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ALMOST IDENTICAL IMITATIONS OF (3, 1)-DIMENSIONAL
MANIFOLD PAIRS

AKIO KAWAUCHI

Dedicated to Professor Fujitsugu Hosokawa on his 60th birthday

(Received March 10, 1989)

By a 3-manifold $M$, we mean a compact connected oriented 3-manifold throughout this paper. Let $\partial_0 M$ be the union of torus components of $\partial M$ and $\partial_1 M = \partial M - \partial_0 M$. In the case that $\partial_1 M = \emptyset$, if $\text{Int} M$ has a complete Riemannian structure with constant curvature $-1$ and with finite volume, then we say that $M$ is hyperbolic and we denote its volume by $\text{Vol} M$. Next we consider the case that $\partial_1 M \neq \emptyset$. Then the double, $D_1 M$, of $M$ pasting two copies of $M$ along $\partial_1 M$ has $\partial_1 D_1 M = \emptyset$. If $D_1 M$ is hyperbolic in the sense stated above, then we say that $M$ is hyperbolic and we define the volume, $\text{Vol} M$, of this $M$ by $\text{Vol} M = \text{Vol} D_1 M / 2$. In this latter case, $M$ is usually said to be hyperbolic with $\partial_1 M$ totally geodesic (cf. [T-1]), but we use this simple terminology throughout this paper. When $M$ is hyperbolic, $\partial M$ has no 2-sphere components and by Mostow rigidity theorem (cf. [T-2], [T-3]), $\text{Vol} M$ is a topological invariant of $M$. By a 1-manifold in $M$, we mean a compact smooth 1-submanifold $L$ of $M$ with $\partial L = L \cap \partial M$ and the pair $(M, L)$ is simply called a (3,1)-manifold pair. A 1-manifold $L$ in $M$ is called a link if $\partial L = \emptyset$, a tangle if $L$ has no loop components, and a good 1-manifold if $|L \cap S^2| \geq 3$ for any 2-sphere component $S^2$ of $\partial M$. A (3,1)-manifold pair $(M, L)$ is also said to be good if $L$ is a good 1-manifold in $M$. In [Kw-1], we defined the notions of imitation, pure imitation and normal imitation for any general manifold pair. In Section 1 we shall define a notion which we call an almost identical imitation $(M, L^*)$ of $(M, L)$, for any good (3,1)-manifold pair $(M, L)$. Roughly speaking, this imitation is a normal imitation with a special property that if $q: (M, L^*) \to (M, L)$ is the imitation map, then $q|(M, L^* - a*) : (M, L^* - a*) \to (M, L - a)$ is $\partial$-relatively homotopic to a diffeomorphism for any connected components $a^*$, $a$ of $L^*$, $L$ with $qa^* = a$. Let $P$ be a polyhedron in a 3-manifold $M$. For a regular neighborhood $N_P$ of $P$ in $M$ (meeting $\partial M$ regularly), the diffeomorphism type of $E(P, M) = \text{cl}_M (M - N_P)$ is uniquely determined by the topological type of the

1 This homotopy can be taken as a one-parameter family of normal imitation maps.
pair \((M, P)\) and we call \(E(P, M)\) the exterior of \(P\) in \(M\). Then our main result of this paper, stated in Theorem 1.1 precisely, asserts the existence of an infinite family of almost identical imitations \((M, L^*)\) of every good \((3,1)\)-manifold pair \((M, L)\) such that the exterior \(E(L^*, M)\) of \(L^*\) in \(M\) is hyperbolic.

The proof of Theorem 1.1 will be given in Section 5. Several applications to spatial graphs, links and 3-manifolds are given throughout Sections 2-4. In Section 2, we prove the existence of an almost trivial spatial \(\Gamma\)-graph, for every planar graph \(\Gamma\) without vertices of degrees \(\leq 1\), affirming a conjecture of Simon and Wolcott. In Section 3, we show a construction of a non-trivial fusion band family from a trivial link to a trivial knot, and a construction of a tangle with hyperbolic exterior in any link. In Section 4, we show that if a closed 3-manifold \(M\) is obtained from a link \(L\) with two or more components by Dehn's surgery, then \(M\) is also obtained from a hyperbolic link \(L^*\), which is a normal link-imitation of \(L\), by Dehn's surgery with the same surgery coefficient data, and that every 3-manifold without 2-sphere boundary component has a hyperbolic 3-manifold as a normal imitation.

This paper is a revised version of a main part of [Kw-0] and a prelude to the principal theorem of [Kw-2] where further consequences are announced.

1. An almost identical imitation of a good \((3,1)\)-manifold pair. Let \(I=[-1, 1]\). For a \((3,1)\)-manifold pair \((M, L)\) we call an element \(\alpha \in \text{Diff}(M, L) \times I\) a reflection in \((M, L) \times I\) if \(\alpha^2 = 1\), \(\alpha(M \times I) = M \times (-1)\) and \(\text{Fix}(\alpha, M \times I)\) is a 3-manifold. In this case, \(\text{Fix}(\alpha, (M, L) \times I)\) is a \((3,1)\)-manifold pair in our sense (See [Kw-1]). We say that a reflection \(\alpha\) in \((M, L) \times I\) is standard if \(\alpha(x, t) = (x, -t)\) for all \((x, t) \in M \times I\), and normal if \(\alpha(x, t) = (x, -t)\) for all \((x, t) \in \partial(M \times I) \cup U_L \times I\), with \(U_L\) a neighborhood of \(L\) in \(M\). A reflection \(\alpha\) in \((M, L) \times I\) is said to be isotopically standard if \(hch^{-1}\) is the standard reflection in \((M, L) \times I\) for an \(h \in \text{Diff}_0((M, L) \times I), \text{rel}\ \partial((M, L) \times I))\). For a good \((3,1)\)-manifold pair \((M, L)\) a reflection \(\alpha\) in \((M, L) \times I\) is isotopically almost standard if \(\phi\) is isotopically standard in \((M, L-a) \times I\) for each connected component \(a\) of \(L\). A smooth embedding \(\phi\) from a \((3,1)\)-manifold pair \((M^*, L^*)\) to \((M, L) \times I\) with \(\phi(M^*, L^*) = \text{Fix}(\alpha, (M, L) \times I)\) is called a reflector of a reflection in \((M, L) \times I\). Let \(p_1: (M, L) \times I \to (M, L)\) be the projection to the first factor. In [Kw-1], we defined that \((M^*, L^*)\) is an imitation (or a normal imitation, respectively) of \((M, L)\), if there is a reflector \(\phi: (M^*, L^*) \to (M, L) \times I\) of a reflection (or normal reflection, respectively) \(\alpha\) in \((M, L) \times I\), and the composite \(q = p_1 \phi: (M^*, L^*) \to (M, L)\) is the imitation map.

**Definition.** A \((3,1)\)-manifold pair \((M^*, L^*)\) is an almost identical imitation\(^2\)  

\(^2\) \text{Diff}_0\) denotes the path connected component of the topological diffeomorphism group \text{Diff} containing \(1\)(cf. [Kw-1]).
of a good (3, l)-manifold pair \((M, L)\) if there is a reflector \(\phi: (M^*, L^*) \to (M, L)\times I\) of an isotopically almost standard normal reflection \(\alpha\) in \((M, L)\times I\), and the composite \(q=p_1\phi: (M^*, L^*) \to (M, L)\) is the imitation map.

In this definition, \((M^*, L^*)\) is also a good (3, l)-manifold pair and \(q|L^*: L^* \to L\) is a diffeomorphism and \(q|(M^*, L^* - a^*): (M^*, L^* - a^*) \to (M, L - a)\) is \(\partial\)-relatively homotopic to a diffeomorphism. We identify \(M^*\) with \(M\) so that \(q|\partial M\) is the identity on \(\partial M\). We may write any almost identical imitation of \((M, L)\) as \((M, L^*)\). We state here our main theorem.

**Theorem 1.1.** For any number \(K > 0\) and any good (3,1)-manifold pair \((M, L)\) there are a number \(K^+ > K\) and an infinite family of almost identical imitations \((M, L^*)\) of \((M, L)\) such that the exterior \(E(L^*, M)\) of \(L^*\) in \(M\) is hyperbolic with \(\text{Vol } E(L^*, M) < K^+\) and \(\text{Sup}_L \text{Vol } E(L^*, M) = K^+\).

2. An almost identical spatial graph imitation. Let \((M^0, L)\) be a good (3,1)-manifold pair such that \(\partial M^0\) has at least one 2-sphere component. For some 2-sphere components \(S_1, S_2, \ldots, S_r\) of \(\partial M^0\), let \((M^0_+, L_+)\) be a pair obtained from \((M^0, L)\) by taking a cone over \((S_i, S_i \cap L)\) for each \(i\). Then note that \(M^0_+\) is a 3-manifold and \(L_+\) is a finite graph which we may consider to be smoothly embedded in \(M^0_+\) except the vertices of degrees \(\geq 3\). We call this pair \((M^0_+, L_+)\) the spherical completion of \((M^0, L)\) associated with the 2-spheres \(S_1, S_2, \ldots, S_r\). A graph \(\Gamma\) embedded in a 3-manifold \(M\) is said to be good if \((M, \Gamma)\) is diffeomorphic to the spherical completion \((M^0_+, L_+)\) of a good (3,1)-manifold pair \((M^0, L)\) associated with some 2-sphere components of \(\partial M^0\).

**Definition.** For good graphs \(\Gamma^*, \Gamma\) in a 3-manifold \(M\) the pair \((M, \Gamma^*)\) is an almost identical imitation of the pair \((M, \Gamma)\) if there are a good (3,1)-manifold pair \((M^0, L)\) and some 2-sphere components \(S_1, S_2, \ldots, S_r\) of \(\partial M^0\) and an almost identical imitation \((M^0, L^*)\) of \((M^0, L)\) such that the spherical completions \((M^0_+, L^*_+)\) and \((M^0_+, L^*_+)\) of \((M^0, L^*)\) and \((M^0, L)\) associated with the 2-spheres \(S_1, S_2, \ldots, S_r\) are diffeomorphic to \((M, \Gamma^*)\) and \((M, \Gamma)\), respectively.

Note that there is a map \(q: (M, \Gamma^*) \to (M, \Gamma)\) uniquely determined by the imitation map \(q^0: (M^0, L^*) \to (M^0, L)\). We also call this map \(q\) the imitation map of the almost identical imitation \((M, \Gamma^*)\) of \((M, \Gamma)\). Since, in this definition, the exterior \(E(\Gamma^*, M)\) of \(\Gamma^*\) in \(M\) is diffeomorphic to \(E(L^*, M^0)\), the following theorem follows directly from Theorem 1.1:

**Theorem 2.1.** For each good graph \(\Gamma\) in a 3-manifold \(M\) and a positive number \(K\), there are a number \(K^+ > K\) and an infinite family of almost identical imitations \((M, \Gamma^*)\) of \((M, \Gamma)\) such that \(E(\Gamma^*, M)\) is hyperbolic with \(\text{Vol } E(\Gamma^*, M) < K^+\) and \(\text{Sup}_M \text{Vol } E(\Gamma^*, M) = K^+\).
Let $\Gamma$ be a finite graph without vertices of degrees $\leq 1$. If a good graph $\Gamma$ in the 3-sphere $S^3$ is obtained by an embedding of $\Gamma$, then we call this $\Gamma$ a spatial $\Gamma$-graph. Two spatial $\Gamma$-graphs $\Gamma'$, $\Gamma''$ are equivalent if there is an orientation-preserving diffeomorphism $h: S^3 \to S^3$ with $h(\Gamma') = \Gamma''$. The occurring equivalence classes of spatial $\Gamma$-graphs are called the knot types of spatial $\Gamma$-graphs. These knot types were studied by Kinoshita, Suzuki (cf. [Su-1]) as a generalization of the usual knot theory and are now studied in a connection with the synthetic study in molecular chemistry by, for example, Walba [Wa], Simon [Si], Summers [Sum]. We say that a finite graph in $S^3$ is trivial if it is on a 2-sphere smoothly embedded in $S^3$. A spatial $\Gamma$-graph $\Gamma$ is said to belong to an almost trivial knot type, if $\Gamma$ is not trivial but the graph in $S^3$ resulting from $\Gamma$ by removing any open arc is necessarily trivial. Simon and Wolcott (cf. [Si]) conjectured that for every planar graph $\Gamma$ without vertices of degrees $\leq 1$, there exists a spatial $\Gamma$-graph belonging to an almost trivial knot type. Several examples supporting this conjecture were given by Kinoshita [Ki], Suzuki [Su-2], M. Hara (unpublished) and Wolcott [Wo]. Theorem 2.1 solves this conjecture affirmatively. In fact, we have the following stronger result:

**Corollary 2.2.** For every planar graph $\Gamma$ without vertices of degrees $\leq 1$ and any number $K > 0$, there are a number $K^+ > K$ and an infinite family of spatial $\Gamma$-graphs $\Gamma^*$ belonging to infinitely many almost trivial knot types such that $E(\Gamma^*, S^3)$ is hyperbolic with $\text{Vol} E(\Gamma^*, S^3) < K^+$ and $\text{Sup}_* \text{Vol} E(\Gamma^*, S^3) = K^+$ and the quotient group $\pi_1(E(\Gamma^*, S^3))$ of $\pi_1(E(\Gamma^*, S^3))$ by the intersection of the derived series of $\pi_1(E(\Gamma^*, S^3))$ is a free group of rank $\beta_1(\Gamma^*)$ with a basis represented by meridians of $\Gamma^*$ in $S^3$, where $\beta_1(\Gamma^*)$ denotes the first Betti number of $\Gamma^*$.

**Proof.** Let $\Gamma$ be a trivial spatial $\Gamma$-graph. By Theorem 2.1, there are a number $K^+ > K$ and an infinite family of almost identical imitations $(S^3, \Gamma^*)$ of $(S^3, \Gamma)$ such that $E(\Gamma^*, S^3)$ is hyperbolic with $\text{Vol} E(\Gamma^*, S^3) < K^+$ and $\text{Sup}_* \text{Vol} E(\Gamma^*, S^3) = K^+$. Clearly, this $\Gamma^*$ belongs to an almost trivial knot type. If $q: (S^3, \Gamma^*) \to (S^3, \Gamma)$ is the imitation map, then $q$ induces a meridian-preserving isomorphism $\pi_1(S^3 - \Gamma^*) \cong \pi_1(S^3 - \Gamma)$ (See [Kw-1]). Since $\pi_1(S^3 - \Gamma)$ is a free group of rank $\beta_1(\Gamma)$ with a basis represented by meridians of $\Gamma$ in $S^3$, we see from [L-S, p. 14] that $\pi(S^3 - \Gamma) = \pi_1(S^3 - \Gamma)$, so that $\pi_1(E(\Gamma^*, S^3)) \cong \pi_1(S^3 - \Gamma^*)$ is a free group with a desired property. This completes the proof.

**3. Applications to links.** We discuss here two applications to links. One concerns a construction of a non-trivial fusion band family from a trivial link to a trivial knot and the other, a construction of a tangle with the exterior hyperbolic in any link. We say that a mutually disjoint band family $\{B_1, B_2, \ldots, B_l\}$ in $S^3$ spanning a trivial link $L_0$ (as 1-handles) is trivial if the union $L_0 \cup B_1 \cup B_2 \cup \cdots \cup B_l$ is on a 2-sphere smoothly embedded in $S^3$. Let a trivial link $L_0$
have \( r+1 \) components. We consider mutually disjoint \( r \) bands \( B_1, B_2, \ldots, B_r \) in \( S^3 \) which give a fusion from \( L_0 \) to a trivial knot (that is to say, which span \( L_0 \) and along which the surgery of \( L_0 \) produces a trivial knot). We say that this family \( \{B_1, B_2, \ldots, B_r\} \) is a fusion band family from \( L_0 \) to a trivial knot. For \( r=1 \), Scharlemann [Sc] proved that any fusion band family \( \{B_i\} \) is necessarily trivial. For \( r=2 \), Howie and Short [H-S] gave an example of a non-trivial fusion band family \( \{B_1, B_2\} \) (cf. [Kw-2, Figure 4]). In their example, the exterior \( E=E(L_0 \cup B_1 \cup B_2, S^3) \) is easily seen to have a solid torus as a disk summand and hence it is not hyperbolic. As a corollary to Theorem 2.1, we have an infinite family of non-trivial fusion band families with such exteriors hyperbolic.

**Corollary 3.1.** For any number \( K>0 \) and any integer \( r \geq 2 \), there are a number \( K^+>K \) and an infinite family of non-trivial fusion band families \( \beta^* = \{B^+_1, B^+_2, \ldots, B^+_r\} \) from an \((r+1)\)-component trivial link \( L_0 \) to a trivial knot such that the exterior \( E_{\beta^*}=E(L_0 \cup B^+_1 \cup B^+_2 \cup \cdots \cup B^+_r, S^3) \) is hyperbolic with \( \text{Vol} E_{\beta^*}<K^+ \) and \( \text{Sup}_{\beta^*} \text{Vol} E_{\beta^*}=K^+ \) and \( \pi_1(E_{\beta^*}) \) is a free group of rank \( r+1 \) with a basis represented by meridians of \( L_0 \).

**Proof.** Consider a trivial fusion band family \( \{B_1, B_2, \ldots, B_r\} \) from \( L_0 \) to a trivial knot. Let \( L'_0 \) be an \( r \)-component trivial link obtained from \( L_0 \) by surgery along \( B_r \). When we regard the band \( B_r \) as a band spanning \( L_0 \), we denote it by \( B'_r \). Note that a spine \( \Gamma = L'_0 \cup b_1 \cup b_2 \cup \cdots \cup b'_r \) of \( L'_0 \cup B_1 \cup B_2 \cup \cdots \cup B'_r \) is a good planar graph in \( S^3 \). By Theorem 2.1, we have a number \( K^+>K \) and an infinite family of almost identical imitations \( q: (S^3, \Gamma^*) \to (S^3, \Gamma) \) such that \( \text{Vol} E(\Gamma^*, S^3)<K^+ \) and \( \text{Sup}_{\Gamma^*} \text{Vol} E(\Gamma^*, S^3)=K^+ \). Regard the bands \( B_1, B_2, \ldots, B'_r \) as very narrow bands. Then since \( r \geq 2 \) and \( q \) is an almost identical imitation map, we may consider that \( q \) defines a map \( (S^3, L'_0 \cup B^+_1 \cup B^+_2 \cup \cdots \cup B^+_{r-1} \cup B^+_r) \to ((S^3, L'_0 \cup B_1 \cup \cdots \cup B_{r-1} \cup B_r), \) where \( B^+_i \) denotes a band given by \( B^+_i = q^{-1}B_i \) for each \( i \leq r-1 \). Then we see that the bands \( B^+_1, B^+_2, \ldots, B^+_r \) with \( B^+_r = B_r \) form a fusion band family from \( L_0 \) to a trivial knot. Clearly, the exterior \( E \) of \( L_0 \cup B^+_1 \cup B^+_2 \cup \cdots \cup B^+_r \) in \( S^3 \) is diffeomorphic to \( E(\Gamma^*) \). By the proof of Corollary 2.2, \( \pi(E) \) is seen to be a desired free group. This completes the proof of Corollary 3.1.

**Remark 3.2.** In the above proof, we can see that the band family \( \{B^+_1, \ldots, B^+_{r-1}, B^+_{r+1}, \ldots, B^+_r\} \) spanning \( L_0 \) is trivial for each \( i \) with \( 1 \leq i \leq r-1 \). In particular, if \( r \geq 3 \), then each band \( B^+_i(1 \leq i \leq r) \) spans \( L_0 \) trivially.

As another application, we shall show the following:

**Corollary 3.3.** For any link \( L \) in \( S^3 \) we take 3-balls \( B, B' \) in \( S^3 \) so that \( B' = S^3 - \text{Int} B \) and \( T = B \cap L \) is a trivial tangle with 2 or more strings in \( B \) and \( T' = B' \cap L \) is a good 1-manifold in \( B' \). Then for any number \( K>0 \), there are a number \( K^+>K \) and an infinite family of almost identical imitations \( (B', T'^*) \).
of \((B', T')\) such that the exterior \(E(T'^*, B')\) is hyperbolic with \(\text{Vol} E(T'^*, B') < K^+\) and \(\text{Supp}_\tau \text{Vol} E(T'^*, B') = K^+\), and the extension \(q'^*: (S^3, L^*) \to (S^3, L)\) of the imitation map \(q': (B', T'^*) \to (B', T')\) by the identity on \((B, T)\) is homotopic to a diffeomorphism.

Proof. Let \(T\) be a good tree graph in \(B\) obtained by joining the components of \(\hat{T}\) by arcs so that \(B\) collapses to \(\hat{T}\), and \(\Gamma\) the union of \(T\) and \(T'\) which is a good graph in \(S^3\). By Theorem 2.1 we have a number \(K^+ > K\) and an infinite family of almost identical imitations \((S^3, \Gamma^*)\) of \((S^3, \Gamma)\) such that the exterior \(E(\Gamma^*, S^3)\) is hyperbolic with \(\text{Vol} E(\Gamma^*, S^3) < K^+\) and \(\text{Supp}_\tau \text{Vol} E(\Gamma^*, S^3) = K^+\). By replacing \(B\) by a slender regular neighborhood of \(\hat{T}\) in \(B\) we can consider that the almost identical imitation map \(q: (S^3, \Gamma^*) \to (S^3, \Gamma)\) induces the identity on \(B\) and the restriction \(q' = q|B'\) induces an almost identical imitation map \((B', T'^*) \to (B', T')\) with \(T'^* = q' - 1 T\). Moreover, we see that the extension \(q'^*: (S^3, L^*) \to (S^3, L)\) of \(q'\) by the identity on \((B, T)\) is homotopic to a diffeomorphism. Noting that \(E(T'^*, B')\) is diffeomorphic to \(E(\Gamma^*, S^3)\), we complete the proof of Corollary 3.3.

This corollary includes a hyperbolic version of Nakanishi's result \([N]\), telling that every link is splittable by a 2-sphere into a prime 1-manifold and a trivial two-string tangle.

4. Applications to 3-manifolds. Let \(T_i, i = 1, 2, \ldots, r\), be mutually disjoint tubular neighborhoods of the components \(k_i, i = 1, 2, \ldots, r\) of a link \(L\) in \(S^3\). Remove \(\text{Int} T_i\) from \(S^3\) for each \(i\) and then attach \(T_i\) again by using an \(h_i \in \text{Diff} \partial T_i\) for each \(i\). By this operation, we obtain from \(S^3\) a closed 3-manifold \(M\). Let \(m_i\) be a meridian of \(T_i\), and \(l_i\) a longitude of \(T_i\) determined by \(T_i \subset S^3\). Write \(h_i[\begin{bmatrix} m \end{bmatrix}] = a_i[l_i] + b_i[\begin{bmatrix} l \end{bmatrix}]\) in \(H(\partial T_i; Z)\) with integers \(a_i, b_i\). Then we see that the diffeomorphism type of \(M\) depends only on the pairs \((k_i, c_i)\) with \(c_i = a_i/b_i \in Q \cup \{\infty\}, i = 1, 2, \ldots, r\), and we say that \(M\) is obtained from \(S^3\) by Dehn's surgery along the knots \(k_i\) with coefficients \(c_i\) \((i = 1, 2, \ldots, r)\) or that \(M\) has a surgery description \((S^3; (k_1, c_1), (k_2, c_2), \ldots, (k_r, c_r))\). It is well known that every closed connected orientable 3-manifold \(M\) has a surgery description \((S^3; (k_1, c_1), (k_2, c_2), \ldots, (k_r, c_r))\) (cf. \([\text{We}], [\text{L}]\)). We obtain from Theorem 1.1 the following:

**Corollary 4.1.** For any number \(K > 0\) and any surgery description \((S^3; (k_1, c_1), (k_2, c_2), \ldots, (k_r, c_r))\) of any closed 3-manifold \(M\) with \(r \geq 2\), there are a number \(K^+ > K\) and an infinite family of normal imitations \((S^3, L^*)\) of \((S^3, L)\) such that the exterior \(E(L^*, S^3)\) is hyperbolic with \(\text{Vol} E(L^*, S^3) < K^+\) and \(\text{Supp}_\tau \text{Vol} E(L^*, S^3) = K^+\) and \((S^3; (k^*_1, c_1), (k^*_2, c_2), \ldots, (k^*_r, c_r))\) is a surgery description of \(M\) with \(k^*_i = q^{-1} k_i, i = 1, 2, \ldots, r\) for the imitation map \(q: (S^3, L^*) \to (S^3, L)\).

Proof. Let \(M'\) be the manifold with surgery description \((S^3; (k_r, c_r))\). Let
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Let \( k_1, k_2, \ldots, k_{r-1} \) be a core of the solid torus in \( M' \) resulting from the Dehn surgery. Regard that \( k_1, k_2, \ldots, k_{r-1} \) are in \( M' \). Let \( L' = k_1 \cup \cdots \cup k_{r-1} \cup k_r' \). By Theorem 1.1, we have a number \( K^+ > K \) and an infinite family of almost identical imitations \((M', L'^*)\) of \((M', L')\) such that \( E(L'^*, M') \) is hyperbolic with \( \text{Vol}_E(L'^*, M') < K^+ \) and \( \text{Sup}_{L^*} \text{Vol}_E(L'^*, M') = K^+ \). Let \( k_i^* = q'^{-1} k_i, i = 1, \ldots, r-1, \) and \( k_r^* = q'^{-1} k_r' \) for the imitation map \( q': (M', L'^*) \rightarrow (M', L') \). Since \( q' \) is an almost identical imitation map, we may consider that \( k_f^* = k_f' \), so that \( q' \) induces a normal imitation map \( q: (S^3, L^*) \rightarrow (S^3, L) \) with \( L^* = \sum k_i^* \cup k_r^* \subset S^3 \) and \( k_r^* = k_r \), such that \((S^3; (k_1^*, c_1), \ldots, (k_{r-1}^*, c_{r-1}), (k_r, r_r))\) is a surgery description of \( M \). Since \( E(L^*, S^3) \) is diffeomorphic to \( E(L'^*, M') \), we complete the proof of Corollary 4.1.

Remark 4.2. In the above proof, the restriction \( q|((S^3; L^* - k_i^*)): (S^3, L^* - k_i^*) \rightarrow (S^3, L - k_i) \) is homotopic to a diffeomorphism for each \( i, 1 \leq i \leq r-1 \). In particular, if \( r \geq 3 \), then \( k_i^* \) and \( k_i \) belong to the same knot type for all \( i, 1 \leq i \leq r \).

As a final application, we have the following:

**Corollary 4.3.** For any number \( K > 0 \) and any 3-manifold \( M \) such that \( \partial M \) has no 2-sphere components, there are a number \( K^+ > K \) and an infinite family of normal imitations \( M^* \) of \( M \) such that \( M^* \) is hyperbolic with \( \text{Vol}_M M^* < K^+ \) and \( \text{Sup}_M \text{Vol}_M M^* = K^+ \).

Proof. For a trivial knot \( O \) in \( \text{Int} M \), we obtain from Theorem 1.1 an almost identical imitation \((M, O^*)\) of the good pair \((M, O)\) such that \( E(O^*, M) \) is hyperbolic with \( \text{Vol}_E(O^*, M) > K \). For an integer \( n \neq 0 \), let \( M_n^* \) be a 3-manifold obtained from \( M \) by Dehn surgery along \( O^* \) with coefficient \( 1/n \). Since the diffeomorphism type of \( M \) is unaffected by Dehn surgery along \( O \) with coefficient \( 1/n \), the imitation map \( q: (M, O^*) \rightarrow (M, O) \) induces a normal imitation map \( q_n^*: M_n^* \rightarrow M \). Let \( K^+ = \text{Vol}_E(O^*, M) \). By Thurston's theorem on hyperbolic Dehn surgery [T-2], [T-3], there is an integer \( N > 0 \) such that \( M_n^* \) is hyperbolic for all \( n \) with \( |n| \geq N \), and for all such \( n \), \( \text{Vol}_n M_n^* < K^+ \) and \( \text{Sup}_n \text{Vol}_n M_n^* = K^+ \). This completes the proof.

5. Proof of Theorem 1.1. We first show that Theorem 1.1 is obtained from the following:

**Lemma 5.1.** For any good \((3,1)\)-manifold pair \((M, L)\), there is an almost identical imitation \((M, L^*)\) of \((M, L)\) such that \( E(L^*, M) \) is hyperbolic.

Proof of Theorem 1.1 assuming Lemma 5.1. We can see from Jørgensen's theorem (cf. [T-2],[T-3]) that for any number \( K > 0 \) there is an integer \( N^k > 0 \) such that every hyperbolic 3-manifold \( M' \) with \( \text{Vol} M' \leq K \) has the homology
group $H_i(M'; Z)$ generated by at most $N'$ elements. Let $L^+ = L \cup L_0$ with $L_0$ an $N'$-component trivial link in $\text{Int}(M-L)$. By Lemma 5.1, there is an almost identical imitation map $q: (M, L^*) \to (M, L^+)$ such that $E(L^*, M)$ is hyperbolic. Let $K^+ = \text{Vol} E(L^*, M)$. Since

$$H_i(E(L^*, M); Z) \cong H_i(E(L^+, M); Z) \cong H_i(E(L, M); Z) \oplus N' Z$$

(cf. [Kw-1]), we see that $H_i(E(L^*, M); Z)$ cannot be generated by $N'$ elements, so that $K^+ > K$. Let $L^* = q^{-1}L$ and $L^*_0 = q^{-1}L_0$. Note that $L^*_0$ is a trivial link in $\text{Int} M$. For an integer $n \neq 0$, let $(M, L^*_n)$ be a good $(3,1)$-manifold pair obtained from $(M, L^*)$ by Dehn surgery of $M$ along each component of $L^*_0$ with coefficient $1/n$. Then $q$ induces an almost identical imitation map $q_n: (M, L^*_n) \to (M, L)$. By Thurston's theorem on hyperbolic Dehn surgery [T-2], [T-3], there is an integer $N > 0$ such that $E(L^*_n, M)$ is hyperbolic for all $n$ with $|n| \geq N$ and, for all such $n$, $\text{Vol} E(L^*_n, M) < K^+$ and $\sup_n \text{Vol} E(L^*_n, M) = K^+$. This completes the proof of Theorem 1.1 assuming Lemma 5.1.

We say that a tangle $T$ in a 3-ball $B$ is trivial if $T$ is on a disk smoothly and properly embedded in $B$.

Proof of Lemma 5.1. We can see from arguments on Heegaard splitting of $M$ and on isotopic deformation of $L$ that $M$ is splitted by a compact connected surface $F$ with $\partial F \cap L = \emptyset$ into two handlebodies $H_i, i = 1, 2,$ of the same genus, say $g$, such that

1. $F_i = \partial H_i - \text{Int} F$ is a planar surface with the same component number as $\partial M$, 
2. Each component of $L$ meets $F$ transversely, 
3. Each disk component of $F_i$ meets $L$, 
4. There is a 3-ball $B_i \subset H_i$ separated by a proper disk $D_i$ such that $T_i = L \cap H_i$ is a trivial tangle of $s_i$ strings in $B_i$ where $s_i \geq 1$ and $g + s_i \geq 3$.

Our desired situation is illustrated in Figure 1. This situation is made up by the following procedure: When $\partial M = \emptyset$, we take any Heegaard splitting $(H_1, H_2, F)$ of $M$. When $\partial M \neq \emptyset$, we split $M$ by a connected surface $F_M$ into two 3-submanifolds $M_i, i = 1, 2$, such that $\partial M_i$ is connected and $\partial M_i - \text{Int} F_M$ is a planar surface with the same component number as $\partial M$. Then note that $\partial M_i, i = 1, 2$ have the same genus. We obtain a Heegaard splitting $(H_1, H_2, F)$ of $M$ with condition (1) from $(M_i, M_2, F_M)$ by boring along 1-handles in $M_i$ attaching to $F_M$. Next, we deform $L$ so that $L$ is disjoint from $\partial F$ and has (2), (3) by an isotopic deformation of $L$ in $M$. Finally, we deform $L$ so that $L$ has (4) by an isotopic deformation of $L$ in $M$ keeping $\partial M$ fixed and increasing the geometric intersection number with $F$. We proceed to the proof of Lemma 5.1 by assuming the following lemma:
Lemma 5.2. For any integer \( r \geq 3 \) let \( T \) be a trivial tangle of \( r \) strings in a 3-ball \( B \). Then there is an almost identical imitation \((B, T^*)\) of \((B, T)\) such that \( E(T^*, B) \) is hyperbolic.

Since \( H_i \) is the exterior of a trivial \( g \)-tangle in a 3-ball and \( \gamma + s_i \geq 3 \), we obtain from Lemma 5.2 an almost identical imitation map \( q_i: (H_i, T_i^*) \rightarrow (H_i, T_i) \) such that \( E(T_i^*, H_i) \) is hyperbolic. Let \( U_L \) be a tubular neighborhood of \( L \) in \( M - \partial F \) meeting \( \partial H_i \) regularly. We can assume that \( U_i = U_L \cap H_i \) is a tubular neighborhood of \( T_i \) in \( B_i - D_i \) and \( E(L, M) = \text{cl}_M(M - U_L) \) and \( E(T_i, H_i) = \text{cl}_M(H_i - U_i) \) and \( E(T_i^*, H_i) = q_i^{-1}E(T_i, H_i) \). Clearly, \( q_1 \) and \( q_2 \) define an almost identical imitation map \( q: (M, L^*) \rightarrow (M, L) \) with \( L^* = T_1^* \cup T_2^* \). Note that \( E(L^*, M) = q^{-1}E(L, M) \) is a union of \( E(T_i^*, H_i) \) and \( E(T_i^*, H_2) \) pasting along a surface \( F^e = \text{cl}_F(F - F \cap U_L) \). Then we see from the following lemma that \( E(L^*, M) \) is hyperbolic:

Lemma 5.3. Let a 3-manifold \( M \) be splitted into two 3-submanifolds \( M_i, i = 1, 2 \), by a proper surface \( F \). If the following conditions are all satisfied, then \( M \) is hyperbolic:

1. \( M_1 \) and \( M_2 \) are hyperbolic,
2. \( F \) has no disk, annulus, torus components,
3. \( F_i = \partial M_i - \text{Int} F \) has no disk components.

This lemma is a direct consequence of Myers' lemmas (Lemmas 2.4, 2.5) in [My] and Thurston's hyperbolization theorem in [T-3], [Mo]. We complete the proof of Lemma 5.1, assuming Lemma 5.2.

Proof of Lemma 5.2. We construct a pure \( r \)-braid \( \sigma \) with strings \( b_1, b_2, \ldots, b_r \) in the 3-cube \( I^3 \) as follows (cf. Kanenobu [Kn]): Take \( b_1 \cup b_2 \cup \cdots \cup b_{r-1} \) to be a trivial \((r-1)\)-braid. Then take \( b_r \) so that \( b_r \) represents the \((r-2)\)th commutator \([x_1, x_2, \ldots, x_{r-1}]\) in the free group \( \pi = \pi_1(S^3 - \hat{b}_1 \cup \hat{b}_2 \cup \cdots \cup \hat{b}_{r-1}, *) \) with a basis \( x_1, x_2, \ldots, x_{r-1} \) represented by meridians of \( \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_{r-1} \), for the closure link.
\( \sigma = b_1 \cup b_2 \cup \cdots \cup b_r \) in \( S^3 \). For \( r = 3, 4 \), we illustrate \( \sigma \) in Figure 2. Note that this \( r \)-braid \( \sigma \) has the following important property: That is, if we drop any one string \( b_i \) from \( \sigma \), then the resulting \((r-1)\)-braid is a trivial braid. The link \( \sigma \) is a typical example of a \textit{link with Brunnian property} (cf. Rolfsen [R]), or in other words, an \textit{almost trivial link} (cf. Milnor [Mi]). From this \( r \)-braid \( \sigma \subset I^3 \) and any two-string tangle \( T \subset B \), we construct a new \( r \)-string tangle \( T^\circ \subset B^\circ \) as it is illustrated in Figure 3.

This construction has been suggested by Kanenobu [Kn, Figure 7]. A two-string tangle \( T \subset B \) is said to be \textit{simple}, if it is a prime tangle and the exterior \( E(T, B) \) has no incompressible torus (cf. [So]) [Note: \( E(T, B) \) may have an essential annulus as we observe in Remark 5.6]. The following lemma is obtained from Kanenobu’s results in [Kn, Theorem 3 and Proposition 4] and Thurston’s hyperbolization theorem [T-3], [Mo]:

**Lemma 5.4.** If a two-string tangle \( T \subset B \) is simple, then the exterior \( E(T^\circ, B^\circ) \) of the resulting new tangle \( T^\circ \subset B^\circ \) is hyperbolic.

Let \( T^A \subset B^A \) be a one-string tangle obtained from a two-string tangle \( T \subset B \) by adding a trivial one-string tangle \( a_0 \subset B_0 \) as it is illustrated in Figure 4(1).
Let $T_Q$ be a trivial two-string tangle illustrated in Figure 4(2). Assume that there is a normal reflection $\alpha$ in $(B, T_0)$ such that $\text{Fix}(\alpha, (B, T_0) \times I) \cong (B, T)$. Let $\alpha^\wedge$ be the normal reflector in $(B^\wedge, T^\wedge_0) \times I$, extending $\alpha$ naturally, so that $\text{Fix}(\alpha^\wedge, (B^\wedge, T^\wedge_0) \times I) \cong (B^\wedge, T^\wedge)$.

If $\alpha^\wedge$ is isotopically standard, then we would have an almost identical imitation map $q : (B^\otimes, T^\otimes) \to (B^\otimes, T^\otimes)$. Since $(B^\otimes, T^\otimes)$ is a trivial tangle, we complete the proof of Lemma 5.2 when we assume the following lemma:

**Lemma 5.5.** There are a simple two-string tangle $T \subset B$ and a normal reflection $\alpha$ in $(B, T_0) \times I$ with $T_0 \subset B$ a trivial two-string tangle such that $(B, T) \cong \text{Fix}(\alpha, (B, T_0) \times I)$ and the extending normal reflection $\alpha^\wedge$ in $(B^\wedge, T^\wedge_0) \times I$ is isotopically standard.

**Proof of Lemma 5.5.** Consider a two-string tangle $T = a_1 \cup a_2 \subset B$ illustrated in Figure 5. Since $a_i$ is a non-trivial arc in $B$ (in fact, $E(a_i, B)$ is diffeomorphic to the exterior of the 11-crossing Kinoshita-Terasaka knot (cf. [K-T], [Kw-1])) and the one-string tangle $T^\wedge \subset B^\wedge$ is trivial, it follows from a result of Nakanishi...
[N, Lemma 5.4] that \((B, T)\) is a prime tangle. This tangle \(T \subset B\) can be obtained from the Kinoshita-Terasaka tangle \(T' = a_1' \cup a_2' \subset B\), illustrated in Figure 6, by sliding a boundary point of \(a_1'\) along \(\partial B\) and \(a_2'\).

This means that \(E(T, B) \cong E(T', B)\), so that \(T \subset B\) is a simple tangle, because \(T' \subset B\) is known to be simple (cf. Soma [So]). Let \(F\) be a union of two proper disks in \(B \times I\) illustrated in Figure 7 by the motion picture method (cf. [K-S-S]). We denote by \(a_0\) the standard reflection in \(B \times I\) and by \(a_0^\circ\) the extension to \(B^\times \times I\). Let \(G\) be a 1-manifold with a band in \(B\) given by \((B, G) \times (1/4) = (B \times I, F) \cap B \times (1/4)\). We take annuli \(A, A'\) in the figure of \(G \subset B\) as we illustrate in Figure 8. In Figure 8, \(\{C_1, C_2\}, \{C_1', C_2'\}\) denote the boundary components of \(A, A'\) and the intersections \(A \cap G, A' \cap G\) denote disks attaching to the circles \(C_1, C_1'\), respectively. Let \((B^\times I, F^\times)\) be a \((4,2)\)-disk pair obtained from \((B \times I, F)\) by adding \((B_0, a_0) \times I\) with \((B_0, a_0)\) in Figure 4(1).

Note that \(C_2, C_2'\) bound disjoint disks \(D, D'\) in \(B^\times - G\) (where \(G^\times = G \cup a_0\)) so that \(\tilde{A} = A \cup D, \tilde{A}' = A' \cup D'\) are disjoint disks in \(B^\times\) with \(\partial \tilde{A} = C_1, \partial \tilde{A}' = C_1'\). Let \(F'\) be a union of two proper disks in \(B \times I\) illustrated in Figure 9, and \((B^\times I, F'^\times)\)
a \((4, 2)\)-disk pair obtained from \((B \times I, F')\) by adding \((B_0, a_0) \times I\). Let \(G'\) be a 1-manifold with a band in \(B\) given by \((B, G') \times \{1/4\} = (B \times I, F') \cap B \times \{1/4\}\). Note that there is an \(f \in \text{Diff}_0(B^\gamma, \text{rel}(B^\gamma - R))\) with \(f(G^\gamma) = G'^\gamma\) for a regular neighborhood \(R\) of \(\bar{A} \cup \bar{A}'\) in \(\text{Int } B^\gamma\) by sliding the disks \(\bar{A} \cap G^\gamma, \bar{A}' \cap G^\gamma\) along the disks \(\bar{A}, \bar{A}'\). This means that there is an \(f \in \text{Diff}(B^\gamma \times I, \text{rel}(B^\gamma \times I - R \times I'))\) with \(I' = [-1/2, 1/2]\) such that \(f\) is \(\alpha_0\)-invariant and \(f(F^\Lambda) = F'^\Lambda\). Next, note that there is a \(g \in \text{Diff}_0(B^\gamma \times I, \text{rel}(B^\gamma \times I - F^\Lambda \cup F'^\Lambda))\) such that \(g((\bar{A} \cup \bar{A}') \times I') \subseteq B \times I\) by pushing \(D \times I', D' \times I'\) into \(B \times (1/2, 3/4)\).

Then we may consider that \(g(R \times I') \subseteq B \times I\). Let \(h = gfg^{-1} \in \text{Diff}(B^\gamma \times I, \text{rel } \partial(B^\gamma \times I))\). Then since \(h(B \times I) = B \times I\), we can define an \(h' \in \text{Diff}(B \times I, \text{rel } \partial(B \times I))\) by \(h' = h|B \times I\). Note that \(h'(F) = F'\). Since the bands appearing in Figure 7 are untied, we see that there is a \(d \in \text{Diff}(B \times I, \text{rel } \partial(B \times I))\) such that \(d\) is \(\alpha_0\)-invariant and \(d(F) = T_0 \times I\), where \(T_0\) is a trivial two-string tangle in \(B\) determined by \(T_0 \times 1 = F \cap B \times 1\). Let \(\alpha_1 = dh'd^{-1} \alpha_d h'd^{-1}\). Then \(\alpha_1\) defines a
reflection in \((B, T_0) \times I\) with \(\text{Fix}(\alpha, (B, T_0) \times I) \simeq (B, T)\). Further, we can find an \(e \in \text{Diff}_0((B, T_0) \times I, \text{rel } \partial(B \times I))\) such that \(\alpha = e \alpha e^{-1}\) is a normal reflection in \((B, T_0) \times I\) by the fact that \(\text{Diff}(D, \text{rel } \partial D) = \text{Diff}_0(D, \text{rel } \partial D)\) for a 2-disk \(D\) and the isotopy extension theorem and the uniqueness of tubular neighborhoods. Then

\[
\text{Fix}(\alpha, (B, T_0) \times I) \simeq (B, T)
\]

and

\[
\alpha^\wedge = e^\wedge d^\wedge h^{-1} \alpha_h^\circ h(\alpha^\wedge)^{-1},
\]

where \(d^\wedge\) and \(e^\wedge\) denote the extension of \(d\) and \(e\) to \(B^\wedge \times I\) by the identity, respectively. Let

\[
h^\ast = e^\wedge d^\wedge h^{-1} f(\alpha^\wedge)^{-1}.
\]

Then

\[
h^\ast = e^\wedge d^\wedge g f^{-1} g^{-1} f(\alpha^\wedge)^{-1} \in \text{Diff}_0((B^\wedge, T^\wedge_0) \times I, \partial (B^\wedge \times I)),
\]

because \(g \in \text{Diff}_0(B^\wedge \times I, \text{rel } \partial (B^\wedge \times I) \cup F^\wedge \cup F'^\wedge),\) and

\[
h^\ast^{-1} \alpha^\wedge h^\ast = d^\wedge f^{-1} \alpha_h^\circ f(\alpha^\wedge)^{-1} = \alpha_h^\circ
\]

because \(f\) and \(d^\wedge\) are \(\alpha_h^\circ\)-invariant. Hence \(\alpha^\wedge\) is isotopically standard. This completes the proof of Lemma 5.5.

Therefore, we complete the proof of Theorem 1.1.

REMARK 5.6. The exterior of the tangle \(T \subset B\) in Figure 5, that is, the exterior of the Kinoshita-Terasaka tangle \(T' \subset B\) in Figure 6 has an essential annulus, as it is illustrated in Figure 10. Hence it is not hyperbolic in our sense.
References


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