that an analytic function is defined in a non compact domain G with analytic relative boundary ∂G. We shall proved the following

Theorem 2.[5] *Let $R \in HNB(0 \leq N \leq \infty)$ and G be a non compact domain. If a connected piece ψ has no common points with the image of ∂G, then ψ covers K except at most a closed set of capacity zero.*

If we apply the above theorem to smaller connected pieces, we have the following

4) 0_{AB} (0_{AD}) means a class of Riemann surface on which there exists no non-constant bounded (Dirichlet bounded) analytic function.
5) See 1).

Corollary.[6] *Let $n(w)$ be the number of times that w is covered by ψ. Then $n(w)=\sup n(w)$ $(\leqq \infty)$ except at most an F_σ of capacity zero.*

Proof. Let $D_n=E[w\,;\,n(w)\geqq n]$. Then $D_1\supset D_2\supset D_3,\cdots$. Assume that the set $F=E[w: n(w)<\sup n(w)]$ is of positive capacity. Put $F_k=F\cap D_k$. Then $F=\sum F_k$. Hence since $\mathrm{Cap}(F_0)=0$ by Theorem 2, there exists a number k such that $\mathrm{Cap}(F_k)>0$. We can suppose, that F_k is closed. Then there exists a point $w^*\in F_k$ such that $\mathrm{Cap}(F_k\cap K')>0$ for any small circle K' about w^*. Since $w^*\in F_k$, w^* is covered k times by ψ, so that there exist k discs $\psi_1^0, \psi_2^0, \cdots, \psi_k^0$ consisting of inner points. Since $1\leqq k\leqq \sup n(w)-1$, there exists another connected piece ψ^0 over K' except $\psi_1^0, \psi_2^0, \cdots, \psi_{k'}^0$. But ψ^0 does not cover $K'\cap F_k$, which contradicts Theorem 2. Hence we have the corollary.

Theorem 3.[5] *Let $R\in HND(0\leqq N\leqq\infty)$ and let G be a non compact domain. If a connected piece ψ has no common points with $f(\partial G)$ and the spherical area of ψ is finite, then ψ covers K except at most a closed set of capacity zero.*

Similarly as the corollary of Theorem 2, we have the following

Corollary.[6] *Let $n(w)$ be the number of times that w is covered by ψ. Then $n(w)=\sup n(w)<\infty$ except at most a closed set of capacity zero. Because $E[w: n(w)<\sup n(w)]=\sum_i \partial D_i$ is closed. Hance if ψ does not cover a set of positive capacity, the spherical area of ψ must be infinite.*

Since K is bounded, the spherical area of ψ is infinite, if and only if the area is infinite. Therefore we consider only the area but spherical area.

Mean covering number $n^*(w')$. Put $\varlimsup_{r\to 0}\dfrac{\iint_{K_r} n(w)\,df}{\pi r^2}=n^*(w')$ and call $n^*(w)$ *the mean covering number* of w, where $K_r=E[w: |w-w'|<r]$.

Theorem 4. *Let $R\in HND(0\leqq N\leqq\infty)$ and let G be a non compact domain. If the area of a connected piece over a circle K is infinite, $\bar{D}_\infty=\overline{\cap D_n}$ is non empty, where $D_n=E[w: n(w)\geqq n]$. Let $\Omega_1, \Omega_2, \cdots$ be components of the open set $K-\bar{D}_\infty$. Then $n(w)=\sup_{\Omega_i} n(w)=n^{\Omega_i}<\infty$ except at most a closed set of capacity zero in Ω_i and $n^*(w)=\infty$ at every point of $\bar{D}_\infty$.*

Proof. Let Ω be one of components and let $G_i=E\left[w\in\Omega,\ \mathrm{dist}\,(w, (\partial\Omega+\partial K))>\dfrac{1}{i}\right]$. Then it is clear that $n(w)<\infty$ for every point w in G_i.

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5) See 1).

6) These are pointed by K. Matsumoto without proof. Matsumoto: Remarks on some Riemann surface. Proc. Acad. Tokyo. 1958.

Hence $G_i=\sum_{j=1}^{\infty}H_j$, where $H_j=E[w: n(w)\leqq j]$. Then by Theorem of Baire, there exists a number j_0 such that H_{j_0} is dense in G_i. Hence by the lower semicontinuity of $n(w)$, $G_i\subseteqq H_{j_0}$, whence $\sup_{G_i} n(w)\leqq j_0<\infty$. Hence the area of ψ over $G_i\leqq j_0\times$area of $G_i<\infty$. Hence $n(w)=n^i$ in G_i except at most a closed set $F_i(=(\sum_{j=1}^{i}\partial D_j\cap\Omega))$ of capacity zero. Consider about G_{i+j}. Then $F_{i+j}=\sum^{i+j}\partial D_k$ is of capacity zero. Assume that $\sup_{G_i} n(w)=n^i<n^{i+j}=\sup_{G_{i+j}} u(w)$. Since $F_{i+j}(\supset F_i)$ is closed and totally disconnected, we can find two points ω_1 and ω_2 such that $\omega_1\in G_i$, $\omega_2\in G_{i+j}$, $n(\omega_1)<n(\omega_2)$ and both dist (ω_1, F_{i+j}) and dist (ω_2, F_{i+j}) are positive. Connect ω_1 with ω_2 by a curve L in $G_{i+j}-F_{i+j}$. Then L must intersect F_{i+j}. This is a contradiction. Hence $n^i=n^{i+j}$. Now since $\bigcup G_i=\Omega$, $\sup_{\Omega} n(w)=n^{\Omega}<\infty$. Next assume $\bar{D}_\infty=0$. Then $\Omega_1=\Omega_2=,\cdots$ and $\sup_{K} n(w)=n^{\Omega_1}<\infty$. This contradicts the infiniteness of the area of ψ. Thus $\bar{D}_\infty\neq 0$. By $\sup_{\Omega_i} n(w)<\infty$ we have by Theorem 3 that $n(w)=n^{\Omega_i}$ except at most a closed set of capacity zero in Ω_i. Assume $n^*(w^*)<\infty$ at a point w^* of $\bar{D}_\infty$. Then there exists a circle $K_\varepsilon: |w-w^*|<\varepsilon$ such that the area of the part of ψ lying over K_ε is $<\infty$. Then by above mentioned $\sup_{K_\varepsilon} n(w)<\infty$. This contradicts $w^*\in\bar{D}_\infty$ and $\sup_{K_\varepsilon} n(w)=\infty$, whence $n^*(w)=\infty$ at every point of $\bar{D}_\infty$.

Theorem 5. *Let $R\in 0^0_{AB}$ and G be a non compact domain. If a connected piece ψ has no common point with ∂G, then ψ cover $K: |w-w_0|<r$ except at most a closed set $F\in\mathfrak{E}_{AB}(=\partial D_1)$, $\sum^{\infty}\partial D_n$ is totally disconnected and $n(w)=\sup n(w)\leqq\infty$ except at most an F_σ $(=\sum\partial D_n)\subset\mathfrak{E}_2$. If $\sup n(w)<\infty$, F_σ reduces to a closed set.*

Assume that ψ does not cover a set $F\not\subset\mathfrak{E}_{AB}$ in K. Then we can find a closed set F' in the interior of K with $F'\not\subset\mathfrak{E}_{AB}$. Hence we can construct a non-constant bounded analytic function $\varphi(w)$ with vanishing real part on ∂K in $K-F'$. Consider $\varphi(z)=\varphi(f(z))$ in $\Delta=f^{-1}(\psi)$ in R. Then $R\notin 0^0_{AB}$. This is a contradiction. Hence ψ covers K except a set $\subset\mathfrak{E}_{AB}$. Assume that $\sum\partial D_i$ is not disconnected. Then there exists a number n_0 such that ∂D_{n_0} has a continuum α. Let w' be a point of α such that a circle $K_\delta: |w-w'|<\delta$ is divided into some number of components by α. Since $\sup_{\alpha} n(w)\leqq n_0-1$ and since $n(w)$ is lower semicontinuous, there exists a point w^* in K_δ such that $n(w^*)=\max_{K_\delta\cap\alpha} n(w)$. Now there exist connected pieces $\psi_1, \psi_2, \cdots, \psi_i$ consisting of inner points over a small circle K_ε: $|w-w^*|<\varepsilon$. But $w^*\in\partial D_{n_0}$ implies that there exists another connected piece ψ_0 over K_ε which does not cover $\alpha\cap K_\varepsilon$. This contradicts that ψ_0 covers K_ε except a set $\subset\mathfrak{C}_{AB}$. Hence $\sum\partial D_n$ is totally disconnected. Next, assume that the measure of $E[w: n(w)<\sup n(w)]$ is positive. Put $F_k=F\cap D_k$. Then there exists a number n_0 such that mes $F_{n_0}>0$. Hence by the method used in Theorem 2 we have that $E[w: n(w)<\sup n(w)]$ is a set of measure zero in replaceing capacity by measure.

Theorem 6. *Let $R\in 0^0_{ASD}(\supset 0^0_{AD})$ and G be a non compact domain and suppose that a connected piece ψ over a circle K has no common point with $f(\partial G)$.*

1) *If the area of ψ is finite, $\sum\partial D_n$ is totally disconnected and $\bar{D}_\infty=\overline{\cap D_n}=0$ or $\bar{D}_\infty=K$. If $\bar{D}_\infty\neq K$, $\sup\limits_K n(w)<\infty$ and $\bar{D}_\infty=0$ and $n(w)=\sup n(w)$ except at most a closed set $\subset\mathfrak{C}_2$ and $n(w)\geqq 1$ except at most a closed set $\subset\mathfrak{C}_{AD}$.*
2) *If the area of ψ is infinite, $\bar{D}_\infty\neq 0$ and $\sum\partial D_n$ is totally disconnected in Ω, $\sup\limits_\Omega n(w)<\infty$, $\sup\limits_\Omega n(w)=n(w)$ except a closed set $\subset\mathfrak{C}_2$ in Ω and $n(w)\geqq 1$ except a closed set $\subset\mathfrak{C}_{AD}$ for $\sup\limits_\Omega n(w)\geqq 1$, where Ω is a component of $K-\bar{D}_\infty$.*

Proof. Let the area of ψ be finite. On the contrary, suppose that $\sum\partial D_n$ is not disconnected, then there exists a number i_0 such that ∂D_{i_0} has a continuum α. Since $\alpha\subset\partial D_{i_0}$ and $n(w)$ is lower semicontinuous, there exists a point w^* in $\alpha\cap\partial D_{i_0}$ such that $n(w)=\max\limits_\alpha n(w)\leqq i-1$. Hence similarly as in Theorem 5, we can find a circle K_ε such that K_ε is divided into some number of components and a connected piece which does not cover any point of $\alpha\cap K_\varepsilon$. We can find at least one connected piece ψ_0 such that ($K_\varepsilon-$projection of ψ_0) has an open set. Hence we can construct an analytic function $\varphi(w)$ in ($K_\varepsilon\cap$proj ψ_0) such that $Re\,\varphi(w)=0$ on the periphery of K_ε and $\left|\frac{d\varphi(w)}{dw}\right|<M$ in (proj ψ_0). Consider $\varphi(z)=\varphi(f(z))$ in $\Delta=f^{-1}(\psi_0)$. Then $D(\varphi(z))<M^2\times$area of ψ. Hence $R\notin 0^0_{AD}$. This is a contradiction. Hence $\sum\partial D_n$ is totally disconnected. Suppose $\bar{D}_\infty\neq K$. Then there exists an open set G in $K-\bar{D}_\infty$. Put $G_j=E\left[w\subset G: \text{dist}\,(w,\partial G+\partial K)>\frac{1}{j}\right]$ and $F_i=E[w: n(w)\leqq i-1]$. Then $G_j=\sum F_i$. Hence there exists an F_{i_0} such that F_{i_0} is dense in G, whence $G_j\subset F_{i_0}$. Hence $\sup\limits_{G_i} n(w)=n_0<\infty$. Put $n(w^*)=n_0$ in G_i. We show $\sup\limits_K n(w)=n_0$. On the contrary, suppose that there exists a point w^{**} in $K-G_i$ such that $n_1=n(w^{**})>n_0$. Since $\sum\limits^{n_1}\partial D_i$ is also closed and totally disconnected and $\sum_{n_0+1}^{n_1} \partial D_i \frown G_i = 0$ and clearly $w^* \notin \sum^{n_0} \partial D_i$, we can connect w^* with w^{**} by a curve L in $K - \sum^{n_1} \partial D$. This implies $n(w^*) \geqq n(w^{**}) = n_1$. This is a contradiction. Hence $\sup_K n(w) = n_0$ and $\bar{D}_\infty = 0$. We shall show that ψ covers K except at most a closed set $\subset \mathfrak{E}_{AD}$. Assume that ψ does not cover a set $F \not\subset \mathfrak{E}_{AD}$. Then we can easily construct an analytic function $\varphi(w)$ in $K - F'(F' \subset F$ and $F' \frown \partial K = 0)$ such that $Re\,\varphi(w) = 0$ on ∂K and $D(\varphi(w)) < \infty$. Put $\varphi(z) = \varphi(f(z))$. Then $D(\varphi(z)) \leqq n_0 D(\varphi(w))$. Hence $R \notin 0^0_{AD}$. This is a contradiction. Hence ψ covers K except at most a closed set $\subset \mathfrak{E}_{AD}$. Assume that $E[w : n(w) < n_0]$ is of positive measure. Then we can find as Theorem 5 a small circle K_ε and a connected piece ψ_0 over K_ε which does not cover a set of positive measure in K. This contradicts that ψ_0 covers except at most a closed set $\subset \mathfrak{E}_{AD}$, because $\mathfrak{E}_{AD} \subset \mathfrak{E}_2$.

Assume that the area of ψ is infinite. Let Ω be one of components of $K - \bar{D}_\infty$. Then we can prove as above that $\sup_\Omega n(w) < \infty$ in Ω. Hence similarly $\sup_\Omega n(w) = n(w)$ except a closed set $\subset \mathfrak{E}_2$ in Ω and $n(w) \geqq 1$ except for a closed set $\subset \mathfrak{E}_{AD}$ for $\sup_\Omega n(w) \geqq 1$.

We consider the topological properties of $\bar{D}_\infty$.

Theorem 7. *Let $R \in 0^0_{AD} (\supset HND)$. If $\bar{D}_\infty$ is not empty and $\sup_i n\Omega_i \leqq n_0$ (specially the number of components of $K - \bar{D}_\infty$ is finite), then $\bar{D}_\infty$ is a closed domain, whence $\bar{D}_\infty$ is not non dense locally, where $n\Omega_i = \sup_{\Omega_i} n(w)$.*

Assume $\bar{D}_{n_0+1} - \bar{D}_\infty \neq 0$, then there exist a point w_0 and a neighbourhood $v(w_0)$ of w_0 such that $v(w_0) \subset \bar{D}_{n_0+1} - \bar{D}_\infty$ and $\sup_{v(w_0)} n(w) \geqq n_0 + 1$. On the other hand, by $\overline{v(w_0)} \frown \bar{D}_\infty = 0$, $v(w_0)$ is contained in a component of $K - \bar{D}_\infty$. This contradicts $n_0 + 1 > \sup_i n\Omega_i \geqq \sup_{v(w_0)} n(w)$. Hence $\bar{D}_{n_0+1} = \bar{D}_{n_0+2} = \bar{D}_{n_0+3}$. Clearly $\bar{D}_\infty = \overline{\frown D_n} \subset \bar{D}_{n_0+1} = \bar{D}_{n_0+2}$. We show $\bar{D}_\infty \supset \bar{D}_{n_0+1}$. Let $w \notin \bar{D}_\infty$. Then there exists a neighbourhood $v(w_0)$ such that $v(w_0) \frown \bar{D}_\infty = 0$ and $v(w_0) \subset K - \bar{D}_\infty$, whence $\sup_{v(w_0)} n(w) \leqq n_0$ and $w_0 \notin D_{n_0+1}$. Hence $\bar{D}_\infty = \bar{D}_{n_0+1}$. Now since D_{n_0+1} is an open set, $\bar{D}_\infty$ is a closed domain and is not non dense locally.

Corollary. *Let $R \in 0^0_{AD}$ $(\supset HND)$ and $\bar{D}_\infty \neq 0$. Then $\bar{D}_\infty$ consists of continuum components. $\bar{n}(w) = \infty$ for every point w of $\bar{D}_\infty$, where $\bar{n}(w^*) = \lim_{r \to 0} (\sup_{v_r} n(w))$: $v_r(w) = E[w : |w - w^*| < r]$. Hence if every component γ_i of $\bar{D}_\infty$ is non dense in an open set G, every point of $\bar{D}_\infty \frown G$ is an accumulation point of $\bar{D}_\infty \frown G = \sum \gamma_i$.*

Assume that $\bar{D}_\infty$ is totally disconnected in an open set G. We can find another open set $G'(\subset G)$ such that $\partial G' \cap \bar{D}_\infty = 0$, $\partial G'$ is contained in some Ω and $\sup_{\Omega - \bar{D}_\infty} n(w) < \infty$. Hence by Theorem 7, $G \cap \bar{D}_\infty$ is not non dense locally. This is a contradiction. Hence $\bar{D}_\infty$ consists of only continuum components. Next suppose that $\bar{D}_\infty$ is non dense locally with $\bar{n}(w') < \infty : w' \in \bar{D}_\infty$. Then by the upper semicontunuity of $\bar{n}(w)$, we can find a neighbourhood $v(w)$ such that $\sup_{v(w)} n(w) < \infty$ and $v(w) \cap \bar{D}_\infty$ is non dense. Hence also by Theorem 7, $\bar{D}_\infty$ is not non dense locally. This is also a contradiction. Hence $\bar{n}(w) = \infty$ for $w \in \bar{D}_\infty$. Suppose that w is not an accumulation point of $\sum \gamma_i$. Then there exists an open set G such that $G \cap \bar{D}_\infty$ is composed of a finite number of components, whence $\bar{n}(w) < \infty$ at $w \in (G \cap \bar{D}_\infty)$. This contradicts the above mentioned. Hence every point of $\bar{D}_\infty \cap G$ is an accumulating point of $\bar{D}_\infty \cap G = \sum \gamma_i$.

3. Behaviour of Riemann surfaces.

Let S be the w-Riemnn sphere. We consider S instead of a circle. Then we have by theorems mentioned before

Theorem 8. *Let $R \in NHB(0 \leqq N \leqq \infty)$. Then $n(w) = \sup n(w) (\leqq \infty)$ except at most an F_σ of capacity zero. If $R \notin 0_g$, then* $\sup n(w) = \infty$.

Theorem 9. *Let $R \in HND(0 \leqq N \leqq \infty)$. Then* $\sup_{\Omega_i} n(w) = n_{\Omega_i}(w) < \infty$ *in Ω_i except at most a closed set of capacity zero, where $\Omega_1, \Omega_2, \cdots$ are components of $C\bar{D}_\infty$. $n^*(w) = \infty$ at every point of $\bar{D}_\infty$. If $R \notin 0_g$, then $\bar{D}_\infty \neq 0$.*

Theorem 10. *Let $R \subset 0^0_{AB}$. Then* $n(w) = \sup n(w) (\leqq \infty)$ *except at most a totally disconnected set of areal measure zero and R covers at least once except a closed set $\subset \mathfrak{E}_{AB}$.*

Theorem 11. *Let $R \in 0^0_{AD}$. If the spherical area of $R < \infty$ (clearly $D(f(z)) = \infty$), $\bar{D}_\infty = S$ or $\bar{D}_\infty = 0$.* $n(w) = \sup_{\Omega_i} n(w) < \infty$ *except at most a closed and totally disconnected set of areal measure zero in every component Ω_i of $S - \bar{D}_\infty$ and $n(w) \geqq 1$ except a closed set $\subset \mathfrak{E}_{AD}$ in Ω for Ω such that* $\sup_\Omega n(w) > 0$. *If the spherical area of R is infinite, $\bar{D}_\infty \neq 0$.*

Theorem 11′. *Let $R \in 0^0_{ASD}$. Then $\bar{D}_\infty \neq 0$ and R has the same properties as in Theorem* 11.

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7) See Theorem 4 and Theorem 8 of on the ideal boundary of Riemann surfaces

8) Z. Kuramochi: Analytic functions in the neighbourhood of the ideal boundary, Proc. Acad. Tokyo, 1957.

3. Behaviour of Riemann surfaces with compact relative boundaries.

The properties $R \in 0_{HAD}$ and 0_{HAB} depend on a neighbourhood of the ideal boundary. It is suitable to consider them in a Riemann surface with compact relative boundary ∂R. Let $\{R_n\}$ be its exhaustion with compact relative boundary $\{\partial R_n\}$ $(n=1, 2, \cdots)$.

Class 0_{HAD} and 0_{HAB}. Let $f(z)$ be a non-constant analytic function of AD (analytic Dirichlet bounded) or AB (analytic bounded) in R. This implies $R^* \notin HND-0_g$ $(HND-0_g)$, where R^* is made of R by adding a compact set R_0 to R so that $R^*=R+R_0$ has no relative boundary.

Hence in this case $R \subset 0_{HAD}$ or 0_{HAB} depends chiefly *on the size of the ideal boundary.*

Theorem 12. *Let $R \in 0_{HAD}(0_{HAB})$ be a Riemann surface with compact relative boundary ∂R. Suppose that R is represented as a covering surface over the w-plane by a non-constant function $f(z)$ of $AD(AB)$. Then $n(w) = \sup_{\Omega_i} n(w) < N < \infty$ except a closed and totally disconnected set $\subset \mathfrak{E}_2$. $n(w) \geqq 1$ except a closed set $\subset \mathfrak{E}_{AD}(\mathfrak{E}_{AB})$ in Ω_i for $\sup_{\Omega_i} n(w) \geqq 1$, where $\Omega_1, \Omega_2, \cdots$ are components of the complementary set of $f(\partial R)$.*

R-maximum principle. Let $g(z)$ be a non-constant function of $AD(AB)$ in $R-F$, where $R \in 0_{HAD}$ and F is a compact set. Then by Theorem 1 $Re\, g(z) = U(z)$, where $U(z)$ is a harmonic function in $R-F$ such that $U(z) = Re\, g(z)$ on $\partial F + \partial R$ and $U(z)$ has M.D.I. Hence the R-maximum principle is valid.

$$\max_{\partial R} Re(g(z)) \geqq \sup_{R} Re(g(z)) > \inf_{R} Re(g(z)) \geqq \min_{\partial R} Re(g(z))\,.$$

Let w_1 be a point such that dist $(w_1, f(R)) > \delta > 0$. Then $\varphi(z) = \dfrac{a-f(z)}{w_0-f(z)} e^{i\theta}$ is of $AD(AB)$. Hence R-maximum principle is also valid for $\varphi(z)$.

G-maximum principle. Let G be a non compact domain in R ($\in 0_{HAD}$). Let $g(z)$ be a function of AD in R. Then $Re\, g(z)$ has M.D.I. over G among all functions with value $Re\, g(z)$ on ∂G. In fact, if there were another harmonic function $V(z)$ in G such that $V(z) = Re\, g(z)$ on ∂G and $D(V(z)) < D(g(z))$. Then by the Dirichlet principle

$$D(g(z)) \gneqq D_G(V(z)) + D_{R-G}(g(z)) \geqq D(g'(z))\,.$$

where $g'(z)$ is obtained by alternierendes Verfahren from $V(z)$ and $g(z)$. This contradicts that $G(z)$ has M.D.I. Hence $Re\, g(z) = \lim U_n(z)$, where $U_n(z)$ is a harmonic function in $R_n \cap G$ such that $U_n(z) = Re\, g(z)$ on $\partial G \cap R_n$ and $\dfrac{\partial U_n(z)}{\partial n} = 0$ on $\partial R_n \cap G$. Hence

$$\sup_{\partial G} Re(g(z)) \geqq \sup_{G} Re\,(g(z)) > \inf_{G} Re(g(z)) \geqq \inf_{\partial G} Re(g(z))\,.$$

It is an essential condition for the validity of G-maximum principle for $Re\, g(z)$ in non compact domain G that $g(z)$ is of AD not only in G but also in a neighbourhood of the ideal boundary of R. i.e. in the complementary set of a compact set F.

1) $f(R)$ is bounded and we can suppose that the number of components $\Omega_1, \Omega_2, \cdots$ of $Cf(\partial R)$ is finite. In fact, put $\varphi(z)=e^{i\theta} f(z)(2\pi>\theta\geqq 0)$. The by the R-maximum principle, $f(R)$ is bounded. By a little deformation of ∂R, we can suppose that ∂R is analytic and $f(z)$ is analytic on ∂R. Hence the number of $\{\Omega_i\}$ is finite. Denote by Ω_∞ the one containing the point at infinity. Then we see by the R-maximum principle with respect to $\varphi(z)=e^{i\theta}\dfrac{1}{f(z)-w_0}$ that $\overline{f(R)}\frown\Omega_\infty=0$.

2) Put $D_n=E[w:n(w)\geqq n]$. Then ∂D_n is totally disconnected in Ω. Let Ω be one of $\{\Omega_i\}$ such that $\sup\limits_{\Omega} n(w)\geqq 1$. First, we shall show that $\Omega-D_n-\partial D_n=0$. On the contrary, assume $\Omega-D_n-\partial D_n=g>0$. Then ∂D_n has a continuum α. Let $p\in \text{int}\,\alpha$ and $V_\delta(p)$; $|w-p|<\delta$ be a circle such that $V_\delta(p)-\alpha$ is divided into components $\phi_1, \phi_2, \cdots$ of number $\geqq 2$. Let ϕ_1 be one of component such that $\phi_1\subset D_n$ and ϕ_2 be another component contained in $\subset CD_n$. Put $G=f^{-1}(\psi_1)$, where ψ_1 is a connected piece over ϕ_1. Then G is a non compact domain in R. Let $V_{\frac{\delta}{6}}(p)$: $|w-p|<\dfrac{\delta}{6}$ and let $w'\in(V_{\frac{\delta}{6}}(p)\frown\phi_2)$. Then there exists a point w'' in $(\alpha\frown V_{\frac{\delta}{3}}(p))$: $V_{\frac{\delta}{3}}(p)$: $|w-p|<\dfrac{\delta}{3}$ such that $|w'-w''|=\text{dist}\,(w',\alpha)$. Let $v_{\frac{\delta}{10}}(w'')$: $|w-w''|<\dfrac{\delta}{10}$. Then $v_{\frac{\delta}{10}}(w'')\frown D_m$ is open, where $n-1\geqq m\geqq 0$. Hence there exists a point w^* in $v_{\frac{\delta}{10}}(w'')$ such that $n(w^*)=\max n(w)\leqq n-1$: $w\in(v_{\frac{\delta}{10}}(w'')\frown CD_n)$. Let w^{**} be a point in α such that $|w^*-w^{**}|=\text{dist}\,(w^*,\alpha)$. Then $w^{**}\in V_{\frac{\delta}{2}}(p)$: $|w-p|<\dfrac{\delta}{2}$. We fix w^* and w^{**} and G. Since $n(w^*)=\max n(w)=n_0$: $w\in(v_{\frac{\delta}{10}}(w'')\frown CD_n)$, there exists a small circle $\bar{K}_\varepsilon$: $|w-w^*|\leqq\varepsilon$ (this is contained in the set $E[w: n(w)=n_0]$) such that every connected piece $\psi_1, \psi_2, \cdots, \psi_{n'}$ over $\bar{K}_\varepsilon$ is compact. Put $\Delta_i=f^{-1}(\psi_i)$. Then $\sum\Delta_i$ is compact in and $\sum\Delta_i\frown G=0$, $\varphi(z)=\dfrac{1}{w-w^*}e^{i\theta}$: $(\theta=-arg|w^*-w^{**}|)$ is AD in $R-\sum\limits^{n'}\Delta_i$. Consider $\varphi(z)$ in G. Then by the G-maximum principle

$$\sup_{\partial G} Re\,\varphi(z)\geqq\sup_{G} Re\,\varphi(z)\,.$$

On the other hand, by $|w^*-w^{**}|=\text{dist}\,(w^*,\alpha)\leqq\text{dist}\,(w^*,(\partial\phi_1-\alpha))$.

$$\sup_{G} Re\,\varphi(z)>\sup_{\partial G} Re\,\varphi(z)\,.$$

This is a contradiction. Hence $G=0$. i.e. D_n is dense in Ω, if $(D_n \cap \Omega) \neq 0$. Next we shall show that ∂D_n is totally disconnected in Ω. On the contrary, assume that ∂D_n has a continuum α. Let $p \in$ int α and $V_\delta(p)$ as above, (i.e. $V_\delta(p)-\alpha$ consists of some number ($\geqq 2$) components). Then there exists a point w' in $V_\delta(p) \cap \partial D_n$ such that $n_0 = n(w') = \max n(w)$ $(w \in V_\delta(p) \cap \alpha) \leqq n-1$. Then there exists a circle $\bar{K}_\varepsilon(w')$: $|w-w'| \leqq \varepsilon$ such that there exist compact connected pieces over $\bar{K}_\varepsilon(w')$ consisting of n leaves. Since D_n is dense and $n(w') < n$, there exist at least one connected pieces $\psi_1, \psi_2, \cdots$ over $\bar{K}_\varepsilon(w)$ which do not cover every point α. Hence ψ_1 composed of at least two components $\psi_{1,1}, \psi_{1,2}, \cdots$. Hence every $\psi_{1,i}$ has its projection of the shape of the moon with eclips. If α is not a straight line, we can find at least a $\psi_{1,i}$ and w^* in Ω_∞ and $w^{**} \in \alpha$ such that dist $(w^*$, proj $\psi_{1i}) =$ dist (w^*, w^{**}) : $w^{**} \in \alpha$. Consider $\varphi(z) = Re \dfrac{1}{f(z)-w^*} e^{i\theta}$ in $\Delta = f^{-1}(\psi_{1,i})$. Then $\varphi(z) \in AD$ in R. Hence we have a contradiction by the G-maximum principle, where $G = f^{-1}(\psi_{1i})$. Similarly, if α is a straight line. Hence ∂D_n has no continuum.

3) $\sup_\Omega n(w) < \infty$. Let Ω be one of $\{\Omega_i\}$ such that $\partial\Omega \cap \partial\Omega_\infty \neq 0$. Since $f(z)$ is analytic on ∂R, we can find a point w_0 in Ω in a neighbourhood of $\partial\Omega_\infty$ such that dist $(w_0, f(R')) > 0$, where R' is obtained from R by a little changing of ∂R. Because $f(R')$ is contained in the domain enclosed by $f(\partial R')$. Hence there exists a number $n(w_0) < \infty$ and a small circle K_ε: $|w-w_0| < \varepsilon$ such that every connected piece over K_ε is compact, whence there exists a constant δ_0 such that dist $(w_0, \partial D_m \geqq \delta_0 > 0$ for every m.

Assume that there exists a point w' in Ω such that $m = n(w') > n(w_0)$. Then $\sum^m \partial D_i$ is closed and totally disconnected. We can connect w' with w_0 by a curve L in $\Omega - \sum^m \partial D_i$. This implies $n(w_0) \geqq n(w')$. This is a contradiction. Hence $\sup_\Omega n(w) < \infty$. Let Ω_2 be another domain such that $\partial\Omega_2 \cap \partial\Omega \neq 0$. Then we have similarly $|\sup_{\Omega_2} n(w) - \sup_\Omega n(w)| < \infty$. But the member of $\{\Omega_i\}$ is finite. Thus $\sup n(w) < N$.

4) $n(w) \geqq 1$ except at most a closed set $\subset \mathfrak{C}_{AD}$, if $\sup_\Omega n(w) \geqq 1$. Assume that there exists a closed set $F \not\subset \mathfrak{C}_{AD}$. Then there exists a point $w_0 \in F$ such that $(K \cap F) \not\subset \mathfrak{C}_{AD}$ for any small circle K. Since $\sum^N \partial D_i$ is totally disconnected, we can find a simply connected domain H with analytic relative boundary ∂H such that $\partial H \cap (\sum^N \partial D_i) = 0$. Hence we can construct a function $g(w)$ of AD with vanishing real part on ∂H. Now a connected piece $\Delta(\subset R)$ over H has compact relative boundary. Since

$D(g(z)) \leqq \sup n(w) \times D(g(z)) < \infty$, $R \notin 0_{HAD}$. This is a contradiction. Hence $n(w) \geqq 1$ Ω except a set $\subset \mathfrak{E}_{AD}$.

5) Similarly as Theorem 6, we can prove that $n(w) = \sup n(w)$ except a set $\subset \mathfrak{E}_2$ by using the total disconnectedness of $\sum^{N} \partial D$ and $\sup n(w) < \infty$.

Let $R \in 0_{HAB}$ and $f(z) \in AHB$. In this case $Re\, f(z) \equiv U(z)$, where $U(z)$ is a harmonic function in R such that $U(z) = Re\, f(z)$ on ∂R and $U(z)$ has M.D.I. $= \int_{\partial R} U(z) \frac{\partial U(z)}{\partial n} ds$ by the regularity of $f(z)$ on ∂R. Because $\sup | U(z) - Re\, f(z) | < \infty$ implies $U(z) \equiv Re\, f(z)$ in $R \in 0_{HAB}$. Hence $D(f(z)) < \infty$. Hence we have the same results except 4) which is replaced by $n(w) \geqq 1$ in Ω except a set $\subset \mathfrak{E}_{AB}$, $\sup_{\Omega} n(w) \geqq 1$ in Ω.

Class $R \in 0^0_{AD}$ or 0^0_{AB}. In these classes the G-maximum principle for $g(z)$ holds for non compact domain G under the condition $g(z) \in AD$ only in G. From this point of view $R \in 0^{\ 0}_{BA}(0^0_{AB})$ is stronger than $R \in 0_{HAD}(0_{HAB})$. Hence we have more simply and simlarly as Theorem 7

Theorem 13. *Let* $R \in 0^0_{AD}(0^0_{AB})$ *with compact relative boundary* ∂R. *Then the same facts as Theorem* 12 *hold.*

From theorem 12 and 13, $\sum^{N} \partial D$: $(N = \sup n(w))$ is closed and totally disconnected. We have

Corollary. *Let* $f(z) \in AD(AB)$ *be a non constant function in* $R \in 0^0_{AD}$ *or* 0_{HAD} $(0^0_{AB}$ *or* $0_{HAB})$ *with compact relative boundary* ∂R. *Then* $f(z)$ *tends to a point as* z *tends to every boundary component.*

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