Doctoral Dissertation

Studies on the Conditional Stationary Distribution in Markov Chains via Systems of Linear Inequalities

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Preface

The theory of Markov chains is one of the fundamental mathematical tools of analyzing stochastic systems. For example, in many queueing models used in the performance evaluation of telecommunications systems, the system behaviors can be formulated as a discrete-time or continuous-time Markov chain on a countably infinite (or finite yet huge) state space. In such a Markov chain, the stationary distribution, if exists, is of primary interest because it represents the occurrence frequency of various events over a long time interval.

Suppose that a Markov chain is irreducible and positive recurrent. The stationary distribution is then given by the unique solution of the system of linear equations, where the number of unknowns is given by the cardinality of the state space. Therefore, if a Markov chain is defined on a huge state space, it is hard to obtain the stationary distribution from a viewpoint of computational cost. To overcome this difficulty, two main approaches are proposed in the literature. One is the matrix-analytic method for structured Markov chains on a countably infinite state space, where the stationary distribution is given in terms of the solution of a certain finite-dimensional matrix-polynomial equation. The other is the augmented truncation approximation for general Markov chains, where the original Markov chain on a countably infinite (or finite yet huge) state space is approximated by another Markov chain on a finite state space of moderate size.

In this study, we consider the conditional stationary distribution in the ergodic, continuous-time Markov chain on the countably infinite state space $\{0, 1, \ldots\}$, given that the state is not greater than a predefined threshold N. By definition, the conditional stationary distribution converges to the (unconditional) stationary distribution element-wise as N goes to infinity. Therefore, if N is sufficiently large, the conditional distribution may work as a good approximation to the stationary distribution.

The staring points of our discussions are systems of linear inequalities that the conditional stationary distribution satisfies. Note that such systems of linear inequalities determine convex regions that the conditional stationary distribution lies in. We study inclusion and limiting properties of those convex regions. Furthermore, based on these results, we consider numerical algorithms for computing the conditional stationary distribution given that the Markov chain is in $\{0, 1, \ldots, N\}$.

To the best of our knowledge, the characterization of the (un)conditional stationary distributions in Markov chains via systems of linear inequalities has not been studied except for a recent paper by Takine in 2016, where a system of infinitely many linear inequalities that the boundary probability vector satisfies is shown for Markov chains of level-dependent M/G/1-type. Furthermore, under the assumption that matrices in the infinitesimal generator, which represent downward jumps, are nonsingular, the vertices

spanning the convex region specified by the system of linear inequalities are obtained in terms of reverse-directional R-matrices.

In this dissertation, we characterize the conditional stationary distribution via systems of linear inequalities in two ways. In the first characterization, we analyze the conditional stationary distribution in general Markov chains by examining reverse-directional R-matrices and discuss computation of the conditional stationary distribution in Markov chains of level-dependent M/G/1-type. In the second characterization, we analyze the conditional stationary distribution by examining northwest corner submatrices of the infinitesimal generator. Furthermore, we consider computation of the conditional stationary distribution based on these characterizations and develop the computational algorithms. Our computational algorithms have a notable feature that we can evaluate error bounds of the computed conditional stationary distribution because the conditional stationary distribution lies in convex regions whose vertices are given by numerically computable probability vectors.

Chapter 1 provides a background of this study and summarizes known approaches to characterizing the stationary distribution in Markov chains on a countably infinite state space. Chapter 2 characterizes the conditional distribution in bivariate Markov chains via systems of linear inequalities and discusses some properties of the convex regions containing the conditional stationary distribution. Chapter 3 considers the computation of the conditional stationary distribution for Markov chains of level-dependent M/G/1-type based on the results in Chapter 2. Chapter 3 also provides some numerical examples. Chapter 4 characterizes the conditional stationary distribution via systems of linear inequalities constructed from northwest corner submatrices of the infinitesimal generator. We introduce a new state transition structure called (K, N)-skip-free sets and show their roles in the characterization. Chapter 5 provides the practical implications of the results in Chapter 4 for the augmented truncation approximation. Furthermore, we analyze a queueing model with disasters, which is a typical example analyzed effectively by our linear-inequality characterizations, and provide numerical examples. Finally, we conclude this dissertation in Chapter 6.

This dissertation summarizes my studies on the stationary distribution of Markov chains in Doctor's Course of Department of Information and Communications Technology, Graduate School of Engineering, Osaka University. The contents of this dissertation (except Chapters 1 and 6) are based on the papers in the publication list as follows.

Chapter 2: Publications A-1 and C-1,

Chapter 3: Publications A-1, C-2, and C-3,

Chapter 4: Publications A-2, C-4, C-5, C-6, and C-7,

Chapter 5: Publications A-3 and C-8.

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Abbreviations and Conventions

Throughout this dissertation, we use the following abbreviations:

ATA	: augmented truncation approximation
MAM	: matrix-analytic method
LD	: level-dependent or level dependence
MAP	: Markovian arrival process
MMAP	: marked Markovian arrival process
MMPP	: Markov modulated Poison process

In addition, we use the following conventions of mathematical notation unless otherwise mentioned:

- Row vectors are denoted by bold-type lower-case Greek letters, or specific Roman letters $\boldsymbol{x}$ and $\boldsymbol{y}$.
- Column vectors are denoted by bold-type lower-case Roman letters, except for row vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.
- Matrices are denoted by bold-type capital letters.
- Inequalities between matrices or vectors imply that they hold element-wise.
- Empty sum is defined as zero.
- Empty product is defined as one.
- The elements of vectors are counted from zero.
- The rows and columns of matrices are counted from zero.

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1 Introduction

Markov chains are stochastic processes with the Markov property, defined on a finite or countably infinite state space. Owing to their simple mechanism of evolution in time, Markov chains are used for describing stochastic systems in various application domains, including the queueing theory. In this study, we restrict our attention to irreducible, positive recurrent, continuous-time Markov chains on a countably infinite state space. Note that continuous-time Markov chains are ergodic if and only if they are irreducible and positive recurrent. Therefore, the unique stationary distribution exists in Markov chains we consider.

The stationary distribution in an ergodic, continuous-time Markov chain is given by the solution of a system of linear equations called the global balance equation. Except for some special cases, however, the explicit expression for the stationary distribution will be messy in nature even if it can be obtained. Therefore, from a viewpoint of applications, the computation of the stationary distribution in Markov chains is important. We thus consider characterizations of the conditional stationary distribution, given that the Markov chain is in a predefined finite subset of the state space, and using it, we develop new algorithms for computing the stationary distribution. In this chapter, we briefly summarize known approaches to the stationary distribution in Markov chains on a countably infinite state space and then, we describe the motivation of this study.

1.1 Infinite-state Markov chains

We consider a time-homogeneous, continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ on the state space $\mathbb{Z}^+ = \{0, 1, \dots\}$, where $\mathbb{Z}^+$ denotes the set of nonnegative integers. In primitive queueing models, $X(t)$ $(t \geq 0)$ represents the number of customers in the system at time t. From the Markov property, $\{X(t)\}_{t\geq 0}$ satisfies for arbitrary $s, t \geq 0$ and $j \in \mathbb{Z}^+$,

$$\Pr(X(t+s) = j \mid X(u)\ (0 \leq u < t), X(t) = i) = \Pr(X(t+s) = j \mid X(t) = i).$$

Let $q_{i,j}$ $(i, j \in \mathbb{Z}^+, i \neq j)$ denote the transition rate from state i to state j. The transition probabilities of $\{X(t)\}_{t\geq 0}$ satisfy

$$\Pr(X(t+s) = j \mid X(t) = i) = \begin{cases} q_{i,j}s + o(s), & j \neq i, \\ 1 - q_i s + o(s), & j = i, \end{cases} \quad i, j \in \mathbb{Z}^+,\ s, t \geq 0,$$

where $o(s)$ denotes the little-o notation such that $\lim_{s\to 0} o(s)/s = 0$, and

$$q_i = \sum_{j\in\mathbb{Z}^+\setminus\{i\}} q_{i,j}, \quad i \in \mathbb{Z}^+.$$

We assume $q_i < \infty$ ($i \in \mathbb{Z}^+$) throughout this dissertation. Let $\boldsymbol{\kappa}(t)$ ($t \geq 0$) denote the transient probability distribution at time t, i.e., the ith ($i \in \mathbb{Z}^+$) element $[\boldsymbol{\kappa}(t)]_i$ of $\boldsymbol{\kappa}(t)$ denotes $\Pr(X(t) = i)$. Note that the transient probability $\Pr(X(t) = i)$ depends on the initial state. If the initial distribution $\boldsymbol{\kappa}(0)$ is given, we obtain $\boldsymbol{\kappa}(t)$ ($t \geq 0$) as

$$\boldsymbol{\kappa}(t) = \boldsymbol{\kappa}(0) \exp[\boldsymbol{Q}t],$$

where $\boldsymbol{Q}$ denotes the infinitesimal generator of the Markov chain $\{X(t)\}_{t\geq 0}$:

$$\boldsymbol{Q} = \begin{pmatrix} -q_0 & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & \cdots \\ q_{1,0} & -q_1 & q_{1,2} & q_{1,3} & q_{1,4} & \cdots \\ q_{2,0} & q_{2,1} & -q_2 & q_{2,3} & q_{2,4} & \cdots \\ q_{3,0} & q_{3,1} & q_{3,2} & -q_3 & q_{3,4} & \cdots \\ q_{4,0} & q_{4,1} & q_{4,2} & q_{4,3} & -q_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.1}$$

and $\exp[\boldsymbol{Q}t]$ denotes the matrix exponential of $(\boldsymbol{Q}t)$:

$$\exp[\boldsymbol{Q}t] = \sum_{k=0}^{\infty} \frac{1}{k!} (\boldsymbol{Q}t)^k.$$

Note that $\boldsymbol{Q}$ has negative diagonal elements and nonnegative off-diagonal elements, and it satisfies

$$\boldsymbol{Q}\boldsymbol{e} = \boldsymbol{0}, \tag{1.2}$$

where $\boldsymbol{e}$ denotes a column vector with an appropriate dimension, whose elements are all equal to one.

Markov chains are classified according to the reachability and the recurrence time. First, a Markov chain is called irreducible if and only if all states belong to one communication class, i.e., every state can be reached from every other state: for any $i, j \in \mathbb{Z}^+$

$$\Pr(X(t) = j \mid X(0) = i) > 0, \quad t > 0.$$

Note that $\Pr(X(t) = j \mid X(0) = i) = [\exp[\boldsymbol{Q}t]]_{i,j}$. Let $\tau_{i,i}$ denote the recurrence time to state i, i.e., the first transition time to state i after visiting other states, starting from state i. A Markov chain is called recurrent if $\Pr(\tau_{i,i} < \infty) = 1$ for all $i \in \mathbb{Z}^+$, otherwise it is called transient. Furthermore a recurrent Markov chain is called positive-recurrent if $\mathrm{E}[\tau_{i,i}] < \infty$ for all $i \in \mathbb{Z}^+$, otherwise it is called null-recurrent. An irreducible and positive-recurrent continue-time Markov chain is ergodic and it has the unique stationary distribution. In what follows, we assume that $\{X(t)\}_{t\geq 0}$ is ergodic, unless otherwise mentioned.

We also assume that the Markov chain $\{X(t)\}_{t\geq 0}$ is stationary (i.e., $\boldsymbol{\kappa}(t) = \boldsymbol{\kappa}(0)$ ($t \geq 0$)) and we define $\boldsymbol{\pi}$ as the stationary distribution in $\{X(t)\}_{t\geq 0}$.

$$\boldsymbol{\pi} = (\pi_0 \ \pi_1 \ \pi_2 \ \ldots), \tag{1.3}$$

where $\pi_i = \Pr(X(0) = i)$ ($i \in \mathbb{Z}^+$). The stationary distribution satisfies

$$\boldsymbol{\pi} \exp[\boldsymbol{Q}t] = \boldsymbol{\pi}, \quad t \geq 0.$$

By definition, $\boldsymbol{\pi}$ expresses the long-term property of Markov chains in two aspects. If a continuous-time Markov chain is ergodic, for an arbitrary initial distribution $\boldsymbol{\kappa}(0) = \boldsymbol{\kappa}^*$, the transient probability distribution $\boldsymbol{\kappa}(t)$ converges to the stationary distribution $\boldsymbol{\pi}$ as t goes to infinity.

$$\lim_{t\to\infty} \boldsymbol{\kappa}(t) = \lim_{t\to\infty} \boldsymbol{\kappa}^* \cdot \exp[\boldsymbol{Q}t] = \boldsymbol{\pi}.$$

The left-hand side $\lim_{t\to\infty} \boldsymbol{\kappa}(t)$ of the equation is called the limiting probability distribution. Furthermore, it is known that the stationary probability π_i equals the limiting time average as follows.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{X(t)=i\}} \mathrm{d}t = \pi_i, \quad a.s.,$$

where $\mathbb{1}_{\{E\}}$ denote the indicator function of event E.

The stationary distribution $\boldsymbol{\pi}$ in the ergodic Markov chain is determined uniquely by the global balance equation:

$$\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0}, \qquad \boldsymbol{\pi}\boldsymbol{e} = 1. \tag{1.4}$$

Note that $\boldsymbol{\pi}$ can be viewed as the left eigenvector of $\boldsymbol{Q}$ associated with the eigenvalue 0 with $\boldsymbol{\pi}\boldsymbol{e} = 1$. Because of the infinite-dimensional $\boldsymbol{\pi}$ and $\boldsymbol{Q}$, it is hard to solve (1.4) directly except for some special classes, e.g., $\boldsymbol{Q}$ is given in a tri-diagonal form. In many practical situations, to obtain concrete information on the stationary distribution, we have to rewrite the global balance equation (1.4) to be a more tractable form.

In the analysis, stochastic models are sometimes formulated as bivariate Markov chains which are defined in terms of the univariate Markov chain $\{X(t)\}_{t\geq 0}$ as follows. We partition the state space $\mathbb{Z}^+$ of $\{X(t)\}_{t\geq 0}$ into finite and disjoint subsets $\mathcal{L}_\ell$ ($\ell \in \mathbb{Z}^+$) called *levels*. We call $\mathcal{L}_\ell$ level ℓ in particular. Let $M_\ell = |\mathcal{L}_\ell|$ ($\ell \in \mathbb{Z}^+$) denote the number of states in level ℓ, where for any set $\mathcal{X}$, $|\mathcal{X}|$ stands for the cardinality of $\mathcal{X}$. Without loss of generality, we assume $\mathcal{L}_\ell = \{L_{\ell-1}, L_{\ell-1}+1, \ldots, L_\ell - 1\}$ ($\ell \in \mathbb{Z}^+$), where $L_{-1} = 0$ and L_ℓ ($\ell \in \mathbb{Z}^+$) denotes the number of states in levels ℓ or lower.

$$L_\ell = \begin{cases} 0, & \ell = -1, \\ \displaystyle\sum_{k=0}^{\ell} M_k, & \ell \in \mathbb{Z}^+. \end{cases}$$

We then partition the infinitesimal generator $\boldsymbol{Q}$ and the stationary distribution $\boldsymbol{\pi}$ in conformance with the levels.

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{Q}_{0,0} & \boldsymbol{Q}_{0,1} & \boldsymbol{Q}_{0,2} & \boldsymbol{Q}_{0,3} & \boldsymbol{Q}_{0,4} & \cdots \\ \boldsymbol{Q}_{1,0} & \boldsymbol{Q}_{1,1} & \boldsymbol{Q}_{1,2} & \boldsymbol{Q}_{1,3} & \boldsymbol{Q}_{1,4} & \cdots \\ \boldsymbol{Q}_{2,0} & \boldsymbol{Q}_{2,1} & \boldsymbol{Q}_{2,2} & \boldsymbol{Q}_{2,3} & \boldsymbol{Q}_{2,4} & \cdots \\ \boldsymbol{Q}_{3,0} & \boldsymbol{Q}_{3,1} & \boldsymbol{Q}_{3,2} & \boldsymbol{Q}_{3,3} & \boldsymbol{Q}_{3,4} & \cdots \\ \boldsymbol{Q}_{4,0} & \boldsymbol{Q}_{4,1} & \boldsymbol{Q}_{4,2} & \boldsymbol{Q}_{4,3} & \boldsymbol{Q}_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}, \tag{1.5}$$

$$\boldsymbol{\pi} = \begin{array}{c} \begin{array}{cccccc} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \boldsymbol{\pi}_3 & \boldsymbol{\pi}_4 & \cdots \end{pmatrix} \end{array}. \tag{1.6}$$

The Markov chain $\{X(t)\}_{t\geq 0}$ whose infinitesimal generator is given in the form of (1.5) can be regarded as a bivariate Markov chain $\{(L(t), J(t))\}_{t\geq 0}$, where $L(t) \in \mathbb{Z}^+$ denotes the level at time t, and given $L(t) = \ell$, $J(t)$ takes a value in $\mathcal{M}_\ell = \{0, 1, \ldots, M_\ell - 1\}$ and denotes the index of $X(t)$ in $\mathcal{L}_\ell$:

$$\begin{aligned} &L(t) = \ell, && \text{if } X(t) \in \mathcal{L}_\ell, \\ &J(t) = X(t) - L_{\ell-1}, && \text{if } X(t) \in \mathcal{L}_\ell. \end{aligned}$$

We call $J(t)$ the *phase*. Note that $(L(t), J(t)) = (\ell, j)$ ($\ell \in \mathbb{Z}^+$, $j \in \mathcal{M}_\ell$) is equavalent to $X(t) = L_{\ell-1} + j$, so that $[\boldsymbol{\pi}_\ell]_j = \Pr(L(0) = \ell, J(t) = j) = \Pr(X(0) = L_{\ell-1} + j)$. Furthermore, $\mathcal{L}_\ell$ is equivalent to $\{(\ell, j);\ j \in \mathcal{M}_\ell\}$. For a given Markov chain, the univariate form $\{X(t)\}_{t\geq 0}$ and the bivariate form $\{(L(t), J(t))\}_{t\geq 0}$ can be used exchangeably.

In many practical queueing models, the level variable $L(t)$ represents the number of customers at time t, and the phase variable $J(t)$ does the auxiliary state interacting with $L(t)$. Typically, $J(t)$ represents the underlying state of a Markovian arrival process (MAP) or the phase of a phase-type service time distribution. The bivariate (i.e., level-partitioned) Markov chains is called *level-independent* if $\boldsymbol{Q}_{k,\ell} = \boldsymbol{A}_{\ell-k}$ ($k, \ell \in \mathbb{N}$) in (1.5), otherwise it is called *level-dependent*, where $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of positive integers. The level independence can be viewed as the block-Toeplitz structure in the infinitesimal generator $\boldsymbol{Q} := \boldsymbol{Q}^{(\mathrm{LI})}$:

$$\boldsymbol{Q}^{(\mathrm{LI})} = \begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \boldsymbol{B}_4 & \cdots \\ \boldsymbol{B}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{B}_{-2} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{B}_{-3} & \boldsymbol{A}_{-2} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \boldsymbol{B}_{-4} & \boldsymbol{A}_{-3} & \boldsymbol{A}_{-2} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}. \tag{1.7}$$

1.2 Preliminaries

Throughout this dissertation, we use the following conventions. We first define $\mathbb{Z}_m^\ell$ ($m, \ell \in \mathbb{Z}^+$) and $\mathbb{Z}_m^\infty$ ($m \in \mathbb{Z}^+$) as

$$\mathbb{Z}_m^\ell = \begin{cases} \{m, m+1, \ldots, \ell\}, & m \leq \ell, \\ \emptyset, & m > \ell, \end{cases} \qquad \mathbb{Z}_m^\infty = \{m, m+1, \ldots\}, \quad m \in \mathbb{Z}^+.$$

Note that $\mathbb{Z}^+ = \mathbb{Z}_0^\infty$, $\mathbb{N} = \mathbb{Z}_1^\infty$, and for any $\ell \in \mathbb{Z}^+$, $\mathbb{Z}^+$ is partitioned into a finite subset $\mathbb{Z}_0^\ell$ and its complement $\mathbb{Z}_{\ell+1}^\infty$. For any row vector $\boldsymbol{x}$, we define $\overline{\boldsymbol{x}}$ as

$$\overline{\boldsymbol{x}} = \begin{cases} \dfrac{\boldsymbol{x}}{\boldsymbol{x}\boldsymbol{e}}, & \boldsymbol{x}\boldsymbol{e} \neq 0, \\ \boldsymbol{0}, & \text{otherwise}, \end{cases}$$

and for any matrix $\boldsymbol{X}$ composed of $M+1$ row vectors $\boldsymbol{x}(i)$ ($i = 0, 1, \ldots, M$), we define $\overline{\boldsymbol{X}}$ as the corresponding matrix composed of $M+1$ row vectors $\overline{\boldsymbol{x}}(i)$ ($i = 0, 1, \ldots, M$). For

any $(M+1)$-dimensional row or column vector $\boldsymbol{x}$, let $\mathrm{diag}(\boldsymbol{x})$ denote an $(M+1)\times(M+1)$ diagonal matrix whose ith diagonal element is given by the ith element $[\boldsymbol{x}]_i$ of $\boldsymbol{x}$. We then have

$$\boldsymbol{x} = \boldsymbol{x}\boldsymbol{e}\cdot\overline{\boldsymbol{x}}, \qquad \boldsymbol{X} = \mathrm{diag}(\boldsymbol{X}\boldsymbol{e})\overline{\boldsymbol{X}}. \tag{1.8}$$

For an arbitrary $1\times(M+1)$ vector $\boldsymbol{x} = (x_0\ x_1\ \cdots\ x_M)$, $\|\boldsymbol{x}\|_1$ is defined as the ℓ_1-norm of $\boldsymbol{x}$.

$$\|\boldsymbol{x}\|_1 = \sum_{j=0}^{M} |x_j|. \tag{1.9}$$

We will apply the same notation $\|\cdot\|_1$ as the ℓ_1-norm in (1.9) to the case of $M=\infty$ (i.e., the total variation norm) as well.

We construct a convex polytope $\mathcal{P}$, by considering the intersection of a polyhedral convex cone $\mathcal{C}$ on the first orthant $\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\geq\boldsymbol{0}\}$ of $\mathbb{R}^{N+1}$ and a hyperplane containing all probability vectors in $\mathbb{R}^{N+1}$. As we will see, $\mathcal{C}$ and $\mathcal{P}$ take the following forms:

$$\begin{aligned}
&\mathcal{C} = \{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\boldsymbol{A}\geq\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{B}=\boldsymbol{0}\},\\
&\mathcal{P} = \mathcal{C}\cap\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\boldsymbol{e}=1\} = \{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\boldsymbol{A}\geq\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{B}=\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e}=1\},
\end{aligned} \tag{1.10}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ denote appropriate matrices with $N+1$ rows. Note that we allow $\boldsymbol{B}=\boldsymbol{O}$ and in that case, we can ignore the constraint $\boldsymbol{x}\boldsymbol{B}=\boldsymbol{0}$ in (1.10). In general, a convex polytope $\mathcal{P}$ can also be represented by a set of convex combinations of vertices of $\mathcal{P}$. Specifically, by using an appropriate nonnegative matrix $\overline{\boldsymbol{C}}$ with $N+1$ columns such that $\overline{\boldsymbol{C}}\boldsymbol{e}=\boldsymbol{e}$, $\mathcal{P}$ in (1.10) can also be represented to be

$$\mathcal{P} = \{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}=\boldsymbol{\alpha}\overline{\boldsymbol{C}},\ \boldsymbol{\alpha}\geq\boldsymbol{0},\ \boldsymbol{\alpha}\boldsymbol{e}=1\}.$$

Note that $\overline{\boldsymbol{C}}$ is composed of $1\times(N+1)$ probability vectors $\overline{\boldsymbol{\gamma}}_i$ $(i=0,1,\ldots,M)$ that span $\mathcal{P}$. If M vectors $\overline{\boldsymbol{\gamma}}_1-\overline{\boldsymbol{\gamma}}_0$, $\overline{\boldsymbol{\gamma}}_2-\overline{\boldsymbol{\gamma}}_0$, …, $\overline{\boldsymbol{\gamma}}_M-\overline{\boldsymbol{\gamma}}_0$ are linearly independent, $\mathcal{P}$ is called an M-simplex. For an arbitrarily convex polytope $\mathcal{P}$, let $\mathrm{ri}\,\mathcal{P}$ denote the relative interior of $\mathcal{P}$ [Roc70, Theorem 6.9].

$$\mathrm{ri}\,\mathcal{P} = \{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\boldsymbol{A}>\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{B}=\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e}=1\} \tag{1.11}$$

$$= \{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}=\boldsymbol{\alpha}\overline{\boldsymbol{C}},\ \boldsymbol{\alpha}>\boldsymbol{0},\ \boldsymbol{\alpha}\boldsymbol{e}=1\}. \tag{1.12}$$

1.3 Known results on the stationary distribution

As stated in Section 1.1, it is hard to solve the global balance equation (1.4) analytically except for some special classes. In the past, this problem has been tackled mainly by two approaches: the matrix-analytic method (MAM) [Lat99, Neu81, Neu89] and the augmented truncation approximation (ATA) [Har12, Liu18, Sen80, Zha04]. In this section, we summarize these two approaches briefly.

1.3.1 The matrix-analytic method (MAM) and its extensions

The MAM targets Markov chains with special transition structures which appear often in applications and gives *semi-explicit* expressions of the stationary distribution $\boldsymbol{\pi}$. In

this subsection, we will describe the MAM for continuous-time Markov chains based on the results in [Joy16, Kli06]. In this subsection, we consider Markov chains in the bivariate form $\{(L(t), J(t))\}_{t\geq 0}$ (c.f. (1.5) and (1.6)).

The standard MAM is applicable to level-independent Markov chains with the following features: direct transitions between levels are skip-free in one direction (or both). In particular, level-independent Markov chains are called M/G/1-type if transitions between levels are skip-free to the left [Neu89] and they are called G/M/1-type if transitions between levels are skip-free to the right [Neu81]. If transitions between levels are skip-free to both directions, they are called quasi birth-and-death (QBD) processes [Bri95, Neu81]. Note that the skip-free property of levels implies $\boldsymbol{A}_k = \boldsymbol{B}_k = \boldsymbol{O}$ $(k \in \mathbb{Z}_2^\infty)$ or $\boldsymbol{A}_{-k} = \boldsymbol{B}_{-k} = \boldsymbol{O}$ $(k \in \mathbb{Z}_2^\infty)$ in (1.7). In other words, the skip-free property of levels can be view as the block-Hessenberg structure in the infinitesimal generator $\boldsymbol{Q}$.

The MAM for Markov chains of M/G/1-type [Neu89, Kli06]: We consider an ergodic and continuous-time Markov chain $\{(L(t), J(t))\}_{t\geq 0}$ of M/G/1-type with $M_\ell = M$ $(\ell \in \mathbb{Z}^+)$. The infinitesimal generator $\boldsymbol{Q} := \boldsymbol{Q}^{(\mathrm{M/G/1})}$ is given by

$$\boldsymbol{Q}^{(\mathrm{M/G/1})} = \begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \boldsymbol{B}_4 & \cdots \\ \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{O} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

We define $\boldsymbol{G}$ as an $M \times M$ probability matrix whose (i,j)th $(i, j \in \mathcal{M} = \{0, 1, \ldots, M-1\})$ element represents the probability that the first passage time from state $(\ell+1, i)$ $(\ell \in \mathbb{Z}^+)$ to level ℓ ends at state (ℓ, j). The matrix $\boldsymbol{G}$ is called G-matrix and it is given by the minimum nonnegative solution $\boldsymbol{X}$ to the following matrix polynomial equation.

$$\boldsymbol{A}_{-1} + \sum_{k=0}^{\infty} \boldsymbol{A}_k \boldsymbol{X}^{k+1} = \boldsymbol{O}. \tag{1.13}$$

We define $\widetilde{\boldsymbol{A}}_\ell$ $(\ell \in \mathbb{Z}^+)$ and $\widetilde{\boldsymbol{B}}_\ell$ $(\ell \in \mathbb{Z}^+)$ as

$$\widetilde{\boldsymbol{A}}_\ell = \sum_{n=\ell}^{\infty} \boldsymbol{A}_n \boldsymbol{G}^{n-\ell}, \quad \widetilde{\boldsymbol{B}}_\ell = \sum_{n=\ell}^{\infty} \boldsymbol{B}_n \boldsymbol{G}^{n-\ell}.$$

It is known that $\boldsymbol{\pi}_\ell$ $(\ell \in \mathbb{N})$ is given in terms of $\boldsymbol{\pi}_0$ (cf. (1.6)).

$$\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_0 \boldsymbol{F}_\ell, \quad \ell \in \mathbb{N}, \tag{1.14}$$

where

$$\boldsymbol{F}_\ell = (\widetilde{\boldsymbol{B}}_\ell + \sum_{i=1}^{\ell-1} \boldsymbol{F}_i \widetilde{\boldsymbol{A}}_{\ell-i})(-\widetilde{\boldsymbol{A}}_0)^{-1}, \quad \ell \in \mathbb{N}.$$

The subvector $\boldsymbol{\pi}_0$ is called the boundary vector. It is also known that the boundary vector $\boldsymbol{\pi}_0$ is determined uniquely by

$$\boldsymbol{\pi}_0 \widetilde{\boldsymbol{B}}_0 = \boldsymbol{0}, \quad \boldsymbol{\pi}_0 \big(\boldsymbol{I} + \sum_{\ell=1}^{\infty} \boldsymbol{F}_\ell\big)\boldsymbol{e} = 1. \tag{1.15}$$

Therefore, the stationary distribution $\boldsymbol{\pi}$ in Markov chains of M/G/1-type is given by the semi-explicit expressions (1.14) with (1.15) in terms of $\boldsymbol{G}$.

Note that the second equation in (1.15) determines $\boldsymbol{\pi}_0\boldsymbol{e} = \Pr(L(0) = 0)$. By applying the results for discrete-time Markov chains in [Neu89] to continuous-time Markov chains via the uniformization technique, $\boldsymbol{\pi}_0\boldsymbol{e}$ is also determined as follows.

$$\boldsymbol{\pi}_0\boldsymbol{e} = \Big(1 + \overline{\pi}_0\big(\sum_{k=0}^{\infty} \boldsymbol{B}_k + \sum_{k=1}^{\infty} k\boldsymbol{B}_k\boldsymbol{e}\cdot\boldsymbol{\gamma}\big)\big(-\sum_{k=-1}^{\infty} \boldsymbol{A}_k - \sum_{k=1}^{\infty} k\boldsymbol{A}_{k-1}\boldsymbol{e}\boldsymbol{\gamma}\big)^{-1}\boldsymbol{e}\Big)^{-1},$$

where $\boldsymbol{\gamma}$ is the unique solution of

$$\boldsymbol{\gamma}\boldsymbol{G} = \boldsymbol{\gamma}, \quad \boldsymbol{\gamma}\boldsymbol{e} = 1.$$

The MAM for Markov chains of G/M/1-type [Neu81, Joy16]: We consider an ergodic and continuous-time Markov chain $\{(L(t), J(t))\}_{t\geq 0}$ of G/M/1-type with $M_\ell = M$ $(\ell \in \mathbb{N})$. The infinitesimal generator $\boldsymbol{Q} := \boldsymbol{Q}^{(\mathrm{G/M/1})}$ of $\{(L(t), J(t))\}_{t\geq 0}$ is given by

$$\boldsymbol{Q}^{(\mathrm{G/M/1})} = \begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_1 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{B}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{B}_{-2} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{O} & \cdots \\ \boldsymbol{B}_{-3} & \boldsymbol{A}_{-2} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \boldsymbol{B}_{-4} & \boldsymbol{A}_{-3} & \boldsymbol{A}_{-2} & \boldsymbol{A}_{-1} & \boldsymbol{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

We define $\boldsymbol{R}$ as an $M \times M$ matrix whose (i,j)th $(i, j \in \mathcal{M} = \{0, 1, \ldots, M-1\})$ element represents the ratio of the mean total sojourn time in state $(\ell+1, j)$ $(\ell \in \mathbb{N})$ in the recurrence time to level ℓ starting from state (ℓ, i) to the mean sojourn time in state (ℓ, i). The matrix $\boldsymbol{R}$ is called the R-matrix and it is given by the minimum nonnegative solution $\boldsymbol{X}$ to the following matrix polynomial equation.

$$\sum_{k=0}^{\infty} \boldsymbol{X}^k \boldsymbol{A}_{1-k} = \boldsymbol{O}. \tag{1.16}$$

By definition, we have $\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_{\ell-1}\boldsymbol{R}$ $(\ell \in \mathbb{N})$. We thus have

$$\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_0 \boldsymbol{R}^\ell, \quad \ell \in \mathbb{N}. \tag{1.17}$$

By substituting (1.17) into (1.4), the boundary vector $\boldsymbol{\pi}_0$ is determined uniquely by

$$\boldsymbol{\pi}_0 \sum_{\ell=0}^{\infty} \boldsymbol{R}^\ell \boldsymbol{B}_{-\ell} = \boldsymbol{0}, \quad \boldsymbol{\pi}_0 \sum_{\ell=0}^{\infty} \boldsymbol{R}^\ell \boldsymbol{e} = 1. \tag{1.18}$$

Therefore, the stationary distribution $\boldsymbol{\pi}$ in Markov chains of G/M/1-type is given by the semi-explicit expressions (1.17) and (1.18) in terms of $\boldsymbol{R}$.

Note that the second equation in (1.18) is also rewritten to be

$$\boldsymbol{\pi}_0(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}=1,$$

because the spectrum radius of $\boldsymbol{R}$ is less than one if the Markov chain is ergodic.

$$\sum_{\ell=0}^{\infty}\boldsymbol{R}^{\ell}=(\boldsymbol{I}-\boldsymbol{R})^{-1}.$$

Roughly speaking, in the standard MAM, the global balance equation is reduced to the matrix polynomial equations determining G-matrix or R-matrix as in (1.13) or (1.16). The stationary distribution in QBD processes is also studied in a similar way [Neu81].

In the literature, extensions of the MAM for level-dependent Markov chains (hereafter referred to as LD-MAM) are studied. Level-dependent Markov chains are called level-dependent M/G/1-type (LD-M/G/1-type) if the transition between levels are skip-free to the left and they are called the level-dependent G/M/1-type (LD-G/M/1-type) if the transition between levels are skip-free to the right. Specifically, the infinitesimal generator $\boldsymbol{Q}:=\boldsymbol{Q}^{(\text{LD-M/G/1})}$ of Markov chains of LD-M/G/1-type is given by

$$\boldsymbol{Q}^{(\text{LD-M/G/1})}=\begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \vdots \end{array}\begin{array}{c} \begin{array}{cccccc} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{Q}_{0,0} & \boldsymbol{Q}_{0,1} & \boldsymbol{Q}_{0,2} & \boldsymbol{Q}_{0,3} & \boldsymbol{Q}_{0,4} & \cdots \\ \boldsymbol{Q}_{1,0} & \boldsymbol{Q}_{1,1} & \boldsymbol{Q}_{1,2} & \boldsymbol{Q}_{1,3} & \boldsymbol{Q}_{1,4} & \cdots \\ \boldsymbol{O} & \boldsymbol{Q}_{2,1} & \boldsymbol{Q}_{2,2} & \boldsymbol{Q}_{2,3} & \boldsymbol{Q}_{2,4} & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{Q}_{3,2} & \boldsymbol{Q}_{3,3} & \boldsymbol{Q}_{3,4} & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{Q}_{4,3} & \boldsymbol{Q}_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}. \tag{1.19}$$

For Markov chains of LD-M/G/1-type, we can obtain a similar expression to (1.14) and (1.15) based on the level-dependent G-matrix $\boldsymbol{G}_\ell$ ($\ell\in\mathbb{N}$) [Hof01, Kli06, Li05]. For Markov chains of LD-G/M/1-type, we can also obtain a similar expression to (1.17) and (1.18) based on the level-dependent R-matrix $\boldsymbol{R}_\ell$ ($\ell\in\mathbb{N}$). The stationary distribution in level-dependent QBD processes is also studied in [Bau10, Bri95, Phu10].

1.3.2 The augmented truncation approximation (ATA)

To obtain the stationary distribution in Markov chains on a countably infinite state space, we need to consider approximation except for cases that the explicit expression of the stationary distribution is available or that the MAM or the LD-MAM is applicable. In this subsection, we consider Markov chains in a univariate form $\{X(t)\}_{t\geq 0}$.

Let $\xi(N)$ ($N\in\mathbb{Z}^+$) denote the tail probability of the stationary distribution.

$$\xi(N)=\sum_{i=N+1}^{\infty}\pi_i=\Pr(X(0)>N).$$

In any ergodic Markov chains, the tail probability $\xi(N)$ monotonically converges to zero as N goes to infinity. Roughly speaking, for a sufficiently large i, the stationary probability π_i is negligible, so that the influence of transition structures in sufficiently large states on $\boldsymbol{\pi}$ may be negligible. In the ATA, we choose a sufficiently large $N \in \mathbb{Z}^+$ and ignore the transition structures on $\mathbb{Z}_{N+1}^{\infty}$. In other words, the ATA attempts to obtain an approximation to $\boldsymbol{\pi}$ from the $(N+1) \times (N+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ of the infinitesimal generator $\boldsymbol{Q}$ (cf. (4.2) in Section 4.2):

$$\boldsymbol{Q}^{(1,1)}(N) = \begin{pmatrix} -q_0 & q_{0,1} & q_{0,2} & q_{0,3} & \cdots & q_{0,N-1} & q_{0,N} \\ q_{1,0} & -q_1 & q_{1,2} & q_{1,3} & \cdots & q_{1,N-1} & q_{1,N} \\ q_{2,0} & q_{2,1} & -q_2 & q_{2,3} & \cdots & q_{2,N-1} & q_{2,N} \\ q_{3,0} & q_{3,1} & q_{3,2} & -q_3 & \cdots & q_{3,N-1} & q_{3,N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{N-1,0} & q_{N-1,1} & q_{N-1,2} & q_{N-1,3} & \cdots & -q_{N-1} & q_{N-1,N} \\ q_{N,0} & q_{N,1} & q_{N,2} & q_{N,3} & \cdots & q_{N,N-1} & -q_N \end{pmatrix}.$$

The northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ of an irreducible Markov chain is a *defective* infinitesimal generator, i.e., $\boldsymbol{Q}^{(1,1)}(N)$ has negative diagonal elements and nonnegative off-diagonal elements, and it satisfies $\boldsymbol{Q}^{(1,1)}(N)\boldsymbol{e} \leq \boldsymbol{0}$ and $\boldsymbol{Q}^{(1,1)}(N)\boldsymbol{e} \neq \boldsymbol{0}$. The approximations based on northwest corner submatrices are initially studied in [Sen67, Sen68, Gol73].

In the ATA proposed in [Gol74], we construct an $(N+1) \times (N+1)$ infinitesimal generator $\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)$, where $\boldsymbol{Q}_{\mathrm{A}}(N)$ is an augmentation matrix such that

$$\boldsymbol{Q}_{\mathrm{A}}(N) \geq \boldsymbol{O}, \quad \boldsymbol{Q}_{\mathrm{A}}(N)\boldsymbol{e} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}. \tag{1.20}$$

This procedure is called the *augmentation*. Usually, $[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)]$ is assumed to be irreducible. Let $\boldsymbol{\pi}^{(i)}(N)$ $(i = 1, 2)$ denote the subvector of $\boldsymbol{\pi}$:

$$\boldsymbol{\pi} = \begin{pmatrix} \overset{\mathbb{Z}_0^N}{\boldsymbol{\pi}^{(1)}(N)} & \overset{\mathbb{Z}_{N+1}^{\infty}}{\boldsymbol{\pi}^{(2)}(N)} \end{pmatrix}. \tag{1.21}$$

We obtain an approximation $\boldsymbol{\pi}^{\mathrm{approx}}(N)$ to $\boldsymbol{\pi}^{(1)}(N)$ by solving

$$\boldsymbol{\pi}^{\mathrm{approx}}(N)[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)] = \boldsymbol{0}, \quad \boldsymbol{\pi}^{\mathrm{approx}}(N)\boldsymbol{e} = 1. \tag{1.22}$$

We then adopt $(\boldsymbol{\pi}^{\mathrm{approx}}(N) \;\; \boldsymbol{0})$ as an approximation to $\boldsymbol{\pi}$. Note that (1.22) is a global balance equation for a Markov chain on $\mathbb{Z}_0^N$. Roughly speaking, the ATA constructs a *finite-state* Markov chain on $\mathbb{Z}_0^N$ with the infinitesimal generator $\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)$ as an approximation to the original Markov chain $\{X(t)\}_{t \geq 0}$ and solves its global balance equation. In the ATA, we approximate $\boldsymbol{\pi}^{(2)}(N)$ to be $\boldsymbol{0}$. Note that $\boldsymbol{\pi}^{(2)}(N)$ $(> \boldsymbol{0})$ is negligible for a sufficiently large N, since $\xi(N) = \boldsymbol{\pi}^{(2)}(N)\boldsymbol{e}$ converges to zero monotonically as N increases.

To obtain a good approximation $(\boldsymbol{\pi}^{\mathrm{approx}}(N) \;\; \boldsymbol{0})$ to $\boldsymbol{\pi}$, it is important whether the approximation $(\boldsymbol{\pi}^{\mathrm{approx}}(N) \;\; \boldsymbol{0})$ converges to $\boldsymbol{\pi}$ as N goes to infinity:

$$\lim_{N \to \infty} \|(\boldsymbol{\pi}^{\mathrm{approx}}(N) \;\; \boldsymbol{0}) - \boldsymbol{\pi}\|_1 = \lim_{N \to \infty} \left(\|\boldsymbol{\pi}^{\mathrm{approx}}(N) - \boldsymbol{\pi}^{(1)}(N)\|_1 + \xi(N) \right) = 0. \tag{1.23}$$

For discrete-time Markov chains, the convergence (1.23) is shown in the case that the state transition probability matrix $\boldsymbol{P}$ is a upper-Hessenberg matrix or $\boldsymbol{P}$ satisfies $\inf_{i\in\mathbb{Z}^+}[\boldsymbol{P}]_{i,j} > 0$ for some $j \in \mathbb{Z}^+$ [Gib87a]. The convergence (1.23) is also known for stochastically monotone or stochastically block-monotone discrete-time Markov chains [Gib87b, Li00].

The approximation $(\boldsymbol{\pi}^{\mathrm{approx}}(N)\ \ \boldsymbol{0})$ clearly depends not only on the truncation point N but also on the selection of the augmentation matrix $\boldsymbol{Q}_{\mathrm{A}}(N)$. In the literature, some strategies for it (hereinafter referred to as augmentation strategies) are studied. The ATA is called *linear* if the rank of $\boldsymbol{Q}_{\mathrm{A}}(N)$ is equal to one [Gib87a]. Specifically, if $\boldsymbol{Q}_{\mathrm{A}}(N) = (-\boldsymbol{Q}(N)\boldsymbol{e})\boldsymbol{\zeta}$ for some $1 \times (N+1)$ probability vector $\boldsymbol{\zeta}$, $\boldsymbol{Q}_{\mathrm{A}}(N)$ is called a linear augmentation matrix and $\boldsymbol{\pi}^{\mathrm{approx}}(N)$ is called a linear ATA solution. Let $\boldsymbol{e}_i$ ($i \in \mathbb{Z}^+$) denote the ith unit column vector. In particular, if $\boldsymbol{Q}_{\mathrm{A}}(N) = (-\boldsymbol{Q}(N)\boldsymbol{e})\boldsymbol{e}_0^T$ (resp. $\boldsymbol{Q}_{\mathrm{A}}(N) = (-\boldsymbol{Q}(N)\boldsymbol{e})\boldsymbol{e}_N^T$), the augmentation is called the first-column augmentation (resp. the last-column augmentation), where T stands for the transpose operator. Furthermore, the last-column-block augmentation [Mas18], the augmentation of diagonal elements [Gib87a], and others [Wol80, Li00] are proposed as augmentation strategies. It is known that (1.23) holds for the first-column augmentation in any ergodic Markov chains [Wol80]. On the other hand, the last-column augmentation requires some technical conditions for the convergence (1.23) [Wol80]. It is, however, claimed that the last-column augmentation is the best augmentation in some stochastically monotone Markov chains [Gib87a], i.e., the approximation converges to $\boldsymbol{\pi}$ faster than any other augmentation strategies. In [Liu10, Section 2], the convergence is shown under well-known technical conditions corresponding to the ergodicity [Bor98, Theorem 1.3, Theorem 2.2].

From a practical point of view, the error $\|(\boldsymbol{\pi}^{\mathrm{approx}}(N)\ \ \boldsymbol{0}) - \boldsymbol{\pi}\|_1$ for a specific N of the approximation is also important as well as the convergence (1.23). Upper bounds of the error for the last-column augmentation and the last-column-block augmentation are shown under some monotonicity and/or ergodicity for discrete-time Markov chains [Liu10, Mas15, Mas16, Twe98]. The upper bounds of the error is also given in terms of parameters related to some drift conditions for discrete-time or continue-time Markov chains [Liu10, Mas18, Liu18]. Note here that the constructions of the drift conditions are difficult issues in general Markov chains.

1.4 Motivation of this study

By definition, the subvector $\boldsymbol{\pi}^{(1)}(N)$ of $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} = 1 - \xi(N)$, where $\boldsymbol{\pi}^{(1)}(N)$ appears in (1.21). In general, the approximation $\boldsymbol{\pi}^{\mathrm{approx}}(N)$ to $\boldsymbol{\pi}^{(1)}(N)$ is normalized such that $\boldsymbol{\pi}^{\mathrm{approx}}(N)\boldsymbol{e} = 1$ as in (1.22) because we cannot obtain the tail probability $\xi(N)$ exactly except for some special classes. Let $\boldsymbol{\pi}(N)$ ($N \in \mathbb{Z}^+$) denote the conditional stationary distribution given $X(0) \in \mathbb{Z}_0^N$.

$$\boldsymbol{\pi}(N) = (\pi_0(N)\ \pi_1(N)\ \cdots\ \pi_N(N)),$$

where $\pi_i(N) = \Pr(X(0) = i \mid X(0) \in \mathbb{Z}_0^N)$ ($i \in \mathbb{Z}_0^N$). By definition, we have

$$\boldsymbol{\pi}(N) = \overline{\boldsymbol{\pi}}^{(1)}(N) = \frac{\boldsymbol{\pi}^{(1)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}}. \tag{1.24}$$

It is clear that $\boldsymbol{\pi}(N)$ is the normalized $\boldsymbol{\pi}^{(1)}(N)$ such that $\boldsymbol{\pi}(N)\boldsymbol{e} = 1$. Therefore, we can regard the ATA solution $\boldsymbol{\pi}^{\mathrm{approx}}(N)$ in (1.22) as an approximation to the *conditional* stationary distribution $\boldsymbol{\pi}(N)$ rather than $\boldsymbol{\pi}^{(1)}(N)$ because of $\boldsymbol{\pi}(N)\boldsymbol{e} = \boldsymbol{\pi}^{\mathrm{approx}}(N)\boldsymbol{e} = 1$. Note here that $\boldsymbol{\pi}(N)$ converges to the (unconditional) stationary distribution $\boldsymbol{\pi}$ element-wise as N goes to infinity.

$$\lim_{N\to\infty} \pi_i(N) = \lim_{N\to\infty} \frac{\pi_i}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}} = \pi_i, \quad i \in \mathbb{Z}^+,$$

because $\lim_{N\to\infty} \boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} = \boldsymbol{\pi}\boldsymbol{e} = 1$.

It is known that the error in an approximation $(\boldsymbol{\pi}^{\mathrm{approx}}(N)\ \ \mathbf{0})$ to $\boldsymbol{\pi}$ is given in terms of the tail probability $\xi(N) = \Pr(X(0) > N)$ and the error $\epsilon(N) = \|\boldsymbol{\pi}^{\mathrm{approx}}(N) - \boldsymbol{\pi}(N)\|_1$ [Zha04]:

$$2\xi(N) \le \|(\boldsymbol{\pi}^{\mathrm{approx}}(N)\ \mathbf{0}) - \boldsymbol{\pi}\|_1 \le 2\xi(N) + \epsilon(N). \tag{1.25}$$

Recall that the tail probability $\xi(N)$ monotonically converges to zero as N increases. To suppress the error bound given in (1.25), we should set N large enough to make the tail probability $\xi(N)$ negligible. Because the exact estimation of the tail probability depends on transition structures on $\mathbb{Z}^{\infty}_{N+1}$, the analytical discussion on the tail probability is hard in the general setting and the selection of N requires model-specific discussions. On the other hand, for a given N, we should suppress the error $\epsilon(N)$ in the computed conditional stationary distribution $\boldsymbol{\pi}^{\mathrm{approx}}(N)$. In this study, we focus on the conditional stationary distribution explicitly.

Recently, characterizations of the (conditional) stationary distribution via systems of linear inequalities have been studied in [Taki16]. Specifically, the stationary distribution in Markov chains of LD-M/G/1-type is characterized as an unique feasible solution of a system of infinitely many linear inequalities. Recall that the infinitesimal generator $\boldsymbol{Q} := \boldsymbol{Q}^{(\text{LD-M/G/1})}$ is given by (1.19).

Proposition 1.1 ([Taki16, Theorem 1])**.** *Consider an ergodic Markov chain of LD-M/G/1-type. For a sequence* $\{\boldsymbol{S}_k;\ k \in \mathbb{Z}^+\}$ *satisfying*

$$\sum_{k=0}^{\ell+1} \boldsymbol{S}_k \boldsymbol{Q}_{k,\ell} = \boldsymbol{O}, \quad \ell \in \mathbb{Z}^+,$$

we assume that there exists a feasible solution $\boldsymbol{x} = \boldsymbol{x}^*$ *of the following system of linear constraints:*

$$\boldsymbol{x}\boldsymbol{S}_\ell > \mathbf{0}\ (\ell \in \mathbb{Z}^+), \quad \boldsymbol{x}\boldsymbol{S}_0\boldsymbol{e} = 1. \tag{1.26}$$

We then have

$$\boldsymbol{\pi}_\ell = p_0 \boldsymbol{x}^* \boldsymbol{S}_\ell, \quad \ell \in \mathbb{Z}^+,$$

where

$$p_0 = \Big(\boldsymbol{x}^* \sum_{k=0}^{\infty} \boldsymbol{S}_k \boldsymbol{e}\Big)^{-1}.$$

Note that $\boldsymbol{S}_\ell$ ($\ell \in \mathbb{Z}^+$) in (1.26) exists but it is not unique. A trivial example is $\boldsymbol{S}_\ell = \boldsymbol{e}\boldsymbol{\pi}_\ell$ ($\ell \in \mathbb{Z}^+$) [Taki16, Section 1]. Note also that Proposition 1.1 does not guarantee the existence of a feasible solution $\boldsymbol{x}^*$. On the other hand, if a feasible solution $\boldsymbol{x}^*$ exists, it is guaranteed to be unique from the global balance equation. Under the assumption that $\boldsymbol{Q}_{\ell,\ell-1}$ ($\ell \in \mathbb{N}$) is nonsingular, [Taki16] obtained a non-trivial $\boldsymbol{S}_\ell$'s and for such $\boldsymbol{S}_\ell$'s, it showed that the system (1.26) of linear inequalities has an unique feasible solution $\boldsymbol{x}^* = \overline{\boldsymbol{\pi}}_0$. Furthermore, under the same assumption, [Taki16] represented the feasible solution $\boldsymbol{x}^* = \overline{\boldsymbol{\pi}}_0$ in terms of *reverse-directional* R-matrices. Finally, it developed a computational algorithm of the conditional stationary distribution in a special class of Markov chains of LD-M/G/1-type, which satisfies that $\boldsymbol{Q}_{\ell,\ell-1}$ ($\ell \in \mathbb{Z}_{L^*}^{\infty}$) is nonsingular for some nonnegative integer L^*.

In this dissertation, we clarify not the exact one-point solution of the conditional stationary distribution, which is not available in the general setting, but regions containing it. For this purpose, we characterize the conditional stationary distribution via systems of linear inequalities, as with [Taki16]. Specifically, we derive systems of linear inequalities that the conditional stationary distribution satisfies. Recall that a system of linear inequalities determines a *convex* region that the conditional stationary distribution lies in. Therefore, we can evaluate error bounds for an approximation to the conditional stationary distribution, using our linear-inequality characterization. To the best of our knowledge, characterizations of the (un)conditional stationary distribution via systems of linear inequalities have not been studied except for [Taki16].

We consider two types of linear-inequality characterizations in this dissertation. We first generalize the results in [Taki16] to general bivariate Markov chains, by eliminating the assumption of the nonsingularity of $\boldsymbol{Q}_{\ell,\ell-1}$'s and the skip-free-to-the-left property posed in [Taki16]. Specifically, we characterize the conditional stationary distribution in general bivariate Markov chains via systems of linear inequalities obtained from reverse-directional R-matrices. We show that the boundary probability vector of the conditional stationary distribution is uniquely determined by infinitely many convex polytopes spanned by reverse-directional R-matrices.

Secondly, we characterize the conditional stationary distribution via systems of linear inequalities obtained from northwest corner submatrices of the infinitesimal generator. We show the minimal convex polytope that contains the conditional stationary distribution given that the state is not greater than N under the condition that the $(K+1) \times (K+1)$ northwest corner submatrix of $\boldsymbol{Q}$ and some information on the transition structure are available. We also show that such convex polytopes monotonically converges to the singleton with the conditional stationary distribution as K goes to infinity.

Although we consider continuous-time infinite-state Markov chains in this dissertation, almost all results in this dissertation, except for them corresponding to limit formulas, are applicable to finite-state Markov chains. Furthermore, the results for continuous-time Markov chains on $\mathbb{Z}^+$ in this dissertation are applicable to time-homogeneous, discrete-time Markov chains on $\mathbb{Z}^+$. Specifically, in a discrete-time Markov chain with a state transient probability matrix $\boldsymbol{P}_D$, the stationary distribution $\boldsymbol{\pi}_D$, if exists, satisfies $\boldsymbol{\pi}_D = \boldsymbol{\pi}_D \boldsymbol{P}_D$ and $\boldsymbol{\pi}_D \boldsymbol{e} = 1$. Therefore, we can study $\boldsymbol{\pi}_D$ by considering the corresponding continuous-time Markov chain with the infinitesimal generator $\boldsymbol{Q} = \boldsymbol{P}_D - \boldsymbol{I}$.

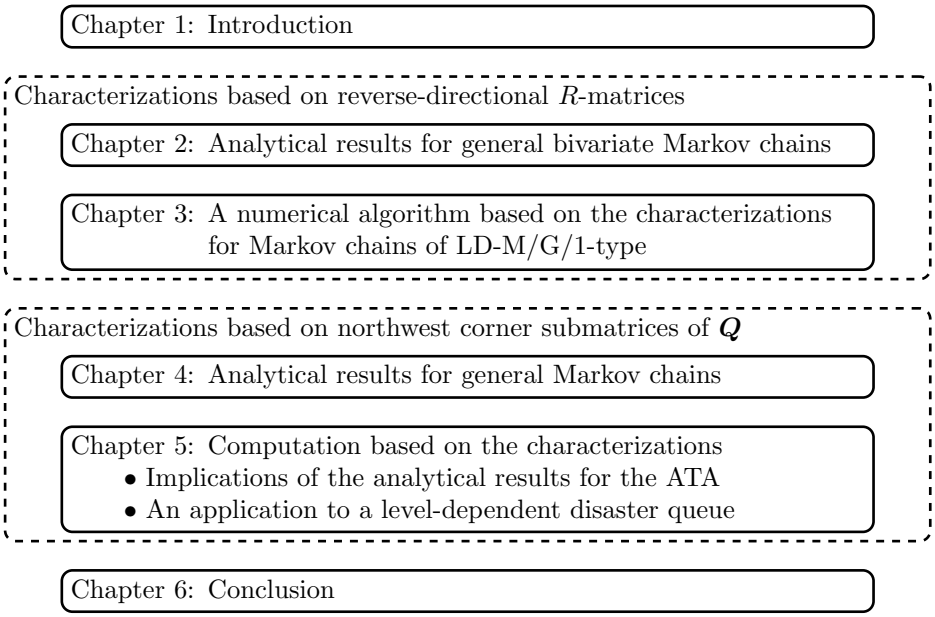


Figure 1.1: Overview of this dissertation.

1.5 Overview of the dissertation

This dissertation is organized as follows (cf. Figure 1.1).

In Chapter 2, we characterize the conditional stationary distribution in bivariate Markov chains via systems of linear inequalities. We first summarize the known results on reverse-directional R-matrices for the special class of Markov chains of LD-M/G/1-type [Taki16]. Next, we generalize the known results to general, bivariate Markov chains. Specifically, we characterize the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(N) := \boldsymbol{\pi}(L_N - 1)$ given $L(0) \in \mathbb{Z}_0^N$ via the infinitely many convex polytopes spanned by the normalized row vectors of reverse-directional R-matrices without any condition of the transition structure. The results imply that for any bivariate Markov chains, the computation of the conditional stationary distribution is reduced to the computation of reverse-directional R-matrices.

In Chapter 3, we consider the computation of the conditional stationary distribution in Markov chains of LD-M/G/1-type, based on the results in Chapter 2. Our algorithm for general Markov chains of LD-M/G/1-type is a slight modification of the algorithm proposed in [Taki16]. The contributions of Chapter 3 are that (i) our algorithm is applicable to any ergodic Markov chains of LD-M/G/1-type, i.e., we eliminate the assumption in [Taki16] of the nonsingularity of matrices $\boldsymbol{Q}_{\ell,\ell-1}$ ($\ell \in \mathbb{N}$) in (1.19) and that (ii) we can set an allowable error bound for the conditional stationary distribution as an input to the algorithm. We also provide some numerical examples.

In Chapter 4, we consider univariate Markov chains without any regular structures. We characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ via systems of linear inequalities, by examining the $(K+1) \times (K+1)$ $(K \geq N)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(K)$ of the infinitesimal generator and a subset of states in $\mathbb{Z}_0^K$, whose members are directly reachable from at least one state in $\mathbb{Z}_{K+1}^{\infty}$. Furthermore, for K such that

$K > N$, we introduce new transition structures called (K, N)-skip-free sets and using them, we obtain the minimum convex polytope that contains $\boldsymbol{\pi}(N)$.

In Chapter 5, we first provide some practical implications of the results in Chapter 4 for the ATA, and next consider the computation of the stationary distribution in a queueing model with disasters as an application of the implications. In queueing systems, a disaster is the event that all customers are removed from the system and it is a typical event violating the skip-free-to-the-left property. We consider a single-server queueing model with level-dependent disasters and generally distributed service times, to which we cannot apply the MAM nor the LD-MAM, and we provide the numerical implementation of its stationary distribution. Using the numerical example for the queueing model, we also discuss our linear-inequality characterizations numerically, and provide some tips for the numerical implementation of the ATA.

Finally, we conclude this dissertation in Chapter 6.

2 Characterizations of the Conditional Stationary Distribution based on Reverse-Directional R-Matrices

2.1 Introduction

This chapter considers a time-homogeneous, continuous-time bivariate Markov chain $\{(L(t), J(t))\}_{t\geq 0}$ stated in Section 1.1. We characterize the conditional stationary distribution via systems of linear inequalities obtained by reverse-directional R-matrices, by generalizing the existing results in [Taki16] for a special class of Markov chains of LD-M/G/1-type.

Recall that the level variable $L(t)$ takes a value in $\mathbb{Z}^+ = \{0, 1, \ldots\}$, and given $L(t) = \ell$ ($\ell \in \mathbb{Z}^+$), the phase variable $J(t)$ takes a value in a finite set $\mathcal{M}_\ell = \{0, 1, \ldots, M_\ell - 1\}$. It is assumed that the Markov chain is ergodic and stationary, and its infinitesimal generator $\boldsymbol{Q}$ and stationary distribution $\boldsymbol{\pi}$ are partitioned in conformance with levels, as in (1.5) and (1.6). Note that $\boldsymbol{Q}_{k,\ell}$ in (1.5) denotes an $M_k \times M_\ell$ matrix representing transitions from level k to level ℓ. We refer to a Markov chain $\{(L(t), J(t))\}_{t\geq 0}$ with an infinitesimal generator of the form in (1.5) as *level-dependent G/G/1-type* (LD-G/G/1-type). Roughly speaking, the LD-G/G/1 type implies that a bivariate Markov chain has no regular structures.

To characterize $\boldsymbol{\pi}_\ell$ ($\ell \in \mathbb{Z}^+$), we consider the recurrence time $T_k(i)$ ($k \in \mathbb{Z}^+$, $i \in \mathcal{M}_k$) of level k, starting from state (k, i).

$$T_k(i) = \min\{t > 0;\ L(t) = k,\ (L(0), J(0)) = (k, i), \\ (L(s), J(s)) \neq (k, i) \text{ for some } s \in [0, t]\}.$$

Let $\boldsymbol{N}_{k,\ell}$ ($k, \ell \in \mathbb{Z}^+$) denote an $M_k \times M_\ell$ matrix whose (i, j)th ($i \in \mathcal{M}_k$, $j \in \mathcal{M}_\ell$) element represents the ratio of the mean total sojourn time in state (ℓ, j) in the recurrence time $T_k(i)$ to the mean sojourn time in state (k, i). By definition, $\boldsymbol{N}_{k,\ell}$ ($k, \ell \in \mathbb{Z}^+$) is a nonnegative matrix, $\boldsymbol{N}_{k,k} = \boldsymbol{I}$ ($k \in \mathbb{Z}^+$), and (cf. Remark 2.1 in Section 2.3)

$$\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_k \boldsymbol{N}_{k,\ell}, \quad k, \ell \in \mathbb{Z}^+. \tag{2.1}$$

It is easy to verify that

$$\boldsymbol{\pi}_\ell = \begin{cases} \boldsymbol{\pi}_k \boldsymbol{R}_{k+1} \boldsymbol{R}_{k+2} \cdot \cdots \cdot \boldsymbol{R}_\ell, & \ell > k, \\ \boldsymbol{\pi}_k \boldsymbol{Z}_{k-1} \boldsymbol{Z}_{k-2} \cdot \cdots \cdot \boldsymbol{Z}_\ell, & \ell < k, \end{cases} \tag{2.2}$$

where

$$\boldsymbol{R}_\ell = \boldsymbol{N}_{\ell-1,\ell} \quad (\ell \in \mathbb{N}), \qquad \boldsymbol{Z}_\ell = \boldsymbol{N}_{\ell+1,\ell} \quad (\ell \in \mathbb{Z}^+).$$

Note that $\boldsymbol{R}_\ell$ is the level-dependent R-matrix [Bau10, Bri95, Li05, Phu10], which has been studied mainly for quasi birth-and-death processes. Conversely, $\boldsymbol{Z}_\ell$ represents the reverse-directional analogue of the R-matrix. In particular, we have

$$\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_{\ell+1}\boldsymbol{Z}_\ell, \quad \ell \in \mathbb{Z}^+. \tag{2.3}$$

In this chapter, we focus on the *multilevel* reverse-directional R-matrix $\boldsymbol{N}_{k,\ell}$ $(k > \ell)$. In what follows, we refer to $\boldsymbol{N}_{k,\ell}$ $(k > \ell)$ as the reverse-directional R-matrix for short.

We consider the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ given $L(0) \in \mathbb{Z}_0^N$.

$$\boldsymbol{\pi}_{\mathrm{lv}}(N) = (\boldsymbol{\pi}_0(N)\ \boldsymbol{\pi}_1(N)\ \cdots\ \boldsymbol{\pi}_N(N)),$$

where $\boldsymbol{\pi}_\ell(N)$ $(\ell \in \mathbb{Z}_0^N)$ denotes a $1 \times M_\ell$ vector whose jth $(j \in \mathcal{M}_\ell)$ element represents $\Pr((L(0), J(0)) = (\ell, j) \mid L(0) \in \mathbb{Z}_0^N)$.

$$\boldsymbol{\pi}_\ell(N) = \frac{\boldsymbol{\pi}_\ell}{\sum_{k=0}^{N} \boldsymbol{\pi}_k \boldsymbol{e}}, \quad \ell \in \mathbb{Z}_0^N. \tag{2.4}$$

Note that $\boldsymbol{\pi}_{\mathrm{lv}}(N)\boldsymbol{e} = 1$ by definition and $\boldsymbol{\pi}_\ell(N) = \boldsymbol{\pi}_N(N) \cdot \boldsymbol{N}_{N,\ell}$ $(\ell \in \mathbb{Z}_0^{N-1})$ because of (2.1) and (2.4). Therefore, $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ is given in terms of the (normalized) boundary vector $\overline{\boldsymbol{\pi}}_N = \boldsymbol{\pi}_N/\boldsymbol{\pi}_N\boldsymbol{e} = \boldsymbol{\pi}_N(N)/\boldsymbol{\pi}_N(N)\boldsymbol{e}$ and $\boldsymbol{N}_{k,\ell}$'s:

$$\boldsymbol{\pi}_{\mathrm{lv}}(N) = \frac{\overline{\boldsymbol{\pi}}_N(\boldsymbol{N}_{N,0}\ \ \boldsymbol{N}_{N,1}\ \ \cdots\ \ \boldsymbol{N}_{N,N})}{\overline{\boldsymbol{\pi}}_N(\boldsymbol{N}_{N,0}\ \ \boldsymbol{N}_{N,1}\ \ \cdots\ \ \boldsymbol{N}_{N,N})\boldsymbol{e}}. \tag{2.5}$$

In this chapter, we characterize the boundary vector $\overline{\boldsymbol{\pi}}_N$ based on the reverse-directional R-matrix $\boldsymbol{N}_{k,\ell}$ $(k > \ell)$. As stated in Section 1.4, the stationary distribution $\boldsymbol{\pi}$ in Markov chains of LD-M/G/1-type was characterized by systems of infinitely many linear inequalities in [Taki16] (cf. Proposition 1.1). Furthermore, that paper represented the conditional stationary distribution $\boldsymbol{\pi}(N)$ in a special class of Markov chains of LD-M/G/1-type, which satisfies that $\boldsymbol{Q}_{\ell,\ell-1}$'s is nonsingular, in terms of the reverse-directional R-matrices. We generalize the results in [Taki16] to Markov chains of LD-G/G/1-type.

The rest of this chapter is organized as follows. In Section 2.2, we summarize the known results on the reverse-directional R-matrices for special Markov chains of LD-M/G/1-type in [Taki16]. In Section 2.3, we characterize the boundary vector based on the reverse-directional R-matrices for Markov chains of LD-G/G/1-type. Finally, we conclude this chapter in Section 2.4.

2.2 Known results on reverse-directional R-matrices

In this section, we summarize known results on $\boldsymbol{N}_{k,\ell}$ $(k, \ell \in \mathbb{Z}^+)$, which is shown in [Taki16]. Let $\boldsymbol{\nu}_{k,\ell}(i)$ $(k, \ell \in \mathbb{Z}^+,\ i \in \mathcal{M}_k)$ denote the ith row vector of $\boldsymbol{N}_{k,\ell}$. Recall that $\overline{\boldsymbol{\nu}}_{k,\ell}(i) = \boldsymbol{\nu}_{k,\ell}(i)/(\boldsymbol{\nu}_{k,\ell}(i)\boldsymbol{e})$ $(i \in \mathcal{M}_k)$ denotes the normalized $\boldsymbol{\nu}_{k,\ell}(i)$ if $\boldsymbol{\nu}_{k,\ell}(i)\boldsymbol{e} \neq 0$, otherwise $\overline{\boldsymbol{\nu}}_{k,\ell}(i) = \boldsymbol{0}$. In [Taki16], the stationary distribution in Markov chains of LD-M/G/1-type is characterized via the system of linear inequalities as shown in Proposition 1.1, where the infinitesimal generator $\boldsymbol{Q} := \boldsymbol{Q}^{(\text{LD-M/G/1})}$ of LD-M/G/1-type is given in (1.19).

For a while, we assume that $\boldsymbol{Q}_{\ell,\ell-1}$ is nonsingular for all $\ell \in \mathbb{N}$. Let $\mathcal{M} := \mathcal{M}_0$ $(= \mathcal{M}_1 = \mathcal{M}_2 = \cdots)$. In [Taki16], an explicit form of $\boldsymbol{S}_\ell$ $(\ell \in \mathbb{Z}^+)$ in Proposition 1.1 is given by $\boldsymbol{S}_0 = \boldsymbol{I}$ and

$$\boldsymbol{S}_\ell = \big(\boldsymbol{Q}_{0,\ell-1} + \sum_{n=1}^{\ell-1} \boldsymbol{S}_n \boldsymbol{Q}_{n,\ell-1}\big)(-\boldsymbol{Q}_{\ell,\ell-1})^{-1}, \quad \ell \in \mathbb{N}. \tag{2.6}$$

Note that (2.6) requires the nonsingularity of $\boldsymbol{Q}_{\ell,\ell-1}$'s. Let $\hat{\mathcal{P}}_k$ $(k \in \mathbb{Z}^+)$ denote a convex polytope determined by $\boldsymbol{x}\boldsymbol{S}_k \geq \boldsymbol{0}$ and $\boldsymbol{x}\boldsymbol{e} = 1$:

$$\hat{\mathcal{P}}_k = \{\boldsymbol{x} \in \mathbb{R}^M;\ \boldsymbol{x}\boldsymbol{S}_k \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\}.$$

It is clear that $\overline{\boldsymbol{\pi}}_0 \in \hat{\mathcal{P}}_k$ $(k \in \mathbb{Z}^+)$ from Proposition 1.1. It is also known that $\boldsymbol{S}_k$ in (2.6) is nonsingular and $\boldsymbol{S}_k^{-1} = \boldsymbol{N}_{k,0}$ $(k \in \mathbb{Z}^+)$. Furthermore, the vertices $\{\overline{\boldsymbol{\nu}}_{k,0}(i);\ i \in \mathcal{M}\}$ of $\hat{\mathcal{P}}_k$ are identified:

$$\hat{\mathcal{P}}_k = \{\boldsymbol{x} \in \mathbb{R}^M;\ \boldsymbol{x} = \sum_{i\in\mathcal{M}} \alpha(i)\overline{\boldsymbol{\nu}}_{k,0}(i),\ \sum_{i\in\mathcal{M}} \alpha(i) = 1,\ \alpha(i) \geq 0\ (i \in \mathcal{M})\}.$$

Roughly speaking, $\overline{\boldsymbol{\pi}}_0$ is characterized via convex polytopes spanned by normalized row vectors of $\boldsymbol{N}_{k,0}$. Furthermore, by generalizing these results, [Taki16] showed the following results for a special class of Markov chains of LD-M/G/1-type.

Proposition 2.1 ([Taki16, Equations (21), (27), and (31) and Theorems 2 and 3])**.** *Consider an ergodic Markov chain of LD-M/G/1-type. For a nonnegative integer N, if every $\boldsymbol{Q}_{\ell,\ell-1}$ $(\ell \in \mathbb{Z}_{N+1}^{\infty})$ is an $M \times M$ nonsingular matrix, we have*

(a) For every integer k $(k > N)$, $\boldsymbol{\nu}_{k,N}(i)\boldsymbol{e} > 0$ for all $i \in \mathcal{M} = \{0, 1, \ldots, M-1\}$ and $\boldsymbol{\nu}_{k,N}(i)$'s $(i \in \mathcal{M})$ are linearly independent,

(b) $\lim_{k\to\infty} \overline{\boldsymbol{\nu}}_{k,N}(i) = \overline{\boldsymbol{\pi}}_N$ for all $i \in \mathcal{M}$,

(c) $(M-1)$-simplices $\mathcal{P}_{k,N}$ $(k > N)$ spanned by $\overline{\boldsymbol{\nu}}_{k,N}(i)$ $(i \in \mathcal{M})$, i.e.,

$$\mathcal{P}_{k,N} = \Big\{\boldsymbol{x} \in \mathbb{R}^M;\ \boldsymbol{x} = \sum_{i\in\mathcal{M}} \alpha(i)\overline{\boldsymbol{\nu}}_{k,N}(i),\ \sum_{i\in\mathcal{M}} \alpha(i) = 1,\ \alpha(i) \geq 0\ (i \in \mathcal{M})\Big\}, \tag{2.7}$$

satisfy $\mathcal{P}_{k+1,N} \subseteq \mathcal{P}_{k,N}$ $(k \in \mathbb{Z}_{N+1}^{\infty})$, and

(d) $\overline{\boldsymbol{\pi}}_N$ is given by

$$\{\overline{\boldsymbol{\pi}}_N\} = \bigcap_{k=N+1}^{\infty} \mathrm{ri}\,\mathcal{P}_{k,N} = \lim_{k\to\infty} \mathrm{ri}\,\mathcal{P}_{k,N}.$$

2.3 Characterizations of the boundary probability vector $\overline{\boldsymbol{\pi}}_N$

In this section, we discuss similar properties of $\boldsymbol{N}_{k,\ell}$ to Proposition 2.1 (b)–(d) in Markov chains of LD-G/G/1-type. Note that Proposition 2.1 (b) is rewritten as follows: For an arbitrary positive ϵ, there exists a natural number K $(K > N)$ such that

$$\|\overline{\boldsymbol{\nu}}_{k,N}(i) - \overline{\boldsymbol{\pi}}_N\|_1 \leq \epsilon, \quad i \in \mathcal{M},\ k \in \mathbb{Z}_K^{\infty}. \tag{2.8}$$

As we will see, even the skip-free-to-the-left property is not used to show the results corresponding to Proposition 2.1 (b) and (d), so that they hold for Markov chains of LD-G/G/1-type.

Note that Proposition 2.1 (a) does not hold in general because it is a consequence of the M/G/1-type structure with nonsingular $\boldsymbol{Q}_{\ell,\ell-1}$ ($\ell \in \mathbb{Z}^{\infty}_{N+1}$). Let $\mathcal{M}^{+}_{k,\ell}$ denote a subset of $\mathcal{M}_k$ such that

$$\mathcal{M}^{+}_{k,\ell} = \{i \in \mathcal{M}_k;\ \overline{\boldsymbol{\nu}}_{k,\ell}(i)\boldsymbol{e} > 0\}, \quad k, \ell \in \mathbb{Z}^{+},\ k \neq \ell.$$

If $\boldsymbol{\nu}_{k,\ell}(i) = \boldsymbol{0}$ ($k \neq \ell$) for all $i \in \mathcal{M}_k$, $\{(L(t), J(t))\}_{t\geq 0}$ could not reach from any state in level k to level ℓ, which contradicts the irreducibility assumption. Therefore, in any irreducible bivariate Markov chain $\{(L(t), J(t))\}_{t\geq 0}$, we have $\mathcal{M}^{+}_{k,\ell} \neq \emptyset$ for arbitrarily $k, \ell \in \mathbb{Z}^{+}$ ($k > \ell$), i.e., every $\boldsymbol{N}_{k,\ell}$ has at least one non-zero row vector.

We first show a convex polytope corresponding to $\mathcal{P}_{k,N}$ in (2.7).

Theorem 2.1. *Consider an ergodic Markov chain of LD-G/G/1-type. We have*

$$\overline{\boldsymbol{\pi}}_{\ell} \in \mathrm{ri}\, \mathcal{P}^{+}_{k,\ell}, \quad \ell \in \mathbb{Z}^{+},\ k \in \mathbb{Z}^{\infty}_{\ell+1}, \tag{2.9}$$

where $\mathrm{ri}\, \mathcal{P}^{+}_{k,\ell}$ *denotes a relative interior of a convex polytope* $\mathcal{P}^{+}_{k,\ell}$ *spanned by* $\overline{\boldsymbol{\nu}}_{k,\ell}(i)$ *(*$i \in \mathcal{M}^{+}_{k,\ell}$*).*

$$\mathcal{P}^{+}_{k,\ell} = \Big\{\boldsymbol{x} \in \mathbb{R}^{M_\ell};\ \boldsymbol{x} = \sum_{i\in\mathcal{M}^{+}_{k,\ell}} \alpha(i)\overline{\boldsymbol{\nu}}_{k,\ell}(i),\ \sum_{i\in\mathcal{M}^{+}_{k,\ell}} \alpha(i) = 1,\ \alpha(i) \geq 0\ (i \in \mathcal{M}^{+}_{k,\ell})\Big\}.$$

Proof. It follows from (1.8) and (2.1) that

$$\overline{\boldsymbol{\pi}}_{\ell} = \frac{\boldsymbol{\pi}_k \boldsymbol{N}_{k,\ell}}{\boldsymbol{\pi}_k \boldsymbol{N}_{k,\ell}\boldsymbol{e}} = \frac{\boldsymbol{\pi}_k\, \mathrm{diag}\,(\boldsymbol{N}_{k,\ell}\boldsymbol{e})\, \overline{\boldsymbol{N}}_{k,\ell}}{\boldsymbol{\pi}_k \boldsymbol{N}_{k,\ell}\boldsymbol{e}} = \boldsymbol{\beta}_{k,\ell}\, \overline{\boldsymbol{N}}_{k,\ell},$$

where $\boldsymbol{\beta}_{k,\ell} = (\beta_{k,\ell}(0)\ \beta_{k,\ell}(1)\ \cdots\ \beta_{k,\ell}(M_k - 1))$ denotes a $1 \times M_k$ vector given by

$$\boldsymbol{\beta}_{k,\ell} = \frac{\boldsymbol{\pi}_k \mathrm{diag}\,(\boldsymbol{N}_{k,\ell}\boldsymbol{e})}{\boldsymbol{\pi}_k \boldsymbol{N}_{k,\ell}\boldsymbol{e}}.$$

Since $\boldsymbol{\pi}_k > \boldsymbol{0}$, it is easy to verify that

$$\beta_{k,\ell}(i) > 0 \quad (i \in \mathcal{M}^{+}_{k,\ell}), \qquad \beta_{k,\ell}(i) = 0 \quad (i \in \mathcal{M}_k \setminus \mathcal{M}^{+}_{k,\ell}). \tag{2.10}$$

We thus have

$$\overline{\boldsymbol{\pi}}_{\ell} = \sum_{i\in\mathcal{M}_k} \beta_{k,\ell}(i)\overline{\boldsymbol{\nu}}_{k,\ell}(i) = \sum_{i\in\mathcal{M}^{+}_{k,\ell}} \beta_{k,\ell}(i)\overline{\boldsymbol{\nu}}_{k,\ell}(i). \tag{2.11}$$

Furthermore, $\overline{\boldsymbol{\pi}}_{\ell}\boldsymbol{e} = 1$ and $\overline{\boldsymbol{\nu}}_{k,\ell}(i)\boldsymbol{e} = 1$ ($i \in \mathcal{M}^{+}_{k,\ell}$) imply

$$\sum_{i\in\mathcal{M}^{+}_{k,\ell}} \beta_{k,\ell}(i) = 1. \tag{2.12}$$

(2.9) follows from (1.12), (2.10), (2.11), and (2.12), which completes the proof. □

We next discuss the property corresponding to Proposition 2.1 (b), i.e., the convergence of $\overline{\boldsymbol{\nu}}_{k,\ell}(i)$'s to $\overline{\boldsymbol{\pi}}_\ell$. We first show a limiting formula under the condition that every $\overline{\boldsymbol{\nu}}_{k,\ell}(0)$ ($k \in \mathbb{Z}_{\ell+1}^{\infty}$) is not equal to $\mathbf{0}$.

Lemma 2.1. *Consider an ergodic Markov chain of LD-G/G/1-type. For an arbitrarily fixed ℓ ($\ell \in \mathbb{Z}^+$), if $\boldsymbol{\nu}_{k,\ell}(0)\boldsymbol{e} > 0$ for all k ($k \in \mathbb{Z}_{\ell+1}^{\infty}$), we have*

$$\lim_{k\to\infty} \overline{\boldsymbol{\nu}}_{k,\ell}(0) = \overline{\boldsymbol{\pi}}_\ell.$$

Proof. Associated with an ergodic Markov chain $\{(L(t), J(t))\}_{t\geq 0}$ of LD-G/G/1-type, we consider an imbedded Markov chain $\{(L_n, J_n)\}_{n=0,1,\ldots}$ immediately after state transitions. Let $\boldsymbol{\delta}_\ell$ ($\ell \in \mathbb{Z}^+$) denote a $1 \times M_\ell$ vector whose jth ($j \in \mathcal{M}_\ell$) element $[\boldsymbol{\delta}_\ell]_j$ represents the rate $-[\boldsymbol{Q}_{\ell,\ell}]_{j,j}$ of leaving from state (ℓ, j). The transition probability matrix $\boldsymbol{P}$ of the imbedded Markov chain $\{(L_n, J_n)\}_{n=0,1,\ldots}$ is then given by

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_{0,0} & \boldsymbol{P}_{0,1} & \boldsymbol{P}_{0,2} & \boldsymbol{P}_{0,3} & \boldsymbol{P}_{0,4} & \cdots \\ \boldsymbol{P}_{1,0} & \boldsymbol{P}_{1,1} & \boldsymbol{P}_{1,2} & \boldsymbol{P}_{1,3} & \boldsymbol{P}_{1,4} & \cdots \\ \boldsymbol{P}_{2,0} & \boldsymbol{P}_{2,1} & \boldsymbol{P}_{2,2} & \boldsymbol{P}_{2,3} & \boldsymbol{P}_{2,4} & \cdots \\ \boldsymbol{P}_{3,0} & \boldsymbol{P}_{3,1} & \boldsymbol{P}_{3,2} & \boldsymbol{P}_{3,3} & \boldsymbol{P}_{3,4} & \cdots \\ \boldsymbol{P}_{4,0} & \boldsymbol{P}_{4,1} & \boldsymbol{P}_{4,2} & \boldsymbol{P}_{4,3} & \boldsymbol{P}_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.13}$$

where $\boldsymbol{P}_{k,\ell}$ ($k, \ell \in \mathbb{Z}^+$) is defined as

$$\boldsymbol{P}_{k,\ell} = \begin{cases} \boldsymbol{I} + \operatorname{diag}^{-1}(\boldsymbol{\delta}_k)\boldsymbol{Q}_{k,k}, & \ell = k, \\ \operatorname{diag}^{-1}(\boldsymbol{\delta}_k)\boldsymbol{Q}_{k,\ell}, & \ell \in \mathbb{Z}^+ \setminus \{k\}. \end{cases}$$

By definition, the imbedded Markov chain $\{(L_n, J_n)\}_{n=0,1,\ldots}$ is irreducible and positive-recurrent. We then define $\boldsymbol{\rho} = (\boldsymbol{\rho}_0\ \boldsymbol{\rho}_1\ \cdots)$ as the stationary probability vector of the imbedded Markov chain $\{(L_n, J_n)\}_{n=0,1,\ldots}$, where $\boldsymbol{\rho}_\ell$ ($\ell \in \mathbb{Z}^+$) denotes a $1 \times M_\ell$ vector whose jth ($j \in \mathcal{M}_\ell$) element represents the stationary probability of state (ℓ, j). Note that $\boldsymbol{\rho}$ is determined uniquely by

$$\boldsymbol{\rho} = \boldsymbol{\rho}\boldsymbol{P}, \qquad \boldsymbol{\rho}\boldsymbol{e} = 1. \tag{2.14}$$

It follows from (1.4) and (2.14) that $\boldsymbol{\pi}_\ell$ ($\ell \in \mathbb{Z}^+$) is given in terms of $\boldsymbol{\rho}_k$'s.

$$\boldsymbol{\pi}_\ell = \frac{\boldsymbol{\rho}_\ell \operatorname{diag}^{-1}(\boldsymbol{\delta}_\ell)}{\displaystyle\sum_{n=0}^{\infty} \boldsymbol{\rho}_n \operatorname{diag}^{-1}(\boldsymbol{\delta}_n)\boldsymbol{e}}, \quad \ell \in \mathbb{Z}^+. \tag{2.15}$$

Let ${}_k\boldsymbol{\tau}_{k,\ell}(i)$ ($k, \ell \in \mathbb{Z}^+$, $k \neq \ell$, $i \in \mathcal{M}_k$) denote a $1 \times M_\ell$ vector whose jth ($j \in \mathcal{M}_\ell$) element represents the probability that the first passage time from state (k, i) to level ℓ ends at state (ℓ, j) without visiting level k. Note that ${}_k\boldsymbol{\tau}_{k,\ell}(i)\boldsymbol{e} > 0 \ \Leftrightarrow\ \boldsymbol{\nu}_{k,\ell}(i)\boldsymbol{e} > 0$ ($i \in \mathcal{M}_k$). We thus have ${}_k\boldsymbol{\tau}_{k,\ell}(0)\boldsymbol{e} > 0$ ($k > \ell$) because of the assumption $\boldsymbol{\nu}_{k,\ell}(0)\boldsymbol{e} > 0$. Let ${}_k\boldsymbol{F}_{\ell,\ell}$ ($k, \ell \in \mathbb{Z}^+$, $k \neq \ell$) denote an $M_\ell \times M_\ell$ matrix whose (i, j)th ($i, j \in \mathcal{M}_\ell$) element represents the conditional probability that the recurrence time to level ℓ ends at state

(ℓ, j) without visiting level k, given that it starts from state (ℓ, i). Note that the jth ($j \in \mathcal{M}_\ell$) element of

$$ {}_k\boldsymbol{\tau}_{k,\ell}(i) \sum_{n=0}^{\infty} {}_k\boldsymbol{F}_{\ell,\ell}^n = {}_k\boldsymbol{\tau}_{k,\ell}(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1} \tag{2.16} $$

represents the mean number of visits to state (ℓ, j) in the recurrence time to level k, given that the recurrence time starts from state (k, i). It then follows that

$$ \boldsymbol{\nu}_{k,\ell}(i) = [\boldsymbol{\delta}_k]_i \cdot {}_k\boldsymbol{\tau}_{k,\ell}(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1}\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell). \tag{2.17} $$

We thus obtain

$$ \overline{\boldsymbol{\nu}}_{k,\ell}(i) = \frac{{}_k\boldsymbol{\tau}_{k,\ell}(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1}\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{{}_k\boldsymbol{\tau}_{k,\ell}(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1}\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}}, \quad \text{if} \quad \boldsymbol{\nu}_{k,\ell}(i)\boldsymbol{e} > 0. \tag{2.18} $$

To analyze $\overline{\boldsymbol{\nu}}_{k,\ell}(i)$ ($k \in \mathbb{Z}_{\ell+1}^{\infty}$, $i \in \mathcal{M}_\ell$), we introduce a discrete-time renewal process $\{X_n(k, \ell, i)\}_{n=0,1,\ldots}$, whose inter-renewal times follow a discrete-time phase-type distribution with representation $(\boldsymbol{e}_i^T, {}_k\boldsymbol{F}_{\ell,\ell})$. Note that immediately after renewals, the underlying phase process for $\{X_n(k, \ell, i)\}_{n=0,1,\ldots}$ is in phase i with probability 1. Associated with $\{X_n(k, \ell, i)\}_{n=0,1,\ldots}$, we define ${}_k\hat{\boldsymbol{F}}_\ell(i)$ ($k \in \mathbb{Z}_{\ell+1}^{\infty}$) as an $M_\ell \times M_\ell$ matrix given by

$$ {}_k\hat{\boldsymbol{F}}_\ell(i) = {}_k\boldsymbol{F}_{\ell,\ell} + (\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})\boldsymbol{e}\boldsymbol{e}_i^T. \tag{2.19} $$

It is easy to verify that ${}_k\hat{\boldsymbol{F}}_\ell(i)$ is a stochastic matrix, more specifically, the transition probability matrix of the underlying phase process for $\{X_n(k, \ell, i)\}_{n=0,1,\ldots}$, which has a single irreducible class composed of phases in $\mathcal{M}_\ell$ reachable from phase i with respect to ${}_k\boldsymbol{F}_{\ell,\ell}$. We then define ${}_k\hat{\boldsymbol{\rho}}_\ell(i)$ ($k \in \mathbb{Z}_{\ell+1}^{\infty}$, $i \in \mathcal{M}_\ell$) as a $1 \times M_\ell$ invariant probability vector of ${}_k\hat{\boldsymbol{F}}_\ell(i)$, which is determined uniquely by

$$ {}_k\hat{\boldsymbol{\rho}}_\ell(i) = {}_k\hat{\boldsymbol{\rho}}_\ell(i){}_k\hat{\boldsymbol{F}}_\ell(i), \qquad {}_k\hat{\boldsymbol{\rho}}_\ell(i)\boldsymbol{e} = 1. \tag{2.20} $$

We now consider the factor $(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1}$ on the right-hand side of (2.18). Substituting (2.19) into (2.20) and rearranging terms yield

$$ {}_k\hat{\boldsymbol{\rho}}_\ell(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell}) = {}_k\hat{\boldsymbol{\rho}}_\ell(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})\boldsymbol{e} \cdot \boldsymbol{e}_i^T, \quad i \in \mathcal{M}_\ell. \tag{2.21} $$

Let ${}_k\hat{\boldsymbol{P}}_\ell$ ($k \in \mathbb{Z}_{\ell+1}^{\infty}$) denote an $M_\ell \times M_\ell$ matrix whose ith ($i \in \mathcal{M}_\ell$) row vector is given by ${}_k\hat{\boldsymbol{\rho}}_\ell(i)$. Note that (2.21) is equivalent to

$$ {}_k\hat{\boldsymbol{P}}_\ell(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell}) = \mathrm{diag}({}_k\boldsymbol{\gamma}_\ell) \cdot \boldsymbol{I}, \quad k > \ell, $$

where ${}_k\boldsymbol{\gamma}_\ell$ ($k \in \mathbb{Z}_{\ell+1}^{\infty}$) denotes an $M_\ell \times 1$ vector whose ith ($i \in \mathcal{M}_\ell$) element $[{}_k\boldsymbol{\gamma}_\ell]_i$ is given by

$$ [{}_k\boldsymbol{\gamma}_\ell]_i = {}_k\hat{\boldsymbol{\rho}}_\ell(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})\boldsymbol{e}, \quad i \in \mathcal{M}_\ell. $$

By definition, $[{}_k\boldsymbol{\gamma}_\ell]_i$ ($i \in \mathcal{M}_\ell$) represents the probability that a renewal happens at a randomly chosen instant in the equilibrium version of $\{X_n(k, \ell, i)\}_{n=0,1,\ldots}$. Therefore, $[{}_k\boldsymbol{\gamma}_\ell]_i > 0$ for all $i \in \mathcal{M}_\ell$ owing to the positive recurrence of the imbedded Markov chain $\{(L_n, J_n)\}_{n=0,1,\ldots}$. We thus obtain

$$ (\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1} = \mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell) \cdot {}_k\hat{\boldsymbol{P}}_\ell, \quad k > \ell. \tag{2.22} $$

With (2.22), (2.18) is rewritten to be

$$\overline{\boldsymbol{\nu}}_{k,\ell}(i) = \frac{{}_k\boldsymbol{\tau}_{k,\ell}(i)\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)\cdot {}_k\hat{\boldsymbol{P}}_\ell\cdot \mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{{}_k\boldsymbol{\tau}_{k,\ell}(i)\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)\cdot {}_k\hat{\boldsymbol{P}}_\ell\cdot \mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}}, \quad \text{if } \boldsymbol{\nu}_{k,\ell}(i)\boldsymbol{e} > 0. \tag{2.23}$$

To proceed further, we need the following limit formula whose proof is given in Appendix A.

$$\lim_{k\to\infty} {}_k\hat{\boldsymbol{\rho}}_\ell(i) = \overline{\boldsymbol{\rho}}_\ell, \quad \ell \in \mathbb{Z}^+,\ i \in \mathcal{M}_\ell. \tag{2.24}$$

It follows from (2.24) and $\overline{\boldsymbol{\rho}}_\ell > \boldsymbol{0}$ ($\ell \in \mathbb{Z}^+$) that for a fixed $\ell \in \mathbb{Z}^+$ and for any $\epsilon > 0$, there exists a natural number $k^* := k^*(\ell, \epsilon)$ such that

$$\left(1 - \frac{\epsilon}{2+\epsilon}\right)\boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell < {}_k\hat{\boldsymbol{P}}_\ell < \left(1 + \frac{\epsilon}{2+\epsilon}\right)\boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell, \quad k \ge k^*. \tag{2.25}$$

Note also that ${}_k\boldsymbol{\tau}_{k,\ell}(i)$, $\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)$, and $\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)$ in (2.23) are nonnegative. It then follows from $\boldsymbol{\nu}_{k,\ell}(0)\boldsymbol{e} > 0$, (2.23), and (2.25) that for $k \ge k^*$,

$$\begin{aligned}
&\frac{{}_k\boldsymbol{\tau}_{k,\ell}(0)\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)\cdot\left(1 - \frac{\epsilon}{2+\epsilon}\right)\boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell\cdot\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{{}_k\boldsymbol{\tau}_{k,\ell}(0)\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)\cdot\left(1 + \frac{\epsilon}{2+\epsilon}\right)\boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell\cdot\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}} \\
&\qquad < \overline{\boldsymbol{\nu}}_{k,\ell}(0) < \frac{{}_k\boldsymbol{\tau}_{k,\ell}(0)\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)\cdot\left(1 + \frac{\epsilon}{2+\epsilon}\right)\boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell\cdot\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{{}_k\boldsymbol{\tau}_{k,\ell}(0)\mathrm{diag}^{-1}({}_k\boldsymbol{\gamma}_\ell)\cdot\left(1 - \frac{\epsilon}{2+\epsilon}\right)\boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell\cdot\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}},
\end{aligned}$$

which is reduced to

$$\frac{1}{1+\epsilon}\cdot\frac{\overline{\boldsymbol{\rho}}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{\overline{\boldsymbol{\rho}}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}} < \overline{\boldsymbol{\nu}}_{k,\ell}(0) < (1+\epsilon)\cdot\frac{\overline{\boldsymbol{\rho}}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{\overline{\boldsymbol{\rho}}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}}.$$

Note here that $(1+\epsilon)^{-1} > 1 - \epsilon$ for $\epsilon > 0$ and that from (2.15),

$$\frac{\overline{\boldsymbol{\rho}}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{\overline{\boldsymbol{\rho}}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}} = \frac{\boldsymbol{\rho}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)}{\boldsymbol{\rho}_\ell\mathrm{diag}^{-1}(\boldsymbol{\delta}_\ell)\boldsymbol{e}} = \overline{\boldsymbol{\pi}}_\ell.$$

We thus obtain for an arbitrary $\epsilon > 0$,

$$(1-\epsilon)\overline{\boldsymbol{\pi}}_\ell < \overline{\boldsymbol{\nu}}_{k,\ell}(0) < (1+\epsilon)\overline{\boldsymbol{\pi}}_\ell, \quad k \ge k^*,$$

which completes the proof. □

Remark 2.1. *For k and ℓ ($k, \ell \in \mathbb{Z}^+,\ k \ne \ell$), (2.1) can be proven as follows. Let $\boldsymbol{V}_{k,\ell}$ ($k, \ell \in \mathbb{Z}^+,\ k \ne \ell$) denote an $M_k \times M_\ell$ matrix whose ith ($i \in \mathcal{M}_k$) row vector is given by ${}_k\boldsymbol{\tau}_{k,\ell}(i)(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})^{-1}$ (see (2.16)). It then follows from [Lat99, Theorem 5.2.1] that $\boldsymbol{\rho}_\ell = \boldsymbol{\rho}_k\boldsymbol{V}_{k,\ell}$ ($k, \ell \in \mathbb{Z}^+,\ k \ne \ell$), which is equivalent to (2.1) because of (2.15) and (2.17).*

The following theorem corresponds to Proposition 2.1 (b), which does not require the condition $\boldsymbol{\nu}_{k,\ell}(0)\boldsymbol{e} > 0$ ($k \in \mathbb{Z}_{\ell+1}^\infty$).

Theorem 2.2. *Consider an ergodic Markov chain of LD-G/G/1-type. For an arbitrarily fixed ℓ ($\ell \in \mathbb{Z}^+$) and ϵ ($\epsilon > 0$), there exists a natural number $K := K(\ell, \epsilon)$ such that for all k ($k \in \mathbb{Z}_K^\infty$),*

$$\|\overline{\boldsymbol{\nu}}_{k,\ell}(i) - \overline{\boldsymbol{\pi}}_\ell\|_1 < \epsilon, \quad i \in \mathcal{M}_{k,\ell}^+. \tag{2.26}$$

Proof. Recall that $\mathcal{M}_{k,\ell}^+ \neq \emptyset$. We then define $i_{k,\ell}^*$ ($k \in \mathbb{Z}_{\ell+1}^\infty$) as

$$i_{k,\ell}^* = \arg\max_{i \in \mathcal{M}_{k,\ell}^+} \|\overline{\boldsymbol{\nu}}_{k,\ell}(i) - \overline{\boldsymbol{\pi}}_\ell\|_1.$$

By definition, we have for all k ($k \in \mathbb{Z}_{\ell+1}^\infty$),

$$\|\overline{\boldsymbol{\nu}}_{k,\ell}(i) - \overline{\boldsymbol{\pi}}_\ell\|_1 \leq \|\overline{\boldsymbol{\nu}}_{k,\ell}(i_{k,\ell}^*) - \overline{\boldsymbol{\pi}}_\ell\|_1, \quad i \in \mathcal{M}_{k,\ell}^+. \tag{2.27}$$

Note here that the labeling of phases in each level can be changed arbitrarily while preserving the G/G/1-type (or M/G/1-type) structure. Therefore, re-labeling every $i_{k,\ell}^*$ ($k \in \mathbb{Z}_{\ell+1}^\infty$) as zero and applying Lemma 2.1, we see that for an arbitrary $\epsilon > 0$, there exists a natural number K such that

$$\|\overline{\boldsymbol{\nu}}_{k,\ell}(i_{k,\ell}^*) - \overline{\boldsymbol{\pi}}_\ell\|_1 < \epsilon, \quad k \geq K. \tag{2.28}$$

(2.26) now follows from (2.27) and (2.28). □

We then obtain the limit formula corresponding to Proposition 2.1 (d).

Theorem 2.3. *Consider an ergodic Markov chain of LD-G/G/1-type. For an arbitrary fixed ℓ ($\ell \in \mathbb{Z}^+$), $\overline{\boldsymbol{\pi}}_\ell$ is given by*

$$\{\overline{\boldsymbol{\pi}}_\ell\} = \bigcap_{k=\ell+1}^{\infty} \mathrm{ri}\, \mathcal{P}_{k,\ell}^+ = \lim_{k\to\infty} \mathrm{ri}\, \mathcal{P}_{k,\ell}^+. \tag{2.29}$$

Proof. It follows from Theorem 2.1 that

$$\overline{\boldsymbol{\pi}}_\ell \in \bigcap_{k=\ell+1}^{\infty} \mathrm{ri}\, \mathcal{P}_{k,\ell}^+. \tag{2.30}$$

On the other hand, we have from Theorem 2.2 and the definition of $\mathrm{ri}\, \mathcal{P}_{k,\ell}^+$,

$$\lim_{k\to\infty} \mathrm{ri}\, \mathcal{P}_{k,\ell}^+ = \{\overline{\boldsymbol{\pi}}_\ell\}. \tag{2.31}$$

(2.29) now follows from (2.30) and (2.31). □

Finally, we discuss the monotonic inclusion property of a sequence $\{\mathcal{P}_{k,\ell}^+;\ k \in \mathbb{Z}_{\ell+1}^\infty\}$, which corresponds to Proposition 2.1 (c).

Theorem 2.4. *Consider an ergodic Markov chain of LD-M/G/1-type. For an arbitrarily fixed ℓ ($\ell \in \mathbb{Z}^+$), we have*

$$\mathcal{P}_{k+1,\ell}^+ \subseteq \mathcal{P}_{k,\ell}^+, \quad k \in \mathbb{Z}_{\ell+1}^\infty. \tag{2.32}$$

Remark 2.2. *In general,* (2.32) *does not necessarily hold for Markov chains of LD-G/G/1-type. Note that the G/G/1-type structure is preserved in any labeling of levels, as well as phases in each level. Therefore, if* (2.32) *holds for any Markov chain of LD-G/G/1-type, we also have* $\mathcal{P}^+_{k+1,\ell} \supseteq \mathcal{P}^+_{k,\ell}$*, by considering the corresponding Markov chain produced by the label exchange of level* k *and level* $k+1$*. As a result,* $\mathcal{P}^+_{k,\ell} = \mathcal{P}^+_{k+1,\ell}$*, which does not necessarily hold clearly. Conversely, if* $\mathcal{P}^+_{k,\ell} = \mathcal{P}^+_{k+1,\ell}$ *holds for all* $k > \ell$ *in a specific Markov chain of LD-G/G/1-type, Theorem 2.3 implies* $\mathcal{P}^+_{k,\ell} = \{\overline{\boldsymbol{\pi}}_\ell\}$ *(*$k > \ell$*).*

Proof. Suppose $\boldsymbol{x} \in \mathcal{P}^+_{k+1,\ell}$ for some k ($k \in \mathbb{Z}^\infty_{\ell+1}$), i.e.,

$$\boldsymbol{x} = \sum_{i \in \mathcal{M}^+_{k+1,\ell}} \alpha_{k+1}(i) \overline{\boldsymbol{\nu}}_{k+1,\ell}(i) = \boldsymbol{\alpha}_{k+1} \overline{\boldsymbol{N}}_{k+1,\ell}, \tag{2.33}$$

where $\boldsymbol{\alpha}_{k+1} = (\alpha_{k+1}(0)\ \alpha_{k+1}(1)\ \cdots\ \alpha_{k+1}(M_{k+1}-1))$ is a nonnegative $1 \times M_{k+1}$ vector satisfying

$$\sum_{i \in \mathcal{M}^+_{k+1,\ell}} \alpha_{k+1}(i) = 1, \qquad \alpha_{k+1}(i) = 0 \quad (i \in \mathcal{M}_{k+1} \setminus \mathcal{M}^+_{k+1,\ell}).$$

In what follows, we will show $\boldsymbol{x} \in \mathcal{P}^+_{k,\ell}$, which implies (2.32).

For an arbitrary vector $\boldsymbol{y}$, let $\mathrm{diag}^*(\boldsymbol{y})$ denote a diagonal matrix whose ith diagonal element $[\mathrm{diag}^*(\boldsymbol{y})]_{i,i}$ is given by

$$[\mathrm{diag}^*(\boldsymbol{y})]_{i,i} = \begin{cases} \dfrac{1}{[\boldsymbol{y}]_i}, & [\boldsymbol{y}]_i \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, $\overline{\boldsymbol{N}}_{k+1,\ell} = \mathrm{diag}^*(\boldsymbol{N}_{k+1,\ell}\boldsymbol{e})\boldsymbol{N}_{k+1,\ell}$. It then follows from (2.33) and $\boldsymbol{N}_{k+1,\ell} = \boldsymbol{Z}_k \boldsymbol{N}_{k,\ell}$ ($k \in \mathbb{Z}^\infty_{\ell+1}$) [Taki16, Lemma 1] that

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{\alpha}_{k+1} \mathrm{diag}^*(\boldsymbol{N}_{k+1,\ell}\boldsymbol{e}) \boldsymbol{N}_{k+1,\ell} \\ &= \boldsymbol{\alpha}_{k+1} \mathrm{diag}^*(\boldsymbol{N}_{k+1,\ell}\boldsymbol{e}) \boldsymbol{Z}_k \mathrm{diag}(\boldsymbol{N}_{k,\ell}\boldsymbol{e}) \overline{\boldsymbol{N}}_{k,\ell} \\ &= \boldsymbol{\alpha}_k \overline{\boldsymbol{N}}_{k,\ell}, \end{aligned} \tag{2.34}$$

where $\boldsymbol{\alpha}_k = (\alpha_k(0)\ \alpha_k(1)\ \cdots\ \alpha_k(M_k-1))$ is given by

$$\boldsymbol{\alpha}_k = \boldsymbol{\alpha}_{k+1} \mathrm{diag}^*(\boldsymbol{N}_{k+1,\ell}\boldsymbol{e}) \boldsymbol{Z}_k \mathrm{diag}(\boldsymbol{N}_{k,\ell}\boldsymbol{e}).$$

It is easy to verify that $\boldsymbol{\alpha}_k \geq \boldsymbol{0}$. Furthermore,

$$\begin{aligned} \boldsymbol{\alpha}_k \boldsymbol{e} = \boldsymbol{\alpha}_{k+1} \mathrm{diag}^*(\boldsymbol{N}_{k+1,\ell}\boldsymbol{e}) \boldsymbol{Z}_k \boldsymbol{N}_{k,\ell} \boldsymbol{e} &= \boldsymbol{\alpha}_{k+1} \mathrm{diag}^*(\boldsymbol{N}_{k+1,\ell}\boldsymbol{e}) \boldsymbol{N}_{k+1,\ell} \boldsymbol{e} \\ &= \sum_{i \in \mathcal{M}^+_{k+1,\ell}} \alpha_{k+1}(i) = 1. \end{aligned} \tag{2.35}$$

The remaining is to show $\alpha_k(i) = 0$ ($i \in \mathcal{M}_k \setminus \mathcal{M}^+_{k,\ell}$). By definition, $\boldsymbol{x}\boldsymbol{e} = 1$. We thus have from (2.34) and $\overline{\boldsymbol{\nu}}_{k,\ell}(i)\boldsymbol{e} = 0$ for $i \in \mathcal{M}_k \setminus \mathcal{M}^+_{k,\ell}$,

$$\begin{aligned} 1 = \boldsymbol{x}\boldsymbol{e} = \boldsymbol{\alpha}_k \overline{\boldsymbol{N}}_{k,\ell} \boldsymbol{e} &= \sum_{i \in \mathcal{M}^+_{k,\ell}} \alpha_k(i) \overline{\boldsymbol{\nu}}_{k,\ell}(i) \boldsymbol{e} + \sum_{i \in \mathcal{M}_k \setminus \mathcal{M}^+_{k,\ell}} \alpha_k(i) \overline{\boldsymbol{\nu}}_{k,\ell}(i) \boldsymbol{e} \\ &= \sum_{i \in \mathcal{M}^+_{k,\ell}} \alpha_k(i), \end{aligned}$$

which implies $\alpha_k(i) = 0$ ($i \in \mathcal{M}_k \setminus \mathcal{M}_{k,\ell}^+$) because $\boldsymbol{\alpha}_k \geq \mathbf{0}$ and $\boldsymbol{\alpha}_k \boldsymbol{e} = 1$. We thus conclude $\boldsymbol{x} \in \mathcal{P}_{k,\ell}^+$. □

2.4 Conclusion

In this chapter, we considered the conditional stationary distribution $\overline{\boldsymbol{\pi}}_\ell$ in a Markov chain of LD-G/G/1-type given that the Markov chain is in a fixed level ℓ. We first obtained the convex polytope $\mathcal{P}_{k,\ell}^+$ containing $\overline{\boldsymbol{\pi}}_\ell$ in terms of the reverse-directional R-matrix $\boldsymbol{N}_{k,\ell}$. We then showed that $\mathcal{P}_{k,\ell}^+$ converges to the singleton $\{\overline{\boldsymbol{\pi}}_\ell\}$ as k goes to infinity. In addition, we showed the inclusion property $\mathcal{P}_{k+1,\ell}^+ \subseteq \mathcal{P}_{k,\ell}^+$ for Markov chains of LD-M/G/1-type. These results are obvious generalizations of the results in [Taki16] (cf. Proposition 2.1).

In this chapter, the state space of the marginal process $\{L(t)\}_{t\geq 0}$ was assumed to be countably infinite. It is easy to verify that Theorems 2.1 and 2.4 hold for Markov chains on a finite state space $\bigcup_{k=0}^{K^*}\{k\} \times \mathcal{M}_k$ for some $K^* \in \mathbb{Z}^+$ such that $\ell < k < K^*$, if we replace $\mathbb{Z}_{\ell+1}^{\infty}$ to $\mathbb{Z}_{\ell+1}^{K^*}$.

If $\overline{\boldsymbol{\pi}}_\ell$ is given for some $\ell = N$, we also have $\overline{\boldsymbol{\pi}}_\ell$ ($\ell = 0, 1, \ldots, N-1$) up to a multiplicative constant by (2.1). Therefore, we can obtain the conditional distribution $\boldsymbol{\pi}(N) = (\pi_0(N)\ \pi_1(N)\ \cdots\ \pi_N(N))$ by normalizing those. These observations naturally lead to a numerical procedure for $\boldsymbol{\pi}(N)$, which will be discussed in the next chapter.

3 Computation of the Conditional Stationary Distribution in Markov Chains of LD-M/G/1-Type

3.1 Introduction

We consider computation of the conditional stationary distribution based on the results in Chapter 2, which showed that the conditional stationary distribution is given in terms of the reverse-directional R-matrices. In general, it is hard to compute the reverse-directional R-matrices because a reverse-directional R-matrix depends on the transition structure among all levels. Note, however, that we can readily compute the reverse-directional R-matrices in Markov chains of LD-M/G/1-type [Taki16].

This chapter considers a time-homogeneous, continuous-time bivariate Markov chain $\{(L(t), J(t))\}_{t\geq 0}$ of LD-M/G/1-type. Recall that the level variable $L(t)$ takes a value in $\mathbb{Z}^+ = \{0, 1, \ldots\}$, and given $L(t) = \ell$ ($\ell \in \mathbb{Z}^+$), the phase variable $J(t)$ takes a value in a finite set $\mathcal{M}_\ell = \{0, 1, \ldots, M_\ell - 1\}$. It is assumed that the Markov chain is ergodic and stationary, and its infinitesimal generator $\boldsymbol{Q} := \boldsymbol{Q}^{(\text{LD-M/G/1})}$ and the stationary distribution $\boldsymbol{\pi}$ are partitioned in conformance with levels, as in (1.6) and (1.19). Note that $\boldsymbol{Q}_{k,\ell}$ in (1.19) denotes an $M_k \times M_\ell$ matrix representing transitions from level k to level ℓ.

In [Taki16], a computational algorithm of the conditional stationary distribution $\boldsymbol{\pi}_{\text{lv}}(N)$ given that the level is not greater than a predefined threshold N has been proposed for special class of Markov chains of LD-M/G/1-type, in which $\boldsymbol{Q}_{\ell,\ell-1}$ ($\ell \in \mathbb{N}$) is assumed to be nonsingular. Takine's algorithm is based on the reverse-directional R-matrices $\boldsymbol{N}_{k,\ell}$ ($k > \ell$) and evaluates the error bound of the computed boundary probability vector $\overline{\boldsymbol{\pi}}_N$ (cf. (2.5)). We develop a computational algorithm of $\boldsymbol{\pi}_{\text{lv}}(N)$ with a predefined threshold N for general Markov chains of LD-M/G/1-type, by modifying Takine's algorithm slightly (i.e., the selection of an approximation to the boundary vector and the error evaluation). In principle, our algorithm is applicable to any ergodic Markov chain of LD-M/G/1-type. Furthermore, as an input to the algorithm, we can set an allowable error bound for the computed conditional stationary distribution instead of that for the computed boundary vector considered in [Taki16]. The error evaluation is a notable feature of our algorithm.

The rest of this chapter is organized as follows. In Section 3.2, we develop a computational algorithm for a conditional stationary distribution with the error evaluation. In Section 3.3, we provide some numerical examples. Finally, we conclude this chapter in Section 3.4.

3.2 A computational algorithm

In this section, we develop a computational algorithm for the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(N) = (\boldsymbol{\pi}_0(N)\ \boldsymbol{\pi}_1(N)\ \cdots\ \boldsymbol{\pi}_N(N))$ $(N \in \mathbb{Z}^+)$ given that the level is not greater than a predefined threshold N. Recall that if $\overline{\boldsymbol{\pi}}_N$ is obtained, $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ can be computed by (cf. (2.2) and (2.4))

$$\boldsymbol{\pi}_\ell(N) = \begin{cases} c^{-1} \cdot \overline{\boldsymbol{\pi}}_N, & \ell = N, \\ c^{-1} \cdot \overline{\boldsymbol{\pi}}_N \boldsymbol{Z}_{N-1} \boldsymbol{Z}_{N-2} \cdot \cdots \cdot \boldsymbol{Z}_\ell, & \ell = 0, 1, \ldots, N-1, \end{cases} \tag{3.1}$$

where c denotes the normalized constant given by

$$c = 1 + \sum_{\ell=0}^{N-1} \overline{\boldsymbol{\pi}}_N \boldsymbol{Z}_{N-1} \boldsymbol{Z}_{N-2} \cdot \cdots \cdot \boldsymbol{Z}_\ell \boldsymbol{e}.$$

As stated in Section 3.1, the main body of our algorithm is identical to the algorithm in [Taki16], which is based on the following result.

Proposition 3.1 ([Taki16, Lemma 1])**.** *Consider an ergodic Markov chain of LD-M/G/1-type.* $\boldsymbol{Z}_k$ $(k \in \mathbb{Z}^+)$ *is then given by*

$$\boldsymbol{Z}_k = \boldsymbol{Q}_{k+1,k}(-\boldsymbol{T}_k)^{-1}, \quad k \in \mathbb{Z}^+, \tag{3.2}$$

where $\boldsymbol{T}_0 = \boldsymbol{Q}_{0,0}$ *and*

$$\boldsymbol{T}_k = \boldsymbol{Q}_{k,k} + \sum_{n=0}^{k-1} \boldsymbol{Z}_{k-1} \boldsymbol{Z}_{k-2} \cdot \cdots \cdot \boldsymbol{Z}_n \boldsymbol{Q}_{n,k}, \quad k \in \mathbb{N}. \tag{3.3}$$

Furthermore, for an arbitrarily fixed, nonnegative integer N, $\boldsymbol{N}_{k,N}$ $(k \in \mathbb{Z}_{N+1}^{\infty})$ *is given by* $\boldsymbol{N}_{N,N} = \boldsymbol{I}$ *and*

$$\boldsymbol{N}_{k,N} = \boldsymbol{Z}_{k-1} \boldsymbol{N}_{k-1,N}, \quad k \in \mathbb{Z}_{N+1}^{\infty}. \tag{3.4}$$

Remark 3.1. *The* $M_k \times M_k$ *matrix* $\boldsymbol{T}_k$ *denotes the northwest corner submatrix of the infinitesimal generator* $\boldsymbol{Q}^{(k)}$ *in the censored process obtained by observing the Markov chain* $\{(L(t), J(t))\}_{t \geq 0}$ *only when* $L(t) \geq k$*. It is clear that* $\boldsymbol{T}_k$ *is a defective infinitesimal generator and therefore it is nonsingular.*

With Proposition 3.1, the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ can be computed as follows [Taki16].

Step 1: Compute $\boldsymbol{Z}_k$ $(k \in \mathbb{Z}_0^{N-1})$ by (3.2) and (3.3).
Step 2: Compute $\boldsymbol{Z}_{k-1}$ and $\boldsymbol{N}_{k,N}$ $(k \in \mathbb{Z}_{N+1}^{\infty})$ successively and obtain $\overline{\boldsymbol{\pi}}_N$ (cf. Theorem 2.2).
Step 3: Compute $\boldsymbol{\pi}_{\mathrm{lv}}(N) = (\boldsymbol{\pi}_0(N)\ \boldsymbol{\pi}_1(N)\ \cdots\ \boldsymbol{\pi}_N(N))$ by (3.1).

The problem is thus how to stop the successive computation of $\boldsymbol{N}_{k,N}$ $(k \in \mathbb{Z}_{N+1}^{\infty})$ in Step 2, which is discussed below. As we will see, all analytical results in this section are valid in Markov chains of LD-G/G/1-type because we will not use the skip-free-to-the-left property.

Recall that $\mathcal{M}_k \setminus \mathcal{M}_{k,\ell}^+ \neq \emptyset$ in general, i.e., $\boldsymbol{\nu}_{k,N}(i) = \mathbf{0}$ can happen for some i ($i \in \mathcal{M}_k \setminus \mathcal{M}_{k,\ell}^+$). Furthermore, $i \in \mathcal{M}_{k,\ell}^+$ ($i \in \mathcal{M}_k \cap \mathcal{M}_{k+1}$) does not imply $i \in \mathcal{M}_{k+1,\ell}^+$ and vice versa. Taking account of these facts, we adopt

$$\overline{\overline{\boldsymbol{\nu}}}_{k,N} = \frac{\boldsymbol{e}^T \boldsymbol{N}_{k,N}}{\boldsymbol{e}^T \boldsymbol{N}_{k,N} \boldsymbol{e}}, \quad k \in \mathbb{Z}_{N+1}^{\infty}, \tag{3.5}$$

as an approximation to $\overline{\boldsymbol{\pi}}_N$. Since every $\boldsymbol{N}_{k,N}$ has at least one non-zero row vector, we have $\boldsymbol{e}^T \boldsymbol{N}_{k,N} \boldsymbol{e} > 0$, so that $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$ is well-defined. The following corollary is obtained from Theorem 2.1.

Corollary 3.1. *Consider an ergodic Markov chain of LD-G/G/1-type. For an arbitrarily fixed N ($N \in \mathbb{Z}^+$), we have*

$$\overline{\overline{\boldsymbol{\nu}}}_{k,N} \in \mathrm{ri}\, \mathcal{P}_{k,N}^+, \quad k \in \mathbb{Z}_{N+1}^{\infty}, \tag{3.6}$$

and

$$\lim_{k \to \infty} \overline{\overline{\boldsymbol{\nu}}}_{k,N} = \overline{\boldsymbol{\pi}}_N. \tag{3.7}$$

Proof. Note first that (3.5) is rewritten to be

$$\overline{\overline{\boldsymbol{\nu}}}_{k,N} = \boldsymbol{\gamma}_{k,N} \overline{\boldsymbol{N}}_{k,N},$$

where $\boldsymbol{\gamma}_{k,N} = (\gamma_{k,N}(0)\ \gamma_{k,N}(1)\ \cdots\ \gamma_{k,N}(M_k - 1))$ denotes a $1 \times M_k$ vector given by

$$\boldsymbol{\gamma}_{k,N} = \frac{\boldsymbol{e}^T \mathrm{diag}(\boldsymbol{N}_{k,N} \boldsymbol{e})}{\boldsymbol{e}^T \boldsymbol{N}_{k,N} \boldsymbol{e}}.$$

It is easy to verify that $\boldsymbol{\gamma}_{k,N} \boldsymbol{e} = 1$, $\gamma_{k,N}(i) > 0$ ($i \in \mathcal{M}_{k,\ell}^+$), and $\gamma_{k,N}(i) = 0$ ($i \in \mathcal{M}_k \setminus \mathcal{M}_{k,\ell}^+$). We thus have (3.6), and (3.7) follows from (2.9) and (3.6). □

For a finite-dimensional matrix $\boldsymbol{X}$, let $\mathrm{rank}(\boldsymbol{X})$ denote the rank of $\boldsymbol{X}$. In an ergodic Markov chain of LD-G/G/1-type, $1 \leq \mathrm{rank}(\boldsymbol{N}_{k,\ell}) \leq \min(M_k, M_\ell)$ ($k, \ell \in \mathbb{Z}^+$, $k \neq \ell$) because $\mathcal{M}_{k,\ell}^+ \neq \emptyset$.

Corollary 3.2. *Consider an ergodic Markov chain of LD-G/G/1-type. For an arbitrarily fixed N ($N \in \mathbb{Z}^+$), if $\mathrm{rank}(\boldsymbol{N}_{k,N}) = 1$ for some k ($k > N$), we have*

$$\overline{\overline{\boldsymbol{\nu}}}_{k,N} = \overline{\boldsymbol{\pi}}_N.$$

Proof. If $\mathrm{rank}(\boldsymbol{N}_{k,N}) = 1$, we have $\boldsymbol{N}_{k,N} = \boldsymbol{N}_{k,N} \boldsymbol{e} \cdot \overline{\overline{\boldsymbol{\nu}}}_{k,N}$, so that

$$\mathcal{P}_{k,N}^+ = \mathrm{ri}\, \mathcal{P}_{k,N}^+ = \{\overline{\overline{\boldsymbol{\nu}}}_{k,N}\}. \tag{3.8}$$

The corollary now follows from (2.9) and (3.8). □

Remark 3.2. *Corollary 3.2 can also be shown algebraically as follows.*

$$\overline{\boldsymbol{\pi}}_N = \frac{\boldsymbol{\pi}_N}{\boldsymbol{\pi}_N \boldsymbol{e}} = \frac{\boldsymbol{\pi}_k \boldsymbol{N}_{k,N}}{\boldsymbol{\pi}_N \boldsymbol{e}} = \frac{\boldsymbol{\pi}_k \boldsymbol{N}_{k,N} \boldsymbol{e} \cdot \overline{\overline{\boldsymbol{\nu}}}_{k,N}}{\boldsymbol{\pi}_N \boldsymbol{e}} = \frac{\boldsymbol{\pi}_N \boldsymbol{e} \cdot \overline{\overline{\boldsymbol{\nu}}}_{k,N}}{\boldsymbol{\pi}_N \boldsymbol{e}} = \overline{\overline{\boldsymbol{\nu}}}_{k,N}.$$

Remark 3.3. *Implications of Corollary 3.2 for computation of the conditional distribution* $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ *in Markov chains of LD-M/G/1-type are as follows. Note first that* $\mathrm{rank}(\boldsymbol{XY}) \leq \min(\mathrm{rank}(\boldsymbol{X}), \mathrm{rank}(\boldsymbol{Y}))$ *for any finite matrices* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *whose product* $\boldsymbol{XY}$ *can be defined. Therefore,* (3.4) *and Corollary 3.2 imply that if* $\mathrm{rank}(\boldsymbol{Z}_{k-1}) = 1$ *for some* k $(k > N)$*, we have* $\mathrm{rank}(\boldsymbol{N}_{k,N}) = 1$ *and in this case, we can exactly compute* $\overline{\boldsymbol{\pi}}_N$ *(and the conditional stationary distribution* $\boldsymbol{\pi}_{\mathrm{lv}}(N)$*). Note further that* $\mathrm{rank}(\boldsymbol{Z}_{k-1}) = 1 \Leftrightarrow \mathrm{rank}(\boldsymbol{Q}_{k,k-1}) = 1$ *because of* (3.2)*. Therefore, we can judge whether a Markov chain of LD-M/G/1-type comes under this special class, by examining the internal structure of* $\boldsymbol{Q}_{k,k-1}$*. We notice that if* $\mathrm{rank}(\boldsymbol{Q}_{k,k-1}) = 1$*, the level-dependent G-matrix from level* k *to level* $k-1$ *can be obtained explicitly, so that* $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ *can also be computed exactly by algorithms in [Kli06, Li05]. Finally, using this special case, we can examine the magnitude of unavoidable numerical errors caused by finite-precision computation, as demonstrated in Section 3.3.2.*

Remark 3.4. *If a Markov chain of LD-M/G/1-type satisfies* $\mathrm{rank}(\boldsymbol{N}_{k,N}) = 1$ *for some* k *and* N*, we have from* (3.4)*,* $\mathrm{rank}(\boldsymbol{N}_{\ell,N}) \leq \min(\mathrm{rank}(\boldsymbol{N}_{\ell,k}), \mathrm{rank}(\boldsymbol{N}_{k,N})) = 1$ *(*$\ell \in \mathbb{Z}_{k+1}^{\infty}$*), so that*

$$\mathrm{rank}(\boldsymbol{N}_{\ell,N}) = 1, \quad \ell \in \mathbb{Z}_{k+1}^{\infty}.$$

Therefore, in Markov chain of LD-M/G/1-type, Corollary 3.2 implies that $\mathcal{P}_{\ell,N}^{+} = \{\overline{\boldsymbol{\pi}}_N\}$ *for all* $\ell \in \mathbb{Z}_k^{\infty}$*.*

In what follows, we implicitly assume $\mathrm{rank}(\boldsymbol{N}_{k,N}) \geq 2$ and discuss error bounds for $\overline{\boldsymbol{\pi}}_N$ and $\boldsymbol{\pi}_{\mathrm{lv}}(N)$.

Lemma 3.1. *Consider an ergodic Markov chain of LD-G/G/1-type. We then have*

$$\|\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N\|_1 \leq e_N(k,N), \quad k \in \mathbb{Z}_{N+1}^{\infty}, \tag{3.9}$$

where

$$e_N(k,N) = \max_{i \in \mathcal{M}_{k,N}^{+}} \|\overline{\boldsymbol{\nu}}_{k,N}(i) - \overline{\overline{\boldsymbol{\nu}}}_{k,N}\|_1, \quad k \in \mathbb{Z}_{N+1}^{\infty}. \tag{3.10}$$

Furthermore, $e_N(k,N)$ *satisfies*

$$\lim_{k \to \infty} e_N(k,N) = 0. \tag{3.11}$$

Proof. We first consider (3.9). It follows from (2.10), (2.11), and (2.12) that

$$\begin{aligned}
\|\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N\|_1 &= \left\| \sum_{i \in \mathcal{M}_{k,N}^{+}} \beta_{k,N}(i)(\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\nu}}_{k,N}(i)) \right\|_1 \\
&\leq \sum_{i \in \mathcal{M}_{k,N}^{+}} \beta_{k,N}(i) \left\| \overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\nu}}_{k,N}(i) \right\|_1 \\
&\leq \max_{i \in \mathcal{M}_{k,N}^{+}} \left\| \overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\nu}}_{k,N}(i) \right\|_1,
\end{aligned}$$

which implies (3.9).

Next we consider (3.11). By definition, $\overline{\boldsymbol{\nu}}_{k,N}(i) \in \mathcal{P}_{k,N}^{+}$ for all $i \in \mathcal{M}_{k,N}^{+}$. Furthermore, since $\mathrm{ri}\, \mathcal{P}_{k,N}^{+} \subseteq \mathcal{P}_{k,N}^{+}$, (3.6) implies $\overline{\overline{\boldsymbol{\nu}}}_{k,N} \in \mathcal{P}_{k,N}^{+}$. We thus have

$$0 \leq e_N(k,N) \leq \mathrm{diam}(\mathcal{P}_{k,N}^{+}), \tag{3.12}$$

where $\mathrm{diam}(\mathcal{P}^+_{k,N})$ denotes the diameter of $\mathcal{P}^+_{k,N}$.

$$\mathrm{diam}(\mathcal{P}^+_{k,N}) = \max_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{P}^+_{k,N}} \|\boldsymbol{x}-\boldsymbol{y}\|_1.$$

Note here that Theorem 2.3 implies

$$\lim_{k\to\infty} \mathrm{diam}(\mathcal{P}^+_{k,N}) = 0. \tag{3.13}$$

(3.11) now follows from (3.12) and (3.13).

□

Lemma 3.1 indicates that during the course of computing $\boldsymbol{N}_{k,N}$ ($k \in \mathbb{Z}^\infty_{N+1}$), we can keep track of the error bound $e_N(k,N)$ for an approximation $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$ to $\overline{\boldsymbol{\pi}}_N$. Keeping this in mind, we consider an error bound for the computed conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ in terms of the error bound for the approximation $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$ to $\overline{\boldsymbol{\pi}}_N$. For simplicity in description, we define $\widetilde{\boldsymbol{N}}_N$ as

$$\widetilde{\boldsymbol{N}}_N = (\boldsymbol{N}_{N,0}\ \boldsymbol{N}_{N,1}\ \cdots\ \boldsymbol{N}_{N,N-1}\ \boldsymbol{I})\,. \tag{3.14}$$

It then follows from (2.1) that

$$\boldsymbol{\pi}_{\mathrm{lv}}(N) = (\boldsymbol{\pi}_0(N)\ \boldsymbol{\pi}_1(N)\ \cdots\ \boldsymbol{\pi}_N(N)) = \frac{\overline{\boldsymbol{\pi}}_N \widetilde{\boldsymbol{N}}_N}{\overline{\boldsymbol{\pi}}_N \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}. \tag{3.15}$$

For $k \in \mathbb{Z}^\infty_{N+1}$, we define $\boldsymbol{\pi}^{\mathrm{approx},k}_{\mathrm{lv}}(N) = (\boldsymbol{\pi}^{\mathrm{approx},k}_0(N)\ \boldsymbol{\pi}^{\mathrm{approx},k}_1(N)\cdots\ \boldsymbol{\pi}^{\mathrm{approx},k}_N(N))$ as an approximation to $\boldsymbol{\pi}_{\mathrm{lv}}(N)$, which is given by

$$\boldsymbol{\pi}^{\mathrm{approx},k}_{\mathrm{lv}}(N) = \frac{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}. \tag{3.16}$$

Theorem 3.1. *Consider an ergodic Markov chain of LD-G/G/1-type. We then have*

$$\|\boldsymbol{\pi}^{\mathrm{approx},k}_{\mathrm{lv}}(N) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \le \frac{e_N(k,N)}{\overline{\overline{\boldsymbol{\nu}}}_{k,N}\, \boldsymbol{w}_N}, \tag{3.17}$$

where $e_N(k,N)$ is given by (3.10) *and $\boldsymbol{w}_N$ is defined as an $M_N \times 1$ vector given by*

$$\boldsymbol{w}_N = \frac{\widetilde{\boldsymbol{N}}_N \boldsymbol{e}}{2\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}.$$

Furthermore, we have

$$\lim_{k\to\infty} \frac{e_N(k,N)}{\overline{\overline{\boldsymbol{\nu}}}_{k,N}\, \boldsymbol{w}_N} = 0. \tag{3.18}$$

Proof. We first consider (3.17). From (3.15), we have $\overline{\boldsymbol{\pi}}_N \widetilde{\boldsymbol{N}}_N = \overline{\boldsymbol{\pi}}_N \widetilde{\boldsymbol{N}}_N \boldsymbol{e} \cdot \boldsymbol{\pi}_{\mathrm{lv}}(N)$. It then follows from (3.16) that

$$\begin{aligned}
&\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},k}(N) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \\
&\quad = \frac{\|\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N - \overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e} \cdot \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \\
&\quad = \frac{\|(\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N)\widetilde{\boldsymbol{N}}_N + \overline{\boldsymbol{\pi}}_N \widetilde{\boldsymbol{N}}_N \boldsymbol{e} \cdot \boldsymbol{\pi}_{\mathrm{lv}}(N) - \overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e} \cdot \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \\
&\quad \le \frac{\|(\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N)\widetilde{\boldsymbol{N}}_N\|_1}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} + \frac{\|(\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N)\widetilde{\boldsymbol{N}}_N \boldsymbol{e} \cdot \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}. \qquad (3.19)
\end{aligned}$$

Note here that for a vector $\boldsymbol{x}$ and a nonnegative matrix $\boldsymbol{X}$ whose product $\boldsymbol{xX}$ can be defined, we have $\|\boldsymbol{xX}\|_1 \le \boldsymbol{e}^T \boldsymbol{X} \boldsymbol{e} \cdot \|\boldsymbol{x}\|_1$ [Bel97, Section 10.3]. It then follows from Lemma 3.1 and (3.19) that

$$\begin{aligned}
&\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},k}(N) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \\
&\quad \le \frac{\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \cdot \|\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N\|_1 + \frac{\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e} \cdot \boldsymbol{\pi}_{\mathrm{lv}}(N) \boldsymbol{e}}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \cdot \|\overline{\overline{\boldsymbol{\nu}}}_{k,N} - \overline{\boldsymbol{\pi}}_N\|_1 \\
&\quad \le 2 \cdot \frac{\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \cdot e_N(k,N) \\
&\quad = \frac{1}{\overline{\overline{\boldsymbol{\nu}}}_{k,N} \boldsymbol{w}_N} \cdot e_N(k,N),
\end{aligned}$$

which shows (3.17).

Next we show (3.18). Recall that $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$ converges to $\overline{\boldsymbol{\pi}}_N$ as k goes to infinity. Note also that from the definition of $\widetilde{\boldsymbol{N}}_N$ in (3.14), we have $\widetilde{\boldsymbol{N}}_N \boldsymbol{e} \ge \boldsymbol{e}$. It then follows that

$$\lim_{k\to\infty} \overline{\overline{\boldsymbol{\nu}}}_{k,N} \boldsymbol{w}_N = \overline{\boldsymbol{\pi}}_N \boldsymbol{w}_N = \frac{\overline{\boldsymbol{\pi}}_N \widetilde{\boldsymbol{N}}_N \boldsymbol{e}}{2\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \ge \frac{\overline{\boldsymbol{\pi}}_N \boldsymbol{e}}{2\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} = \frac{1}{2\boldsymbol{e}^T \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} > 0. \qquad (3.20)$$

(3.18) now follows from (3.11) and (3.20). □

Remark 3.5. *For any probability vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ on the same state space, $\|\boldsymbol{x}-\boldsymbol{y}\|_1 \le 2$, so that $e_N(k,N) \le 2$ (cf. (3.9)). On the other hand, $\overline{\overline{\boldsymbol{\nu}}}_{k,N} \boldsymbol{w}_N$ can be smaller than one. Therefore, $e_N(k,N)/(\overline{\overline{\boldsymbol{\nu}}}_{k,N} \boldsymbol{w}_N)$ can be greater than 2 for a finite k, and if this is indeed the case, the error bound in Theorem 3.1 is trivial.*

Remark 3.6. *The rate of convergence in* (3.18) *depends on the whole structure of $\boldsymbol{Q}$, so that we cannot discuss it in the general setting.*

Let $\boldsymbol{\pi}^{\mathrm{approx},k,N}$ denote an approximation to the stationary distribution $\boldsymbol{\pi}$ obtained from $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},k}(N)$ such that

$$\boldsymbol{\pi}^{\mathrm{approx},k,N} = (\boldsymbol{\pi}_0^{\mathrm{approx},k}(N)\ \boldsymbol{\pi}_1^{\mathrm{approx},k}(N)\ \cdots\ \boldsymbol{\pi}_N^{\mathrm{approx},k}(N)\ \mathbf{0}\ \mathbf{0}\ \cdots).$$

From an adaptation of (1.25), we have

$$2\xi_{\mathrm{lv}}(N) \le \|\boldsymbol{\pi}^{\mathrm{approx},k,N} - \boldsymbol{\pi}\|_1 \le 2\xi_{\mathrm{lv}}(N) + \epsilon_{\mathrm{lv}}(k,N), \qquad (3.21)$$

where

$$\xi_{\rm lv}(N) = \Pr(L(0) > N) = \sum_{\ell=N+1}^{\infty} \boldsymbol{\pi}_\ell \boldsymbol{e}, \qquad \epsilon_{\rm lv}(k,N) = \|\boldsymbol{\pi}_{\rm lv}^{{\rm approx},k}(N) - \boldsymbol{\pi}_{\rm lv}(N)\|_1.$$

Note that we obtain an upper bound of $\epsilon_{\rm lv}(k,N)$ in Theorem 3.1. Theorem 3.1 and (3.21) imply that the error bound for the approximation $\boldsymbol{\pi}^{{\rm approx},k,N}$ to the unconditional stationary distribution $\boldsymbol{\pi}$ can be obtained if the upper bound for the tail probability $\xi_{\rm lv}(N)$ of the level variable is available.

We now summarize our algorithm in Figure 3.1, where the input ϵ denotes the maximum allowable error in the computed conditional stationary distribution $\boldsymbol{\pi}_{\rm lv}^{{\rm approx},k}(N)$ and $\overline{\boldsymbol{\nu}}(i)$ $(i \in \mathcal{M}_{k,N}^{+})$ in Step 2-2 denotes the ith row vector of $\overline{\boldsymbol{N}}$. Note that Step 1 and Step 3 are identical to those in the algorithm of [Taki16]. In Step 2, Takine's algorithm stops the computation of $\boldsymbol{N}$ $(= \boldsymbol{N}_{k,N})$ when the upper bound of the boundary vector in Lemma 3.1 becomes less than ϵ for the first time. In Step 2 of our algorithm, we stop the computation of $\boldsymbol{N}$ when the upper bound of *the conditional stationary distribution* in Theorem 3.1 becomes less than ϵ for the first time. Except for the computation of inverse matrices in Steps 1 and 2-2 and e_N in Step 2-2, the algorithm performs only addition and multiplication of positive numbers. Furthermore, the computation of the inverse matrix $(-\boldsymbol{T})^{-1}$ is implemented without subtraction as follows.

$$(-\boldsymbol{T})^{-1} = \frac{1}{\theta} \sum_{k=0}^{\infty} (\boldsymbol{I} + \boldsymbol{T}/\theta)^k,$$

where θ denotes the maximum of the absolute values of the diagonal elements of $\boldsymbol{T}$. Note that $\boldsymbol{I} + \boldsymbol{T}/\theta \geq \boldsymbol{O}$. Therefore, our algorithm is numerically stable in principle.

Before closing this section, we briefly discuss the computational complexity of our algorithm in Figure 3.1, where the algorithm is assumed to stop with $k = K$. The most computationally intensive part of the algorithm is the repeated computation of $\boldsymbol{T}$ $(= \boldsymbol{T}_k)$ in (A) and their inverses, and the total number of multiplications is in $O(K^2 \cdot M_{\max}^3(K))$, where

$$M_{\max}(K) = \max\{M_\ell;\ \ell = 0, 1, \ldots, K\}.$$

On the other hand, the space complexity of our algorithm is in $O(K \cdot M_{\max}^2(K))$ because we have to store $\boldsymbol{Z}_\ell$ $(\ell = 0, 1, \ldots, K-1)$ during the course of the algorithm. Therefore, the computational complexity of our algorithm is comparable to the existing methods [Kli06, Li05, Mas18] for Markov chains of LD-M/G/1-type. Note that our algorithm is also applicable to level-dependent QBD processes, and in this case, (A) is reduced to be

$$\boldsymbol{T} := \boldsymbol{Q}_{k,k} + \boldsymbol{Z}_{k-1}\boldsymbol{Q}_{k-1,k}.$$

Therefore, for level-dependent QBD processes, the total number of multiplications is in $O(K \cdot M_{\max}^3(K))$ and the space complexity is in $O(N \cdot M_{\max}^2(K))$, which are comparable to the existing algorithms [Bau10, Bri95, Phu10] for level-dependent QBD processes. Note here that the computational complexities are given in terms of the output value K that depends on the model.

Input: $\boldsymbol{Q}_{k,\ell}$ $(k, \ell \in \mathbb{Z}^+)$, N, and ϵ.
Output: K and $\boldsymbol{\pi}_{\rm lv}(N) = (\boldsymbol{\pi}_0(N)\ \boldsymbol{\pi}_1(N)\ \ldots\ \boldsymbol{\pi}_N(N))$.

Step 1: Computing $\boldsymbol{Z}_\ell$ $(\ell = 0, 1, \ldots, N-1)$.
Let $\boldsymbol{T} := \boldsymbol{Q}_{0,0}$.
for $k = 1$ to N **do**
Let $\boldsymbol{Z}_{k-1} := \boldsymbol{Q}_{k,k-1}(-\boldsymbol{T})^{-1}$.
Compute $\boldsymbol{T}$ by

$$\boldsymbol{T} := \boldsymbol{Q}_{k,k} + \sum_{n=0}^{k-1} \boldsymbol{Z}_{k-1}\boldsymbol{Z}_{k-2} \cdots \cdots \boldsymbol{Z}_n \boldsymbol{Q}_{n,k}. \qquad \text{(A)}$$

endfor
Let $\boldsymbol{N} := \boldsymbol{I}$; $k := N$.

Step 2: Determining the normalized boundary vector $\overline{\boldsymbol{\pi}}_N$.
Step 2-1: Setup for the stopping criterion.
Compute $\boldsymbol{z}$ by

$$\boldsymbol{z} := \boldsymbol{e} + \sum_{n=0}^{N-1} \boldsymbol{Z}_{N-1}\boldsymbol{Z}_{N-2} \cdots \cdots \boldsymbol{Z}_n \boldsymbol{e}.$$

Let $\boldsymbol{w} := \boldsymbol{z}/(2\boldsymbol{e}^T\boldsymbol{z})$.
Step 2-2: Computing $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$.
Let $k := k+1$; $\boldsymbol{Z}_{k-1} := \boldsymbol{Q}_{k,k-1}(-\boldsymbol{T})^{-1}$; $\boldsymbol{N} := \boldsymbol{Z}_{k-1}\boldsymbol{N}$.
Let $\overline{\overline{\boldsymbol{\nu}}} := \boldsymbol{e}^T\boldsymbol{N}/(\boldsymbol{e}^T\boldsymbol{N}\boldsymbol{e})$; $e_N := \max_{i \in \mathcal{M}^+_{k,N}} \|\overline{\boldsymbol{\nu}}(i) - \overline{\overline{\boldsymbol{\nu}}}\|_1$.
if $e_N/(\overline{\overline{\boldsymbol{\nu}}}\boldsymbol{w}) \geq \epsilon$ **then**
Compute $\boldsymbol{T}$ by (A) and return to Step 2-2.
endif
Let $K := k$.

Step 3: Computing $\boldsymbol{\pi}_\ell(N)$ $(\ell = 0, 1, \ldots, N)$.
Let $\boldsymbol{\alpha}_N := \overline{\overline{\boldsymbol{\nu}}}$; $c := 1$.
for $\ell = N-1$ to 0 by -1 **do**
Let $\boldsymbol{\alpha}_\ell := \boldsymbol{\alpha}_{\ell+1}\boldsymbol{Z}_\ell$; $c := c + \boldsymbol{\alpha}_\ell\boldsymbol{e}$.
endfor
for $\ell = 0$ to N **do**
Let $\boldsymbol{\pi}_\ell(N) := \boldsymbol{\alpha}_\ell/c$.
endfor

Figure 3.1: A computational algorithm for the conditional stationary distribution $\boldsymbol{\pi}_{\rm lv}(N)$ in Markov chains of LD-M/G/1-type.

3.3 Numerical examples

In this section, we present some numerical examples. Since the main body of our algorithm is identical to the algorithm in [Taki16], two algorithms share the following features:

(i) The main source of numerical errors lies in the computation of the inverse of $\boldsymbol{T}_k$,

(ii) the numerically achievable minimum value of $e_N(k, N)$ increases with the number of phases in each level, and

(iii) Markov chains with one hundred phases in each level can be handled and if N is large enough, the output $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ gives a good approximation to $\boldsymbol{\pi}$.

As for (i) and (ii), some tips for program implementation are given in [Taki16, Appendix 3], which we also follow in this section. We summarize the tips in Appendix B briefly.

We implement our algorithm in C and conduct numerical experiments by a PC with Intel®Core™i7-7700 (3.60GHz) and 16GB memory. In what follows, we first explain models used in numerical examples and then, examine the magnitude of unavoidable numerical errors caused by finite-precision computation. Next, we examine the convergence of Step 2 in our algorithm and the tightness of the upper bound in Theorem 3.1. Finally, we present some numerical examples for large-scale models.

3.3.1 Models for numerical examples

We consider a single-server queue, where both the batch arrival rate and the service rate depend on the queue length. In particular, we set those rates in such a way that the system is overloaded when the queue length is less than a certain threshold. Customers arrive according to a batch Markov-modulated Poisson process (B-MMPP) characterized by $\{\boldsymbol{D}_k;\ k = 0, 1, \ldots\}$, where the underlying Markov chain $\{J_{\mathrm{A}}(t)\}_{t\geq 0}$ of the B-MMPP is assumed to be a J_{A}-state birth-and-death process with identical transition rates α. When $J_{\mathrm{A}}(t) = j$ $(j \in \{0, 1, \ldots, J_{\mathrm{A}} - 1\})$, batches of customers arrive according to a Poisson process with rate λ_j. Batch sizes B are independent and identically distributed (i.i.d.) according to a (truncated) geometric distribution with mean $\mathrm{E}[B]$, independent of $J_{\mathrm{A}}(t)$. $\{\boldsymbol{D}_k; k = 0, 1, \ldots\}$ is then given by

$$\boldsymbol{D}_0 = \begin{pmatrix} \beta_0 & \alpha & 0 & \cdots & 0 & 0 \\ \alpha & \beta_1 & \alpha & \cdots & 0 & 0 \\ 0 & \alpha & \beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{J_{\mathrm{A}}-2} & \alpha \\ 0 & 0 & 0 & \cdots & \alpha & \beta_{J_{\mathrm{A}}-1} \end{pmatrix}, \quad \boldsymbol{D}_k = b(k) \cdot \mathrm{diag}\big((\lambda_0\ \lambda_1\ \cdots\ \lambda_{J_{\mathrm{A}}-1})\big),$$

where

$$\beta_j = \begin{cases} -(\alpha + \lambda_j), & j = 0, J_{\mathrm{A}} - 1, \\ -(2\alpha + \lambda_j), & j = 1, 2, \ldots, J_{\mathrm{A}} - 2, \end{cases}$$

and

$$b(k) = \Pr(B = k) = \begin{cases} \dfrac{1}{\mathrm{E}[B]} \left(1 - \dfrac{1}{\mathrm{E}[B]}\right)^{k-1}, & k = 1, 2, \ldots, N_B - 1, \\ \left(1 - \dfrac{1}{\mathrm{E}[B]}\right)^{N_B - 1}, & k = N_B, \end{cases}$$

with $N_B = \min(k \in \mathbb{Z}^+;\ (1 - 1/\mathrm{E}[B])^k < 10^{-20})$. When the overall arrival rate λ_B of batches is given, λ_j is set to be

$$\lambda_j = \frac{\lambda_B}{2} \left(1 + \frac{2j}{J_\mathrm{A} - 1}\right), \quad j = 0, 1, \ldots, J_\mathrm{A} - 1,$$

so that $\lambda_B/2 = \lambda_0 < \lambda_1 < \cdots < \lambda_{J_\mathrm{A}-1} = 3\lambda_B/2$ if $J_\mathrm{A} \geq 2$. When $J_\mathrm{A} = 1$, the arrival process is reduced to a batch Poisson process with arrival rate λ_B of batches.

$$\boldsymbol{D}_0 = \big(-\lambda_B\big), \qquad \boldsymbol{D}_k = b(k) \cdot \big(\lambda_B\big) \quad (k \in \mathbb{N}).$$

Unless otherwise stated, the amounts of service requirements of customers are assumed to be i.i.d. according to a J_S-stage Erlang distribution with mean one, which has the phase-type representation $(\boldsymbol{\zeta}, \boldsymbol{S})$, where $\boldsymbol{\zeta}$ and $\boldsymbol{S}$ are a $1 \times J_\mathrm{S}$ vector and a $J_\mathrm{S} \times J_\mathrm{S}$ matrix given by

$$\boldsymbol{\zeta} = (1\ 0\ 0\ \ldots\ 0), \quad \boldsymbol{S} = \begin{pmatrix} -J_\mathrm{S} & J_\mathrm{S} & 0 & \cdots & 0 & 0 \\ 0 & -J_\mathrm{S} & J_\mathrm{S} & \cdots & 0 & 0 \\ 0 & 0 & -J_\mathrm{S} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -J_\mathrm{S} & J_\mathrm{S} \\ 0 & 0 & 0 & \cdots & 0 & -J_\mathrm{S} \end{pmatrix}.$$

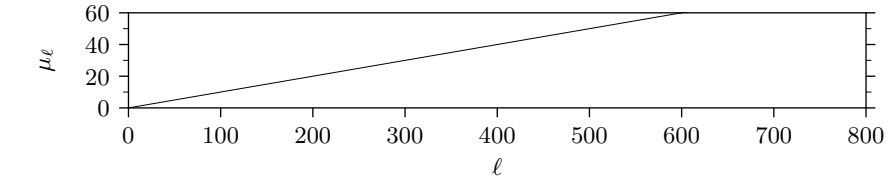
Let $\boldsymbol{s} = -\boldsymbol{S}\boldsymbol{e} = (0\ 0\ \cdots\ 0\ J_\mathrm{S})^T$. Customers are served on a FIFO basis by a single server, where the service speed depends on the number of customers in the system. Let μ_ℓ $(\ell \in \mathbb{Z}^+)$ denote the service speed when there are ℓ customers in the system.

The queue length process in the above-mentioned queue can be formulated as a bivariate Markov chain $\{(L(t), J(t))\}_{t \geq 0}$ of LD-M/G/1-type, where $L(t)$ denotes the number of customers in the system at time t and $J(t)$ denotes the combination of arrival and service phases at time t. It is easy to verify that block matrices in the infinitesimal generator $\boldsymbol{Q}$ are given by

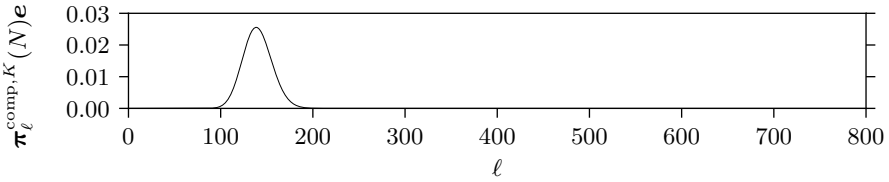
$$\boldsymbol{Q}_{k,k-1} = \begin{cases} \boldsymbol{I}_\mathrm{A} \otimes (\mu_k \boldsymbol{s}), & k = 1, \\ \boldsymbol{I}_\mathrm{A} \otimes (\mu_k \boldsymbol{s})\boldsymbol{\zeta}, & k = 2, 3, \ldots, \end{cases} \quad \boldsymbol{Q}_{k,k} = \begin{cases} \boldsymbol{D}_0, & k = 0, \\ \boldsymbol{D}_0 \oplus (\mu_k \boldsymbol{S}), & k = 1, 2, \ldots, \end{cases}$$

$$\boldsymbol{Q}_{k,\ell} = \begin{cases} \boldsymbol{D}_{\ell-k} \otimes \boldsymbol{\zeta}, & \ell > k,\ k = 0, \\ \boldsymbol{D}_{\ell-k} \otimes \boldsymbol{I}_\mathrm{S}, & \ell > k,\ k = 1, 2, \ldots, \end{cases}$$

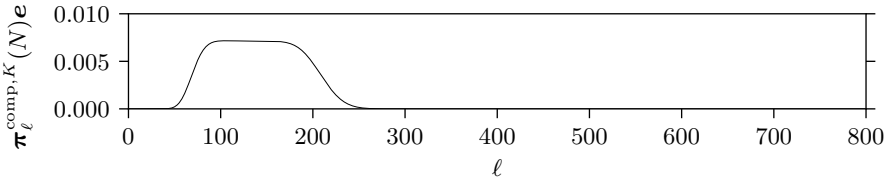
where $\boldsymbol{I}_\mathrm{A}$ and $\boldsymbol{I}_\mathrm{S}$ denote $J_\mathrm{A} \times J_\mathrm{A}$ and $J_\mathrm{S} \times J_\mathrm{S}$ unit matrices. Note that $M_0 = J_\mathrm{A}$ and $M_\ell = J_\mathrm{A} J_\mathrm{S}$ $(\ell \in \mathbb{N})$. Unless otherwise stated, we fix $\alpha = 0.1$, $\mathrm{E}[B] = 2.0$ $(N_B = 67)$, and $\lambda_B = 7.0$. The offered load ρ is then given by $\rho = 14.0$. In Sections 3.3.2–3.3.4, we consider the following two cases of the service speed μ_ℓ $(\ell \in \mathbb{N})$.



(a) Service speed μ_ℓ.



(b) Stationary distribution ($J_{\mathrm{A}} = 1$, $K = 801$).



(c) Stationary distribution ($J_{\mathrm{A}} = 3$, $K = 867$).

Figure 3.2: Models for numerical examples: Case 1 ($J_{\mathrm{S}} = 2$, $N = 800$, $\epsilon = 10^{-13}$).

Case 1: We set $\mu_\ell = \ell/10$ ($\ell \in \mathbb{N}$), so that the mean drift of $L(t)$ is negative when $L(t) > 140$.

Case 2: We set

$$\mu_\ell = \begin{cases} \ell/10, & \ell = 1, 2, \ldots, 139, \\ 14, & \ell = 140, 141, \ldots 400, \\ (\ell - 260)/10, & \ell = 401, 402, \ldots, \end{cases}$$

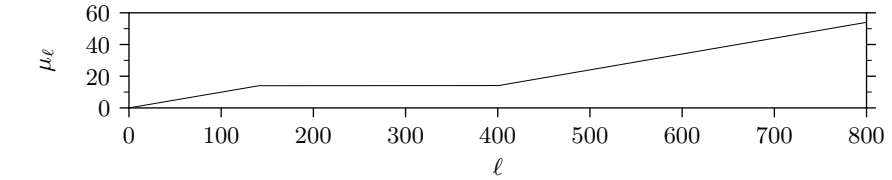
so that the mean drift of $L(t)$ is equal to zero when $L(t) \in [140, 400]$, and it is negative when $L(t) > 400$.

Note that the system is stable in the both cases, because the service speed increases unboundedly, whereas the instantaneous offered load is bounded above by $(3\lambda_B/2)\mathrm{E}[B] \cdot 1 = 21.0$.

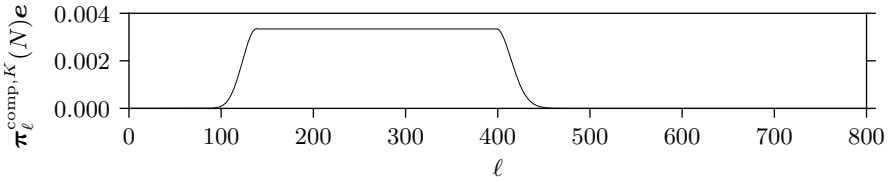
Let $\overline{\overline{\boldsymbol{\nu}}}_{k,N}^{\mathrm{comp}}$ and $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N)$ ($N \in \mathbb{Z}^+$, $k \in \mathbb{Z}_{N+1}^{\infty}$) denote the computed $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$ and $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},k}(N)$. To gain a picture of the (unconditional) stationary distribution, we plot μ_ℓ and $\boldsymbol{\pi}_\ell^{\mathrm{comp},K}(N)\boldsymbol{e}$ ($\simeq \Pr(L(0) = \ell)$) in Figures 3.2 and 3.3.

3.3.2 The magnitude of unavoidable numerical errors

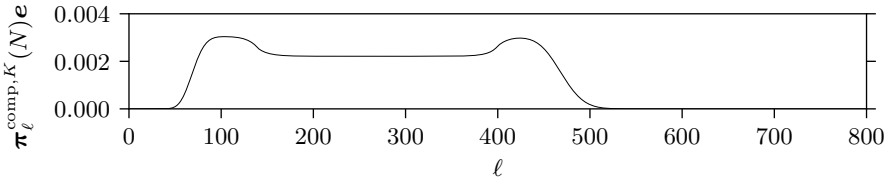
As stated in Remark 3.3, we examine the unavoidable numerical errors by using the case of $J_{\mathrm{A}} = 1$. When $J_{\mathrm{A}} = 1$, the arrival process is reduced to a batch Poisson process, and $\boldsymbol{Z}_k$ ($k \in \mathbb{N}$) has only one non-zero row. Therefore, if $J_{\mathrm{A}} = 1$, $\mathrm{rank}(\boldsymbol{N}_{k,\ell}) = 1$ ($\ell \in \mathbb{Z}^+$, $k \in \mathbb{Z}_{\ell+1}^{\infty}$) and it follows from Corollary 3.2 that $\overline{\boldsymbol{\pi}}_N^{\mathrm{comp},k}(N) = \overline{\overline{\boldsymbol{\nu}}}_{k,N} = \overline{\boldsymbol{\pi}}_N$ for all k ($k > N$). Note, however, that $\overline{\boldsymbol{\pi}}_N^{\mathrm{comp},k}(N) \neq \overline{\boldsymbol{\pi}}_N^{\mathrm{comp},N+1}(N)$ ($k \in \mathbb{Z}_{N+2}^{\infty}$) in general, due



(a) Service speed μ_ℓ.



(b) Stationary distribution ($J_{\mathrm{A}} = 1$, $K = 801$).



(c) Stationary distribution ($J_{\mathrm{A}} = 3$, $K = 880$).

Figure 3.3: Models for numerical examples: Case 2 ($J_{\mathrm{S}} = 2$, $N = 800$, $\epsilon = 10^{-13}$).

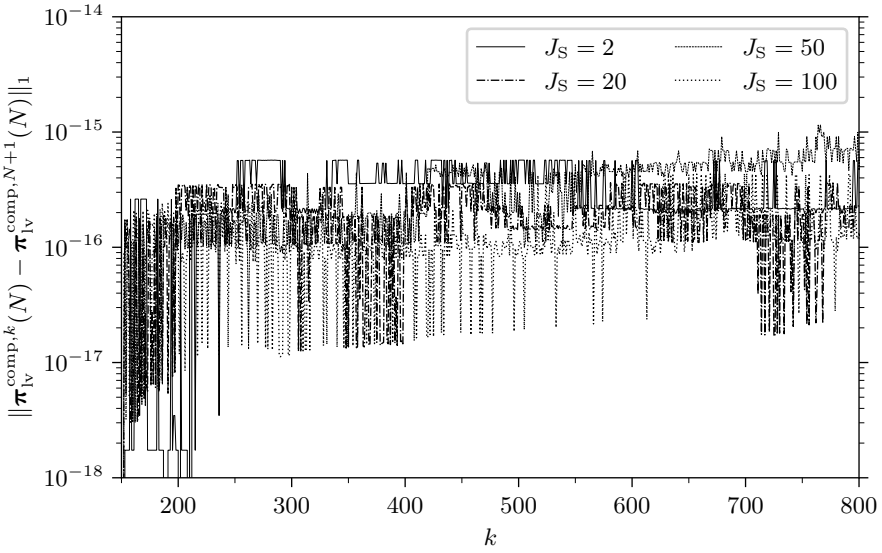
to finite-precision computation.

Figure 3.4 shows the 1-norm $\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N) - \boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},N+1}(N)\|_1$ of the difference of $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N)$ from $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},N+1}(N)$ as a function of k ($k > N$), where $J_{\mathrm{A}} = 1$. We observe that (i) the conditional stationary distribution in Case 2 contains a larger magnitude of error than that in Case 1, (ii) the magnitude of the error tends to increase with the number of phases, and (iii) in all cases, the magnitude of error is less than 10^{-14}.

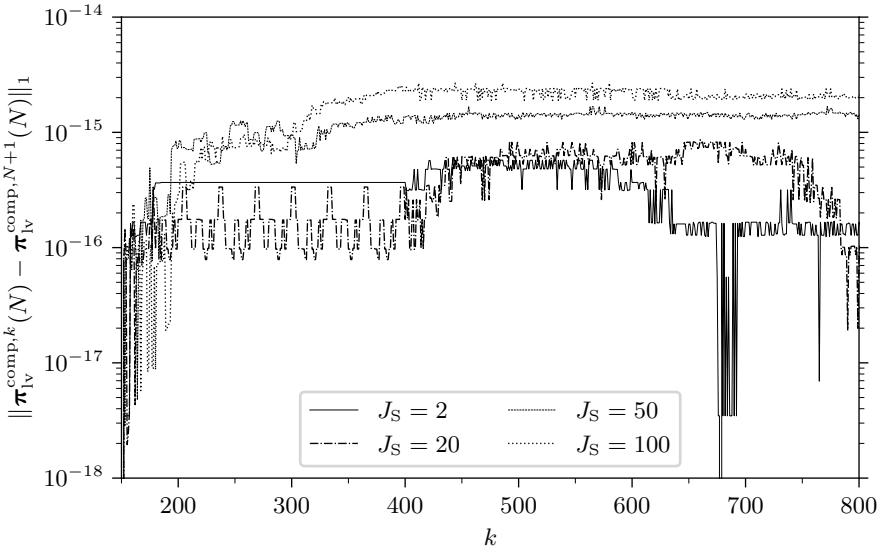
3.3.3 Convergence characteristics

Next we discuss the convergence of the procedure at Step 2 in our algorithm, where we set $J_{\mathrm{A}} = 3$. Figure 3.5 shows the upper bound $e_N(k, N)/(\overline{\overline{\boldsymbol{\nu}}}_{k,N}\, \boldsymbol{w}_N)$ for the error in the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N)$ as a function of k. In any case, the upper bound decreases monotonically with k, even though their convergence rates are not necessarily constant. From Figures 3.2 (c) and 3.3 (c), we observe that the probability mass function $\boldsymbol{\pi}_\ell(N)\boldsymbol{e}$ takes conspicuous values in the range of $\ell \in [50, 250]$ in Case 1 and $\ell \in [50, 500]$ in Case 2, and in those ranges, the convergence becomes slow.

Figure 3.6 shows upper bounds for errors and differences of the first order in the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N)$ and the normalized boundary vector $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$, as a function of k, where $N = 150$. It is interesting to observe that differences of the first order in $\{\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N);\ k \in \mathbb{Z}_{N+1}^{\infty}\}$ and $\{\overline{\overline{\boldsymbol{\nu}}}_{k,N};\ k \in \mathbb{Z}_{N+1}^{\infty}\}$ do not necessarily converge monotonically, and therefore they are not suitable for stopping criteria.

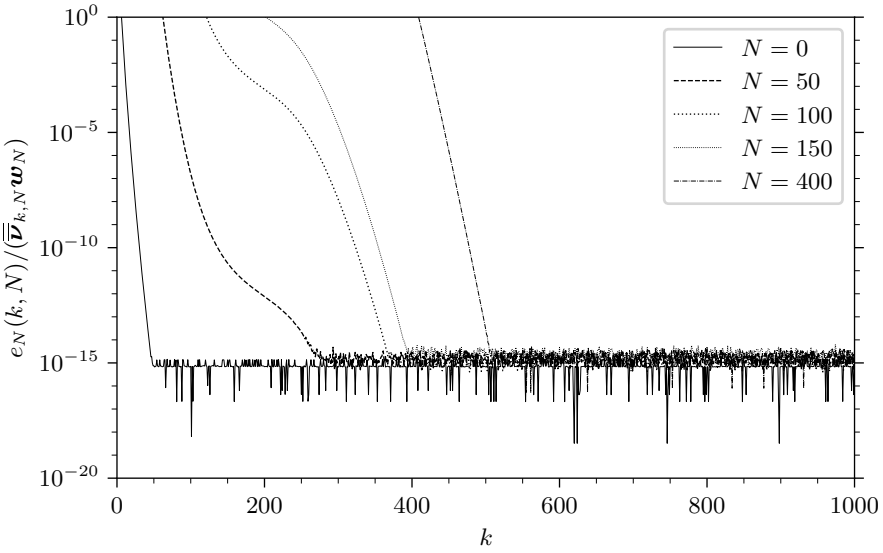


(a) Case 1.

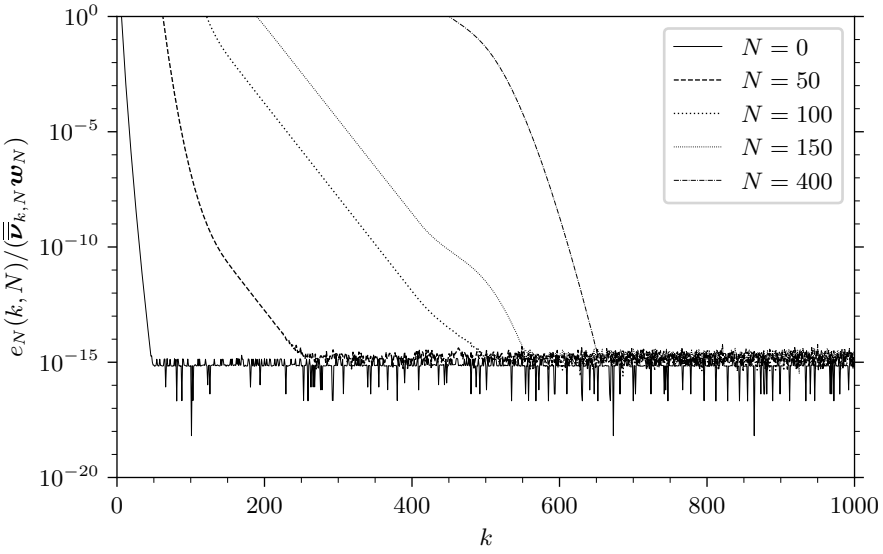


(b) Case 2.

Figure 3.4: The magnitude of unavoidable numerical errors ($J_{\mathrm{A}} = 1$, $N = 150$).

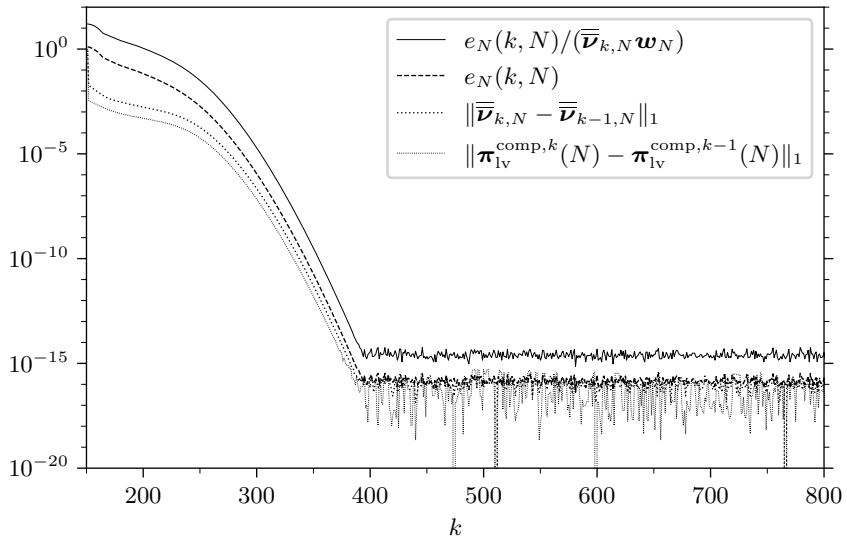


(a) Case 1.

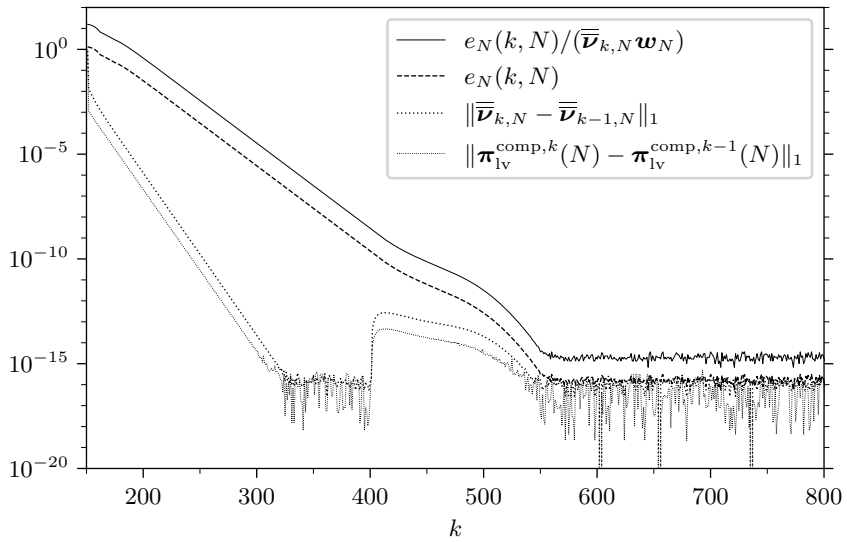


(b) Case 2.

Figure 3.5: Error bounds in $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N)$ ($J_{\mathrm{A}}=3$, $J_{\mathrm{S}}=2$).



(a) Case 1.



(b) Case 2.

Figure 3.6: Error bounds and differences of the first order in $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(N)$ and $\overline{\overline{\boldsymbol{\nu}}}_{k,N}$ ($J_{\mathrm{A}} = 3$, $J_{\mathrm{S}} = 2$, $N = 150$).

Table 3.1: Two approximations $\boldsymbol{\pi}_{\rm lv}^{{\rm comp},K}(N)$ and $\boldsymbol{\pi}_{\rm lv}^{{\rm comp},K^*}(N)$ whose errors are bounded above by 10^{-13} ($J_{\rm A}=3$, $J_{\rm S}=2$).

Case	N	K	K^*	$e^*(K^*,N)$
1	0	41	33	1.4×10^{-11}
	50	234	119	1.6×10^{-9}
	100	353	341	1.9×10^{-12}
	150	381	369	2.3×10^{-12}
	300	425	382	3.4×10^{-8}
	400	497	401	1.2×10^{1}
	500	583	501	1.1×10^{1}
	600	676	601	1.1×10^{1}
	800	867	801	9.9×10^{0}

Case	N	K	K^*	$e^*(K^*,N)$
2	0	41	33	1.4×10^{-11}
	50	207	120	1.4×10^{-09}
	100	431	255	9.9×10^{-07}
	150	531	497	4.2×10^{-12}
	300	603	584	7.9×10^{-12}
	400	639	622	8.0×10^{-12}
	500	657	637	3.0×10^{-11}
	600	711	640	4.0×10^{-4}
	800	880	801	1.1×10^{1}

3.3.4 Tightness of the upper bound

We discuss the tightness of the upper bound in Theorem 3.1, where we set $J_{\rm A}=3$. For this purpose, we prepare two approximations to the conditional stationary distributions, whose errors are bounded above by 10^{-13}.

(i) We compute $\boldsymbol{\pi}_{\rm lv}^{{\rm comp},k}(N)$ for $\epsilon=10^{-13}$, and obtain K.

(ii) We first compute $\boldsymbol{\pi}_{\rm lv}^{*}(N):=\boldsymbol{\pi}_{\rm lv}^{{\rm comp},k}(N)$ for $\epsilon=0.5\times10^{-13}$, and then, we compute $\{\boldsymbol{\pi}_{\rm lv}^{{\rm comp},k}(N);\ k\in\mathbb{Z}_{N+1}^{\infty}\}$ and obtain K^*, where

$$K^*:=\min\{k>N;\ \|\boldsymbol{\pi}_{\rm lv}^{{\rm comp},k}(N)-\boldsymbol{\pi}_{\rm lv}^{*}(N)\|_1<0.5\times10^{-13}\}.$$

Finally, we compute the error bound $e^*(K^*,N)$ of the computed conditional stationary distribution for $k=K^*$ (Theorem 3.1), where

$$e^*(K^*,N):=\frac{e_N(K^*,N)}{\overline{\overline{\boldsymbol{\nu}}}_{K^*,N}\,\boldsymbol{w}_N}.$$

From Theorem 3.1, $\|\boldsymbol{\pi}_{\rm lv}^{{\rm comp},K^*}(N)-\boldsymbol{\pi}_{\rm lv}(N)\|_1\le e^*(K^*,N)$. Besides, we have

$$\|\boldsymbol{\pi}_{\rm lv}^{{\rm comp},K^*}(N)-\boldsymbol{\pi}_{\rm lv}(N)\|_1\le\|\boldsymbol{\pi}_{\rm lv}^{{\rm comp},K^*}(N)-\boldsymbol{\pi}_{\rm lv}^{*}(N)\|_1+\|\boldsymbol{\pi}_{\rm lv}^{*}(N)-\boldsymbol{\pi}_{\rm lv}(N)\|_1<10^{-13}.$$

Therefore, $\boldsymbol{\pi}_{\rm lv}^{{\rm comp},K^*}(N)$ always has the target accuracy 10^{-13}, regardless of the value of $e^*(K^*,N)$. Table 3.1 shows K, K^*, and $e^*(K^*,N)$ for various N. We observe that $K>K^*$ in all cases and in most cases, $e^*(K^*,N)$ is fairly large, compared with 10^{-13}. This implies that the upper bound in Theorem 3.1 can be loose and the stopping criterion of our algorithm is conservative. In particular, as stated in Remark 3.5, the upper bound $e^*(K^*,N)$ is trivial for $N=400$, 500, 600, and 800 in Case 1 and for $N=800$ in Case 2.

It is also interesting to observe that $K^*=N+1$ when N is sufficiently large. We discuss this phenomenon below. For a natural number N^* such that $N^*<N$, we partition $\boldsymbol{\pi}_{\rm lv}^{{\rm approx},k}(N)$ $(k>N)$ into two parts.

$$\boldsymbol{\pi}_{\rm lv}^{{\rm approx},k}(N)=\Big(\ \overset{\mathbb{Z}_0^{N^*}}{\boldsymbol{\pi}_{-}^{{\rm approx},k}(N;N^*)}\quad\overset{\mathbb{Z}_{N^*+1}^{N}}{\boldsymbol{\pi}_{+}^{{\rm approx},k}(N;N^*)}\ \Big).$$

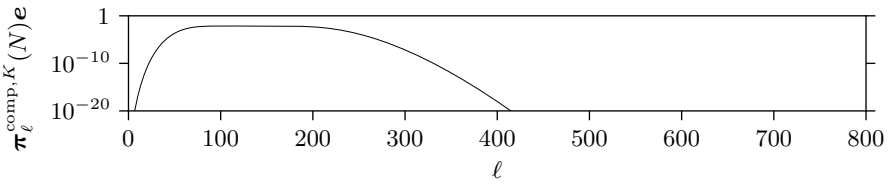


Figure 3.7: The stationary distribution in Case 1 ($J_{\mathrm{A}} = 3$, $J_{\mathrm{S}} = 2$, $N = 800$, $\epsilon = 10^{-13}$, $K = 867$).

Lemma 3.2. *For N^* such that $N^* < N$, suppose*

$$\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},N+1}(N^*) - \boldsymbol{\pi}_{\mathrm{lv}}(N^*)\|_1 < \varepsilon_1, \quad \boldsymbol{\pi}_+^{\mathrm{approx},N+1}(N;N^*)\boldsymbol{e} = \varepsilon_2.$$

We then have

$$\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},N+1}(N) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \le \varepsilon_1 + 2\varepsilon_2 + 2\Pr(L(0) > N^* \mid L(0) \le N). \tag{3.22}$$

Furthermore, for all k ($k \in \mathbb{Z}_{N+1}^{\infty}$),

$$\Pr(L(0) > N^* \mid L(0) \le N) \le \frac{e_N(k,N)}{\overline{\overline{\boldsymbol{\nu}}}_{k,N}\,\boldsymbol{w}_N} + \boldsymbol{\pi}_+^{\mathrm{approx},k}(N;N^*)\boldsymbol{e}. \tag{3.23}$$

Lemma 3.2 implies that we may have $K^* = N + 1$ if the right-hand side of (3.22) is small enough.

To confirm the above discussion, we consider $N = 500$ in Case 1. Figure 3.7 shows the computed stationary distribution in a logarithmic scale, which is the same result as in Figure 3.2 (c). For $N^* = 400$, we have

$$\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},501}(400) - \boldsymbol{\pi}_{\mathrm{lv}}(400)\|_1 \le \frac{e_N(501,400)}{\overline{\overline{\boldsymbol{\nu}}}_{501,400}\,\boldsymbol{w}_{400}} = \varepsilon_1 < 1.8 \times 10^{-14}, \tag{3.24}$$

$$\boldsymbol{\pi}_+^{\mathrm{comp},501}(500;400)\boldsymbol{e} = \varepsilon_2 < 2.7 \times 10^{-18}. \tag{3.25}$$

On the other hand, for $\epsilon = 10^{-14}$, we have $K = 589$, $e_N(K,500)/\overline{\overline{\boldsymbol{\nu}}}_{K,500}\,\boldsymbol{w}_{500} \le 10^{-14}$, and $\boldsymbol{\pi}_+^{\mathrm{comp},K}(500;400)\boldsymbol{e} < 2.7 \times 10^{-18}$, so that $\Pr(L(0) > 400 \mid L(0) \le 500) \le 10^{-14} + 2.7 \times 10^{-18}$. It then follows from Lemma 3.2 that

$$\begin{aligned}
&\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},501}(500) - \boldsymbol{\pi}_{\mathrm{lv}}(500)\|_1 \\
&\qquad \le \varepsilon_1 + 2\varepsilon_2 + 2\Pr(L(0) > 400 \mid L(0) \le 500) \\
&\qquad \le 1.8 \times 10^{-14} + 2 \times 2.7 \times 10^{-18} + 2 \times (10^{-14} + 2.7 \times 10^{-18}) \\
&\qquad < 10^{-13}.
\end{aligned}$$

The above inequality shows that $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},501}(500)$ is a good approximation to $\boldsymbol{\pi}(500)$ under the criterion of 10^{-13}.

3.3.5 Numerical examples for large-scale models

Finally, we present numerical examples for large-scale models, so as to confirm the applicability of our algorithm to such models. We assume $J_{\mathrm{A}} = 11$. We also assume

Table 3.2: Values of K and computation time ($N = 4000$, $\epsilon = 10^{-12}$).

	Exp.	Hyper-Exp.	Erlang
K	4374	4609	4327
time (second)	10.20	24.41	983.22

$\lambda_B = 500$ and $\mathrm{E}[B] = 2$, so that $\rho = 1000$. Furthermore, the service speed μ_ℓ ($\ell \in \mathbb{N}$) is set to be

$$\mu_\ell = \begin{cases} \ell, & \ell = 1, 2, \ldots, 1000, \\ 1000 + 0.25(\ell - 1000), & \ell = 1001, 1002, \ldots, \end{cases}$$

and N is set to be $N = 4000$.

In this setting, we consider three service time distributions with mean one and different coefficient $\mathrm{C_v}$ of variation: 10-stage Erlang distribution ($\mathrm{C_v} = 1/\sqrt{10}$), exponential distribution ($\mathrm{C_v} = 1.0$), and balanced hyper-exponential distribution with $\mathrm{C_v} = 2.0$. Note that in the case of 10-stage Erlang service times, $J_\mathrm{A} \times J_\mathrm{S} \times (N+1) = 440,110$, so that $\boldsymbol{\pi}(4000)$ is a $1 \times 440,110$ vector.

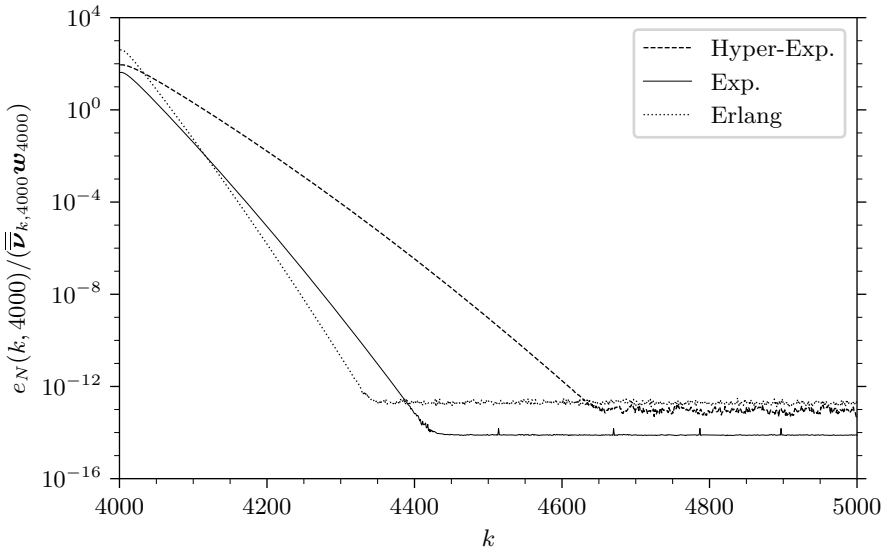
Figure 3.8 shows the upper bound $e_N(k, 4000)/(\overline{\overline{\boldsymbol{\nu}}}_{k,4000}\, \boldsymbol{w}_{4000})$ for the error in the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(4000)$ as a function of k and the computed conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(4000)$. We observe that the shape of $\boldsymbol{\pi}_{\mathrm{lv}}(4000)$ becomes smooth as $\mathrm{C_v}$ increases. Table 3.2 shows the value of K and the computation time for these examples.

3.4 Conclusion

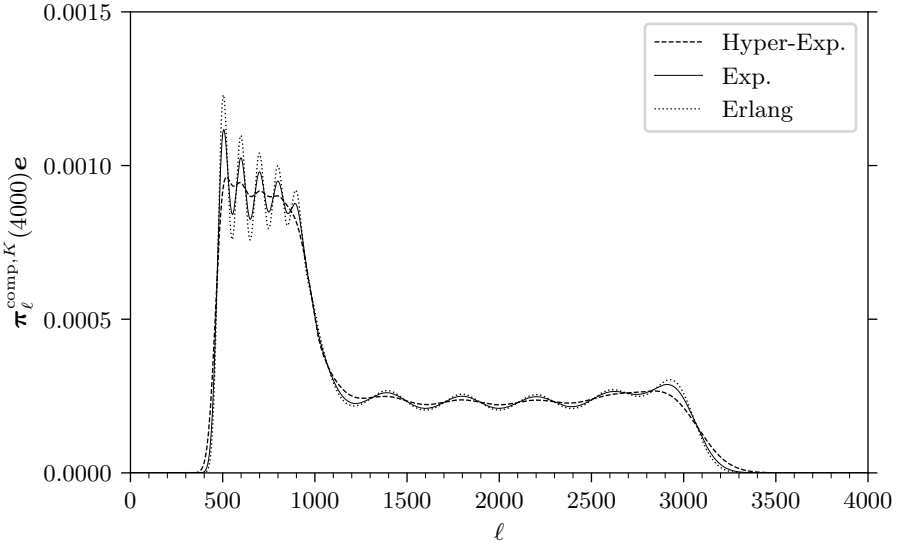
In this chapter, we presented a computational algorithm for the conditional stationary distribution in Markov chains of LD-M/G/1-type, given that the level is not greater than a predefined threshold. The main body of our algorithm is identical to that in [Taki16], which had been developed under the assumption that matrices representing downward jumps are nonsingular. To eliminate this assumption, we utilized the results for Markov chains of LD-G/G/1-type, which is given in Chapter 2.

We first showed the error bound of the computed boundary vector as well as the result in [Taki16] for a special class of Markov chains of LD-M/G/1-type. Furthermore, we derived the error bound of the computed conditional stationary distribution. Therefore, in our algorithm, we can set an error bound for the computed conditional stationary distribution, which distinguishes our algorithm from others.

Through numerical examples, we also showed that (i) unavoidable numerical errors can be estimated, (ii) the algorithm converges monotonically, (iii) the stopping criterion of our algorithm is conservative, and (iv) if N is sufficiently large, we can obtain a good approximation to the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(N)$ by performing the iterative procedure (Step 2-2 in the algorithm) for the computation of the boundary vector $\overline{\boldsymbol{\pi}}_N$ only once.



(a) Upper bounds for errors in $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},k}(4000)$.



(b) Conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{comp},K}(4000)$.

Figure 3.8: Large-scale examples ($J_{\mathrm{A}} = 11$, $N = 4000$, $\epsilon = 10^{-12}$).

4 Characterizations of the Conditional Stationary Distribution based on Northwest Corner Submatrices of the Infinitesimal Generator

4.1 Introduction

This chapter considers an ergodic, continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ on the state space $\mathbb{Z}^+ = \{0, 1, \dots\}$. Recall that its infinitesimal generator $\boldsymbol{Q}$ and stationary distribution $\boldsymbol{\pi}$ are given in (1.1) and (1.3). It is assumed that $q_i < \infty$ ($i \in \mathbb{Z}^+$) and the chain is stationary. We consider the conditional stationary distribution based not on reverse-directional R-matrices but on northwest corner submatrices of the infinitesimal generator $\boldsymbol{Q}$.

The purpose of this chapter is to characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ given that $X(0) \in \mathbb{Z}_0^N = \{0, 1, \dots, N\}$ via systems of linear inequalities, without assuming any regular structures. In other words, we will attempt to find minimum regions on the first orthant $\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} \geq \boldsymbol{0}\}$ of $\mathbb{R}^{N+1}$, which contain $\boldsymbol{\pi}(N)$. Specifically, we first consider $\boldsymbol{\pi}(N)$ based only on the $(N+1) \times (N+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ of the infinitesimal generator $\boldsymbol{Q}$ and we obtain a system of $N+1$ linear inequalities that $\boldsymbol{\pi}(N)$ satisfies. It immediately follows that $\boldsymbol{\pi}(N)$ lies in an N-simplex $\mathcal{P}(N)$ on the first orthant of $\mathbb{R}^{N+1}$, where vertices of $\mathcal{P}(N)$ are determined only by $\boldsymbol{Q}^{(1,1)}(N)$. As we will see, the derivation of these results is absurdly simple. Nonetheless, the $N+1$ vertices of the N-simplex $\mathcal{P}(N)$ (i.e., $N+1$ linearly independent probability vectors that span $\mathcal{P}(N)$) are essential for $\boldsymbol{\pi}(N)$; in any ergodic, continuous-time Markov chain with $\boldsymbol{Q}^{(1,1)}(N)$, $\boldsymbol{\pi}(N)$ can be expressed as a convex combination of those vertices. We also provide a probabilistic interpretation of this result.

Next we consider the case that a subset

$$\mathcal{J}(N) = \{j \in \mathbb{Z}_0^N;\ q_{i,j} > 0 \text{ for some } i \in \mathbb{Z}_{N+1}^{\infty}\} \tag{4.1}$$

of $\mathbb{Z}_0^N$ is available, as well as $\boldsymbol{Q}^{(1,1)}(N)$. Note that $\mathcal{J}(N)$ can be regarded as a sort of structural information because it is a subset of states in $\mathbb{Z}_0^N$, which are directly reachable from at least one state in $\mathbb{Z}_{N+1}^{\infty}$. We show that $\mathcal{J}(N)$ fills the role of eliminating redundant vertices of $\mathcal{P}(N)$ and we obtain a $(|\mathcal{J}(N)|-1)$-simplex $\mathcal{P}^+(N)$ whose relative interior contains $\boldsymbol{\pi}(N)$. In other words, when $\mathcal{J}(N)$ is available, $\boldsymbol{\pi}(N)$ is given by a convex combination of $|\mathcal{J}(N)|$ vertices of $\mathcal{P}^+(N)$ with *positive* weights.

Next we consider an extension of the above results, using state transitions in peripheral states of $\mathbb{Z}_0^N$. Specifically, for a fixed K ($K > N$), we obtain an N-simplex $\widetilde{\mathcal{P}}(K, N)$ ($\subseteq \mathcal{P}(N)$) that contains $\boldsymbol{\pi}(N)$, which is determined only by the $(K+1) \times (K+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(K)$ of $\boldsymbol{Q}$. Furthermore, when both $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available, we show that redundant vertices of $\widetilde{\mathcal{P}}(K, N)$ can be eliminated.

The common feature of all the above results is that we characterize $\boldsymbol{\pi}(N)$ by specifying simplices on the first orthant of $\mathbb{R}^{N+1}$, which contain $\boldsymbol{\pi}(N)$. In order to refine those results further, we consider general convex polytopes, using $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$. For this purpose, we introduce a new structure called (K, N)-*skip-free sets* and using them, we obtain the minimum convex polytope that contains $\boldsymbol{\pi}(N)$, under the condition that only $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available.

The rest of this chapter is organized as follows. In Section 4.2, we obtain simplices that contain $\boldsymbol{\pi}(N)$, based on $\boldsymbol{Q}^{(1,1)}(N)$ (and $\mathcal{J}(N)$). In Section 4.3, we extend the result in the preceding section, using state transitions in peripheral states. In Section 4.4, we introduce skip-free sets and obtain the minimum convex polytope that contains $\boldsymbol{\pi}(N)$. Finally, we conclude this chapter in Section 4.5.

4.2 Characterizations of the conditional stationary distribution $\boldsymbol{\pi}(N)$: Fundamental results

In this section, we characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ via systems of linear inequalities, based on the $(N+1) \times (N+1)$ northwest corner submatrix of the infinitesimal generator $\boldsymbol{Q}$.

4.2.1 Characterization in terms of $\boldsymbol{Q}^{(1,1)}(N)$

In this subsection, we assume that for an arbitrarily fixed $N \in \mathbb{Z}^+$, the stationary distribution $\boldsymbol{\pi}$ of an ergodic, continuous-time Markov chain $\{X(t)\}_{t \geq 0}$ is partitioned as in (1.21) and the infinitesimal generator $\boldsymbol{Q}$ is partitioned as follows.

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^\infty \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^\infty \end{array} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) & \boldsymbol{Q}^{(1,2)}(N) \\ \boldsymbol{Q}^{(2,1)}(N) & \boldsymbol{Q}^{(2,2)}(N) \end{pmatrix} \end{array}. \tag{4.2}$$

We define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ as a collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$, whose infinitesimal generators have a specific $(N+1) \times (N+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$.

We first characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ of Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$. Owing to the ergodicity, the northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ is nonsingular regardless of whether it is irreducible or not. We then define $\boldsymbol{H}(N)$ as

$$\boldsymbol{H}(N) = (-\boldsymbol{Q}^{(1,1)}(N))^{-1}. \tag{4.3}$$

By definition, $\boldsymbol{H}(N) \geq \boldsymbol{O}$ and $\boldsymbol{H}(N)\boldsymbol{e} > \boldsymbol{0}$. Let $\overline{\boldsymbol{H}}(N)$ denote an $(N+1) \times (N+1)$ matrix obtained by normalizing each row of $\boldsymbol{H}(N)$ in such a way that $\overline{\boldsymbol{H}}(N)\boldsymbol{e} = \boldsymbol{e}$.

$$\overline{\boldsymbol{H}}(N) = \mathrm{diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{H}(N).$$

We also define $\Gamma(n)$ ($n \in \mathbb{Z}^+$) as a set of all $(n+1)$-dimensional probability vectors.

$$\Gamma(n) = \{\boldsymbol{\alpha} \in \mathbb{R}^{n+1};\ \boldsymbol{\alpha} \geq \mathbf{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1\}, \quad n \in \mathbb{Z}^+. \tag{4.4}$$

Theorem 4.1. *For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$, we have*

$$\boldsymbol{\pi}(N) \in \mathcal{P}(N), \quad N \in \mathbb{Z}^+, \tag{4.5}$$

where $\mathcal{P}(N)$ ($N \in \mathbb{Z}^+$) denotes an N-simplex on the first orthant of $\mathbb{R}^{N+1}$, which is given by

$$\mathcal{P}(N) = \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \tag{4.6}$$

$$= \left\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \Gamma(N)\right\}. \tag{4.7}$$

Proof. The starting point of the proof is the global balance equation (1.4) for $\boldsymbol{\pi}^{(1)}(N)$.

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N) = \mathbf{0}. \tag{4.8}$$

It then follows from (1.24) and (4.8) that

$$\boldsymbol{\pi}(N)(-\boldsymbol{Q}^{(1,1)}(N)) = \frac{\boldsymbol{\pi}^{(2)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}} \cdot \boldsymbol{Q}^{(2,1)}(N). \tag{4.9}$$

Because $\boldsymbol{\pi}^{(2)}(N)/\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} > \mathbf{0}$ and $\boldsymbol{Q}^{(2,1)}(N) \geq \boldsymbol{O}$, $\boldsymbol{\pi}(N)$ satisfies

$$\boldsymbol{\pi}(N)(-\boldsymbol{Q}^{(1,1)}(N)) \geq \mathbf{0}, \qquad \boldsymbol{\pi}(N)\boldsymbol{e} = 1,$$

from which (4.5) with (4.6) follows.

Next, we show the equivalence between (4.6) and (4.7). We rewrite (4.6) to be

$$\begin{aligned}
&\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{y},\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}(N),\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\}.
\end{aligned}$$

Let $\boldsymbol{\alpha} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})$. Since $\boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N)$, we have $\boldsymbol{x}\boldsymbol{e} = \boldsymbol{\alpha}\boldsymbol{e}$. Furthermore, if $\boldsymbol{y} \geq \mathbf{0}$, we have $\boldsymbol{\alpha} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}) \geq \mathbf{0}$ and vice versa, since $\boldsymbol{H}(N)\boldsymbol{e} > \mathbf{0}$. We thus have

$$\begin{aligned}
&\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\qquad = \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \geq \mathbf{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1\}.
\end{aligned} \tag{4.10}$$

Because $\overline{\boldsymbol{H}}(N)$ is composed of $N+1$ linearly independent probability vectors, $\mathcal{P}(N)$ is an N-simplex on the first orthant of $\mathbb{R}^{N+1}$. □

Theorem 4.1 implies that the conditional stationary distribution $\boldsymbol{\pi}(N)$ of *any* Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ is given by a convex combination of row vectors of $\overline{\boldsymbol{H}}(N)$, i.e., there exists $\boldsymbol{\alpha}^*(N)$ such that

$$\boldsymbol{\pi}(N) = \boldsymbol{\alpha}^*(N)\overline{\boldsymbol{H}}(N), \quad \boldsymbol{\alpha}^*(N) \in \Gamma(N). \tag{4.11}$$

Remark 4.1. *Since $\overline{\boldsymbol{H}}(N)$ is nonsingular, there is a one-to-one correspondence between $\boldsymbol{\pi}(N)$ and $\boldsymbol{\alpha}^*(N)$, i.e., $\boldsymbol{\alpha}^*(N) = \boldsymbol{\pi}(N)(-\boldsymbol{Q}^{(1,1)}(N))\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})$. Consequently, obtaining the conditional stationary distribution $\boldsymbol{\pi}(N)$ in an ergodic, continuous-time Markov chain is equivalent to obtaining the nonnegative weight vector $\boldsymbol{\alpha}^*(N)$.*

The following corollary can be used to evaluate the accuracy of an approximation $\boldsymbol{x}$ to $\boldsymbol{\pi}(N)$.

Corollary 4.1. *For an arbitrary probability vector $\boldsymbol{x} \in \Gamma(N)$, we have*

$$\|\boldsymbol{x} - \boldsymbol{\pi}(N)\|_1 \le \max_{i \in \mathbb{Z}_0^N} \|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i(N)\|_1, \tag{4.12}$$

where $\overline{\boldsymbol{\eta}}_i(N)$ $(i \in \mathbb{Z}_0^N)$ denotes the ith row vector of $\overline{\boldsymbol{H}}(N)$. Furthermore, if $\boldsymbol{x} \in \mathcal{P}(N)$, we have

$$\max_{i \in \mathbb{Z}_0^N} \|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i(N)\|_1 \le \max_{i,j \in \mathbb{Z}_0^N} \|\overline{\boldsymbol{\eta}}_j(N) - \overline{\boldsymbol{\eta}}_i(N)\|_1, \quad \boldsymbol{x} \in \mathcal{P}(N). \tag{4.13}$$

Proof. (4.12) immediately follows from Theorem 4.1 and the convexity of $\mathcal{P}(N)$. Furthermore, if $\boldsymbol{x} \in \mathcal{P}(N)$, there exists a probability vector $\boldsymbol{\beta} = (\beta_0\ \beta_1\ \cdots\ \beta_N) \in \Gamma(N)$ such that

$$\|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i(N)\|_1 = \|\sum_{j \in \mathbb{Z}_0^N} \beta_j \big(\overline{\boldsymbol{\eta}}_j(N) - \overline{\boldsymbol{\eta}}_i(N)\big)\|_1 \le \sum_{j \in \mathbb{Z}_0^N} \beta_j \|\overline{\boldsymbol{\eta}}_j(N) - \overline{\boldsymbol{\eta}}_i(N)\|_1,$$

from which (4.13) follows. □

We now provide a probabilistic interpretation of (4.11). For the sake of convenience, we assume that the Markov chain starts at time $-\infty$ in the rest of this subsection. We then partition the state space $\mathbb{Z}^+$ into $\mathbb{Z}_0^N$ and $\mathbb{Z}_{N+1}^\infty$ and regard $\{X(t)\}_{t \in (-\infty,\infty)}$ as an alternating Markov-renewal process. For an integer $n \in \mathbb{Z}$, let $T_n^{(1)}$ (resp. $T_n^{(2)}$) denote the nth time instant at which $\{X(t)\}_{t \in (-\infty,\infty)}$ enters into $\mathbb{Z}_0^N$ from $\mathbb{Z}_{N+1}^\infty$ (resp. into $\mathbb{Z}_{N+1}^\infty$ from $\mathbb{Z}_0^N$), where we assume $T_n^{(1)} < T_n^{(2)} < T_{n+1}^{(1)}$ without loss of generality. We define $I(t)$ $(t \in (-\infty, \infty))$ as the state at the last renewal epoch before time t, i.e.,

$$I(t) = \begin{cases} X(T_n^{(1)}), & T_n^{(1)} \le t < T_n^{(2)}, \\ X(T_n^{(2)}), & T_n^{(2)} \le t < T_{n+1}^{(1)}. \end{cases}$$

We then consider the joint process $\{(X(t), I(t))\}_{t \in (-\infty,\infty)}$, which is assumed to be stationary.

Corollary 4.2. *For any Markov chain $\{X(t)\}_{t \in (-\infty,\infty)}$ in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$, the (i,j)th $(i,j \in \mathbb{Z}_0^N)$ element of $\overline{\boldsymbol{H}}(N)$ is given by*

$$[\overline{\boldsymbol{H}}(N)]_{i,j} = \Pr(X(0) = j \mid I(0) = i), \quad i,j \in \mathbb{Z}_0^N, \tag{4.14}$$

and the ith $(i \in \mathbb{Z}_0^N)$ element of $\boldsymbol{\alpha}^*(N)$ is given by

$$[\boldsymbol{\alpha}^*(N)]_i = \Pr(I(0) = i \mid X(0) \in \mathbb{Z}_0^N). \tag{4.15}$$

We thus interpret (4.11) *as*

$$[\boldsymbol{\pi}(N)]_j = \sum_{i \in \mathbb{Z}_0^N} \Pr(I(0) = i \mid X(0) \in \mathbb{Z}_0^N) \cdot \Pr(X(0) = j \mid I(0) = i), \quad j \in \mathbb{Z}_0^N. \tag{4.16}$$

The proof of Corollary 4.2 is given in Appendix D.

Note that for a given northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$, $\mathcal{P}(N)$ in Theorem 4.1 may not be tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$, i.e., $\mathcal{P}(N)$ may contain redundant vectors. For example, $\mathcal{P}(N)$ may contain $\boldsymbol{x} \in \mathbb{R}^{N+1}$ such that $[\boldsymbol{x}]_i = 0$ for some $i \in \mathbb{Z}_0^N$, whereas $\boldsymbol{\pi}(N) > \boldsymbol{0}$ since Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ are ergodic. In the next subsection, we eliminate redundancy in $\mathcal{P}(N)$ using structural information $\mathcal{J}(N)$ in (4.1).

4.2.2 Characterization in terms of $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$

For specific $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$, we define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ as a collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$, whose infinitesimal generators have $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$. By definition,

$$\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N)) = \bigcup_{\mathcal{J}(N) \in 2^{\mathbb{Z}_0^N} \setminus \{\emptyset\}} \mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)).$$

Note here that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ may be empty for some pairs of $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$. For example, if $\boldsymbol{Q}^{(1,1)}(N)$ is a diagonal matrix, $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) = \emptyset$ unless $\mathcal{J}(N) = \mathbb{Z}_0^N$, owing to the ergodicity. In the rest of this subsection, we assume that if $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) \neq \emptyset$, $\mathcal{J}(N) = \{0, 1, \ldots, |\mathcal{J}(N)| - 1\}$ without loss of generality.

If $\mathcal{J}(N) \neq \mathbb{Z}_0^N$, we partition $\boldsymbol{Q}^{(1,1)}(N)$ and $\boldsymbol{Q}^{(2,1)}(N)$ in (4.2) into two matrices.

$$\begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) \\ \boldsymbol{Q}^{(2,1)}(N) \end{pmatrix} = \begin{matrix} \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^\infty \end{matrix} \overset{\begin{matrix} \mathcal{J}(N) & \mathbb{Z}_0^N \setminus \mathcal{J}(N) \end{matrix}}{\begin{pmatrix} \boldsymbol{Q}_+^{(1,1)}(N) & \boldsymbol{Q}_0^{(1,1)}(N) \\ \boldsymbol{Q}_+^{(2,1)}(N) & \boldsymbol{O} \end{pmatrix}}. \tag{4.17}$$

(4.9) is then rewritten to be

$$\boldsymbol{\pi}(N) \cdot \overset{\begin{matrix} \mathcal{J}(N) & \mathbb{Z}_0^N \setminus \mathcal{J}(N) \end{matrix}}{\begin{pmatrix} -\boldsymbol{Q}_+^{(1,1)}(N) & -\boldsymbol{Q}_0^{(1,1)}(N) \end{pmatrix}} = \overset{\begin{matrix} \mathcal{J}(N) & \mathbb{Z}_0^N \setminus \mathcal{J}(N) \end{matrix}}{\begin{pmatrix} \dfrac{\boldsymbol{\pi}^{(2)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}} \cdot \boldsymbol{Q}_+^{(2,1)}(N) & \boldsymbol{0} \end{pmatrix}}. \tag{4.18}$$

We define $\Gamma^+(N)$ as

$$\Gamma^+(N) = \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{N+1};\ \boldsymbol{\alpha} \geq \boldsymbol{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1,\ [\boldsymbol{\alpha}]_i = 0\ (i \in \mathbb{Z}_0^N \setminus \mathcal{J}(N)) \right\}. \tag{4.19}$$

Theorem 4.2. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ is not empty. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, we have*

$$\boldsymbol{\pi}(N) \in \mathrm{ri}\, \mathcal{P}^+(N), \quad N \in \mathbb{Z}^+, \tag{4.20}$$

where $\mathcal{P}^+(N)$ $(N \in \mathbb{Z}^+)$ denotes a $(|\mathcal{J}(N)| - 1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$, which is given by $\mathcal{P}^+(N) = \mathcal{P}(N)$ if $\mathcal{J}(N) = \mathbb{Z}_0^N$ and otherwise,

$$\mathcal{P}^+(N) = \left\{ \boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N)) \geq \boldsymbol{0},\ \boldsymbol{x}(-\boldsymbol{Q}_0^{(1,1)}(N)) = \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1 \right\} \tag{4.21}$$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \Gamma^+(N) \right\}. \tag{4.22}$$

Proof. If $\mathcal{J}(N) = \mathbb{Z}_0^N$, we have $\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N) > \mathbf{0}$ since $\boldsymbol{\pi}^{(2)}(N) > \mathbf{0}$. Theorem 4.2 for $\mathcal{J}(N) = \mathbb{Z}_0^N$ then follows from (1.11), (4.9), and Theorem 4.1. On the other hand, if $\mathcal{J}(N) \neq \mathbb{Z}_0^N$, $\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}_+^{(2,1)}(N) > \mathbf{0}$. (4.20) with (4.21) then follows from (1.11) and (4.18). The equivalence between (4.21) and (4.22) for $\mathcal{J}(N) \neq \mathbb{Z}_0^N$ can be shown in a way similar to the proof of Theorem 4.1, as shown in Appendix E. Since $\overline{\boldsymbol{H}}(N)$ is nonnegative and nonsingular, $\mathcal{P}^+(N)$ is a $(|\mathcal{J}(N)| - 1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$. □

By definition, we have $\Gamma^+(N) \subseteq \Gamma(N)$ and therefore,

$$\mathcal{P}^+(N) \subseteq \mathcal{P}(N), \tag{4.23}$$

where $\mathcal{P}^+(N) = \mathcal{P}(N)$ iff $\mathcal{J}(N) = \mathbb{Z}_0^N$. Theorem 4.2 indicates the importance of the structural information $\mathcal{J}(N)$ about direct transitions from $\mathbb{Z}_{N+1}^\infty$ to $\mathbb{Z}_0^N$. For example, if $\mathcal{J}(N)$ is a singleton for a certain N, say, $\mathcal{J}(N) = \{i\}$ ($i \in \mathbb{Z}_0^N$), $\boldsymbol{\pi}(N)$ is given by the ith row vector of $\overline{\boldsymbol{H}}(N)$, as pointed out in [Zha04, Corollary 3].

The following corollary follows from Theorem 4.2, whose proof is omitted because it is almost the same as Corollary 4.1.

Corollary 4.3. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ is not empty. For an arbitrary probability vector $\boldsymbol{x} \in \Gamma(N)$, we have*

$$\|\boldsymbol{x} - \boldsymbol{\pi}(N)\|_1 \leq \max_{i \in \mathcal{J}(N)} \|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i(N)\|_1.$$

Furthermore, if $\boldsymbol{x} \in \mathcal{P}^+(N)$, we have

$$\max_{i \in \mathcal{J}(N)} \|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i(N)\|_1 \leq \max_{i,j \in \mathcal{J}(N)} \|\overline{\boldsymbol{\eta}}_j(N) - \overline{\boldsymbol{\eta}}_i(N)\|_1, \quad \boldsymbol{x} \in \mathcal{P}^+(N).$$

Example 4.1. *Consider an ergodic, level-dependent $M^X/M/1$ queue with disasters, whose infinitesimal generator $\boldsymbol{Q}$ takes the following form.*

$$\boldsymbol{Q} = \begin{pmatrix} -q_0 & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & q_{0,5} & \cdots \\ q_{1,0} & -q_1 & q_{1,2} & q_{1,3} & q_{1,4} & q_{1,5} & \cdots \\ q_{2,0} & q_{2,1} & -q_2 & q_{2,3} & q_{2,4} & q_{2,5} & \cdots \\ q_{3,0} & 0 & q_{3,2} & -q_3 & q_{3,4} & q_{3,5} & \cdots \\ q_{4,0} & 0 & 0 & q_{4,3} & -q_4 & q_{4,5} & \cdots \\ q_{5,0} & 0 & 0 & 0 & q_{5,4} & -q_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $q_{i,0} > 0$ and $q_{i,i-1} > 0$ ($i \in \mathbb{N}$). We then have

$$\mathcal{J}(N) = \{0, N\},$$

because the state immediately after a downward transition from state i ($i \geq N+1$) is either state 0 or state $i-1$. It then follows that

$$\Gamma^+(N) = \big\{\begin{pmatrix} \alpha_0 & 0 & 0 & \cdots & 0 & 1-\alpha_0 \end{pmatrix} \in \mathbb{R}^{N+1};\ \alpha_0 \in [0,1]\big\},$$

and

$$\boldsymbol{\pi}(N) \in \operatorname{ri} \mathcal{P}^+(N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \alpha_0 \overline{\boldsymbol{\eta}}_0(N) + (1-\alpha_0)\overline{\boldsymbol{\eta}}_N(N),\ \alpha_0 \in (0,1)\big\}.$$

Note here that $\overline{\boldsymbol{\eta}}_i(N)$ $(i \in \mathbb{Z}_0^N)$ *can be obtained by (i) solving* $\boldsymbol{x}_i(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{e}_i^T$ *for* $\boldsymbol{x}_i \in \mathbb{R}^{N+1}$ *and (ii) let* $\overline{\boldsymbol{\eta}}_i(N) = \boldsymbol{x}_i/\boldsymbol{x}_i\boldsymbol{e}$. *Recall that* $\boldsymbol{e}_i$ *denotes the* i*th unit column vector. The simplest way to obtain an approximation* $\boldsymbol{\pi}^{\mathrm{approx}}(N) \in \mathrm{ri}\,\mathcal{P}^+(N)$ *to* $\boldsymbol{\pi}(N)$ *might be to solve* $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}) = \beta_0\boldsymbol{e}_0^T + (1-\beta_0)\boldsymbol{e}_N^T$ *for an arbitrary* $\beta_0 \in (0,1)$ *and let* $\boldsymbol{\pi}^{\mathrm{approx}}(N) = \boldsymbol{x}/\boldsymbol{x}\boldsymbol{e}$.

We now show that $\mathcal{P}^+(N)$ is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, i.e., $\mathcal{P}^+(N)$ does not contain any redundant vectors.

Lemma 4.1. *If* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) \neq \emptyset$ *and* $\boldsymbol{x} \in \mathrm{ri}\,\mathcal{P}^+(N)$, *we have* $\boldsymbol{x} > \boldsymbol{0}$.

Proof. We assume $[\boldsymbol{x}]_{j^*} = 0$ for some $j^* \in \mathbb{Z}_0^N$ and show a contradiction. Since $\boldsymbol{x} = \boldsymbol{\alpha}(\boldsymbol{x})\overline{\boldsymbol{H}}(N)$ for some $\boldsymbol{\alpha}(\boldsymbol{x}) \in \mathrm{ri}\,\Gamma^+(N)$, $[\boldsymbol{x}]_{j^*} = 0$ is equivalent to $[\boldsymbol{\alpha}(\boldsymbol{x})\overline{\boldsymbol{H}}(N)]_{j^*} = 0$ for some $\boldsymbol{\alpha}(\boldsymbol{x}) \in \mathrm{ri}\,\Gamma^+(N)$. This implies that $[\overline{\boldsymbol{H}}(N)]_{i,j^*} = 0$ for all $i \in \mathcal{J}(N)$ since $[\boldsymbol{\alpha}(\boldsymbol{x})]_i > 0$ for all $i \in \mathcal{J}(N)$ and $\overline{\boldsymbol{H}}(N) \geq \boldsymbol{O}$. It then follows from (4.14) that $\Pr(X(0) = j^* \mid I(0) \in \mathcal{J}(N)) = 0$ for any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$. Furthermore, $\Pr(I(0) \in \mathbb{Z}_0^N \setminus \mathcal{J}(N)) = 0$ and $\Pr(X(0) = j^* \mid I(0) \in \mathbb{Z}_{N+1}^\infty) = 0$ by definition. We thus have

$$\begin{aligned}\Pr(X(0) = j^*) &= \Pr(I(0) \in \mathcal{J}(N)) \Pr(X(0) = j^* \mid I(0) \in \mathcal{J}(N)) \\ &\qquad + \Pr(I(0) \in \mathbb{Z}_{N+1}^\infty) \Pr(X(0) = j^* \mid I(0) \in \mathbb{Z}_{N+1}^\infty) \\ &= 0,\end{aligned}$$

for any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, which contradicts the ergodicity. □

Theorem 4.3. *Suppose that* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ *is not empty. For any* $\boldsymbol{x} \in \mathrm{ri}\,\mathcal{P}^+(N)$, *there exists a Markov chain in* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, *whose* $\boldsymbol{\pi}(N)$ *is given by* $\boldsymbol{x}$.

Proof. It follows from (4.23) that if $\boldsymbol{x} \in \mathrm{ri}\,\mathcal{P}^+(N)$, we have $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \boldsymbol{0}$ and $\boldsymbol{x}\boldsymbol{e} = 1$ because of $\boldsymbol{x} \in \mathcal{P}(N)$. Note also that $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \neq \boldsymbol{0}$ because if $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{0}$ held, we would have $\boldsymbol{x} = \boldsymbol{0} \cdot (-\boldsymbol{Q}^{(1,1)}(N))^{-1} = \boldsymbol{0}$, which contradicts $\boldsymbol{x}\boldsymbol{e} = 1$. We then define $\boldsymbol{\zeta}(\boldsymbol{x})$ as

$$\boldsymbol{\zeta}(\boldsymbol{x}) = \frac{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}. \tag{4.24}$$

It is easy to see that $\boldsymbol{\zeta}(\boldsymbol{x}) \in \mathrm{ri}\,\Gamma^+(N)$ for $\boldsymbol{x} \in \mathrm{ri}\,\mathcal{P}^+(N)$. For an arbitrarily chosen $\boldsymbol{x} \in \mathrm{ri}\,\mathcal{P}^+(N)$, we consider a Markov chain whose infinitesimal generator $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^\infty \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^\infty \end{array} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) & \boldsymbol{Q}^{(1,2)}(N) \\ (-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x}) & \boldsymbol{Q}^{(2,2)}(N) \end{pmatrix} \end{array}. \tag{4.25}$$

Its global balance equation is then given by

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{\pi}^{(2)}(N)(-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x}) = \boldsymbol{0}, \tag{4.26}$$

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{Q}^{(1,2)}(N) + \boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,2)}(N) = \boldsymbol{0}. \tag{4.27}$$

From (4.24) and (4.26), we observe that the special $\boldsymbol{Q}^{(2,1)}(N) = (-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x})$ ensures $\boldsymbol{\pi}^{(1)}(N) = c_1\boldsymbol{x}$ for some positive constant c_1. We now set

$$\boldsymbol{Q}^{(1,2)}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{z}, \qquad \boldsymbol{Q}^{(2,2)}(N) = -\boldsymbol{I}, \tag{4.28}$$

where $\boldsymbol{z}$ denotes a $1 \times \infty$ positive probability vector. (4.27) is then reduced to

$$\boldsymbol{\pi}^{(1)}(N)(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{z} - \boldsymbol{\pi}^{(2)}(N) = \boldsymbol{0}, \tag{4.29}$$

which indicates that (4.28) ensures $\boldsymbol{\pi}^{(2)}(N) = c_2\boldsymbol{z} > \boldsymbol{0}$ for some positive constant c_2. In fact, solving (4.26) and (4.29) with $\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} + \boldsymbol{\pi}^{(2)}(N)\boldsymbol{e} = 1$, we obtain

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)}(N) & \boldsymbol{\pi}^{(2)}(N) \end{pmatrix} = \frac{1}{1 + \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \begin{pmatrix} \boldsymbol{x} & \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \boldsymbol{z} \end{pmatrix} > \boldsymbol{0}.$$

We thus conclude that the Markov chain with the infinitesimal generator $\boldsymbol{Q}$ defined by (4.25) and (4.28) is a member of $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ and it has $\boldsymbol{\pi}(N) = \boldsymbol{x}$. □

Example 4.2 (cont'd)**.** *Consider $\boldsymbol{Q}$ in Example 4.1 of Section 4.2.2. Theorem 4.2 implies that there exists $\alpha_0 = \alpha_0^* \in (0,1)$ such that $\boldsymbol{\pi}(N) = \alpha_0^*\overline{\boldsymbol{\eta}}_0(N) + (1-\alpha_0^*)\overline{\boldsymbol{\eta}}_N(N)$. As stated in Theorem 4.3, however, $\mathrm{ri}\,\mathcal{P}^+(N)$ is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$. Therefore, we cannot identify the exact $\alpha_0^* \in (0,1)$ if only $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N) = \{0, N\}$ are available.*

4.3 Extensions using state transitions in peripheral states

In this section, we consider extensions of the results in Section 4.2 by using the $(K+1) \times (K+1)$ northwest corner submatrix of the infinitesimal generator $\boldsymbol{Q}$, where $K > N$. For this purpose, we partition $\boldsymbol{\pi}$ and $\boldsymbol{Q}$ as follows.

$$\begin{aligned}
\boldsymbol{\pi} &= \begin{array}{c} \quad\mathbb{Z}_0^K \qquad\quad \mathbb{Z}_{K+1}^{\infty} \\ \begin{pmatrix} \boldsymbol{\pi}^{(1)}(K) & \boldsymbol{\pi}^{(2)}(K) \end{pmatrix} \end{array} = \begin{array}{c} \quad\mathbb{Z}_0^N \qquad\quad \mathbb{Z}_{N+1}^K \qquad\quad \mathbb{Z}_{K+1}^{\infty} \\ \left(\begin{array}{cc|c} \boldsymbol{\pi}^{(1)}(K,N) & \boldsymbol{\pi}^{(2)}(K,N) & \boldsymbol{\pi}^{(3)}(K,N) \end{array}\right) \end{array}, \\
\boldsymbol{Q} &= \begin{array}{c} \\ \mathbb{Z}_0^K \\ \mathbb{Z}_{K+1}^{\infty} \end{array} \begin{array}{c} \quad\mathbb{Z}_0^K \qquad\quad \mathbb{Z}_{K+1}^{\infty} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(K) & \boldsymbol{Q}^{(1,2)}(K) \\ \boldsymbol{Q}^{(2,1)}(K) & \boldsymbol{Q}^{(2,2)}(K) \end{pmatrix} \end{array} \\
&= \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^K \\ \hline \mathbb{Z}_{K+1}^{\infty} \end{array} \begin{array}{c} \quad\mathbb{Z}_0^N \qquad\quad \mathbb{Z}_{N+1}^K \qquad\quad \mathbb{Z}_{K+1}^{\infty} \\ \left(\begin{array}{cc|c} \boldsymbol{Q}^{(1,1)}(K,N) & \boldsymbol{Q}^{(1,2)}(K,N) & \boldsymbol{Q}^{(1,3)}(K,N) \\ \boldsymbol{Q}^{(2,1)}(K,N) & \boldsymbol{Q}^{(2,2)}(K,N) & \boldsymbol{Q}^{(2,3)}(K,N) \\ \hline \boldsymbol{Q}^{(3,1)}(K,N) & \boldsymbol{Q}^{(3,2)}(K,N) & \boldsymbol{Q}^{(3,3)}(K,N) \end{array}\right) \end{array}.
\end{aligned} \tag{4.30}$$

Note here that from (1.21) and (4.2),

$$\begin{aligned}
\boldsymbol{\pi}^{(1)}(K,N) &= \boldsymbol{\pi}^{(1)}(N), & \boldsymbol{\pi}^{(3)}(K,N) &= \boldsymbol{\pi}^{(2)}(K), \\
\boldsymbol{Q}^{(1,1)}(K,N) &= \boldsymbol{Q}^{(1,1)}(N), & \boldsymbol{Q}^{(3,3)}(K,N) &= \boldsymbol{Q}^{(2,2)}(K).
\end{aligned}$$

In order to utilize the results in Section 4.2, we consider the censored Markov chain $\{\widetilde{X}(t)\}_{t\geq 0}$ obtained by observing the Markov chain $\{X(t)\}_{t\geq 0}$ only when $X(t) \in \mathbb{Z}_0^N \cup \mathbb{Z}_{K+1}^{\infty}$. The infinitesimal generator of $\{\widetilde{X}(t)\}_{t\geq 0}$ is then given by

$$\widetilde{\boldsymbol{Q}}(K,N) = \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{K+1}^{\infty} \end{array} \begin{array}{c} \quad\mathbb{Z}_0^N \qquad\quad \mathbb{Z}_{K+1}^{\infty} \\ \begin{pmatrix} \widetilde{\boldsymbol{Q}}^{(1,1)}(K,N) & \widetilde{\boldsymbol{Q}}^{(1,3)}(K,N) \\ \widetilde{\boldsymbol{Q}}^{(3,1)}(K,N) & \widetilde{\boldsymbol{Q}}^{(3,3)}(K,N) \end{pmatrix} \end{array}, \tag{4.31}$$

where for $i, j = 1, 3$,

$$\widetilde{\boldsymbol{Q}}^{(i,j)}(K,N) = \boldsymbol{Q}^{(i,j)}(K,N) + \boldsymbol{Q}^{(i,2)}(K,N)(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,j)}(K,N). \quad (4.32)$$

In particular, we have

$$\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N) = \boldsymbol{Q}^{(1,1)}(N)(\boldsymbol{I} - \boldsymbol{R}(K,N)), \quad (4.33)$$

where

$$\boldsymbol{R}(K,N) = (-\boldsymbol{Q}^{(1,1)}(N))^{-1}\boldsymbol{Q}^{(1,2)}(K,N)(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N). \quad (4.34)$$

Note that $(\boldsymbol{I} - \boldsymbol{R}(K,N))$ in (4.33) is nonsingular because $\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N)$ and $\boldsymbol{Q}^{(1,1)}(N)$ are nonsingular.

Apparently, Theorem 4.1 is applicable to the censored Markov chain $\{\widetilde{X}(t)\}_{t\geq 0}$ on $\mathbb{Z}_0^N \cup \mathbb{Z}_{K+1}^{\infty}$. Note also that the conditional stationary distribution of $\{\widetilde{X}(t)\}_{t\geq 0}$ given $\widetilde{X}(0) \in \mathbb{Z}_0^N$ is identical to $\boldsymbol{\pi}(N)$. To proceed further, we define $\widetilde{\Gamma}(K,N)$ $(K > N)$ as

$$\widetilde{\Gamma}(K,N) = \big\{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}; \boldsymbol{\alpha} = \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}), \\ \boldsymbol{\beta} \in \Gamma(N)\big\}, \quad (4.35)$$

where

$$\widetilde{\boldsymbol{H}}(K,N) = (-\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N))^{-1}. \quad (4.36)$$

Remark 4.2. *It can be verified that $\widetilde{\boldsymbol{H}}(K,N)$ is identical to the $(N+1)\times(N+1)$ northwest corner submatrix of the $(K+1)\times(K+1)$ matrix $\boldsymbol{H}(K) = (-\boldsymbol{Q}^{(1,1)}(K))^{-1}$.*

For a specific $(K+1)\times(K+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(K)$, we define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K))$ as a collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$, whose infinitesimal generators have $\boldsymbol{Q}^{(1,1)}(K)$.

Theorem 4.4. *For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K))$, we have*

$$\boldsymbol{\pi}(N) \in \widetilde{\mathcal{P}}(K,N), \quad K, N \in \mathbb{Z}^+,\ K > N,$$

where $\widetilde{\mathcal{P}}(K,N)$ denotes an N-simplex on the first orthant of $\mathbb{R}^{N+1}$, which is given by

$$\widetilde{\mathcal{P}}(K,N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))(\boldsymbol{I} - \boldsymbol{R}(K,N)) \geq \boldsymbol{0}, \boldsymbol{x}\boldsymbol{e} = 1\big\} \quad (4.37)$$

$$= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \widetilde{\Gamma}(K,N)\}, \quad (4.38)$$

with $\boldsymbol{R}(K,N)$ in (4.34).

Proof. Associated with $\widetilde{\boldsymbol{H}}(K,N)$ in (4.36), we define $\overline{\widetilde{\boldsymbol{H}}}(K,N)$ as

$$\overline{\widetilde{\boldsymbol{H}}}(K,N) = \mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})\widetilde{\boldsymbol{H}}(K,N). \quad (4.39)$$

Applying Theorem 4.1 to (4.31), we obtain $\boldsymbol{\pi}(N) \in \widehat{\mathcal{P}}(K,N)$, where $\widehat{\mathcal{P}}(K,N)$ has two equivalent expressions:

$$\widehat{\mathcal{P}}(K,N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N)) \geq \boldsymbol{0}, \boldsymbol{x}\boldsymbol{e} = 1\big\}$$

$$= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\beta}\overline{\widetilde{\boldsymbol{H}}}(K,N),\ \boldsymbol{\beta} \in \Gamma(N)\big\}. \quad (4.40)$$

Because of (4.33), $\widetilde{\mathcal{P}}(K,N)$ in (4.37) is identical to $\widehat{\mathcal{P}}(K,N)$.

Next, we show the equivalence between (4.38) and (4.40). It follows from (4.33) and (4.36) that

$$\begin{aligned}\widetilde{\boldsymbol{H}}(K,N) = (-\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N))^{-1} &= (\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}(-\boldsymbol{Q}^{(1,1)}(N))^{-1} \\ &= (\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\boldsymbol{H}(N). \end{aligned} \tag{4.41}$$

Therefore, for an arbitrary $\boldsymbol{\beta} \in \Gamma(N)$, we have from (4.39) and (4.41),

$$\begin{aligned}\boldsymbol{\beta}\widetilde{\overline{\boldsymbol{H}}}(K,N) &= \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\boldsymbol{H}(N) \\ &= \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N) \\ &= \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N), \end{aligned} \tag{4.42}$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}), \tag{4.43}$$

which shows the equivalence between (4.38) and (4.40). We thus conclude that

$$\widetilde{\mathcal{P}}(K,N) = \widehat{\mathcal{P}}(K,N). \tag{4.44}$$

$\widetilde{\mathcal{P}}(K,N)$ is an N-simplex on the first orthant of $\mathbb{R}^{N+1}$ because of the nonsingularity of $\widetilde{\overline{\boldsymbol{H}}}(K,N) \geq \boldsymbol{O}$, (4.40), and (4.44). □

Corollary 4.4. $\widetilde{\Gamma}(K,N)$ *in* (4.35) *is a subset of* $\Gamma(N)$.

$$\widetilde{\Gamma}(K,N) \subseteq \Gamma(N), \quad N,K \in \mathbb{Z}^+,\ N<K. \tag{4.45}$$

Furthermore,

$$\widetilde{\Gamma}(K_2,N) \subseteq \widetilde{\Gamma}(K_1,N), \quad N,K_1,K_2 \in \mathbb{Z}^+,\ N<K_1<K_2, \tag{4.46}$$

and therefore

$$\widetilde{\mathcal{P}}(K_2,N) \subseteq \widetilde{\mathcal{P}}(K_1,N) \subseteq \mathcal{P}(N), \tag{4.47}$$

where $\mathcal{P}(N)$ *is given in Theorem 4.1.*

The proof of Corollary 4.4 is given in Appendix F.

Next, we consider an extension of the results in Section 4.2.2. For specific $(K+1)\times(K+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(K)$ and structural information $\mathcal{J}(K)$ $(K>N)$, we define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K),\mathcal{J}(K))$ as a collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$, whose infinitesimal generators have $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$. For Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K),\mathcal{J}(K))$, we define $\widetilde{\mathcal{J}}(K,N)$ $(K>N)$ as a set of states in $\mathbb{Z}_0^N$, which is reachable from some states in $\mathbb{Z}_{K+1}^{\infty}$ directly or via only some states in $\mathbb{Z}_{N+1}^K$.

$$\widetilde{\mathcal{J}}(K,N) = \{j \in \mathbb{Z}_0^N;\ [\boldsymbol{e}^T\widetilde{\boldsymbol{Q}}^{(3,1)}(K,N)]_j > 0\}. \tag{4.48}$$

Note here that $\widetilde{\mathcal{J}}(K,N) \neq \emptyset$ because of the ergodicity. Note also that $\widetilde{\mathcal{J}}(K,N)$ is determined completely by $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$, because from (4.1), (4.30), and (4.32),

$$\begin{aligned}\mathcal{J}(K) &= \{j \in \mathbb{Z}_0^N;\ \left[\boldsymbol{e}^T\boldsymbol{Q}^{(3,1)}(K,N)\right]_j > 0\} \cup \{j \in \mathbb{Z}_{N+1}^K;\ \left[\boldsymbol{e}^T\boldsymbol{Q}^{(3,2)}(K,N)\right]_j > 0\}, \\ \widetilde{\boldsymbol{Q}}^{(3,1)}(K,N) &= \boldsymbol{Q}^{(3,1)}(K,N) + \boldsymbol{Q}^{(3,2)}(K,N)\cdot(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N). \end{aligned}$$

Furthermore, by definition,

$$\widetilde{\mathcal{J}}(K,N) \subseteq \mathcal{J}(N), \tag{4.49}$$

and states in $\mathcal{J}(N) \setminus \widetilde{\mathcal{J}}(K,N)$, if any, can be reached from $\mathbb{Z}_{K+1}^{\infty}$ only after visiting some states in $\widetilde{\mathcal{J}}(K,N)$.

Example 4.3 (cont'd)**.** *Consider $\boldsymbol{Q}$ in Example 4.1 of Section 4.2.2. By definition, $\mathcal{J}(K) = \{0, K\}$ and $\widetilde{\mathcal{J}}(K,N) = \{0, N\} = \mathcal{J}(N)$.*

Without loss of generality, we assume $\widetilde{\mathcal{J}}(K,N) = \{0, 1, \ldots, |\widetilde{\mathcal{J}}(K,N)|-1\}$. We then partition $\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N)$ in (4.31) into two matrices:

$$\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N) = \begin{pmatrix} \overset{\widetilde{\mathcal{J}}(K,N)}{\widetilde{\boldsymbol{Q}}_{+}^{(1,1)}(K,N)} & \overset{\mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K,N)}{\widetilde{\boldsymbol{Q}}_{0}^{(1,1)}(K,N)} \end{pmatrix}, \tag{4.50}$$

and define $\widetilde{\Gamma}^{+}(K,N)$ as

$$\widetilde{\Gamma}^{+}(K,N) = \big\{\boldsymbol{\alpha} \in \mathbb{R}^{N+1};\ \boldsymbol{\alpha} = \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}),$$
$$\boldsymbol{\beta} \in \Gamma(N),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K,N))\big\}. \tag{4.51}$$

Theorem 4.5. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ is not empty. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ $(K > N)$, we have*

$$\boldsymbol{\pi}(N) \in \mathrm{ri}\,\widetilde{\mathcal{P}}^{+}(K,N), \quad K, N \in \mathbb{Z}^{+},\ K > N, \tag{4.52}$$

where $\widetilde{\mathcal{P}}^{+}(K,N)$ denotes a $(|\widetilde{\mathcal{J}}(K,N)|-1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$, which is given by $\widetilde{\mathcal{P}}(K,N)$ if $\widetilde{\mathcal{J}}(K,N) = \mathbb{Z}_0^N$ and otherwise,

$$\widetilde{\mathcal{P}}^{+}(K,N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\widetilde{\boldsymbol{Q}}_{+}^{(1,1)}(K,N)) \geq \boldsymbol{0},$$
$$\boldsymbol{x}(-\widetilde{\boldsymbol{Q}}_{0}^{(1,1)}(K,N)) = \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\big\} \tag{4.53}$$
$$= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \widetilde{\Gamma}^{+}(K,N)\big\}. \tag{4.54}$$

Proof. If $\widetilde{\mathcal{J}}(K,N) = \mathbb{Z}_0^N$, Theorem 4.5 can be shown in the same way as Theorem 4.2 for $\mathcal{J}(N) = \mathbb{Z}_0^N$. We thus assume that $\widetilde{\mathcal{J}}(K,N) \neq \mathbb{Z}_0^N$. Applying Theorem 4.2 to the censored Markov chain with infinitesimal generator $\widetilde{\boldsymbol{Q}}(K,N)$ in (4.31) with partition (4.50), we obtain (4.52) with (4.53), where (4.53) can also be expressed as follows.

$$\widetilde{\mathcal{P}}^{+}(K,N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\beta}\overline{\overline{\boldsymbol{H}}}(K,N),$$
$$\boldsymbol{\beta} \in \Gamma(N),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K,N))\big\}, \tag{4.55}$$

where $\overline{\overline{\boldsymbol{H}}}(K,N)$ is given by (4.39). (4.54) now follows from (4.42) and (4.55). Since $\overline{\overline{\boldsymbol{H}}}(K,N)$ is nonnegative and nonsingular, (4.55) implies that $\widetilde{\mathcal{P}}^{+}(K,N)$ is a $(|\widetilde{\mathcal{J}}(K,N)|-1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$. □

By definition,

$$\widetilde{\Gamma}^{+}(K,N) \subseteq \widetilde{\Gamma}(K,N), \tag{4.56}$$

where $\widetilde{\Gamma}(K,N)$ is given by (4.35). We thus have

$$\widetilde{\mathcal{P}}^{+}(K,N) \subseteq \widetilde{\mathcal{P}}(K,N), \tag{4.57}$$

where $\widetilde{\mathcal{P}}(K,N)$ is given in Theorem 4.4.

Corollary 4.5. *$\widetilde{\Gamma}^+(K,N)$ in (4.51) is a subset of $\Gamma^+(N)$ in (4.19).*

$$\widetilde{\Gamma}^+(K,N) \subseteq \Gamma^+(N). \tag{4.58}$$

Furthermore,

$$\widetilde{\Gamma}^+(K_2,N) \subseteq \widetilde{\Gamma}^+(K_1,N), \quad N,K_1,K_2 \in \mathbb{Z}^+,\ N<K_1<K_2, \tag{4.59}$$

and therefore

$$\widetilde{\mathcal{P}}^+(K_2,N) \subseteq \widetilde{\mathcal{P}}^+(K_1,N) \subseteq \mathcal{P}^+(N). \tag{4.60}$$

The proof of Corollary 4.5 is given in Appendix G. Since $\widetilde{\mathcal{P}}(K,N)$ in Theorem 4.4 and $\widetilde{\mathcal{P}}^+(K,N)$ in Theorem 4.5 are compact, Corollaries 4.4 and 4.5 suggest that those simplices converge to certain sets as K goes to infinity. In fact, we have the following proposition.

Proposition 4.1 ([Mas18, Theorem 2.3])**.** *In any ergodic, continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}^+$,*

$$\lim_{K\to\infty} \overline{\widetilde{\boldsymbol{H}}}(K,N) = \boldsymbol{e}\boldsymbol{\pi}(N), \quad N \in \mathbb{Z}^+.$$

The following corollary immediately follows from (4.40), (4.44), (4.47), (4.55), (4.57), (4.60), and Proposition 4.1.

Corollary 4.6. *In any ergodic, continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}^+$,*

$$\bigcap_{K=N+1}^{\infty} \widetilde{\mathcal{P}}(K,N) = \lim_{K\to\infty} \widetilde{\mathcal{P}}(K,N) = \{\boldsymbol{\pi}(N)\}, \quad N \in \mathbb{Z}^+,$$

and

$$\bigcap_{K=N+1}^{\infty} \widetilde{\mathcal{P}}^+(K,N) = \lim_{K\to\infty} \widetilde{\mathcal{P}}^+(K,N) = \{\boldsymbol{\pi}(N)\}, \quad N \in \mathbb{Z}^+.$$

Since $\mathrm{ri}\,\widetilde{\mathcal{P}}^+(K,N) \subseteq \widetilde{\mathcal{P}}^+(K,N)$, $\mathrm{ri}\,\widetilde{\mathcal{P}}^+(K,N)$ also converges to $\{\boldsymbol{\pi}(N)\}$ as K goes to infinity. Note here that $\mathrm{ri}\,\widetilde{\mathcal{P}}^+(K,N)$ may not be tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K),\mathcal{J}(K))$ for a finite K. For example, we consider $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N+1),\mathcal{J}(N+1))$, where $K=N+1$, $\mathcal{J}(N+1)=\{N+1\}$, and $q_{N+1,j}>0$ for all $j\in\mathbb{Z}_0^N$. We then have $\widetilde{\mathcal{J}}(N+1,N)=\mathbb{Z}_0^N$ and therefore $\widetilde{\mathcal{P}}^+(N+1,N)$ is an N-simplex spanned by $(N+1)$ row vectors of $\overline{\widetilde{\boldsymbol{H}}}(N+1,N)$, as shown in the proof of Theorem 4.5. On the other hand, it follows from Theorem 4.2 that $\mathcal{P}^+(N+1)$ is given by a singleton with the $(N+1)$st row vector $\overline{\boldsymbol{\eta}}_{N+1}(N+1)$ of $\overline{\boldsymbol{H}}(N+1)$. In other words, all Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N+1),\{N+1\})$ have the same $\boldsymbol{\pi}(N)$ (i.e., the normalized vector of the first $N+1$ elements of $\overline{\boldsymbol{\eta}}_{N+1}(N+1)$), so that $\mathrm{ri}\,\widetilde{\mathcal{P}}^+(N+1,N)$ is not tight. In the next section, we find minimum convex polytopes that contain $\boldsymbol{\pi}(N)$ in a sense that it is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K),\mathcal{J}(K))$.

4.4 Minimum convex polytopes for $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$

In this section, we consider minimum convex polytopes that contain $\boldsymbol{\pi}(N)$ for Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$. The outline of our approach can be summarized as follows. For convenience, we partition $\boldsymbol{H}(K) = (-\boldsymbol{Q}^{(1,1)}(K))^{-1}$ and $\boldsymbol{\pi}(K)$ as follows.

$$\begin{aligned} \boldsymbol{H}(K) &= \begin{array}{c} \mathbb{Z}_0^N \quad\quad\quad\quad \mathbb{Z}_{N+1}^K \\ \big(\ \boldsymbol{H}^{(*,1)}(K;\, N) \quad \boldsymbol{H}^{(*,2)}(K;\, N) \ \big) \end{array}, \\ \boldsymbol{\pi}(K) &= \begin{array}{c} \mathbb{Z}_0^N \quad\quad\quad\quad \mathbb{Z}_{N+1}^K \\ \big(\ \boldsymbol{\pi}^{(1)}(K;\, N) \quad \boldsymbol{\pi}^{(2)}(K;\, N) \ \big) \end{array}. \end{aligned} \tag{4.61}$$

Since $\boldsymbol{\pi}(K) \in \mathrm{ri}\, \mathcal{P}^+(K)$ by Theorem 4.2, $\boldsymbol{\pi}(N) = \boldsymbol{\pi}^{(1)}(K;\, N)/\boldsymbol{\pi}^{(1)}(K;\, N)\boldsymbol{e}$ is given by a convex combination of normalized row vectors of $\boldsymbol{H}^{(*,1)}(K;\, N)$ corresponding to $\mathcal{J}(K)$.

Note here that in all sample paths of the first passage from $\mathbb{Z}_{K+1}^{\infty}$ to $\mathbb{Z}_0^N$, $\{X(t)\}_{t\geq 0}$ must visit at least one state in $\mathcal{J}(K)$. In view of this fact, we first introduce (K,N)-*skip-free* sets. Roughly speaking, a (K,N)-skip-free set $\mathcal{X}$ is a proper subset of $\mathbb{Z}_0^K$, where in all sample paths of the first passage from $\mathbb{Z}_{K+1}^{\infty}$ to $\mathbb{Z}_0^K$, $\{X(t)\}_{t\geq 0}$ must visit at least one state in $\mathcal{X}$. We then show in Theorem 4.6 that $\boldsymbol{\pi}(N) = \boldsymbol{\pi}^{(1)}(K;\, N)/\boldsymbol{\pi}^{(1)}(K;\, N)\boldsymbol{e}$ is given by a convex combination of normalized row vectors of $\boldsymbol{H}^{(*,1)}(K;\, N)$ corresponding to members in $\mathcal{X}$.

To find the minimum convex polytopes that contain $\boldsymbol{\pi}(N)$, we introduce a partial order among (K,N)-skip-free sets, considering the first passage time from $\mathbb{Z}_{K+1}^{\infty}$ to those. We then show the inclusion relation among convex polytopes associated with (K,N)-skip-free sets in Lemma 4.2. Finally, we find the *smallest* (K,N)-skip-free set and the corresponding minimum convex polytope in Theorem 4.7, which is shown to be tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

4.4.1 (K,N)-skip-free sets

For an arbitrary, non-empty subset $\mathcal{B}$ of $\mathbb{Z}^+$, we define $F(\mathcal{B})$ as the first passage time to $\mathcal{B}$.

$$F(\mathcal{B}) = \inf\{t \geq 0;\ X(t) \in \mathcal{B}\}, \qquad \mathcal{B} \subseteq \mathbb{Z}^+.$$

Definition 4.1 ((K,N)-skip-free set)**.** *For arbitrarily fixed $K, N \in \mathbb{Z}^+$ $(K > N)$, consider a Markov chain $\{X(t)\}_{t\geq 0}$ in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$. We refer to a subset $\mathcal{X}$ of $\mathbb{Z}_0^K$ as a (K,N)-skip-free set if $\{X(t)\}_{t\geq 0}$ starting from any state in $\mathbb{Z}_{K+1}^{\infty}$ must visit $\mathcal{X}$ by the first passage time to $\mathbb{Z}_0^N$, i.e.,*

$$\Pr(X(t) \in \mathcal{X} \text{ for some } t \in (0, F(\mathbb{Z}_0^N)] \mid X(0) \in \mathbb{Z}_{K+1}^{\infty}) = 1.$$

In particular, a (K,N)-skip-free set $\mathcal{X}$ is called proper if for any proper subset $\mathcal{Y}$ of $\mathcal{X}$,

$$\Pr(X(t) \in \mathcal{Y} \text{ for some } t \in (0, F(\mathbb{Z}_0^N)] \mid X(0) \in \mathbb{Z}_{K+1}^{\infty}) < 1.$$

Let $\mathcal{S}(K,N)$ denote the family of all (K,N)-skip-free sets and $\mathcal{S}^*(K,N)$ denote the family of all proper (K,N)-skip-free sets.

Remark 4.3. *By definition,*

(i) (K,N)*-skip-free sets are determined only by* $\boldsymbol{Q}^{(1,1)}(K)$ *and* $\mathcal{J}(K)$*,*

(ii) $\mathcal{S}^*(K,N) \subseteq \mathcal{S}(K,N)$*,*

(iii) $\mathcal{J}(K) \in \mathcal{S}(K,N)$ *and* $\widetilde{\mathcal{J}}(K,N) \in \mathcal{S}^*(K,N)$*, where* $\widetilde{\mathcal{J}}(K,N)$ *is given by* (4.48)*, and*

(iv) $\mathcal{J}(K) \cap \mathbb{Z}_0^N \subseteq \mathcal{X}$ *for all* $\mathcal{X} \in \mathcal{S}(K,N)$*.*

We can provide an equivalent definition of (K,N)-skip-free sets by considering a directed graph associated with $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$. Specifically, for arbitrarily fixed $K, N \in \mathbb{Z}^+$ $(K > N)$, we define

$$G(K,N) = (V(K,N), E(K,N)) \tag{4.62}$$

as a directed graph, where $V(K,N)$ and $E(K,N)$ denote sets of nodes and arcs defined as

$$\begin{aligned} V(K,N) &= \{s,t\} \cup \mathbb{Z}_0^K, \\ E(K,N) &= \{(s,j);\ j \in \mathcal{J}(K)\} \cup \{(i,t);\ i \in \mathbb{Z}_0^N\} \\ &\qquad \cup \{(i,j);\ i \in \mathbb{Z}_{N+1}^K,\ j \in \mathbb{Z}_0^K, q_{i,j} > 0\}. \end{aligned}$$

Node s represents the set of states in $\mathbb{Z}_{K+1}^\infty$ and node t is a virtual terminal node for the first passage to $\mathbb{Z}_0^N$. It is readily seen from Definition 4.1 that a subset $\mathcal{X}$ of $\mathbb{Z}_0^K$ is a (K,N)-skip-free set iff $\mathcal{X}$ is an (s,t)-*node cut* in $G(K,N)$ and that $\mathcal{X}$ is proper iff it is a *minimal* (s,t)-node cut.

For a given $G(K,N)$, nodes in $\mathbb{Z}_0^K$ can be classified into two classes, depending on if they appear on the way of at least one path from node s to node t.

$$\mathbb{Z}_0^K = \mathcal{N}_+(K,N) \cup \mathcal{N}_0(K,N), \qquad \mathcal{N}_+(K,N) \cap \mathcal{N}_0(K,N) = \emptyset,$$

where nodes in $\mathcal{N}_+(K,N)$ are reachable from node s and each of them have at least one path to node t. Note here that $\mathcal{N}_0(K,N) = \mathbb{Z}_0^K \setminus \mathcal{N}_+(K,N)$ is given by

$$\mathcal{N}_0(K,N) = \mathcal{N}_0^{(1)}(K,N) \cup \mathcal{N}_0^{(2)}(K,N) \cup \mathcal{N}_0^{(3)}(K,N),$$

where $\mathcal{N}_0^{(i)}(K,N)$ $(i = 1,2,3)$ are defined as

$$\begin{aligned} \mathcal{N}_0^{(1)}(K,N) &= \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K,N) = \{i \in \mathbb{Z}_0^N;\ \Pr(X(F(\mathbb{Z}_0^N)) = i \mid X(0) \in \mathbb{Z}_{K+1}^\infty) = 0\}, \\ \mathcal{N}_0^{(2)}(K,N) &= \{i \in \mathbb{Z}_{N+1}^K; \\ &\qquad \Pr(X(t) \in \mathbb{Z}_{N+1}^\infty \text{ for all } t \in (0, F(\{i\})] \mid X(0) \in \mathbb{Z}_{K+1}^\infty) = 0\}, \\ \mathcal{N}_0^{(3)}(K,N) &= \{i \in \mathbb{Z}_{N+1}^K;\ \Pr(X(t) \in \mathbb{Z}_{N+1}^K \text{ for all } t \in (0, F(\mathbb{Z}_0^N)] \mid X(0) = i) = 0\}. \end{aligned}$$

Roughly speaking, states in $\mathcal{N}_0^{(1)}(K,N) \subseteq \mathbb{Z}_0^N$ and states in $\mathcal{N}_0^{(2)}(K,N) \subseteq \mathbb{Z}_{N+1}^K$ can be reached from any states in $\mathbb{Z}_{K+1}^\infty$ only after visiting $\widetilde{\mathcal{J}}(K,N) \subseteq \mathbb{Z}_0^N$. On the other hand, states in $\mathcal{N}_0^{(3)}(K,N) \in \mathbb{Z}_{N+1}^K$ have no paths to $\mathbb{Z}_0^N$ without visiting $\mathbb{Z}_{K+1}^\infty$.

By definition, $\mathcal{X} \cap \mathcal{N}_0(K,N) = \emptyset$ if $\mathcal{X} \in \mathcal{S}^*(K,N)$, whereas there may exist $\mathcal{X} \in \mathcal{S}(K,N)$ such that $\mathcal{X} \cap \mathcal{N}_0(K,N) \neq \emptyset$. As we will see, all states in $\mathcal{N}_0(K,N)$ can be

excluded from consideration when we study the conditional stationary distribution $\boldsymbol{\pi}(N)$ for Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

Before considering minimum convex polytopes for Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, we provide a remark on (K,N)-skip-free sets. Note that (K,N)-skip-free sets can be regarded as a natural extension of levels with the skip-free-to-the-left property in Markov chains of LD-M/G/1-type. Suppose a Markov chain $\{X(t)\}_{t\geq 0}$ is of LD-M/G/1-type, i.e., its state space $\mathbb{Z}^+$ is partitioned into *disjoint* levels $\{\mathcal{L}_m;\ m \in \mathbb{Z}^+\}$ and transitions between two levels are skip-free to the left. We now consider the first passage time from $\mathcal{L}_{K+1}$ to $\mathcal{L}_N$ $(K > N)$. It then follows that each level $\mathcal{L}_m$ $(m = N, N+1, \ldots, K)$ can be regarded as a $(\hat{K}, \hat{N})$-skip-free set, where $\hat{K}$ and $\hat{N}$ denote the greatest state in $\cup_{m=0}^{K}\mathcal{L}_m$ and $\cup_{m=0}^{N}\mathcal{L}_m$, i.e., $\hat{K} = L_K - 1$ and $\hat{N} = L_N - 1$. One of the most notable differences between (K,N)-skip-free sets and levels with the skip-free-to-the-left property is that (K,N)-skip-free sets are *not* disjoint, while levels are disjoint.

Example 4.4 (cont'd)**.** *Consider $\boldsymbol{Q}$ in Example 4.1 of Section 4.2.2. $\mathcal{S}^*(K,N)$ is given by*

$$\mathcal{S}^*(K,N) = \big\{\{0,k\};\ k \in \mathbb{Z}_N^K\big\}.$$

4.4.2 Minimum convex polytopes

We consider convex polytopes associated with (K,N)-skip-free sets. We then partition $\boldsymbol{H}(K) = (-\boldsymbol{Q}^{(1,1)}(K))^{-1}$ into two matrices as in (4.61). Note here that

$$[\boldsymbol{H}^{(*,1)}(K;\ N)\boldsymbol{e}]_i = 0, \quad i \in \mathcal{N}_0^{(3)}(K,N), \tag{4.63}$$

and $[\boldsymbol{H}^{(*,1)}(K;\ N)\boldsymbol{e}]_i > 0$ for all $i \in \mathbb{Z}_0^K \setminus \mathcal{N}_0^{(3)}(K,N)$. For any $(K+1)$-dimensional vector $\boldsymbol{x}$, we define $\mathrm{diag}^*(\boldsymbol{x})$ as a $(K+1)$-dimensional diagonal matrix whose ith $(i \in \mathbb{Z}_0^K)$ diagonal element is given by

$$[\mathrm{diag}^*(\boldsymbol{x})]_{i,i} = \begin{cases} \dfrac{1}{[\boldsymbol{x}]_i}, & [\boldsymbol{x}]_i \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We then define $\Gamma(K,N;\ \mathcal{B})$ $(K,N \in \mathbb{Z}^+,\ K > N,\ \mathcal{B} \subseteq \mathbb{Z}_0^K)$ as

$$\Gamma(K,N;\ \mathcal{B}) = \big\{\boldsymbol{\alpha} \in \mathbb{R}^{N+1};\ \boldsymbol{\alpha} = \boldsymbol{\beta}\boldsymbol{U}(K,N),\ \boldsymbol{\beta} \in \Gamma(K), \\ [\boldsymbol{\beta}]_i \geq 0\ (i \in \mathcal{B}),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{B})\big\},$$

where $\Gamma(n)$ $(n \in \mathbb{Z}^+)$ is defined in (4.4) and $\boldsymbol{U}(K,N)$ is given by

$$\boldsymbol{U}(K,N) = \mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;\ N)\boldsymbol{e}) \begin{pmatrix} \boldsymbol{I} \\ (-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N) \end{pmatrix} \\ \cdot (\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}).$$

Moreover, for $\mathcal{X} \in \mathcal{S}(K,N)$, we define $\mathcal{X}^*$ as

$$\mathcal{X}^* = \mathcal{X} \setminus \mathcal{N}_0(K,N).$$

Note that

$$\mathcal{X}^* \in \mathcal{S}(K,N) \ \text{ if } \ \mathcal{X} \in \mathcal{S}(K,N), \qquad \mathcal{X}^* = \mathcal{X} \ \text{ if } \ \mathcal{X} \in \mathcal{S}^*(K,N). \tag{4.64}$$

We then have the following theorem, whose proof is given in Appendix H.

Theorem 4.6. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ is not empty. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, we have for $\mathcal{X} \in \mathcal{S}(K,N)$ $(K > N)$,*

$$\boldsymbol{\pi}(N) \in \mathcal{P}^+(K,N;\, \mathcal{X}^*),$$

where $\mathcal{P}^+(K,N;\, \mathcal{X}^*)$ denotes a convex polytope on the first orthant of $\mathbb{R}^{N+1}$, which is given by

$$\begin{aligned}\mathcal{P}^+(K,N;\, \mathcal{X}^*) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ & \big[(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\big]_i \ge 0 \ (i \in \mathcal{X}^*), \\ & \big[(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\big]_i = 0 \ (i \in \mathbb{Z}_0^K \setminus \mathcal{X}^*), \\ & \boldsymbol{x}\boldsymbol{e} = 1,\ \boldsymbol{y} \ge \boldsymbol{0}\big\} \qquad (4.65) \\ = \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ & \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \Gamma(K,N;\, \mathcal{X}^*)\}. \qquad (4.66)\end{aligned}$$

In particular,

$$\boldsymbol{\pi}(N) \in \mathrm{ri}\, \mathcal{P}^+(K,N;\, \mathcal{X}^*) \ \text{ if } \ \mathcal{X}^* \in \mathcal{S}^*(K,N).$$

Remark 4.4. *Consider two (K,N)-skip-free sets $\mathcal{X}_A, \mathcal{X}_B \in \mathcal{S}(K,N)$. If $\mathcal{X}_A \subseteq \mathcal{X}_B$, we have $\mathcal{X}_A^* \subseteq \mathcal{X}_B^*$ by definition. Therefore, it follows from (4.65) that $\mathcal{P}^+(K,N;\, \mathcal{X}_A^*) \subseteq \mathcal{P}^+(K,N;\, \mathcal{X}_B^*)$ if $\mathcal{X}_A \subseteq \mathcal{X}_B$.*

Lemma 4.2. *For Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ with $\mathcal{X}_A, \mathcal{X}_B \in \mathcal{S}(K,N)$, we consider the first passage times $F(\mathcal{X}_A^*)$, $F(\mathcal{X}_B^*)$, and $F(\mathbb{Z}_0^N)$. If the statements*

$$F(\mathcal{X}_A^*) \le F(\mathcal{X}_B^*) \ \text{ given that } \ X(0) \in \mathbb{Z}_{K+1}^\infty$$

and

$$F(\mathcal{X}_B^*) \le F(\mathbb{Z}_0^N) \ \text{ given that } \ X(0) \in \mathcal{X}_A^*, \tag{4.67}$$

hold sample path-wise, we have

$$\mathcal{P}(K,N;\, \mathcal{X}_A^*) \subseteq \mathcal{P}(K,N;\, \mathcal{X}_B^*).$$

The proof of Lemma 4.2 is given in Appendix I. By definition, $\widetilde{\mathcal{J}}(K,N) \in \mathcal{S}^*(K,N)$ and $\mathcal{P}^+(K,N;\, \widetilde{\mathcal{J}}(K,N))$ is identical to $\widetilde{\mathcal{P}}^+(K,N)$ in (4.53). Recall that $\widetilde{\mathcal{J}}(K,N) \subseteq \mathbb{Z}_0^N$. Therefore, for any $\mathcal{X} \in \mathcal{S}(K,N)$, $F(\mathcal{X}) \le F(\widetilde{\mathcal{J}}(K,N))$ holds sample path-wise, given $X(0) \in \mathbb{Z}_{K+1}^\infty$. On the other hand, $\mathcal{J}(K) \in \mathcal{S}(K,N)$ and each state in $\mathcal{J}(K)$ is reachable from some states in $\mathbb{Z}_{K+1}^\infty$ by direct transitions. Note here that

$$\mathcal{J}^*(K) = \mathcal{J}(K) \setminus \mathcal{N}_0(K,N) = \mathcal{J}(K) \setminus \mathcal{N}_0^{(3)}(K,N). \tag{4.68}$$

We thus have $F(\mathcal{J}^*(K)) \le F(\mathcal{X}^*)$ sample path-wise, given $X(0) \in \mathbb{Z}_{K+1}^\infty$. The following corollary follows from Theorem 4.6 and the above observations.

Corollary 4.7. *Among (K,N)-skip-free sets, $\widetilde{\mathcal{J}}(K,N)$ in (4.48) gives a maximum convex polytope, i.e.,*

$$\mathcal{P}^+(K,N;\,\mathcal{X}^*) \subseteq \mathcal{P}^+(K,N;\,\widetilde{\mathcal{J}}(K,N)), \quad \mathcal{X} \in \mathcal{S}(K,N).$$

On the other hand, among (K,N)-skip-free sets, $\mathcal{J}^*(K)$ gives a minimum convex polytope, i.e.,

$$\mathcal{P}^+(K,N;\,\mathcal{J}^*(K)) \subseteq \mathcal{P}^+(K,N;\,\mathcal{X}^*), \quad \mathcal{X} \in \mathcal{S}(K,N).$$

Note here that $\mathcal{J}^*(K)$ is not proper in general. We thus consider a *proper* (K,N)-skip-free set that gives the minimum convex polytope $\mathcal{P}^+(K,N;\,\mathcal{J}^*(K))$. Specifically, we introduce a subset $\mathcal{D}(K,N)$ of $\mathcal{J}(K)$, which is generated by the following procedure.

Procedure 4.1 ([Nag18])**.**

(i) We obtain a directed graph $G^*(K,N)$ from $G(K,N)$ in (4.62) by removing all incoming edges to nodes in $\mathcal{J}(K)$ and

(ii) find a subset $\mathcal{D}(K,N)$ of nodes in $\mathcal{J}(K)$, from which node t is reachable in $G^*(K,N)$.

Note that Step (ii) can be performed with $O(K)$ operations, by considering reverse paths from node t to node s in graph $G^*(K,N)$. The subset $\mathcal{D}(K,N)$ gives the minimum convex polytope.

Lemma 4.3 ([Nag18])**.** *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ is not empty. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ $(K > N)$, we have $\mathcal{D}(K,N) \in \mathcal{S}^*(K,N)$. Furthermore, $\mathcal{D}(K,N) \subseteq \mathcal{X}$ for all $\mathcal{X} \subseteq \mathcal{J}(K)$ such that $\mathcal{X} \in \mathcal{S}(K,N)$.*

The proof of Lemma 4.3 is given in Appendix J.

Lemma 4.4. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ is not empty. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ $(K > N)$, we have*

$$\mathcal{P}^+(K,N;\,\mathcal{D}(K,N)) = \mathcal{P}^+(K,N;\,\mathcal{J}^*(K)), \quad K,N \in \mathbb{Z}^+,\ K > N,$$

where $\mathcal{J}^*(K)$ is given by (4.68).

Proof. Since $\mathcal{D}(K,N) \in \mathcal{S}^*(K,N)$, we have $\mathcal{D}^*(K,N) = \mathcal{D}(K,N)$ from (4.64). It then follows from Remark 4.4 and $\mathcal{D}(K,N) \subseteq \mathcal{J}(K)$ that $\mathcal{P}^+(K,N;\,\mathcal{D}(K,N)) = \mathcal{P}^+(K,N;\,\mathcal{D}^*(K,N)) \subseteq \mathcal{P}^+(K,N;\,\mathcal{J}^*(K))$. On the other hand, it follows from Lemma 4.3 that $F(\mathcal{J}^*(K)) \leq F(\mathcal{D}(K,N))$ if $X(0) \in \mathbb{Z}_{K+1}^{\infty}$ and $F(\mathcal{D}(K,N)) \leq F(\mathbb{Z}_0^N)$ if $X(0) \in \mathcal{J}^*(K)$. We thus have $\mathcal{P}^+(K,N;\,\mathcal{J}^*(K)) \subseteq \mathcal{P}^+(K,N;\,\mathcal{D}(K,N))$ from Lemma 4.2, which completes the proof. □

The following theorem shows that the minimum convex polytope $\mathcal{P}^+(K,N;\,\mathcal{D}(K,N))$ is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

Theorem 4.7. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ is not empty. For any probability vector $\boldsymbol{x} \in \mathrm{ri}\,\mathcal{P}^+(K,N;\,\mathcal{D}(K,N))$, there exists a Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, whose $\boldsymbol{\pi}(N)$ is given by $\boldsymbol{x}$.*

Table 4.1: The worst-case error bounds in Example 4.5 ($N = 50$).

K	worst-case error bound
50	$1.4822557975239 \times 10^{-1}$
60	$2.8325135949980 \times 10^{-2}$
70	$3.3500314471736 \times 10^{-3}$
80	$2.3592892522569 \times 10^{-4}$
90	$1.0840482894389 \times 10^{-5}$
100	$3.5265317430058 \times 10^{-7}$
110	$8.6146805062734 \times 10^{-9}$
120	$1.6515206910372 \times 10^{-10}$
130	$2.5711073894930 \times 10^{-12}$
140	$3.2661373605691 \times 10^{-14}$
150	$2.0140139556091 \times 10^{-15}$

The proof of Theorem 4.7 is given in Appendix K. As shown in the proof, if $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$ and $\boldsymbol{x} \in \mathrm{ri}\, \mathcal{P}^+(K, N; \mathcal{D}(K, N))$, we have $\boldsymbol{x} > \boldsymbol{0}$. The following corollary comes from the fact that $\mathcal{P}^+(K, N;\ \mathcal{D}(K, N))$ is a convex polytope spanned by $\overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)$ for $i \in \mathcal{D}(K, N)$ (cf. (A.18)), where $\overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)$ ($i \in \mathbb{Z}_0^K$) denotes the ith normalized row vector of $\boldsymbol{H}^{(*,1)}(K;\, N)$ in (4.61).

Corollary 4.8. *Suppose that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ is not empty. For an arbitrary probability vector $\boldsymbol{x} \in \Gamma(N)$, we have*

$$\|\boldsymbol{x} - \boldsymbol{\pi}(N)\|_1 \leq \max_{i \in \mathcal{D}(K,N)} \|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)\|_1.$$

Furthermore, if $\boldsymbol{x} \in \mathcal{P}^+(K, N; \mathcal{D}(K, N))$, we have

$$\max_{i \in \mathcal{D}(K,N)} \|\boldsymbol{x} - \overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)\|_1 \leq \max_{i,j \in \mathcal{D}(K,N)} \|\overline{\boldsymbol{\eta}}_j^{(*,1)}(K;\, N) - \overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)\|_1, \quad \boldsymbol{x} \in \mathcal{P}^+(K, N; \mathcal{D}(K, N)).$$

Example 4.5 (cont'd)**.** *Consider $\boldsymbol{Q}$ in Example 4.1 of Section 4.2.2. Note that $\mathcal{J}(N) = \{0, N\}$ and $\mathcal{J}(K) = \mathcal{J}^*(K) = \mathcal{D}(K, N) = \{0, K\}$. We set $q_{i,j} = \lambda_i b_{j-i}$ ($i \in \mathbb{Z}^+, j > i$), $q_{i,i-1} = \mu_i$ ($i \in \mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$), and $q_{i,0} = \gamma_i$ ($i \in \mathbb{N}$), where*

$$\lambda_i = 2 - \frac{1}{i+1} \quad (i \in \mathbb{Z}^+), \qquad b_i = \frac{1}{2^i} \quad (i \in \mathbb{N}),$$
$$\mu_i = \frac{i}{10} \quad (i \in \mathbb{N}), \qquad \gamma_i = \frac{\lambda_i}{i} \quad (i \in \mathbb{Z}^+ \setminus \{0, 1\}).$$

Table 4.1 shows the worst-case error bound $\max_{i,j \in \mathcal{D}(K,N)} \|\overline{\boldsymbol{\eta}}_j^{(*,1)}(K;\, N) - \overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)\|_1 = \|\overline{\boldsymbol{\eta}}_0^{(*,1)}(K;\, N) - \overline{\boldsymbol{\eta}}_K^{(*,1)}(K;\, N)\|_1$ in Corollary 4.8, where $N = 50$ and the result for $K = N$ indicates the worst-case error bound $\max_{i,j \in \mathcal{J}(N)} \|\overline{\boldsymbol{\eta}}_j(N) - \overline{\boldsymbol{\eta}}_i(N)\|_1 = \|\overline{\boldsymbol{\eta}}_0(N) - \overline{\boldsymbol{\eta}}_N(N)\|_1$ in Corollary 4.3. We observe that the worst-case error bound steadily decreases as K increases and that if $K = 150$, any probability vector $\boldsymbol{x} \in \mathrm{ri}\, \mathcal{P}^+(K, N; \mathcal{D}(K, N))$ is a good approximation to $\boldsymbol{\pi}(N)$ in this specific case.

4.5 Conclusion

In this chapter, we considered ergodic, continuous-time Markov chains on $\mathbb{Z}^+$ and characterized the conditional stationary distribution $\boldsymbol{\pi}(N)$ given that the Markov chain is in $\mathbb{Z}_0^N$ via systems of linear inequalities. Specifically, we first obtained an N-simplex $\mathcal{P}(N)$ that contains $\boldsymbol{\pi}(N)$, assuming that we know only the $(N+1)\times(N+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ of the infinitesimal generator. We then refined this result and eliminated unnecessary vertices from $\mathcal{P}(N)$, using the structural information $\mathcal{J}(N)$. Next, we extended those results to the cases that the $(K+1)\times(K+1)$ $(K>N)$ northwest corner submatrix (and $\mathcal{J}(K)$) are available. Furthermore, we introduced a new state transition structure called (K,N)-skip-free sets and obtained a tight convex polytope that contains $\boldsymbol{\pi}(N)$, under the condition that only $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available. Recall that the state space of a Markov chain $\{X(t)\}_{t\geq 0}$ was assumed to be countably infinite. Except for Corollary 4.6, however, all the results in this chapter also hold for Markov chains with finite state space $\mathbb{Z}_0^{K^*}$ for some $K^*\in\mathbb{Z}^+$ such that $N<K<K^*$, if we replace $\mathbb{Z}_N^\infty$ and $\mathbb{Z}_K^\infty$ by $\mathbb{Z}_N^{K^*}$ and $\mathbb{Z}_K^{K^*}$.

We scarcely ever discussed one-point approximations to the conditional stationary distribution $\boldsymbol{\pi}(N)$. In view of the error bounds in Corollaries 4.3 and 4.8, it seems to be reasonable to choose a center of a convex polytope that contains $\boldsymbol{\pi}(N)$ as a one-point approximation to $\boldsymbol{\pi}(N)$. Note, however, that we cannot identify the exact location of $\boldsymbol{\pi}(N)$ within the convex polytopes, unless further information is available. Therefore, for a specific problem, the choice of the center may or may not work well. We will discuss the selections of one-point approximations in the next chapter.

In Chapter 2, we characterized the conditional stationary distribution in terms of the reverse-directional R-matrices. On the other hand, this chapter characterized the conditional stationary distribution in general Markov chains based on the northwest corner submatrices of the infinitesimal generator. Therefore, all the results in this chapter are also applicable to Markov chains of LD-M/G/1-type and in this special case, both characterizations are different but closely related. See Appendix L for details.

5 Implications for the Augmented Truncation Approximation and their Application to Disaster Queues

5.1 Introduction

In this chapter, we discuss numerical computations of the (conditional) stationary distribution based on the results in Chapter 4. We first provide practical implications of results in Chapter 4 for the augmented truncation approximation (ATA). As shown in Section 1.3, the standard ATA constructs a finite-state Markov chain based on the northwest corner submatrix of the infinitesimal generator and then, it computes the stationary distribution in the finite-state Markov chain so as to obtain a one-point approximation of the conditional stationary distribution. We provide numerical implementations of the conditional stationary distribution based on the standard ATA, which enable us to evaluate the error bound of the computed conditional stationary distribution.

Next, we analyze a specific queueing model with disasters by applying the above-mentioned numerical implementation. In queueing systems, a *disaster* is the event that all customers are removed from the system [Jai96, Kum08]. For instance, resets in computer systems, which arise from overloads or virus infections, can be modeled by disasters. We consider a queueing model with queue-length-dependent disasters and generally distributed service times. In such a model, the stochastic process of the number of customers is not formulated as a continuous-time Markov chain because of the generally distributed service times, so that we analyze the model using an imbedded Markov chain. Specifically, (i) we construct the imbedded Markov chain associated with the original stochastic process of the number of customers, i.e, compute the state transition probability matrix, (ii) compute the stationary distribution in the imbedded chain, and (iii) convert the stationary distribution in the imbedded chain to the stationary distribution in the original continuous-time process. We first discuss Step (i) and then, we study the error in Step (i) and its propagation on the Steps (ii)–(iii). Finally, we develop the numerical computation of the stationary distribution of queue-length-dependent disaster queues and provide its numerical examples.

The rest of this chapter is organized as follows. In Section 5.2, we provide practical implications for the ATA. In Section 5.3, we describe the model. In Section 5.4, we give the overview of the approach to computing the stationary distribution. In Section 5.5, we consider the numerical computations of the state transition probability matrix of the imbedded Markov chain. In Section 5.6, we briefly discuss the selection of the augmentation matrix and the error bound of the ATA solution in disaster queues. In

Section 5.7, we discuss the error in the approximation to the stationary distribution. In Section 5.8, we show some numerical examples to demonstrate that our numerical procedure works well. Finally, we conclude this chapter in Section 5.9.

5.2 Implications for the augmented truncation approximation

In this section, we consider an ergodic, time-homogeneous, and continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}^+ = \{0, 1, \ldots\}$ described in Chapter 1. It is assumed that $\{X(t)\}_{t\geq 0}$ is stationary and the infinitesimal generator $\boldsymbol{Q}$ and the stationary distribution $\boldsymbol{\pi}$ are partitioned in conformance with $\mathbb{Z}^+ = \mathbb{Z}_0^N \cup \mathbb{Z}_{N+1}^\infty$, as in (1.21) and (4.2). We provide some practical implications of the results in Chapter 4 for the ATA. As mentioned in Section 1.3, for a given N, an ATA solution $\boldsymbol{\pi}^{\rm approx}(N)$ to the conditional stationary distribution $\boldsymbol{\pi}(N)$ is given by the solution of (1.22). Recall that the central topic in the literature on the ATA is the convergence of $(\boldsymbol{\pi}^{\rm approx}(N)\ \boldsymbol{0})$ to the stationary distribution $\boldsymbol{\pi}$ as N goes to infinity. Note that our interest is different from it, that is to say, we are interested in what kind of augmentation matrices $\boldsymbol{Q}_{\rm A}(N)$ is *reasonable* for a given N.

Although $\boldsymbol{Q}_{\rm A}(N)$ is usually chosen in such a way that $[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\rm A}(N)]$ is irreducible, we allow reducible $[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\rm A}(N)]$ as well. Recall that the error bound for the approximation $(\boldsymbol{\pi}^{\rm approx}(N)\ \boldsymbol{0})$ to the stationary distribution $\boldsymbol{\pi}$ is given by the tail probability $\xi(N) = \Pr(X(0) > N)$ and the error $\epsilon(N) = \|\boldsymbol{\pi}^{\rm approx}(N) - \boldsymbol{\pi}(N)\|_1$ in the approximation $\boldsymbol{\pi}^{\rm approx}(N)$ to the conditional stationary distribution $\boldsymbol{\pi}(N)$ (cf. (1.25)). It is clear that for a given N, the tail probability $\xi(N)$ is an unknown but positive fixed value and $(\boldsymbol{\pi}^{\rm approx}(N)\ \boldsymbol{0})$ will be the best approximation to $\boldsymbol{\pi}$ when $\boldsymbol{\pi}^{\rm approx}(N) = \boldsymbol{\pi}(N)$ (i.e., $\epsilon(N) = 0$) [Zha04]. In what follows, we first consider some implications of our results in Section 4.2.1 for the ATA under the condition that only $(N+1) \times (N+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ is available.

We define $\mathcal{T}(N)$ as a set of all possible ATA solutions satisfying (1.22).

$$\mathcal{T}(N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}\big[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\rm A}(N)\big] = \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1,\ \boldsymbol{Q}_{\rm A}(N) \in \mathcal{A}(N)\big\},$$

where $\mathcal{A}(N)$ denotes a set of all possible augmentation matrices.

$$\mathcal{A}(N) = \big\{\boldsymbol{X} \in \mathbb{R}^{(N+1)\times(N+1)};\ \boldsymbol{X} \geq \boldsymbol{O},\ \boldsymbol{X}\boldsymbol{e} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\big\}.$$

In the literature, ATA is called *linear* if the rank of $\boldsymbol{Q}_{\rm A}(N)$ is equal to one [Gib87a]. We then define $\mathcal{T}_{\rm L}(N)$ as a set of all possible approximations obtained by the linear ATA.

$$\mathcal{T}_{\rm L}(N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}\big[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\rm A}(N)\big] = \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1,\ \boldsymbol{Q}_{\rm A}(N) \in \mathcal{A}_{\rm L}(N)\big\},$$

where $\mathcal{A}_{\rm L}(N)$ denotes a set of all possible linear augmentation matrices.

$$\mathcal{A}_{\rm L}(N) = \big\{\boldsymbol{X} \in \mathbb{R}^{(N+1)\times(N+1)};\ \boldsymbol{X} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta},\ \boldsymbol{\zeta} \in \Gamma(N)\big\},$$

where $\Gamma(N)$ denotes the set of all $1 \times (N+1)$ probability vectors, which is defined in (4.4).

Lemma 5.1. *Consider an ergodic Markov chain $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}^+$. We have*

$$\mathcal{T}(N) = \mathcal{T}_{\rm L}(N) = \mathcal{P}(N), \quad N \in \mathbb{Z}^+,$$

where $\mathcal{P}(N)$ is given by (4.7).

The proof of Lemma 5.1 is given in Appendix M. Lemma 5.1 implies the following.

Implication 5.1. *The linear ATA solution has the same degree of freedom as the general ATA solution does. Specifically, for any general augmentation, there is a linear augmentation which constructs the same ATA solution.*

We thus restrict our attention to the linear ATA with $\boldsymbol{Q}_{\rm A}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}$, where $\boldsymbol{\zeta} \in \Gamma(N)$. In this case, (1.22) is reduced to

$$\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}] = \boldsymbol{0}, \quad \boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})\boldsymbol{e} = 1. \tag{5.1}$$

Note here that $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ in (5.1) is closely related to $\boldsymbol{\pi}(N)$ in a Markov chain with the infinitesimal generator $\boldsymbol{Q}$ in (4.25). Specifically, using the cut-flow balance equation

$$\boldsymbol{\pi}^{(1)}(N)(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} = \boldsymbol{\pi}^{(2)}(N)(-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e},$$

we can rewrite (4.26) to be

$$\boldsymbol{\pi}^{(1)}(N)\big[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x})\big] = \boldsymbol{0}.$$

We thus have $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) = \boldsymbol{\pi}(N) = \boldsymbol{x} \in \mathcal{P}(N)$ if we set $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{x})$ in (5.1).

Implication 5.2. *The linear ATA solution $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ for a specific $\boldsymbol{\zeta}$ is identical to $\boldsymbol{\pi}(N)$ in an ergodic Markov chain whose infinitesimal generator takes the following form:*

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^\infty \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^\infty \end{array} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) & \boldsymbol{Q}^{(1,2)}(N) \\ (-\boldsymbol{Q}^{(2,2)}(N)\boldsymbol{e})\boldsymbol{\zeta} & \boldsymbol{Q}^{(2,2)}(N) \end{pmatrix} \end{array}.$$

Since $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) \in \mathcal{T}_{\rm L}(N) = \mathcal{P}(N)$, it follows from Theorem 4.1 that

$$\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) = \boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})\overline{\boldsymbol{H}}(N), \tag{5.2}$$

for some $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}) \in \Gamma(N)$. Recall that there is a one-to-one correspondence between $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ and $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$, as stated in Remark 4.1. Note also that there are one-to-one correspondences between $\boldsymbol{\zeta}$ and $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$:

$$\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}) = \frac{\boldsymbol{\zeta}{\rm diag}(\boldsymbol{H}(N)\boldsymbol{e})}{\boldsymbol{\zeta}{\rm diag}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}}, \qquad \boldsymbol{\zeta} = \frac{\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}){\rm diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})}{\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}){\rm diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}}, \tag{5.3}$$

and between $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ and $\boldsymbol{\zeta}$:

$$\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) = \frac{\boldsymbol{\zeta}\boldsymbol{H}(N)}{\boldsymbol{\zeta}\boldsymbol{H}(N)\boldsymbol{e}}, \qquad \boldsymbol{\zeta} = \frac{\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N))}{\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}.$$

It is readily seen that the vertex $\overline{\boldsymbol{\eta}}_i(N)$ of the convex polytope $\mathcal{P}(N)$ $(= \mathcal{T}_{\rm L}(N))$ is given by a linear ATA solution with a single-column augmentation.

$$\overline{\boldsymbol{\eta}}_i(N) = \boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{e}_i^T). \tag{5.4}$$

From (5.2), we also observe that $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ is given by a convex combination of row vectors $\overline{\boldsymbol{\eta}}_i(N)$'s ($i \in \mathbb{Z}_0^N$) of $\overline{\boldsymbol{H}}(N)$ with the nonnegative weight vector $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$. It then follows from (4.12) that

$$\begin{aligned}\|\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) - \boldsymbol{\pi}(N)\|_1 &\le \max_{i\in\mathbb{Z}_0^N} \|\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) - \overline{\boldsymbol{\eta}}_i(N)\|_1, \\ &= \max_{i\in\mathbb{Z}_0^N} \|\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta}) - \boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{e}_i^T)\|_1.\end{aligned} \tag{5.5}$$

This error bound tempts us to set $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$ in such a way that $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ is located at the *center* of $\mathcal{P}(N)$ spanned by $\overline{\boldsymbol{\eta}}_i(N)$'s ($i \in \mathbb{Z}_0^N$). For example, if we set $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}) = \boldsymbol{e}^T/(N+1)$, $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ is given by the center of gravity of $\mathcal{P}(N)$. Note here that (5.2) is equivalent to $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}){\rm diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})$.

Implication 5.3. *If we have a desirable $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$ rather than $\boldsymbol{\zeta}$ itself, $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ for such an $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$ can be computed as follows:*

(i) We first obtain $\boldsymbol{h}(N) := \boldsymbol{H}(N)\boldsymbol{e}$ by solving $(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{h}(N) = \boldsymbol{e}$ and then

(ii) obtain $\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})$ by solving

$$\boldsymbol{\pi}^{\rm approx}(N;\,\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{\alpha}(N;\,\boldsymbol{\zeta}){\rm diag}^{-1}(\boldsymbol{h}(N)).$$

In the above procedure, we have to solve two systems of linear equations with $-\boldsymbol{Q}^{(1,1)}(N)$. Therefore, comparing with the procedure for solving (5.1) directly for a specific $\boldsymbol{\zeta}$, the computational cost increases by $(N+1)^2$ in terms of the number of multiplications/divisions, if we utilize the LU decomposition. This increase can be regarded as a cost of specifying the weight vector $\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})$ directly, instead of $\boldsymbol{\zeta}$.

Next we consider Markov chains under the condition that the northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$ and the structural information $\mathcal{J}(N)$ are available. Note that $\mathcal{J}(N)$ is given in (4.1) and represents a subset of states in $\mathbb{Z}_0^N$, which are directly reachable from at least one state in $\mathbb{Z}_{N+1}^{\infty}$.

It follows from (5.3) that for all $i \in \mathbb{Z}_0^N$,

$$[\boldsymbol{\alpha}(N;\,\boldsymbol{\zeta})]_i = 0 \;\Leftrightarrow\; [\boldsymbol{\zeta}]_i = 0. \tag{5.6}$$

We then define $\mathcal{T}_{\rm L}^+(N)$ as the set of all possible approximations obtained by the linear ATA when $\mathcal{J}(N)$ is available.

$$\mathcal{T}_{\rm L}^+(N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\; \boldsymbol{x}\big[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}\big] = \boldsymbol{0},\; \boldsymbol{x}\boldsymbol{e} = 1,\; \boldsymbol{\zeta} \in \Gamma^+(N)\big\},$$

where $\Gamma^+(N)$ is given by (4.19). By definition, we have $\mathcal{T}_{\rm L}^+(N) \subseteq \mathcal{T}_{\rm L}(N)$.

Lemma 5.2. *Consider an ergodic Markov chain $\{X(t)\}_{t\ge 0}$ on $\mathbb{Z}^+$. We have*

$$\mathcal{T}_{\rm L}^+(N) = \mathcal{P}^+(N), \quad N \in \mathbb{Z}^+,$$

where $\mathcal{P}^+(N)$ is given by (4.22).

Input: $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$.
Output: $\boldsymbol{\pi}^{\mathrm{approx}}(N) = \boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta})$ and $\varepsilon(N)$.

Compute $\overline{\boldsymbol{\eta}}_i(N) = \boldsymbol{x}_i / \boldsymbol{x}_i \boldsymbol{e}$ for all $i \in \mathcal{J}(N)$,
where $\boldsymbol{x}_i$ is the unique solution of $\boldsymbol{x}_i(-\boldsymbol{Q}(N)) = \boldsymbol{e}_i^T$.
Compute $\boldsymbol{\pi}^{\mathrm{approx}}(N) = |\mathcal{J}(N)|^{-1} \sum_{i \in \mathcal{J}(N)} \overline{\boldsymbol{\eta}}_i(N)$.
Compute the error bound $\varepsilon(N) = \max_{i \in \mathcal{J}(N)} \|\boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta}) - \overline{\boldsymbol{\eta}}_i(N)\|_1$.

Figure 5.1: A computational algorithm for an ATA solution $\boldsymbol{\pi}^{\mathrm{approx}}(N)$ and its error bound $\varepsilon(N)$.

The proof of Lemma 5.2 is given in Appendix N. We thus have $\boldsymbol{\pi}(N) \in \mathrm{ri}\, \mathcal{T}_{\mathrm{L}}^{+}(N)$ from Theorem 4.2. It also follows from (5.6) that if $[\boldsymbol{\zeta}]_i > 0$ for some $i \in \mathbb{Z}_0^N \setminus \mathcal{J}(N)$, we have $\boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta}) \notin \mathcal{P}^{+}(N)$ because $\overline{\boldsymbol{\eta}}_j(N)$'s ($j \in \mathbb{Z}_0^N$) are linearly independent. Moreover, it follows from Corollary 4.3 that

$$\begin{aligned} \|\boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta}) - \boldsymbol{\pi}(N)\|_1 &\leq \max_{i \in \mathcal{J}(N)} \|\boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta}) - \overline{\boldsymbol{\eta}}_i(N)\|_1, \qquad (5.7) \\ &= \max_{i \in \mathcal{J}(N)} \|\boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta}) - \boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{e}_i^T)\|_1, \end{aligned}$$

which is tighter than (5.5). In summary, we have:

Implication 5.4. *If the structural information $\mathcal{J}(N)$ is available, it is natural that we should choose $\boldsymbol{\zeta}$ from $\mathrm{ri}\, \Gamma^{+}(N)$ in obtaining a linear ATA solution, where $\Gamma^{+}(N)$ is given by* (4.19)*. For example, we may choose $\boldsymbol{\zeta}$ whose ith ($i \in \mathbb{Z}_0^N$) element is given by $1/|\mathcal{J}(N)|$ if $i \in \mathcal{J}(N)$ and otherwise it is given by 0.*

Implication 5.4 indicates that the last-column augmentation $\boldsymbol{\zeta} = (0\ 0\ \cdots\ 0\ 1)$, which is one of the common augmentation strategies in the literature [Gib87a, Wol80], may not be effective unless $\mathcal{J}(N) = \{N\}$, because if $\mathcal{J}(N) \neq \{N\}$, we have $\boldsymbol{\zeta} = (0\ 0\ \cdots\ 0\ 1) \notin \mathrm{ri}\, \Gamma^{+}(N)$ and therefore $\boldsymbol{\pi}^{\mathrm{approx}}(N;\, \boldsymbol{\zeta}) \notin \mathrm{ri}\, \mathcal{P}^{+}(N)$, while $\boldsymbol{\pi}(N) \in \mathrm{ri}\, \mathcal{P}^{+}(N)$. In Figure 5.1, we summarize the numerical implementation of the conditional stationary distribution $\boldsymbol{\pi}(N)$ with an error evaluation according to our implications.

Remark 5.1. *As shown in* (A.9)*, we have*

$$\boldsymbol{\zeta}^* = \frac{\boldsymbol{\pi}^{(2)}(N) \boldsymbol{Q}^{(2,1)}(N)}{\boldsymbol{\pi}^{(2)}(N) \boldsymbol{Q}^{(2,1)}(N) \boldsymbol{e}},$$

which satisfies $\boldsymbol{\pi}(N)[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N)) \boldsymbol{e} \boldsymbol{\zeta}^*] = \boldsymbol{0}$. Therefore, in order to obtain a good one-point approximation to the conditional stationary distribution $\boldsymbol{\pi}(N)$, we need some information about $\boldsymbol{\pi}^{(2)}(N)$, as in the Iterative Aggregation/Disaggregation methods for finite-state Markov chains [Ste94, Taka75].

As we will see in Section 5.8, when K is sufficiently large, a pathological phenomenon can be observed in the tail of an approximation $\boldsymbol{\pi}^{\mathrm{approx}}(K;\, \boldsymbol{\zeta})$ to $\boldsymbol{\pi}(K)$ because the truncation at K may have a strong impact on the tail of $\boldsymbol{\pi}^{\mathrm{approx}}(K;\, \boldsymbol{\zeta})$. If this is the case, we might adopt an approximation $\boldsymbol{\pi}^{\mathrm{approx}}(K, N;\, \boldsymbol{\zeta})$ to $\boldsymbol{\pi}(N)$ by truncating $\boldsymbol{\pi}^{\mathrm{approx}}(K;\, \boldsymbol{\zeta})$ and normalizing it.

Specifically, we partition $\boldsymbol{\pi}^{\mathrm{approx}}(K;\,\boldsymbol{\zeta})$ as follows.

$$\boldsymbol{\pi}^{\mathrm{approx}}(K;\,\boldsymbol{\zeta}) = \begin{array}{c} \mathbb{Z}_0^N \\ \big(\end{array} \boldsymbol{\pi}^{\mathrm{approx},(1)}(K,N;\,\boldsymbol{\zeta}) \quad \begin{array}{c} \mathbb{Z}_{N+1}^K \\ \boldsymbol{\pi}^{\mathrm{approx},(2)}(K,N;\,\boldsymbol{\zeta}) \big). \end{array}$$

Let $\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta})$ denote an approximation to the conditional stationary distribution $\boldsymbol{\pi}(N)$ obtained from $\boldsymbol{\pi}^{\mathrm{approx}}(K;\,\boldsymbol{\zeta})$.

$$\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) = \frac{\boldsymbol{\pi}^{\mathrm{approx},(1)}(K,N;\,\boldsymbol{\zeta})}{\boldsymbol{\pi}^{\mathrm{approx},(1)}(K,N;\,\boldsymbol{\zeta})\boldsymbol{e}}.$$

It is easy to verify that we have $\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) \in \widetilde{\mathcal{P}}(K,N)$, where $\widetilde{\mathcal{P}}(K,N)$ is given in (4.38), and we have

$$\begin{aligned}\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) &\in \mathrm{ri}\,\mathcal{P}^+(K,N;\mathcal{D}(K,N)) \\ &\qquad \text{if } [\boldsymbol{\zeta}]_i > 0 \ (i \in \mathcal{D}(K,N)) \text{ and } [\boldsymbol{\zeta}]_i = 0 \ (i \in \mathbb{Z}_0^K \setminus \mathcal{D}(K,N)).\end{aligned}$$

Note that $\mathcal{D}(K,N)$ is given by Procedure 4.1 and $\mathcal{P}^+(K,N;\mathcal{D}(K,N))$ denotes the minimum convex polytope containing $\boldsymbol{\pi}(N)$ under the condition that only $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available (cf. Corollary 4.7 and Lemma 4.4). It then follows from Corollary 4.8 that

$$\begin{aligned}\|\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) - \boldsymbol{\pi}(N)\|_1 &\le \max_{i\in\mathcal{D}(K,N)} \|\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) - \overline{\boldsymbol{\eta}}_i^{(*,1)}(K;N)\|_1 \qquad (5.8)\\ &= \max_{i\in\mathcal{D}(K,N)} \|\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) - \boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{e}_i^T)\|_1.\end{aligned}$$

Note that $\boldsymbol{\eta}_i^{(*,1)}(K;N)$ denotes the $1\times(N+1)$ subvector composed of the first $N+1$ elements of $\boldsymbol{\eta}_i(K)$. It follows from Corollaries 4.6 and 4.7 and Lemma 4.4 that

$$\lim_{K\to\infty} \mathrm{ri}\,\mathcal{P}^+(K,N;\mathcal{D}(K,N)) = \{\boldsymbol{\pi}(N)\},$$

and therefore the error bound given in (5.8) converges to zero as K goes to infinity if $\boldsymbol{\pi}^{\mathrm{approx}}(K,N;\,\boldsymbol{\zeta}) \in \mathrm{ri}\,\mathcal{P}^+(K,N;\mathcal{D}(K,N))$ for all $K \in \mathbb{Z}_{N+1}^{\infty}$, i.e., $[\boldsymbol{\zeta}]_i > 0$ $(i \in \mathcal{D}(K,N))$ and $[\boldsymbol{\zeta}]_i = 0$ $(i \in \mathbb{Z}_0^K \setminus \mathcal{D}(K,N))$. It is, however, hard to discuss the rate of the convergence of the error bound because it depends on the model.

5.3 A single-server queue with disasters

In this section, we describe the queueing model with the queue-length-dependent disasters. Recall that a *disaster* is the event that all customers are removed from the system. Such events are also called clearing [Box01], (total) catastrophe [Cha03, Kum08], or mass exodus [Che97] in the literature. The stationary workload [Box01, Ino14, Jai96] and the stationary queue length [Dud99, Dud04, Li06, Shi04] in single-server queues with (queue-length-independent) disasters and generally distributed service times have been studied in the literature.

Queueing systems whose arrival rates and/or occurrence rates of disasters depend on the queue length are termed state-dependent queues or level-dependent queues. Such queueing systems naturally arise in practical modeling. In many situations, a long

waiting line discourages prospective customers from joining the queue, so that the net arrival rate depends on the queue length. Furthermore, a huge backlog may cause a fatal fault or a compulsory reset. To the best of our knowledge, single-server queues with generally distributed service times and level-dependent disasters have not been studied so far.

We consider a single-server queue, where arrivals and disasters occur according to a level-dependent marked Markovian arrival process (LD-MMAP) described in later. We assume that customers are served on a FIFO basis and that their service times are independent and identically distributed (i.i.d.) according to a general distribution $H(x)$ $(x \geq 0)$ with finite mean $\mathrm{E}[H]$.

We define $L(t)$ $(t \geq 0)$ as the number of customers in the system at time t and $S(t) \in \mathcal{M} = \{0, 1, \ldots, M-1\}$ $(t \geq 0)$ as the state of the auxiliary variable at time t. We call $L(t)$ level at time t and $S(t)$ phase at time t. We also define level k as the subset $\{k\} \times \mathcal{M}$ $(k \in \mathbb{Z}^+)$ of the state space $\{0, 1, \ldots\} \times \mathcal{M}$ of $\{(L(t), S(t))\}_{t \geq 0}$. Let $N(s, t]$ $(s < t)$ denote the number of customers entering the system in time interval $(s, t]$. We then define $\boldsymbol{C}_k$, $\boldsymbol{D}_k$, and $\boldsymbol{\Gamma}_k$ $(k \in \mathbb{Z}^+)$ as $M \times M$ matrices whose elements are given by

$$[\boldsymbol{C}_k]_{i,j} = \lim_{\Delta t \to 0} \frac{\Pr\bigl(N(t, t+\Delta t] = 0, S(t+\Delta t) = j \mid (L(t), S(t)) = (k, i)\bigr)}{\Delta t}, \quad i, j \in \mathcal{M},\ i \neq j,$$

$$[\boldsymbol{D}_k]_{i,j} = \lim_{\Delta t \to 0} \frac{\Pr\bigl(N(t, t+\Delta t] = 1, S(t+\Delta t) = j \mid (L(t), S(t)) = (k, i)\bigr)}{\Delta t}, \quad i, j \in \mathcal{M},$$

$$[\boldsymbol{\Gamma}_k]_{i,j} = \lim_{\Delta t \to 0} \frac{\Pr\bigl(\text{a disaster occurs in } (t, t+\Delta t], S(t+\Delta t) = j \mid (L(t), S(t)) = (k, i)\bigr)}{\Delta t}, \quad i, j \in \mathcal{M},$$

and

$$[\boldsymbol{C}_k]_{i,i} = -\Bigl(\sum_{\substack{j \in \mathcal{M} \\ j \neq i}} [\boldsymbol{C}_k]_{i,j} + \sum_{j \in \mathcal{M}} \bigl([\boldsymbol{D}_k]_{i,j} + [\boldsymbol{\Gamma}_k]_{i,j}\bigr)\Bigr), \quad i \in \mathcal{M}.$$

We assume $\boldsymbol{\Gamma}_0 = \boldsymbol{O}$ without loss of generality. We also assume $\boldsymbol{D}_k \neq \boldsymbol{O}$ $(k \in \mathbb{Z}^+)$ and $0 < [-\boldsymbol{C}_k]_{i,i} < \infty$ $(k \in \mathbb{Z}^+,\ i \in \mathcal{M})$. By definition, $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$ describes the arrival/disaster process. Roughly speaking, when k $(k \in \mathbb{Z}^+)$ customers stay in the system (i.e., $L(t) = k$), an arrival (resp. a disaster) occurs along with a transition driven by $\boldsymbol{D}_k$ (resp. $\boldsymbol{\Gamma}_k$). By definition, $(\boldsymbol{C}_k + \boldsymbol{D}_k + \boldsymbol{\Gamma}_k)$ $(k \in \mathbb{Z}^+)$ represents the infinitesimal generator of the underlying phase process $\{S(t)\}_{t \geq 0}$ when $L(t) = k$. Note that

$$(\boldsymbol{C}_k + \boldsymbol{D}_k + \boldsymbol{\Gamma}_k)\boldsymbol{e} = \boldsymbol{0}, \quad k \in \mathbb{Z}^+. \tag{5.9}$$

Note also that when a service completes, $L(t)$ decreases by one, while $S(t)$ remains unchanged.

We are interested in the stationary distribution in $\{(L(t), S(t))\}_{t \geq 0}$. We thus assume that for any pair of states (k_1, j_1) and (k_2, j_2) $(k_1, k_2 \in \mathbb{Z}^+,\ j_1, j_2 \in \mathcal{M})$, there exists a sample path from (k_1, j_1) to (k_2, j_2). Unfortunately, the necessary and sufficient stability condition for queueing systems with level-dependent arrivals and disasters has not been clarified. In the rest of this chapter, we assume the following sufficient condition for the stability.

Assumption 5.1. *$\{(L(t), S(t))\}_{t\geq 0}$ satisfies either of the following two conditions.*

(i) There exists $K_1^\dagger \in \mathbb{Z}^+$ and $\alpha > 0$ such that $\boldsymbol{D}_k\boldsymbol{e} - k\boldsymbol{\Gamma}_k\boldsymbol{e} \leq -\alpha$ $(k \in \mathbb{Z}^\infty_{K_1^\dagger})$, or

(ii) there exists $K_2^\dagger \in \mathbb{Z}^+$ and $\alpha > 0$ such that $\boldsymbol{D}_k\boldsymbol{e} - \mu\boldsymbol{e} \leq -\alpha$ $(k \in \mathbb{Z}^\infty_{K_2^\dagger})$, where $\mu = 1/\mathrm{E}[H]$.

Assumption 5.1 (i) is a drift condition ignoring service completions, while Assumption 5.1 (ii) is a drift condition ignoring disasters. Note that under Assumption 5.1, the arrival/disaster process characterized by $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$ is non-explosive, i.e., for an arbitrary finite $T > 0$ and an arbitrary initial state $(L(0), S(0))$, the number $N(0, T]$ of arrivals in time interval $(0, T]$ is finite with probability 1.

Remark 5.2. *The stability under Assumption 5.1 (i) can be shown as follows. For a while, we assume that services never complete. In this case, $\{(L(t), S(t))\}_{t\geq 0}$ forms a continuous-time Markov chain and it is readily shown by Pakes's Lemma [Bre99, Corollary 1.1 of Chap. 5] that $\{(L(t), S(t))\}_{t\geq 0}$ is positive-recurrent under Assumption 5.1 (i). Therefore, in the model with a finite $\mathrm{E}[H]$ described above, the mean first passage time from any state to the empty system is finite, so that $\{(L(t), S(t))\}_{t\geq 0}$ is stable. On the other hand, the stability under Assumption 5.1 (ii) can be shown by considering a modified model in which all customers remain in the system at the occurrence of disasters (i.e., the ordinary LD-MAP/G/1 queueing model without disasters), and applying Theorem 3.3-3 in [Hof01].*

5.4 Overview of the standard approach to the single-server queue with disasters

Let $\boldsymbol{\varpi} = (\boldsymbol{\varpi}_0\ \boldsymbol{\varpi}_1\ \cdots)$ denote the stationary distribution in $\{(L(t), S(t))\}_{t\geq 0}$, where $\boldsymbol{\varpi}_\ell$ denotes a $1 \times M$ vector whose jth $(j \in \mathcal{M})$ element is given by $\lim_{t\to\infty} \Pr\big((L(t), S(t) = (\ell, j)\big)$. Since service times are generally distributed, the process $\{(L(t), S(t))\}_{t\geq 0}$ is not Markovian. We consider an imbedded Markov chain associated with $\{(L(t), S(t))\}_{t\geq 0}$. Specifically, the standard approach to the analysis of the stationary distribution $\boldsymbol{\varpi}$ is outlined as follows.

Step 1: Construct an imbedded Markov chain: We choose time instants at which services complete or disasters occur as imbedded Markov points. We define τ_n $(n = 0, 1, \ldots)$ as the nth imbedded Markov point, where $\tau_0 = 0$. Let $(L_n, S_n) = (L(\tau_n), S(\tau_n))$ $(n = 0, 1, \ldots)$. It is clear that $\{(L_n, S_n)\}_{n=0,1,\ldots}$ forms a discrete-time Markov chain whose state transition probability matrix $\boldsymbol{P}$ takes the following form:

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_{0,0} & \boldsymbol{P}_{0,1} & \boldsymbol{P}_{0,2} & \boldsymbol{P}_{0,3} & \boldsymbol{P}_{0,4} & \boldsymbol{P}_{0,5} & \cdots \\ \boldsymbol{P}_{1,0} & \boldsymbol{P}_{1,1} & \boldsymbol{P}_{1,2} & \boldsymbol{P}_{1,3} & \boldsymbol{P}_{1,4} & \boldsymbol{P}_{1,5} & \cdots \\ \boldsymbol{P}_{2,0} & \boldsymbol{P}_{2,1} & \boldsymbol{P}_{2,2} & \boldsymbol{P}_{2,3} & \boldsymbol{P}_{2,4} & \boldsymbol{P}_{2,5} & \cdots \\ \boldsymbol{P}_{3,0} & \boldsymbol{O} & \boldsymbol{P}_{3,2} & \boldsymbol{P}_{3,3} & \boldsymbol{P}_{3,4} & \boldsymbol{P}_{3,5} & \cdots \\ \boldsymbol{P}_{4,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{P}_{4,3} & \boldsymbol{P}_{4,4} & \boldsymbol{P}_{4,5} & \cdots \\ \boldsymbol{P}_{5,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{P}_{5,4} & \boldsymbol{P}_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.10}$$

where $\boldsymbol{P}_{k,0}$ ($k \in \mathbb{Z}^+$) and $\boldsymbol{P}_{k,\ell}$ ($k, \ell \in \mathbb{Z}^+$, $\ell \geq k-1$) are $M \times M$ matrices.

Step 2: Compute the stationary distribution in the imbedded Markov chain: We define $\boldsymbol{\pi} = (\boldsymbol{\pi}_0 \ \boldsymbol{\pi}_1 \ \cdots)$ as the stationary distribution in the imbedded Markov chain $\{(L_n, S_n)\}_{n=0,1,\ldots}$, where $\boldsymbol{\pi}_\ell$ ($\ell \in \mathbb{Z}^+$) denotes a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element is given by $\lim_{n\to\infty} \Pr((L_n, S_n) = (\ell, j))$. By definition, $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$.

Step 3: Compute the stationary distribution $\boldsymbol{\varpi} = (\boldsymbol{\varpi}_0 \ \boldsymbol{\varpi}_1 \ \cdots)$ of $\{(L(t), S(t))\}_{t\geq 0}$: $\boldsymbol{\varpi}$ is given in terms of $\boldsymbol{\pi}_k$ ($k \in \mathbb{Z}^+$) as follows.

$$\boldsymbol{\varpi}_0 = \frac{\boldsymbol{\pi}_0(-\boldsymbol{C}_0)^{-1}}{\mathrm{E}[T]}, \qquad \boldsymbol{\varpi}_\ell = \frac{\boldsymbol{\pi}_0(-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{W}_{1,\ell}}{\mathrm{E}[T]} + \sum_{k=1}^{\ell} \frac{\boldsymbol{\pi}_k \boldsymbol{W}_{k,\ell}}{\mathrm{E}[T]} \ (\ell \in \mathbb{N}), \quad (5.11)$$

where $\mathrm{E}[T]$ denotes the average of $\tau_{n+1} - \tau_n$ ($n = 0, 1, \ldots$) and $\boldsymbol{W}_{k,\ell}$ ($k, \ell \in \mathbb{N}$, $\ell \geq k$) denotes an $M \times M$ matrix whose (i,j)th ($i, j \in \mathcal{M}$) element represents the mean total sojourn time in state (ℓ, j) in $(\tau_n, \tau_{n+1}]$, starting from $(L_n, S_n) = (k, i)$.

The numerical implementation of the above three steps in the LD-MMAP/G/1 queue with level-dependent disasters is not straightforward. In what follows, we identify the problems to be resolved in each step.

In Step 1, we have to compute $\boldsymbol{P}_{k,\ell}$'s in $\boldsymbol{P}$. As we will see in (5.20), $\boldsymbol{P}_{k,\ell}$'s are given in terms of $M \times M$ matrices $\boldsymbol{B}_{\mathrm{P},k,\ell}$'s in $\boldsymbol{B}_{\mathrm{P}}$:

$$\boldsymbol{B}_{\mathrm{P}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{B}_{\mathrm{P},1,0} & \boldsymbol{B}_{\mathrm{P},1,1} & \boldsymbol{B}_{\mathrm{P},1,2} & \boldsymbol{B}_{\mathrm{P},1,3} & \boldsymbol{B}_{\mathrm{P},1,4} & \cdots \\ \boldsymbol{B}_{\mathrm{P},2,0} & \boldsymbol{O} & \boldsymbol{B}_{\mathrm{P},2,2} & \boldsymbol{B}_{\mathrm{P},2,3} & \boldsymbol{B}_{\mathrm{P},3,4} & \cdots \\ \boldsymbol{B}_{\mathrm{P},3,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_{\mathrm{P},3,3} & \boldsymbol{B}_{\mathrm{P},3,4} & \cdots \\ \boldsymbol{B}_{\mathrm{P},4,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_{\mathrm{P},4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \int_0^\infty \exp[\boldsymbol{U}t]\mathrm{d}H(t), \quad (5.12)$$

where $\boldsymbol{U}$ denotes the infinitesimal generator of an absorbing Markov chain that represents the arrival process until a disaster occurs.

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{\Gamma}_1 & \boldsymbol{C}_1 & \boldsymbol{D}_1 & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{\Gamma}_2 & \boldsymbol{O} & \boldsymbol{C}_2 & \boldsymbol{D}_2 & \boldsymbol{O} & \cdots \\ \boldsymbol{\Gamma}_3 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_3 & \boldsymbol{D}_3 & \cdots \\ \boldsymbol{\Gamma}_4 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (5.13)$$

In what follows, we use the following convention, as with $\boldsymbol{B}_{\mathrm{P}}$ in (5.12). For any matrix $\boldsymbol{X}$ composed of $M \times M$ block matrices, let $\boldsymbol{X}_{k,\ell}$ ($k, \ell \geq 0$) denote the (k,ℓ)th block matrix of $\boldsymbol{X}$. For example, for $\boldsymbol{U}$ in (5.13), $\boldsymbol{U}_{k,k} = \boldsymbol{C}_k$ ($k \in \mathbb{N}$).

Remark 5.3. *Note that* $\exp[\boldsymbol{U}t]$ *($t \geq 0$) can be regarded as a transition probability matrix, i.e.,* $\exp[\boldsymbol{U}t] \geq \boldsymbol{O}$ *and* $\exp[\boldsymbol{U}t]\boldsymbol{e} = \boldsymbol{e}$. *Therefore, all elements in* $\boldsymbol{B}_{\mathrm{P}}$ *are finite because* $\int_0^\infty \exp[\boldsymbol{U}t]\mathrm{d}H(t)\boldsymbol{e} = \int_0^\infty \mathrm{d}H(t)\boldsymbol{e} = \boldsymbol{e}$.

The stable and accurate numerical computation of block matrices $\boldsymbol{B}_{\mathrm{P},k,\ell}$ in $\boldsymbol{B}_{\mathrm{P}}$ is crucial and it is one of the main topics of the following sections. We define θ as

$$\theta = \sup_{k\in\mathbb{N}} \big(\max_{i\in\mathcal{M}} [-\boldsymbol{C}_k]_{i,i} \big).$$

If θ is finite, the uniformization technique is directly applicable and $\boldsymbol{B}_{\mathrm{P},k,\ell}$ ($k \in \mathbb{N}$, $\ell \in \{0\} \cup \mathbb{Z}_k^{\infty}$) is given by the weighted sum of infinitely many $M \times M$ non-negative matrices, where the weights are given in terms of θ and the service time distribution $H(x)$. The advantages of the uniformization technique are that (i) the procedure is numerically stable because it does not involve subtractions and that (ii) for a given error bound ϵ, we can set an appropriate truncation point $n := n^*(\epsilon)$ of the infinite sum representing $\boldsymbol{B}_{\mathrm{P},k,\ell}$.

Note that θ can be infinite and in this case, there are no standard methods for computing block matrices $\boldsymbol{B}_{\mathrm{P},k,\ell}$ in $\boldsymbol{B}_{\mathrm{P}}$ in (5.12), which guarantee error bounds. We consider computing an approximation to $\boldsymbol{B}_{\mathrm{P},k,\ell}$ by using only the $(m+1)M \times (m+1)M$ northwest corner submatrix $\boldsymbol{U}(m)$ of $\boldsymbol{U}$ because $\exp[\boldsymbol{U}(m)t]$ is always uniformizable since $[-\boldsymbol{C}_k]_{i,i} < \infty$ ($i \in \mathcal{M}$) for all $k \in \mathbb{N}$.

The truncation of $\boldsymbol{U}$ at level m does not affect the accuracy of computed $\boldsymbol{B}_{\mathrm{P},k,\ell}$ ($\ell \in \mathbb{Z}_k^m$) because it can be represented completely in terms of $\boldsymbol{U}(m)$. Contrarily, the accuracy of computed $\boldsymbol{B}_{\mathrm{P},k,0}$ is affected by the truncation of $\boldsymbol{U}$ by the following reason. Note that $\boldsymbol{B}_{\mathrm{P},k,0}$ includes the occurrence probability that a disaster occurs before a service completion given that the service starts with $L_n = k$. Because level $L(\tau_{n+1}-)$ immediately before the occurrence of the disaster is unbounded, the exact expression of $\boldsymbol{B}_{\mathrm{P},k,0}$ is given in terms of the whole $\boldsymbol{U}$. Therefore, if we compute $\boldsymbol{B}_{\mathrm{P},k,0}$ using the truncated $\boldsymbol{U}(m)$ of $\boldsymbol{U}$ at level m, we will obtain an approximation $\boldsymbol{B}_{\mathrm{P},k,0}(m)$ to $\boldsymbol{B}_{\mathrm{P},k,0}$, which ignores all sample paths with $L(\tau_{n+1}-) > m$. Consequently, in terms of an approximation $\boldsymbol{P}_{k,0}(m)$ to $\boldsymbol{P}_{k,0}$, we have for $k \in \mathbb{Z}_0^m$ and $i, j \in \mathcal{M}$,

$$[\boldsymbol{P}_{k,0}(m)]_{i,j} = \Pr\big((L_{n+1}, S_{n+1}) = (0, j), L(\tau_{n+1}-) \le m \mid (L_n, S_n) = (k, i)\big).$$

The above discussion makes it clear that our approach has two sources of errors: one is the truncation of $\boldsymbol{U}$ at level m and the other is the truncation of the weighted infinite sum obtained by uniformization. While the latter is controllable, the error control of the former is not straightforward because it requires information beyond level m.

To control errors inherent in our approach, we adopt a *cross-layer* design of the computational procedure. Specifically, we assume that the augmented truncation approximation (ATA) is employed in Step 2, as we will explain below, where the truncation level K in the ATA is assumed to be given in advance. We then utilize this information in Step 1, i.e., we consider a computational procedure for $\boldsymbol{P}_{k,0}$ only for $k \in \mathbb{Z}_0^K$, which satisfies a predefined error bound under Assumption 5.1.

In Step 2, we compute the stationary distribution $\boldsymbol{\pi}$ in the imbedded Markov chain $\{(L_n, S_n)\}_{n=0,1,\ldots}$. Owing to the level dependence of arrivals and disasters, however, it is hard to obtain the analytical expression of the stationary distribution $\boldsymbol{\pi}$. We thus adopt the ATA, which attempts to compute an approximation to the conditional stationary distribution $\boldsymbol{\pi}_{\mathrm{lv}}(K) = (\boldsymbol{\pi}_0(K)\ \boldsymbol{\pi}_1(K)\ \cdots\ \boldsymbol{\pi}_K(K))$ for an appropriate K, where $\boldsymbol{\pi}_\ell(K)$ ($\ell \in \mathbb{Z}_0^K$) denotes a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element is given by $\lim_{n\to\infty} \Pr((L_n, S_n) = (\ell, j) \mid L_n \le K)$. Specifically, we compute an ATA solution

$\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ as an approximation to $\boldsymbol{\pi}_{\rm lv}(K)$ by solving

$$\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) = \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)[\boldsymbol{P}(K) + \boldsymbol{P}_{\rm A}(K)], \qquad \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)\boldsymbol{e} = 1, \tag{5.14}$$

where $\boldsymbol{P}(K)$ denotes the $(K+1)M \times (K+1)M$ northwest corner submatrix of $\boldsymbol{P}$:

$$\boldsymbol{P}(K) = \begin{pmatrix} \boldsymbol{P}_{0,0} & \boldsymbol{P}_{0,1} & \boldsymbol{P}_{0,2} & \boldsymbol{P}_{0,3} & \cdots & \boldsymbol{P}_{0,K-1} & \boldsymbol{P}_{0,K} \\ \boldsymbol{P}_{1,0} & \boldsymbol{P}_{1,1} & \boldsymbol{P}_{1,2} & \boldsymbol{P}_{1,3} & \cdots & \boldsymbol{P}_{1,K-1} & \boldsymbol{P}_{1,K} \\ \boldsymbol{P}_{2,0} & \boldsymbol{P}_{2,1} & \boldsymbol{P}_{2,2} & \boldsymbol{P}_{2,3} & \cdots & \boldsymbol{P}_{2,K-1} & \boldsymbol{P}_{2,K} \\ \boldsymbol{P}_{3,0} & \boldsymbol{O} & \boldsymbol{P}_{3,2} & \boldsymbol{P}_{3,3} & \cdots & \boldsymbol{P}_{3,K-1} & \boldsymbol{P}_{3,K} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{P}_{K-1,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{P}_{K-1,K-1} & \boldsymbol{P}_{K-1,K} \\ \boldsymbol{P}_{K,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{P}_{K,K-1} & \boldsymbol{P}_{K,K} \end{pmatrix},$$

and $\boldsymbol{P}_{\rm A}(K)$ denotes an augmentation matrix such that $\boldsymbol{P}_{\rm A}(K) \geq \boldsymbol{O}$ and $[\boldsymbol{P}(K) + \boldsymbol{P}_{\rm A}(K)]\boldsymbol{e} = \boldsymbol{e}$.

If the numerical errors in elements of $\boldsymbol{P}(K)$ are negligible, the error in an approximation $\boldsymbol{\pi}^{\rm approx} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)\ \boldsymbol{0})$ to $\boldsymbol{\pi}$ obtained by the ATA can be evaluated by the tail probability and the error in the ATA solution $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ (cf. (1.25)). Let

$$\xi_{\rm lv}(K) = \lim_{n\to\infty} \Pr(L_n > K), \qquad \epsilon_{\rm lv}(K) = \|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) - \boldsymbol{\pi}_{\rm lv}(K)\|_1.$$

We then have

$$2\xi_{\rm lv}(K) \leq \|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1 \leq 2\xi_{\rm lv}(K) + \epsilon_{\rm lv}(K). \tag{5.15}$$

Recall that the tail probability $\xi_{\rm lv}(K)$ decreases monotonically as K increases. On the other hand, the error $\epsilon_{\rm lv}(K)$ in the ATA solution $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ depends on the selection of the augmentation matrix $\boldsymbol{P}_{\rm A}(K)$. As shown in Section 1.4, the implications in Section 5.2 are applicable to the discrete-time Markov chain. We discuss *reasonable* selections of the augmentation matrix $\boldsymbol{P}_{\rm A}(K)$ and upper bounds of $\epsilon_{\rm lv}(K)$, and therefore we apply the discrete-time version of the procedure in Figure 5.1 to the imbedded Markov chain.

In Step 3, we compute the approximation $\boldsymbol{\varpi}^{\rm approx} = (\boldsymbol{\varpi}_0^{\rm approx}\ \boldsymbol{\varpi}_1^{\rm approx}\ \cdots)$ to the stationary distribution $\boldsymbol{\varpi}$ using $\boldsymbol{\pi}^{\rm approx} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)\ \boldsymbol{0})$. It follows from (5.11) that $\boldsymbol{\varpi}_\ell^{\rm approx}$ $(\ell = 0, 1, \ldots)$ is given in terms of $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$:

$$\boldsymbol{\varpi}_0^{\rm approx} = \frac{\boldsymbol{\pi}_0^{\rm approx}(K)(-\boldsymbol{C}_0)^{-1}}{c}, \tag{5.16}$$

$$\boldsymbol{\varpi}_\ell^{\rm approx} = \frac{\boldsymbol{\pi}_0^{\rm approx}(K)(-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{W}_{1,\ell}}{c} + \sum_{k=1}^{\min(\ell,K)} \frac{\boldsymbol{\pi}_k^{\rm approx}(K)\boldsymbol{W}_{k,\ell}}{c}, \quad \ell \in \mathbb{N}, \tag{5.17}$$

where c denotes the normalizing constant such that $\boldsymbol{\varpi}^{\rm approx}\boldsymbol{e} = 1$. Note here that $\boldsymbol{W}_{k,\ell}$'s in (5.17) are given by block matrices in $\boldsymbol{W}$ defined as

$$\boldsymbol{W} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{W}_{1,0} & \boldsymbol{W}_{1,1} & \boldsymbol{W}_{1,2} & \boldsymbol{W}_{1,3} & \boldsymbol{W}_{1,4} & \cdots \\ \boldsymbol{W}_{2,0} & \boldsymbol{O} & \boldsymbol{W}_{2,2} & \boldsymbol{W}_{2,3} & \boldsymbol{W}_{2,4} & \cdots \\ \boldsymbol{W}_{3,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{W}_{3,3} & \boldsymbol{W}_{3,4} & \cdots \\ \boldsymbol{W}_{4,0} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{W}_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \int_0^\infty \exp[\boldsymbol{U}t]\big(1 - H(t)\big)\mathrm{d}t.$$

It then follows that

$$\boldsymbol{W} = \mathrm{E}[H]\boldsymbol{B}_{\mathrm{W}},$$

where

$$\boldsymbol{B}_{\mathrm{W}} = \int_0^\infty \exp[\boldsymbol{U}t]\frac{1-H(t)}{\mathrm{E}[H]}\mathrm{d}t. \tag{5.18}$$

Because $(1-H(t))/\mathrm{E}[H]$ is the probability density function of elapsed service times, $\boldsymbol{B}_{\mathrm{W}}$ can be computed in the same way as $\boldsymbol{B}_{\mathrm{P}}$ in (5.12).

If the numerical errors in $\boldsymbol{W}_{k,\ell}$ ($k \in \mathbb{Z}_1^K$, $\ell \in \mathbb{Z}_k^\infty$) are negligible, the error in $\boldsymbol{\varpi}^{\mathrm{approx}}$ can be evaluated as follows.

$$\|\boldsymbol{\varpi} - \boldsymbol{\varpi}^{\mathrm{approx}}\|_1 \le c_p\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\mathrm{approx}}\|_1 + o(\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\mathrm{approx}}\|_1), \tag{5.19}$$

where c_p is a finite coefficient. (5.19) shows that we can control the error in $\boldsymbol{\varpi}^{\mathrm{approx}}$ through the error in $\boldsymbol{\pi}^{\mathrm{approx}}$.

In summary, to guarantee a sufficient accuracy of $\boldsymbol{\varpi}^{\mathrm{approx}}$, we have to pay attention to the followings.

(i) K should be set appropriately to make the tail probability $\xi_{\mathrm{lv}}(K)$ negligible,

(ii) the numerical errors in $\boldsymbol{P}_{k,\ell}$'s and $\boldsymbol{W}_{k,\ell}$'s should be sufficiently small, and

(iii) the error $\epsilon_{\mathrm{lv}}(K)$ in $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx}}(K)$ due to the selection of $\boldsymbol{P}_{\mathrm{A}}(K)$ should be small as much as possible.

In what follows, we mainly consider the last two points. In Section 5.5, we consider the point (ii). In Section 5.6, we consider the point (iii). Furthermore, the error propagation in Step 3, i.e., the equation (5.19), is discussed through Section 5.7. The point (i) is briefly discussed in numerical examples in Section 5.8.

5.5 Computation of $\boldsymbol{P}_{k,\ell}$'s and $\boldsymbol{W}_{k,\ell}$'s

As stated in Section 5.4, we adopt the ATA in Step 2. We thus consider the numerical computation of the northwest corner submatrix $\boldsymbol{P}(K)$ of $\boldsymbol{P}$ for a given K. By definition, block matrices $\boldsymbol{P}_{k,\ell}$'s ($k, \ell \in \mathbb{Z}_0^K$) in $\boldsymbol{P}(K)$ are given in terms of $M \times M$ block matrices in $\boldsymbol{B}_{\mathrm{P}}$ defined by (5.12).

$$\boldsymbol{P}_{k,\ell} = \begin{cases} (-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{P}_{1,\ell}, & k=0,\ \ell=0,1,\ldots,K, \\ \boldsymbol{B}_{\mathrm{P},1,0} + \boldsymbol{B}_{\mathrm{P},1,1}, & k=1,\ \ell=0, \\ \boldsymbol{B}_{\mathrm{P},1,\ell+1}, & k=1,\ \ell=1,2,\ldots,K, \\ \boldsymbol{B}_{\mathrm{P},k,0}, & k=2,3,\ldots,K,\ \ell=0, \\ \boldsymbol{B}_{\mathrm{P},k,\ell+1}, & k=2,3,\ldots,K,\ \ell=k-1,k,\ldots,K, \\ \boldsymbol{O}, & \text{otherwise.} \end{cases} \tag{5.20}$$

Therefore, we consider the computation of $\boldsymbol{B}_{\mathrm{P},k,\ell}$'s below. We also discuss the computation of $\boldsymbol{W}_{k,\ell}$ in this section because $\boldsymbol{W} = \mathrm{E}[H]\boldsymbol{B}_{\mathrm{W}}$ and block matrices in $\boldsymbol{B}_{\mathrm{W}}$ can be computed in the same way as those in $\boldsymbol{B}_{\mathrm{P}}$.

5.5.1 The uniformizable case

We first consider the case that $\exp[\boldsymbol{U}t]$ is uniformizable, i.e., there exists a finite θ such that $[-\boldsymbol{C}_k]_{i,i} \le \theta$ $(k \in \mathbb{N}, i \in \mathcal{M})$. In this case, by applying the uniformization technique to $\boldsymbol{B}_{\rm P}$ in (5.12) and $\boldsymbol{B}_{\rm W}$ in (5.18), we can rewrite them to be

$$\boldsymbol{B}_{\rm P} = \int_0^\infty e^{-\theta t} \exp[(\boldsymbol{I} + \theta^{-1}\boldsymbol{U})\theta t] {\rm d}H(t) = \sum_{n=0}^\infty \gamma_n \boldsymbol{V}^n,$$
$$\boldsymbol{B}_{\rm W} = \int_0^\infty e^{-\theta t} \exp[(\boldsymbol{I} + \theta^{-1}\boldsymbol{U})\theta t] \frac{1-H(t)}{{\rm E}[H]} {\rm d}t = \sum_{n=0}^\infty \eta_n \boldsymbol{V}^n,$$

where γ_n and η_n $(n = 0, 1, \ldots)$ denote probability functions given by

$$\gamma_n = \int_0^\infty e^{-\theta t} \frac{(\theta t)^n}{n!} {\rm d}H(t), \qquad \eta_n = \int_0^\infty e^{-\theta t} \frac{(\theta t)^n}{n!} \cdot \frac{1-H(t)}{{\rm E}[H]} {\rm d}t,$$

and $\boldsymbol{V} = \boldsymbol{I} + \theta^{-1}\boldsymbol{U} \ge \boldsymbol{O}$. Note here that $\boldsymbol{V}^0 = \boldsymbol{I}$ and $\boldsymbol{V}\boldsymbol{e} = \boldsymbol{e}$. Block matrices $\boldsymbol{V}_{k,\ell}^{(n)}$ in $\boldsymbol{V}^n$ $(n = 2, 3, \ldots)$ can be computed recursively by $\boldsymbol{V}_{k,\ell}^{(1)} = \boldsymbol{V}_{k,\ell}$ and for $n = 2, 3, \ldots$,

$$\boldsymbol{V}_{k,\ell}^{(n)} = \begin{cases} \boldsymbol{V}_{k,0}^{(n-1)} + \displaystyle\sum_{r=k}^{k+n-1} \boldsymbol{V}_{k,r}^{(n-1)} \theta^{-1} \boldsymbol{\Gamma}_r, & \ell = 0, \\ \boldsymbol{V}_{k,k}^{(n-1)} (\boldsymbol{I} + \theta^{-1}\boldsymbol{C}_k), & \ell = k, \\ \boldsymbol{V}_{k,\ell-1}^{(n-1)} \theta^{-1} \boldsymbol{D}_{\ell-1} + \boldsymbol{V}_{k,\ell}^{(n-1)} \theta^{-1} (\boldsymbol{I} + \theta^{-1}\boldsymbol{C}_\ell), & \ell = k+1, k+2, \ldots, K+1, \\ \boldsymbol{O}, & \text{otherwise.} \end{cases}$$

In numerical computation, we assume that the recursion of $\boldsymbol{V}_{k,\ell}^{(n)}$ for $\boldsymbol{B}_{{\rm P},k,\ell}$ stops at $n = n_{\rm P}$ and that for $\boldsymbol{B}_{{\rm W},k,\ell}$ at $n = n_{\rm W}$.

$$\boldsymbol{B}_{\rm P}^{{\rm comp},n_{\rm P}} = \sum_{n=0}^{n_{\rm P}} \gamma_n \boldsymbol{V}^n, \qquad \boldsymbol{B}_{\rm W}^{{\rm comp},n_{\rm W}} = \sum_{n=0}^{n_{\rm W}} \eta_n \boldsymbol{V}^n.$$

We then adopt $\boldsymbol{B}_{\rm P}^{{\rm comp},n_{\rm P}}$ as an approximation to $\boldsymbol{B}_{\rm P}$ and $\boldsymbol{B}_{\rm W}^{{\rm comp},n_{\rm W}}$ as an approximation to $\boldsymbol{B}_{\rm W}$, where $\boldsymbol{O} \le \boldsymbol{B}_{\rm P}^{{\rm comp},n_{\rm P}} \le \boldsymbol{B}_{\rm P}$ and $\boldsymbol{O} \le \boldsymbol{B}_{\rm W}^{{\rm comp},n_{\rm W}} \le \boldsymbol{B}_{\rm W}$. In what follows, we determine $n_{\rm P} = n_{\rm P}^*$ and $n_{\rm W} = n_{\rm W}^*$ separately in such a way that for $k = 0, 1, \ldots, K$,

$$\sum_{\ell=0}^{K} (\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}) \boldsymbol{e} \le \epsilon_{\rm P} \boldsymbol{e}, \qquad \sum_{\ell=k}^{\infty} (\boldsymbol{W}_{k,\ell} - \boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}^*}) \boldsymbol{e} \le \epsilon_{\rm W} \boldsymbol{e}, \tag{5.21}$$

where $\epsilon_{\rm P}$ and $\epsilon_{\rm W}$ are sufficiently small positive constants.

It follows from (5.20) that the inequality for $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}$ $(k = 0, 1, \ldots, K)$ in (5.21) is equivalent to

$$\sum_{\ell=0}^{K+1} (\boldsymbol{B}_{{\rm P},k,\ell} - \boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}^*}) \boldsymbol{e} \le \epsilon_{\rm P} \boldsymbol{e}, \quad k \in \mathbb{Z}_1^K. \tag{5.22}$$

Note here that

$$\sum_{\ell=0}^{K+1} (\boldsymbol{B}_{{\rm P},k,\ell} - \boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}) \boldsymbol{e} \le \sum_{\ell=0}^{\infty} (\boldsymbol{B}_{{\rm P},k,\ell} - \boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}) \boldsymbol{e} = \sum_{n=n_{\rm P}+1}^{\infty} \gamma_n \sum_{\ell=0}^{\infty} \boldsymbol{V}_{k,\ell}^{(n)} \boldsymbol{e} = \sum_{n=n_{\rm P}+1}^{\infty} \gamma_n \boldsymbol{e}.$$

Therefore, we set $n_{\mathrm{P}} = n_{\mathrm{P}}^*$ in such a way that

$$\sum_{n=0}^{n_{\mathrm{P}}^*} \gamma_n \geq 1 - \epsilon_{\mathrm{P}},$$

which guarantees (5.22).

Similarly, we have for $\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}$,

$$\begin{aligned}\sum_{\ell=k}^{\infty}(\boldsymbol{W}_{k,\ell} - \boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}})\boldsymbol{e} &\leq \sum_{\ell=0}^{\infty}(\boldsymbol{W}_{k,\ell} - \boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}})\boldsymbol{e} \\ &= \mathrm{E}[H]\sum_{n=n_{\mathrm{W}}+1}^{\infty}\eta_n\sum_{\ell=0}^{\infty}\boldsymbol{V}_{k,\ell}^{(n)}\boldsymbol{e} = \mathrm{E}[H]\sum_{n=n_{\mathrm{W}}+1}^{\infty}\eta_n\boldsymbol{e}.\end{aligned}$$

Therefore, we set $n_{\mathrm{W}} = n_{\mathrm{W}}^*$ in such a way that

$$\sum_{n=0}^{n_{\mathrm{W}}^*}\eta_n \geq 1 - \frac{\epsilon_{\mathrm{W}}}{\mathrm{E}[H]},$$

which guarantees the inequality for $\boldsymbol{W}_{k,k+\ell}^{\mathrm{comp},n_{\mathrm{W}}^*}$ in (5.21). Note that for each $k \in \mathbb{Z}_1^K$, the computation of $\boldsymbol{B}_{\mathrm{W},k,\ell}^{\mathrm{comp},n_{\mathrm{W}}^*}$ automatically stops at $\ell = k + n_{\mathrm{W}}^*$ and $\boldsymbol{B}_{\mathrm{W},k,\ell}^{\mathrm{comp},n_{\mathrm{W}}^*} = \boldsymbol{O}$ for all $\ell > k + n_{\mathrm{W}}^*$.

5.5.2 The non-uniformizable case

In this subsection, we consider the computation of approximations to $\boldsymbol{P}_{k,\ell}$'s and $\boldsymbol{W}_{k,\ell}$'s under the assumption that $\exp[\boldsymbol{U}t]$ is non-uniformizable, i.e.,

$$\sup_{k\in\mathbb{N}}\left(\max_{i\in\mathcal{M}}[-\boldsymbol{C}_k]_{i,i}\right) = \infty. \tag{5.23}$$

We define $\boldsymbol{U}(m)$ $(m \in \mathbb{N})$ as

$$\boldsymbol{U}(m) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{\Gamma}_1 & \boldsymbol{C}_1 & \boldsymbol{D}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{\Gamma}_2 & \boldsymbol{O} & \boldsymbol{C}_2 & \boldsymbol{D}_2 & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{\Gamma}_3 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_3 & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{\Gamma}_{m-1} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{C}_{m-1} & \boldsymbol{D}_{m-1} \\ \boldsymbol{\Gamma}_m & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{C}_m \end{pmatrix}.$$

By definition, $\boldsymbol{U}(m)$ $(m \in \mathbb{N})$ is a defective infinitesimal generator. Let θ_m $(m \in \mathbb{N})$ denote the maximum of the absolute values of the diagonal elements of $\boldsymbol{U}(m)$.

$$\theta_m = \max_{\substack{k=1,2,\ldots,m \\ i\in\mathcal{M}}}[-\boldsymbol{C}_k]_{i,i}. \quad m \in \mathbb{N}.$$

In what follows, we first consider $\boldsymbol{P}_{k,\ell}$'s and then consider $\boldsymbol{W}_{k,\ell}$'s.

We assume that $\boldsymbol{U}$ is truncated at level $m_{\rm P}$ in computing $\boldsymbol{P}_{k,\ell}$'s, where $m_{\rm P} \geq K+1$. We then define $\boldsymbol{B}_{\rm P}(m_{\rm P})$ as

$$\boldsymbol{B}_{\rm P}(m_{\rm P}) = \int_0^\infty \exp[\boldsymbol{U}(m_{\rm P})t]{\rm d}H(t) = \sum_{n=0}^\infty \gamma_n(m_{\rm P})\boldsymbol{V}^n(m_{\rm P}), \tag{5.24}$$

where $\gamma_n(m_{\rm P})$ $(n=0,1,\ldots)$ denotes a probability function given by

$$\gamma_n(m_{\rm P}) = \int_0^\infty e^{-\theta_{m_{\rm P}}t}\frac{(\theta_{m_{\rm P}}t)^n}{n!}{\rm d}H(t),$$

and $\boldsymbol{V}(m_{\rm P}) = \boldsymbol{I} + \theta_{m_{\rm P}}^{-1}\boldsymbol{U}(m_{\rm P})$. Note that $\boldsymbol{V}(m_{\rm P})$ is substochastic, i.e., $\boldsymbol{V}(m_{\rm P})\boldsymbol{e} \leq \boldsymbol{e}$. For any $m \in \mathbb{N}$, block matrices $\boldsymbol{V}_{k,\ell}^{(n)}(m)$ in $\boldsymbol{V}^n(m)$ $(n=1,2,\ldots)$ can be computed recursively by $\boldsymbol{V}_{k,\ell}^{(1)}(m) = \boldsymbol{V}_{k,\ell}(m)$ and for $n=2,3,\ldots$,

$$\boldsymbol{V}_{k,\ell}^{(n)}(m) = \begin{cases} \boldsymbol{V}_{k,0}^{(n-1)}(m) + \displaystyle\sum_{r=k}^{\min(m,k+n-1)} \boldsymbol{V}_{k,r}^{(n-1)}(m)\theta_m^{-1}\boldsymbol{\Gamma}_r, & \ell = 0, \\ \boldsymbol{V}_{k,k}^{(n-1)}(m)(\boldsymbol{I} + \theta_m^{-1}\boldsymbol{C}_k), & \ell = k, \\ \boldsymbol{V}_{k,\ell-1}^{(n-1)}(m)\theta_m^{-1}\boldsymbol{D}_{\ell-1} + \boldsymbol{V}_{k,\ell}^{(n-1)}(m)(\boldsymbol{I} + \theta_m^{-1}\boldsymbol{C}_\ell), & \\ \qquad\qquad \ell = k+1, k+2, \ldots, \min(m, k+n), & \\ \boldsymbol{O}, & \text{otherwise.} \end{cases} \tag{5.25}$$

In numerical computation, we have to truncate the infinite sum in (5.24), i.e.,

$$\boldsymbol{B}_{\rm P}^{{\rm comp},n_{\rm P}}(m_{\rm P}) = \sum_{n=0}^{n_{\rm P}} \gamma_n(m_{\rm P})\boldsymbol{V}^n(m_{\rm P}).$$

We adopt $\boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})$ as an approximation to $\boldsymbol{B}_{{\rm P},k,\ell}$. Specifically, for $k \in \mathbb{Z}_1^K$, an approximation $\boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})$ to $\boldsymbol{B}_{{\rm P},k,\ell}$ is given by

$$\boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P}) = \begin{cases} \displaystyle\sum_{n=1}^{n_{\rm P}} \gamma_n(m_{\rm P})\boldsymbol{V}_{k,0}^{(n)}(m_{\rm P}), & \ell = 0, \\ \displaystyle\sum_{n=\ell-k}^{n_{\rm P}} \gamma_n(m_{\rm P})\boldsymbol{V}_{k,\ell}^{(n)}(m_{\rm P}), & \ell = k, k+1, \ldots, \min(m_{\rm P}, k+n_{\rm P}), \\ \boldsymbol{O}, & \text{otherwise.} \end{cases} \tag{5.26}$$

Furthermore, it follows from (5.20) that an approximation $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})$ to $\boldsymbol{P}_{k,\ell}$ is given by

$$\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P}) = \begin{cases} (-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{P}_{1,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P}), & k=0,\ \ell = 0,1,\ldots,m_{\rm P}-1, \\ \boldsymbol{B}_{{\rm P},1,0}^{{\rm comp},n_{\rm P}}(m_{\rm P}) + \boldsymbol{B}_{{\rm P},1,1}^{{\rm comp},n_{\rm P}}(m_{\rm P}), & k=1,\ \ell = 0, \\ \boldsymbol{B}_{{\rm P},1,\ell+1}^{{\rm comp},n_{\rm P}}(m_{\rm P}), & k=1,\ \ell = 1,2,\ldots,m_{\rm P}-1, \\ \boldsymbol{B}_{{\rm P},k,0}^{{\rm comp},n_{\rm P}}(m_{\rm P}), & k=2,3,\ldots,K,\ \ell = 0, \\ \boldsymbol{B}_{{\rm P},k,\ell+1}^{{\rm comp},n_{\rm P}}(m_{\rm P}), & k=2,3,\ldots,K, \\ & \qquad \ell = k-1, k, \ldots, m_{\rm P}-1, \\ \boldsymbol{O}, & \text{otherwise.} \end{cases} \tag{5.27}$$

Note here that

$$\boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}}) \le \boldsymbol{P}_{k,\ell}, \quad k \in \mathbb{Z}_0^K,\ \ell \in \mathbb{Z}_0^{m_{\mathrm{P}}-1}. \tag{5.28}$$

For a given $\epsilon_{\mathrm{P}} > 0$, we would like to find $m_{\mathrm{P}} = m_{\mathrm{P}}^*$ and $n_{\mathrm{P}} = n_{\mathrm{P}}^*$ that satisfy

$$\sum_{\ell=0}^{K}\big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}^*}(m_{\mathrm{P}}^*)\big)\boldsymbol{e} \le \epsilon_{\mathrm{P}}\boldsymbol{e}, \quad k \in \mathbb{Z}_0^K. \tag{5.29}$$

Remark 5.4. *It follows from (5.28) that for $k \in \mathbb{Z}_0^K$,*

$$\begin{aligned}\sum_{\ell=0}^{K}\big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\big)\boldsymbol{e} &\le \sum_{\ell=0}^{m_{\mathrm{P}}-1}\big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\big)\boldsymbol{e} \\ &\le \sum_{\ell=0}^{\infty}\boldsymbol{P}_{k,\ell}\boldsymbol{e} - \sum_{\ell=0}^{m_{\mathrm{P}}-1}\boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\boldsymbol{e} = \boldsymbol{e} - \sum_{\ell=0}^{m_{\mathrm{P}}-1}\boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\boldsymbol{e}.\end{aligned} \tag{5.30}$$

Therefore, if m_{P} and n_{P} are fixed, we can evaluate the upper bound of the left-hand side of (5.29) by computing $\boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})$ ($k \in \mathbb{Z}_0^K$, $\ell \in \mathbb{Z}_0^{m_{\mathrm{P}}-1}$).

This approach has two potential sources of errors: the truncation of $\boldsymbol{U}$ at level m_{P} and the truncation of the infinite sum in (5.24) at $n = n_{\mathrm{P}}$. Note here that

$$\sum_{\ell=0}^{m_{\mathrm{P}}}\boldsymbol{V}_{k,\ell}^{(n)}(m_{\mathrm{P}})\boldsymbol{e} = \boldsymbol{e}, \quad k \in \mathbb{Z}_1^K,\ n = 0, 1, \ldots, m_{\mathrm{P}} - k.$$

Therefore, if $n_{\mathrm{P}} \le m_{\mathrm{P}} - K$,

$$\sum_{\ell=0}^{m_{\mathrm{P}}-1}\boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\boldsymbol{e} = \sum_{\ell=0}^{m_{\mathrm{P}}}\boldsymbol{B}_{\mathrm{P},k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}\boldsymbol{e} = \sum_{\ell=0}^{m_{\mathrm{P}}}\sum_{n=0}^{n_{\mathrm{P}}}\gamma_n(m_{\mathrm{P}})\boldsymbol{V}_{k,\ell}^{(n)}(m_{\mathrm{P}})\boldsymbol{e} = \sum_{n=0}^{n_{\mathrm{P}}}\gamma_n(m_{\mathrm{P}})\boldsymbol{e},$$

for all $k \in \mathbb{Z}_1^K$. It then follows from (5.30) that

$$\sum_{\ell=0}^{K}\big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\big)\boldsymbol{e} \le \Big(1 - \sum_{n=0}^{n_{\mathrm{P}}}\gamma_n(m_{\mathrm{P}})\Big)\boldsymbol{e} \quad \text{if} \quad n_{\mathrm{P}} \le m_{\mathrm{P}} - K. \tag{5.31}$$

Because the right-hand side of (5.31) takes the minimum value at $n_{\mathrm{P}} = m_{\mathrm{P}} - K$, (5.29) would hold if we could find $m_{\mathrm{P}} = m_{\mathrm{P}}^*$ such that

$$\Big(1 - \sum_{n=0}^{m_{\mathrm{P}}^*-K}\gamma_n(m_{\mathrm{P}}^*)\Big)\boldsymbol{e} = \Pr(N_{\theta_{m_{\mathrm{P}}^*}}(H) > m_{\mathrm{P}}^* - K) \le \epsilon_{\mathrm{P}}, \tag{5.32}$$

where $N_{\theta_{m_{\mathrm{P}}^*}}(H)$ denotes the number of Poisson arrivals with rate $\theta_{m_{\mathrm{P}}^*}$ during a randomly chosen service time H.

Note, however, that m_{P}^* satisfying (5.32) does not necessary exist in general. The reason is that (i) for a fixed x, $\Pr(N_{\theta_{m_{\mathrm{P}}}}(H) > x)$ is an increasing function of $\theta_{m_{\mathrm{P}}}$ and (ii) $\theta_{m_{\mathrm{P}}}$ is a non-decreasing function of m_{P}, which imply that $\Pr(N_{\theta_{m_{\mathrm{P}}}}(H) > m_{\mathrm{P}} - K)$ may or may not decrease as m_{P} increases. Besides, $\Pr(N_{\theta_{m_{\mathrm{P}}}}(H) > x)$ depends on the distribution of H. For example, if $\theta_{m_{\mathrm{P}}}/m_{\mathrm{P}}$ is equal to zero in the limit $m_{\mathrm{P}} \to \infty$, it

can be shown that $m_{\rm P} - K > {\rm E}[N_{\theta_{m_{\rm P}}}(H)] = \theta_{m_{\rm P}}{\rm E}[H]$ for a sufficient large $m_{\rm P}$ and in this case, the one-side Chebyshev's inequality (also called Cantelli's inequality) [Usp37, pp.198–199] yields

$$\Pr(N_{\theta_{m_{\rm P}}}(H) > m_{\rm P} - K) \le \frac{\theta_{m_{\rm P}}^2 {\rm Var}[H] + \theta_{m_{\rm P}}{\rm E}[H]}{\theta_{m_{\rm P}}^2 {\rm Var}[H] + \theta_{m_{\rm P}}{\rm E}[H] + (m_{\rm P} - K - \theta_{m_{\rm P}}{\rm E}[H])^2}, \quad (5.33)$$

where the variance ${\rm Var}[H]$ of service times is assumed to be finite. Furthermore, it can be readily verified that the right-hand side of (5.33) is equal to zero in the limit $m_{\rm P} \to \infty$ if $\theta_{m_{\rm P}}/m_{\rm P}$ is equal to zero in the limit $m_{\rm P} \to \infty$. Another example is that if the service time distribution belongs to a certain class of long-tailed distributions, we have [Asm99, Eq. (1.1)]

$$\lim_{x\to\infty} \frac{\Pr(N_{\theta_{m_{\rm P}}}(H) > x)}{\Pr(H > x/\theta_{m_{\rm P}})} = 1.$$

These examples suggest that for the existence of $m_{\rm P} = m_{\rm P}^*$ satisfying (5.32), we need some additional assumptions on the service time distribution and/or the asymptotic property of the sequence of $\theta_{m_{\rm P}}$'s. In what follows, we develop a numerical procedure that always works under Assumption 5.1 introduced in Section 5.3.

Theorem 5.1. *Suppose the arrival/disaster process of* $\{(L(t), S(t))\}_{t\ge 0}$ *is non-explosive, i.e.,* $\sum_{\ell=0}^{\infty} \boldsymbol{P}_{k,\ell}\boldsymbol{e} = \boldsymbol{e}$. *We then have for* $k \in \mathbb{Z}_0^K$,

$$\sum_{\ell=0}^{K}\big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e} \le \max_{i\in\mathcal{M}}\big(b_{{\rm P},(K,i)}^{+}(m_{\rm P})\big)\boldsymbol{e} + \Big(1 - \sum_{n=0}^{n_{\rm P}}\gamma_n(m_{\rm P})\Big)\boldsymbol{e}, \quad (5.34)$$

where $b_{{\rm P},(k,i)}^{+}(m_{\rm P})$ *(*$k \in \mathbb{Z}_0^K$, $i \in \mathcal{M}$*) is defined as*

$$b_{{\rm P},(k,i)}^{+}(m_{\rm P}) = \Pr\big(L(\tau_{n+1}-) > m_{\rm P} \mid (L_n, S_n) = (k,i)\big). \quad (5.35)$$

Proof. Note first that for an arbitrary $\epsilon > 0$,

$$\sum_{\ell=0}^{K}\big(\boldsymbol{P}_{1,\ell} - \boldsymbol{P}_{1,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e} \le \epsilon\boldsymbol{e} \;\Rightarrow\; \sum_{\ell=0}^{K}(\boldsymbol{P}_{0,\ell} - \boldsymbol{P}_{0,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P}))\boldsymbol{e} \le (-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0 \cdot \epsilon\boldsymbol{e} = \epsilon\boldsymbol{e}.$$

We thus consider (5.34) for $k \in \mathbb{Z}_1^K$.

It follows from (5.20) and $\boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P}) \le \boldsymbol{B}_{{\rm P},k,\ell}$ ($k \in \mathbb{Z}_1^K$, $\ell \in \mathbb{Z}_0^{m_{\rm P}}$) that for $k \in \mathbb{Z}_1^K$,

$$\begin{aligned}\sum_{\ell=0}^{K}\big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e} &= \sum_{\ell=0}^{K+1}\big(\boldsymbol{B}_{{\rm P},k,\ell} - \boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e} \\ &\le \sum_{\ell=0}^{\infty}\big(\boldsymbol{B}_{{\rm P},k,\ell} - \boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e},\end{aligned}$$

where $\boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P}) = \boldsymbol{O}$ for $\ell > m_{\rm P}$. In what follows, we will show for $k \in \mathbb{Z}_1^K$,

$$\sum_{\ell=0}^{\infty}\big(\boldsymbol{B}_{{\rm P},k,\ell} - \boldsymbol{B}_{{\rm P},k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e} \le \max_{i\in\mathcal{M}}\big(b_{{\rm P},(K,i)}^{+}(m_{\rm P})\big)\boldsymbol{e} + \Big(1 - \sum_{n=0}^{n_{\rm P}}\gamma_n(m_{\rm P})\Big)\boldsymbol{e}. \quad (5.36)$$

By definition, $\boldsymbol{B}_{\mathrm{P},k,\ell}(m_{\mathrm{P}}) = \boldsymbol{B}_{\mathrm{P},k,\ell}$ ($k \in \mathbb{Z}_1^K$, $\ell \in \mathbb{Z}_1^{m_{\mathrm{P}}}$), where $m_{\mathrm{P}} \geq K+1$. It then follows that for $k \in \mathbb{Z}_1^K$,

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\big(\boldsymbol{B}_{\mathrm{P},k,\ell} - \boldsymbol{B}_{\mathrm{P},k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\big)\boldsymbol{e} \\
&\quad = \big(\boldsymbol{B}_{\mathrm{P},k,0} - \boldsymbol{B}_{\mathrm{P},k,0}(m_{\mathrm{P}})\big)\boldsymbol{e} + \sum_{\ell=0}^{m_{\mathrm{P}}}\big(\boldsymbol{B}_{\mathrm{P},k,\ell}(m_{\mathrm{P}}) - \boldsymbol{B}_{\mathrm{P},k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\big)\boldsymbol{e} + \sum_{\ell=m_{\mathrm{P}}+1}^{\infty}\boldsymbol{B}_{\mathrm{P},k,\ell}\boldsymbol{e}.
\end{aligned} \tag{5.37}$$

Note here that for $k \in \mathbb{Z}_0^K$ and $j \in \mathcal{M}$,

$$\Big[\big(\boldsymbol{B}_{\mathrm{P},k,0} - \boldsymbol{B}_{\mathrm{P},k,0}(m_{\mathrm{P}})\big)\boldsymbol{e} + \sum_{\ell=m_{\mathrm{P}}+1}^{\infty}\boldsymbol{B}_{\mathrm{P},k,\ell}\boldsymbol{e}\Big]_j = b_{\mathrm{P},(k,j)}^{+}(m_{\mathrm{P}}) \leq \max_{i\in\mathcal{M}}\big(b_{\mathrm{P},(K,i)}^{+}(m_{\mathrm{P}})\big). \tag{5.38}$$

On the other hand, we have for $k \in \mathbb{Z}_1^K$,

$$\sum_{\ell=0}^{m_{\mathrm{P}}}\big(\boldsymbol{B}_{\mathrm{P},k,\ell}(m_{\mathrm{P}}) - \boldsymbol{B}_{\mathrm{P},k,\ell}^{\mathrm{comp},n_{\mathrm{P}}}(m_{\mathrm{P}})\big)\boldsymbol{e} = \sum_{\ell=0}^{m_{\mathrm{P}}}\sum_{n=n_{\mathrm{P}}+1}^{\infty}\gamma_n(m_{\mathrm{P}})\boldsymbol{V}_{k,\ell}^{(n)}(m_{\mathrm{P}})\boldsymbol{e} \leq \sum_{n=n_{\mathrm{P}}+1}^{\infty}\gamma_n(m_{\mathrm{P}})\boldsymbol{e}. \tag{5.39}$$

(5.36) now follows from (5.37), (5.38), and (5.39), which completes the proof. □

By definition, $\max_{i\in\mathcal{M}}\big(b_{\mathrm{P},(K,i)}^{+}(m_{\mathrm{P}})\big)$ on the right-hand side of (5.34) monotonically converges to zero as m_{P} goes to infinity and for a fixed m_{P}, the second term also converges to zero monotonically as n_{P} goes to infinitely. Under Assumption 5.1 (i), we have the following theorem, where

$$\boldsymbol{b}_{\mathrm{P},K}^{+}(m_{\mathrm{P}}) = \big(b_{\mathrm{P},(K,1)}^{+}(m_{\mathrm{P}})\ b_{\mathrm{P},(K,2)}^{+}(m_{\mathrm{P}})\ \cdots\ b_{\mathrm{P},(K,M)}^{+}(m_{\mathrm{P}})\big)^T.$$

Theorem 5.2. *We consider $\{(L(t), S(t))\}_{t\geq 0}$, where (5.23) is assumed to hold. For arbitrary positive integers K and m_{P} ($m_{\mathrm{P}} \geq K+1$), we have*

$$\boldsymbol{b}_{\mathrm{P},K}^{+}(m_{\mathrm{P}}) \leq (-\boldsymbol{C}_K)^{-1}\boldsymbol{D}_K(-\boldsymbol{C}_{K+1})^{-1}\boldsymbol{D}_{K+1}\cdots(-\boldsymbol{C}_{m_{\mathrm{P}}})^{-1}\boldsymbol{D}_{m_{\mathrm{P}}}\boldsymbol{e}. \tag{5.40}$$

Furthermore, for arbitrary K and $\epsilon > 0$, there exists $m_{\mathrm{P}} = m_{\mathrm{P}}^*$ satisfying

$$(-\boldsymbol{C}_K)^{-1}\boldsymbol{D}_K(-\boldsymbol{C}_{K+1})^{-1}\boldsymbol{D}_{K+1}\cdots(-\boldsymbol{C}_{m_{\mathrm{P}}})^{-1}\boldsymbol{D}_{m_{\mathrm{P}}^*}\boldsymbol{e} \leq \epsilon\boldsymbol{e}, \tag{5.41}$$

under Assumption 5.1 (i).

Proof. We first prove (5.40) by a probabilistic argument, even though it can also be shown algebraically. It follows from (5.35) that

$$\begin{aligned}
b_{\mathrm{P},(K,i)}^{+}(m_{\mathrm{P}}) &= \Pr(L(\tau_{n+1}-) > m_{\mathrm{P}} \mid (L_n, S_n) = (K,i)) \\
&= \Pr(L(t) = m_{\mathrm{P}}+1 \text{ for some } t \in (\tau_n, \tau_{n+1}) \mid (L_n, S_n) = (K,i)).
\end{aligned}$$

Note here that $\tau_{n+1} = \min(d_{n+1}, \tau_n + H_n)$, where d_{n+1} denotes the first occurrence time of a disaster after time τ_n and H_n denotes the service time starting at time τ_n.

Let $\tilde{N}(s,t)$ $(s<t)$ denote the number of arrivals during an interval (s,t) under the assumption that the service never completes. We then have

$$\begin{aligned} b^{+}_{\mathrm{P},(K,i)}(m_{\mathrm{P}}) &= \Pr(L(t)=m_{\mathrm{P}}+1 \text{ for some } t\in(\tau_n,\tau_{n+1}) \mid (L_n,S_n)=(K,i)) \\ &\le \Pr(L(t)=m_{\mathrm{P}}+1 \text{ for some } t\in(\tau_n,d_{n+1}) \mid (L_n,S_n)=(K,i)) \\ &= \Pr(\tilde{N}(\tau_n,d_{n+1})>m_{\mathrm{P}}-K \mid (L_n,S_n)=(K,i)) \\ &= [(-\boldsymbol{C}_K)^{-1}\boldsymbol{D}_K(-\boldsymbol{C}_{K+1})^{-1}\boldsymbol{D}_{K+1}\cdots(-\boldsymbol{C}_{m_{\mathrm{P}}})^{-1}\boldsymbol{D}_{m_{\mathrm{P}}}\boldsymbol{e}]_i, \end{aligned}$$

from which (5.40) follows.

Next we consider (5.41) under Assumption 5.1 (i). It follows from (5.9) and Assumption 5.1 (i) that $(-\boldsymbol{C}_k)^{-1}\boldsymbol{D}_k\boldsymbol{e}<\{k/(k+1)\}\cdot\boldsymbol{e}$ $(k\in\mathbb{Z}^{\infty}_{K_1^{\dagger}})$. We then have

$$\begin{aligned} &\lim_{m_{\mathrm{P}}\to\infty}(-\boldsymbol{C}_{K_1^{\dagger}})^{-1}\boldsymbol{D}_{K_1^{\dagger}}(-\boldsymbol{C}_{K_1^{\dagger}+1})^{-1}\boldsymbol{D}_{K_1^{\dagger}+1}\cdots(-\boldsymbol{C}_{m_{\mathrm{P}}})^{-1}\boldsymbol{D}_{m_{\mathrm{P}}}\boldsymbol{e} \\ &\qquad < \lim_{m_{\mathrm{P}}\to\infty}\frac{K_1^{\dagger}}{K_1^{\dagger}+1}\cdot\frac{K_1^{\dagger}+1}{K_1^{\dagger}+2}\cdots\frac{m_{\mathrm{P}}}{m_{\mathrm{P}}+1}\cdot\boldsymbol{e}=\lim_{m_{\mathrm{P}}\to\infty}\frac{K_1^{\dagger}}{m_{\mathrm{P}}+1}\cdot\boldsymbol{e}=\boldsymbol{0}, \end{aligned}$$

so that for an arbitrary ϵ $(0<\epsilon<1)$, $m_{\mathrm{P}}=m_{\mathrm{P}}^{*}$ satisfying (5.41) exists. □

Next we consider Assumption 5.1 (ii). Preliminary to it, we show the following lemma.

Lemma 5.3. *Let* $\boldsymbol{C}_k^{\natural}=\boldsymbol{C}_k+\mathrm{diag}(\boldsymbol{\Gamma}_k\boldsymbol{e})$.

(i) For any $t\ge 0$, *we have*

$$\exp[\boldsymbol{C}_k t]\le\exp[\boldsymbol{C}_k^{\natural}t],\quad k\in\mathbb{N}. \tag{5.42}$$

(ii) For any $M\times 1$ *vector* $\boldsymbol{x}\ge\boldsymbol{0}$, *we have*

$$(-\boldsymbol{C}_k^{\natural})^{-1}\boldsymbol{D}_k=\big[-\big(\boldsymbol{C}_k^{\natural}-\mathrm{diag}(\boldsymbol{x})\big)\big]^{-1}[\boldsymbol{D}_k+\mathrm{diag}(\boldsymbol{x})(-\boldsymbol{C}_k^{\natural})^{-1}\boldsymbol{D}_k]. \tag{5.43}$$

(iii) For any $M\times 1$ *vector* $\boldsymbol{x}\ge\boldsymbol{0}$ *and* $\tau\ge 0$, *we have*

$$\int_0^{\tau}\exp[\boldsymbol{C}_k^{\natural}t]\boldsymbol{D}_k\mathrm{d}t\le\int_0^{\tau}\exp\big[\big(\boldsymbol{C}_k^{\natural}-\mathrm{diag}(\boldsymbol{x})\big)t\big][\boldsymbol{D}_k+\mathrm{diag}(\boldsymbol{x})(-\boldsymbol{C}_k^{\natural})^{-1}\boldsymbol{D}_k]\mathrm{d}t. \tag{5.44}$$

Proof. We first consider (5.42). Since $\mathrm{diag}(\boldsymbol{\Gamma}_k\boldsymbol{e})\ge\boldsymbol{O}$, we have $\boldsymbol{C}_k\le\boldsymbol{C}_k+\mathrm{diag}(\boldsymbol{\Gamma}\boldsymbol{e})=\boldsymbol{C}_k^{\natural}$. It then follows that $[\boldsymbol{I}+\theta_{m_{\mathrm{P}}}^{-1}\boldsymbol{C}_k]^n\le[\boldsymbol{I}+\theta_{m_{\mathrm{P}}}^{-1}\boldsymbol{C}_k^{\natural}]^n$ and therefore

$$\begin{aligned} \exp[\boldsymbol{C}_k t]=e^{-\theta_{m_{\mathrm{P}}}t}\sum_{n=0}^{\infty}\frac{(\theta_{m_{\mathrm{P}}}t)^n}{n!}[\boldsymbol{I}+\theta_{m_{\mathrm{P}}}^{-1}\boldsymbol{C}_k]^n &\le e^{-\theta_{m_{\mathrm{P}}}t}\sum_{n=0}^{\infty}\frac{(\theta_{m_{\mathrm{P}}}t)^n}{n!}[\boldsymbol{I}+\theta_{m_{\mathrm{P}}}^{-1}\boldsymbol{C}_k^{\natural}]^n \\ &=\exp[\boldsymbol{C}_k^{\natural}t]. \end{aligned}$$

(5.43) also follows from $\big[-\big(\boldsymbol{C}_k^\natural - \mathrm{diag}(\boldsymbol{x})\big)\big](-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k = \boldsymbol{D}_k + \mathrm{diag}(\boldsymbol{x})(-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k$. Finally, (5.44) follows from

$$\begin{aligned}
&\int_0^\tau \exp\big[\big(\boldsymbol{C}_k^\natural - \mathrm{diag}(\boldsymbol{x})\big)t\big][\boldsymbol{D}_k + \mathrm{diag}(\boldsymbol{x})(-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k]\mathrm{d}t \\
&\quad = \big(\boldsymbol{I} - \exp\big[\big(\boldsymbol{C}_k^\natural - \mathrm{diag}(\boldsymbol{x})\big)\tau\big]\big)\big[-\big(\boldsymbol{C}_k^\natural - \mathrm{diag}(\boldsymbol{x})\big)\big]^{-1}[\boldsymbol{D}_k + \mathrm{diag}(\boldsymbol{x})(-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k] \\
&\quad = \big(\boldsymbol{I} - \exp\big[\big(\boldsymbol{C}_k^\natural - \mathrm{diag}(\boldsymbol{x})\big)\tau\big]\big)(-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k \\
&\quad \geq (\boldsymbol{I} - \exp[\boldsymbol{C}_k^\natural\tau])(-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k = \int_0^\tau \exp[\boldsymbol{C}_k^\natural t]\boldsymbol{D}_k\mathrm{d}t,
\end{aligned}$$

where (5.43) is used in the second equality and the inequality can be shown in the same way as (5.42) because $(-\boldsymbol{C}_k^\natural)^{-1}\boldsymbol{D}_k \geq \boldsymbol{O}$. □

Theorem 5.3. *We consider $\{(L(t), S(t))\}_{t\geq 0}$ under Assumption 5.1 (ii), where* (5.23) *is assumed to hold. For an arbitrary integer $K > 0$, we have*

$$\max_{i\in\mathcal{M}}\big(b^+_{\mathrm{P},(K,i)}(m_\mathrm{P})\big) \leq 1 - \sum_{n=0}^{m_\mathrm{P}-\max(K,K_2^\dagger)} \int_0^\infty e^{-\mu t}\frac{(\mu t)^n}{n!}\mathrm{d}H(t), \quad m_\mathrm{P} \geq \max(K, K_2^\dagger),$$

where $\mu = 1/\mathrm{E}[H]$ and $K_2^\dagger$ is given in Assumption 5.1 (ii).

Because the proof of Theorem 5.3 is a bit lengthy, it is given in Appendix O. Theorems 5.1–5.3 enable us to compute the approximation $\boldsymbol{P}_{k,\ell}^{\mathrm{approx},n_\mathrm{P}}(m_\mathrm{P})$ $(k, \ell \leq K)$ to $\boldsymbol{P}_{k,\ell}$, which satisfies (5.29) under Assumption 5.1. We summarize the procedures in Figure 5.2.

Next, we consider the computation of an approximation to $\boldsymbol{W}_{k,\ell}$'s using $\boldsymbol{U}(m_\mathrm{W})$. We first define $\boldsymbol{W}(m_\mathrm{W})$ $(m_\mathrm{W} > K)$ as

$$\boldsymbol{W}(m_\mathrm{W}) = \int_0^\infty \exp[\boldsymbol{U}(m_\mathrm{W})t]\big(1 - H(t)\big)\mathrm{d}t = \mathrm{E}[H]\cdot\boldsymbol{B}_\mathrm{W}(m_\mathrm{W}),$$

where

$$\boldsymbol{B}_\mathrm{W}(m_\mathrm{W}) = \int_0^\infty \exp[\boldsymbol{U}(m_\mathrm{W})t]\frac{1 - H(t)}{\mathrm{E}[H]}\mathrm{d}t.$$

Note here that $\big(1 - H(t)\big)/\mathrm{E}[H]$ is the probability density function of the equilibrium random variable for service times and therefore, we have

$$\boldsymbol{B}_\mathrm{W}(m_\mathrm{W}) = \sum_{n=0}^\infty \eta_n(m_\mathrm{W})\boldsymbol{V}^n(m_\mathrm{W}), \tag{5.45}$$

where $\eta_n(m_\mathrm{W})$ $(n = 0, 1, \ldots)$ denotes a probability function given by

$$\eta_n(m_\mathrm{W}) = \int_0^\infty e^{-\theta_{m_\mathrm{W}}t}\frac{(\theta_{m_\mathrm{W}}t)^n}{n!}\frac{1 - H(t)}{\mathrm{E}[H]}\mathrm{d}t,$$

and $\boldsymbol{V}(m_\mathrm{W}) = \boldsymbol{I} + \theta_{m_\mathrm{W}}^{-1}\boldsymbol{U}(m_\mathrm{W})$. In numerical computation, we truncate the infinite sum in (5.45) at $n = n_\mathrm{W}$.

$$\boldsymbol{B}_\mathrm{W}^{\mathrm{comp},n_\mathrm{W}}(m_\mathrm{W}) = \sum_{n=0}^{n_\mathrm{W}} \eta_n(m_\mathrm{W})\boldsymbol{V}^n(m_\mathrm{W}).$$

Input: $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$, K, and $\epsilon_{\rm P}$.
Output: $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}$ $(k,\ell = 0,1,\dots,K)$.

Let $m_{\rm P}^* := \min\Big(m_{\rm P} > K;\ (-\boldsymbol{C}_K)^{-1}\boldsymbol{D}_K(-\boldsymbol{C}_{K+1})^{-1}\boldsymbol{D}_{K+1}\cdots\boldsymbol{D}_{m_{\rm P}}\boldsymbol{e} \le \frac{\epsilon_{\rm P}}{2}\boldsymbol{e}\Big)$.
Let $n_{\rm P}^* := \min\Big(n_{\rm P} \ge 0;\ \sum_{n=0}^{n_{\rm P}} \gamma_n(m_{\rm P}^*) \ge 1 - \frac{\epsilon_{\rm P}}{2}\Big)$.
Compute $\boldsymbol{V}_{k,\ell}^{(n)}(m_{\rm P}^*)$'s $(n = 0,1,\dots,n_{\rm P}^*)$ by (5.25).
Compute $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}(m_{\rm P}^*)$ $(k,\ell = 0,1,\dots,K)$ by (5.27) with (5.26),
where $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}(m_{\rm P}^*) = \boldsymbol{O}$ $(k = 3,4,\dots,K,\ \ell = 1,2,,\dots,k-2)$.

(a) Under Assumption 5.1 (i).

Input: $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$, K, and $\epsilon_{\rm P}$.
Output: $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}$ $(k,\ell = 0,1,\dots,K)$.

Let $m_{\rm P}^* := \min\Big(m_{\rm P} > K;\ \sum_{n=0}^{m_{\rm P}-\max(K,K_2^\dagger)} \int_0^\infty e^{-\mu t}\frac{(\mu t)^n}{n!}{\rm d}H(t) \ge 1 - \frac{\epsilon_{\rm P}}{2}\Big)$.
Let $n_{\rm P}^* := \min\Big(n_{\rm P} \ge 0;\ \sum_{n=0}^{n_{\rm P}} \gamma_n(m_{\rm P}^*) \ge 1 - \frac{\epsilon_{\rm P}}{2}\Big)$.
Compute $\boldsymbol{V}_{k,\ell}^{(n)}(m_{\rm P}^*)$'s $(n = 0,1,\dots,n_{\rm P}^*)$ by (5.25).
Compute $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}(m_{\rm P}^*)$ $(k,\ell = 0,1,\dots,K)$ by (5.27) with (5.26),
where $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}^*}(m_{\rm P}^*) = \boldsymbol{O}$ $(k = 3,4,\dots,K,\ \ell = 1,2,,\dots,k-2)$.

(b) Under Assumption 5.1 (ii).

Figure 5.2: Computational algorithms for $\boldsymbol{P}_{k,\ell}$'s in the non-uniformizable case.

It then follows that for $k \in \mathbb{Z}_1^K$, an approximation $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})$ to $\boldsymbol{W}_{k,\ell}$ is given by

$$\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W}) = \begin{cases} {\rm E}[H]\displaystyle\sum_{n=1}^{n_{\rm W}} \eta_n(m_{\rm W})\boldsymbol{V}_{k,0}^{(n)}(m_{\rm W}), & \ell = 0, \\ {\rm E}[H]\displaystyle\sum_{n=\ell-k}^{n_{\rm W}} \eta_n(m_{\rm W})\boldsymbol{V}_{k,\ell}^{(n)}(m_{\rm W}), & \ell = k,k+1,\dots,\min(m_{\rm W},k+n_{\rm W}), \\ \boldsymbol{O}, & \text{otherwise.} \end{cases} \tag{5.46}$$

We would like to find $m_{\rm W} = m_{\rm W}^*$ and $n_{\rm W} = n_{\rm W}^*$ such that for a given $\epsilon_{\rm W} > 0$,

$$\sum_{\ell=k}^{\infty}\big(\boldsymbol{W}_{k,\ell} - \boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})\big)\boldsymbol{e} = {\rm E}[H]\sum_{\ell=k}^{\infty}\big(\boldsymbol{B}_{{\rm W},k,\ell} - \boldsymbol{B}_{{\rm W},k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})\big)\boldsymbol{e} \le \epsilon_{\rm W}\boldsymbol{e}, \tag{5.47}$$

for all $k \in \mathbb{Z}_1^K$. Note here that $\boldsymbol{B}_{{\rm W},k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W}) = \boldsymbol{O}$ for all $\ell > \min(m_{\rm W}, k+n_{\rm W})$. Since $\boldsymbol{B}_{{\rm W},k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W}) \le \boldsymbol{B}_{{\rm W},k,\ell}$, we have

$$\sum_{\ell=k}^{\infty}\big(\boldsymbol{B}_{{\rm W},k,\ell} - \boldsymbol{B}_{{\rm W},k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})\big)\boldsymbol{e} \le \sum_{\ell=0}^{\infty}\big(\boldsymbol{B}_{{\rm W},k,\ell} - \boldsymbol{B}_{{\rm W},k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})\big)\boldsymbol{e}.$$

Therefore, (5.47) holds if

$$\sum_{\ell=0}^{\infty}\big(\boldsymbol{B}_{\mathrm{W},k,\ell}-\boldsymbol{B}_{\mathrm{W},k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}(m_{\mathrm{W}})\big)\boldsymbol{e}\le\frac{\epsilon_{\mathrm{W}}}{\mathrm{E}[H]}\boldsymbol{e},\quad k\in\mathbb{Z}_1^K. \tag{5.48}$$

Because the left-hand side of (5.48) takes the same form as that of (5.36), we can obtain the following corollary according to the same line of discussions as for Theorems 5.1–5.3.

Corollary 5.1. *For arbitrary K, m_{W} $(0<K<m_{\mathrm{W}})$ and $n_{\mathrm{W}}>0$, we have*

$$\sum_{\ell=0}^{\infty}\big(\boldsymbol{B}_{\mathrm{W},k,\ell}-\boldsymbol{B}_{\mathrm{W},k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}(m_{\mathrm{W}})\big)\boldsymbol{e}\le\max_{i\in\mathcal{M}}\big(b_{\mathrm{W},(K,i)}^{+}(m_{\mathrm{P}})\big)\boldsymbol{e}+\big(1-\sum_{n=0}^{n_{\mathrm{W}}}\eta_n(m_{\mathrm{W}})\big)\boldsymbol{e},\ k\in\mathbb{Z}_1^K,$$

where $\boldsymbol{b}_{\mathrm{W},K}^{+}(m_{\mathrm{W}})=\big(b_{\mathrm{W},(K,1)}^{+}(m_{\mathrm{W}})\ b_{\mathrm{W},(K,2)}^{+}(m_{\mathrm{W}})\ \cdots\ b_{\mathrm{W},(K,i)}^{+}(m_{\mathrm{W}})\big)^T$ *is given by*

$$\boldsymbol{b}_{\mathrm{W},K}^{+}(m_{\mathrm{W}})=\big(\boldsymbol{B}_{\mathrm{W},k,0}-\boldsymbol{B}_{\mathrm{W},k,0}(m_{\mathrm{W}})\big)\boldsymbol{e}+\sum_{\ell=m_{\mathrm{W}}+1}^{\infty}\boldsymbol{B}_{\mathrm{W},k,\ell}\boldsymbol{e}.$$

Under Assumption 5.1 (i), for a given $\epsilon>0$, we have

$$\boldsymbol{b}_{\mathrm{W},K}^{+}(m_{\mathrm{W}})\le\epsilon\boldsymbol{e}, \tag{5.49}$$

if m_{W} satisfies $(-\boldsymbol{C}_K)^{-1}\boldsymbol{D}_K(-\boldsymbol{C}_{K+1})^{-1}\boldsymbol{D}_{K+1}\cdots(-\boldsymbol{C}_{m_{\mathrm{W}}})^{-1}\boldsymbol{D}_{m_{\mathrm{W}}}\boldsymbol{e}\le\epsilon\boldsymbol{e}$. On the other hand, under Assumption 5.1 (ii), (5.49) holds for a given $\epsilon>0$ if m_{W} satisfies $m_{\mathrm{W}}\ge\max(K,K_2^{\dagger})$ and

$$1-\sum_{n=0}^{m_{\mathrm{W}}-\max(K,K_2^{\dagger})}\int_0^{\infty}e^{-\mu t}\frac{(\mu t)^n}{n!}\cdot\frac{1-H(t)}{\mathrm{E}[H]}\mathrm{d}t\le\epsilon,$$

where $\mu=1/\mathrm{E}[H]$ and $K_2^{\dagger}$ is given in Assumption 5.1 (ii).

Figure 5.3 shows the computational procedures for $\boldsymbol{W}_{k,\ell}$ ($k\in\mathbb{Z}_1^K$, $\ell\in\{0\}\cup\mathbb{Z}_k^{\min(m_{\mathrm{W}},k+n_{\mathrm{W}})}$) satisfying (5.47). Once we compute $\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}(m_{\mathrm{W}})$'s, we can estimate the error contained in those by

$$\sum_{\ell=k}^{\infty}\big(\boldsymbol{W}_{k,\ell}-\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}(m_{\mathrm{W}})\big)\boldsymbol{e}\le\mathrm{E}[H]\boldsymbol{e}-\sum_{\ell=0}^{m_{\mathrm{W}}}\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}(m_{\mathrm{W}})\boldsymbol{e}, \tag{5.50}$$

which can be derived in the same way as (5.30).

5.6 The augmented truncation approximation to the imbedded Markov chain

In this section, we discuss the augmented truncation approximation (ATA) to the imbedded Markov chain with the truncated transition probability matrix $\boldsymbol{P}(K)$. In the ATA, the selection of the augmentation matrix $\boldsymbol{P}_{\mathrm{A}}(K)$ in (5.14) is crucial and we briefly summarize how to manage it according to Section 5.2. First of all, we restrict our attention

Input: $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$, K, and $\epsilon_{\rm W}$.
Output: $m^*_{\rm W}$ and $\boldsymbol{W}^{{\rm comp},n^*_{\rm W}}_{k,\ell}$ $(k = 1, 2, \ldots, K,\ \ell = 0, k, k+1, \ldots, m^*_{\rm W})$.

Let $m^*_{\rm W} := \min\Big(m_{\rm W} > K;\ (-\boldsymbol{C}_K)^{-1}\boldsymbol{D}_K(-\boldsymbol{C}_{K+1})^{-1}\boldsymbol{D}_{K+1}\cdots\boldsymbol{D}_{m_{\rm W}}\boldsymbol{e} \le \dfrac{\epsilon_{\rm W}}{2{\rm E}[H]}\boldsymbol{e}\Big)$.
Let $n^*_{\rm W} := \min\Big(n_{\rm W} \ge 0;\ \displaystyle\sum_{n=0}^{n_{\rm W}} \eta_n(m^*_{\rm W}) \ge 1 - \frac{\epsilon_{\rm W}}{2{\rm E}[H]}\Big)$.
Compute $\boldsymbol{V}^{(n)}_{k,\ell}(m^*_{\rm W})$'s $(n = 0, 1, \ldots, n^*_{\rm W})$ by (5.25).
Compute $\boldsymbol{W}^{{\rm comp},n^*_{\rm W}}_{k,\ell}(m^*_{\rm W})$ $(k = 1, 2, \ldots, K,\ \ell = 0, k, k+1, \ldots, \min(m^*_{\rm W}, k + n^*_{\rm W}))$.

(a) Under Assumption 5.1 (i).

Input: $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$, K, and $\epsilon_{\rm W}$.
Output: $m^*_{\rm W}$ and $\boldsymbol{W}^{{\rm comp},n^*_{\rm W}}_{k,\ell}$ $(k = 1, 2, \ldots, K,\ \ell = 0, k, k+1, \ldots, m^*_{\rm W})$.

Let $m^*_{\rm W} := \min\Big(m_{\rm W} > K;\ \displaystyle\sum_{n=0}^{m_{\rm W}-\max(K,K_2^\dagger)} \int_0^\infty e^{-\mu t}\frac{(\mu t)^n}{n!}\frac{1-H(t)}{{\rm E}[H]}{\rm d}t \ge 1 - \frac{\epsilon_{\rm W}}{2{\rm E}[H]}\Big)$.
Let $n^*_{\rm W} := \min\Big(n_{\rm P} \ge 0;\ \displaystyle\sum_{n=0}^{n_{\rm W}} \eta_n(m^*_{\rm W}) \ge 1 - \frac{\epsilon_{\rm W}}{2{\rm E}[H]}\Big)$.
Compute $\boldsymbol{V}^{(n)}_{k,\ell}(m^*_{\rm W})$'s $(n = 0, 1, \ldots, n^*_{\rm W})$ by (5.25).
Compute $\boldsymbol{W}^{{\rm comp},n^*_{\rm W}}_{k,\ell}(m^*_{\rm W})$ $(k = 1, 2, \ldots, K,\ \ell = 0, k, k+1, \ldots, \min(m^*_{\rm W}, k + n^*_{\rm W}))$.

(b) Under Assumption 5.1 (ii).

Figure 5.3: Computational algorithms for $\boldsymbol{W}_{k,\ell}$'s in the non-uniformizable case.

to the linear augmentation, i.e., $\boldsymbol{P}_{\rm A}(K) = (\boldsymbol{I} - \boldsymbol{P}(K))\boldsymbol{e}\boldsymbol{\zeta}$ for some $1 \times (K+1)M$ probability vector $\boldsymbol{\zeta}$, which does not lose generality (cf. Implication 5.1). We then define $\boldsymbol{\pi}^{\rm approx}_{\rm lv}(K;\boldsymbol{\zeta})$ as the linear ATA solution obtained by

$$\boldsymbol{\pi}^{\rm approx}_{\rm lv}(K;\boldsymbol{\zeta}) = \boldsymbol{\pi}^{\rm approx}_{\rm lv}(K;\boldsymbol{\zeta})\big[\boldsymbol{P}(K) + (\boldsymbol{I} - \boldsymbol{P}(K))\boldsymbol{e}\boldsymbol{\zeta}\big], \qquad \boldsymbol{\pi}^{\rm approx}_{\rm lv}(K;\boldsymbol{\zeta})\boldsymbol{e} = 1.$$

Let $[\boldsymbol{\zeta}]_{(k,i)}$ $(k \in \mathbb{Z}^+,\ i \in \mathcal{M})$ denote the $(kM+i)$th element of $\boldsymbol{\zeta}$ and let $[\boldsymbol{P}]_{(k,i),(\ell,j)}$ $(k, \ell \in \mathbb{Z}^+,\ i, j \in \mathcal{M})$ denote the $(kM+i, \ell M+j)$th element of $\boldsymbol{P}$. We then define $\Gamma^+_{\rm lv}(K)$ as

$$\Gamma^+_{\rm lv}(K) = \big\{\boldsymbol{\zeta} \in \mathbb{R}^{(K+1)M};\ \boldsymbol{\zeta} \ge \boldsymbol{0},\ \boldsymbol{\zeta}\boldsymbol{e} = 1,\ [\boldsymbol{\zeta}]_{(k,i)} > 0 \text{ if } (k,i) \in \mathcal{J}_{\rm lv}(K),\ [\boldsymbol{\zeta}]_{(k,i)} = 0 \text{ if } (k,i) \notin \mathcal{J}_{\rm lv}(K)\big\},$$

where $\mathcal{J}_{\rm lv}(K)$ denotes a subset of levels zero to K, whose states are directly reachable from at least one state in levels $K+1$ or higher.

$$\mathcal{J}_{\rm lv}(K) = \Big\{(0,i);\ i \in \mathcal{M},\ \sum_{k=K+1}^{\infty}[\boldsymbol{e}^T\boldsymbol{\Gamma}_k]_i > 0\Big\} \cup \big\{(K,i);\ i \in \mathcal{M}\big\}. \tag{5.51}$$

Note that $\Gamma^+_{\rm lv}(K)$ and $\mathcal{J}_{\rm lv}(K)$ are modified versions of $\Gamma^+(K)$ and $\mathcal{J}(K)$ considered in Chapter 4 for the bivariate imbedded Markov chains (cf. (4.1) and (4.19)). Note also

Input: $\boldsymbol{P}(K)$ and $\mathcal{J}_{\rm lv}(K)$.
Output: $\boldsymbol{\pi}^{\rm approx} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) \ \ \boldsymbol{0})$ and $\varepsilon_{\rm lv}(K)$.

Compute $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T) = \boldsymbol{x}_{(k,i)}/\boldsymbol{x}_{(k,i)}\boldsymbol{e}$ for all $(k,i) \in \mathcal{J}_{\rm lv}(K)$,
where $\boldsymbol{x}_{(k,i)}$ is the unique solution of $\boldsymbol{x}_{(k,i)}(\boldsymbol{I} - \boldsymbol{P}(K)) = \boldsymbol{e}_{(k,i)}^T$.
Compute $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ by (5.53) and set $\boldsymbol{\pi}^{\rm approx} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) \ \ \boldsymbol{0})$.
Compute the error bound $\varepsilon_{\rm lv}(K)$ by (5.54).

Figure 5.4: A computational algorithm for $\boldsymbol{\pi}^{\rm approx} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) \ \ \boldsymbol{0})$ in the imbedded Markov chain.

that there exists $\boldsymbol{\zeta}^* \in \Gamma_{\rm lv}^+(K)$ such that $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{\zeta}^*) = \boldsymbol{\pi}_{\rm lv}(K)$ because of Theorem 4.2 and Lemma 5.2.

Furthermore, for an arbitrary $\boldsymbol{\zeta} \in \Gamma_{\rm lv}^+(K)$, $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{\zeta})$ is given by a convex combination of $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)$'s ($(k,i) \in \mathcal{J}_{\rm lv}(K)$) with positive weights (cf. (5.2) and (5.4)), where $\boldsymbol{e}_{(k,i)}^T$ denotes the unit row vector whose $(kM+i)$th element is equal to one. Specifically,

$$\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{\zeta}) = \sum_{(k,i) \in \mathcal{J}_{\rm lv}(K)} [\boldsymbol{\alpha}(K;\, \boldsymbol{\zeta})]_{(k,i)} \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T),$$

where $\boldsymbol{\alpha}(K;\, \boldsymbol{\zeta}) \in \Gamma_{\rm lv}^+(K)$ and there is a one-to-one correspondence between $\boldsymbol{\zeta}$ and $\boldsymbol{\alpha}(K;\, \boldsymbol{\zeta})$ (cf. (5.3)). It then follows from (5.7) that

$$\|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\boldsymbol{\zeta}) - \boldsymbol{\pi}_{\rm lv}(K)\|_1 \le \max_{(k,i) \in \mathcal{J}_{\rm lv}(K)} \|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\boldsymbol{\zeta}) - \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)\|_1. \quad (5.52)$$

Based on this observation and Implication 5.4, we set $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ as an approximation to the conditional stationary distribution $\boldsymbol{\pi}_{\rm lv}(K)$ as follows:

$$\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) = \sum_{(k,i) \in \mathcal{J}_{\rm lv}(K)} \frac{1}{|\mathcal{J}_{\rm lv}(K)|} \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T), \quad (5.53)$$

which presumes that the transition structure in levels higher than K is unavailable, and the error bound in (5.52) is given by

$$\|\boldsymbol{\pi}_{\rm lv}(K) - \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)\|_1 \le \varepsilon_{\rm lv}(K) = \max_{(k,i) \in \mathcal{J}_{\rm lv}(K)} \|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K) - \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)\|_1. \quad (5.54)$$

In summary, we compute the approximation $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ and its error bound $\varepsilon_{\rm lv}(K)$ by the procedure in Figure 5.4.

5.7 The error bound in the approximation to $\boldsymbol{\varpi}$

If we obtain an approximation $\boldsymbol{\pi}^{\rm approx}$ to the stationary distribution $\boldsymbol{\pi}$ of the imbedded Markov chain $\{(L_n, S_n)\}_{n=0,1,\ldots}$ and $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})$'s, we can compute an approximation $\boldsymbol{\varpi}^{\rm approx} = (\boldsymbol{\varpi}_0^{\rm approx} \ \boldsymbol{\varpi}_1^{\rm approx} \ \cdots)$ to the stationary distribution $\boldsymbol{\varpi}$ by (5.16) and (5.17), where $\boldsymbol{W}_{k,\ell}$ is replaced by $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})$. Since $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W}) = \boldsymbol{O}$ ($\ell > \min(m_{\rm W}, k + n_{\rm W})$), the computation of $\boldsymbol{\varpi}_k^{\rm approx}$'s stops at $\ell = \min(m_{\rm W}, K + n_{\rm W})$.

Recall that for an arbitrary $\epsilon_{\rm W} > 0$, $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})$'s satisfy (5.47). We thus discuss the error in $\boldsymbol{\varpi}^{\rm approx}$, assuming that errors in $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})$'s are negligible, i.e.,

$$\boldsymbol{\varpi} = \frac{\boldsymbol{\pi}\widehat{\boldsymbol{W}}}{\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}} = \frac{\boldsymbol{\pi}\widehat{\boldsymbol{W}}}{{\rm E}[T]}, \qquad \boldsymbol{\varpi}^{\rm approx} = \frac{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}},$$

where

$$\widehat{\boldsymbol{W}} = \begin{pmatrix} (-\boldsymbol{C}_0)^{-1} & (-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{W}_{1,1} & (-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{W}_{1,2} & (-\boldsymbol{C}_0)^{-1}\boldsymbol{D}_0\boldsymbol{W}_{1,3} & \cdots \\ \boldsymbol{O} & \boldsymbol{W}_{1,1} & \boldsymbol{W}_{1,2} & \boldsymbol{W}_{1,3} & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{W}_{2,2} & \boldsymbol{W}_{2,3} & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{W}_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and ${\rm E}[T] = \boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}$ denotes the average length of two consecutive imbedded points.

Theorem 5.4. *If errors in $\boldsymbol{W}_{k,\ell}^{{\rm comp},n_{\rm W}}(m_{\rm W})$'s are negligible, we have*

$$\|\boldsymbol{\varpi} - \boldsymbol{\varpi}^{\rm approx}\|_1 \le \frac{T_{\max}^{\rm upper}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1\Big(1 + \frac{T_{\max}^{\rm upper}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e} - T_{\max}^{\rm upper}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1}\Big). \tag{5.55}$$

where $T_{\max}^{\rm upper} = {\rm E}[H] + \max_{i\in\mathcal{M}}[(-\boldsymbol{C}_0)^{-1}\boldsymbol{e}]_i$.

Remark 5.5. *(5.19) comes from (5.55) and*

$$\begin{aligned} &\frac{T_{\max}^{\rm upper}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1\Big(1 + \frac{T_{\max}^{\rm upper}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e} - T_{\max}^{\rm upper}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1}\Big) \\ &\qquad = \frac{T_{\max}^{\rm upper}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\Big(1 + \frac{T_{\max}^{\rm upper}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\Big)\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1 + o(\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1). \end{aligned}$$

Proof. Let $T_{\max} = \max_{(k,i)\in\mathbb{Z}^+\times\mathcal{M}}[\widehat{\boldsymbol{W}}\boldsymbol{e}]_{(k,i)}$. Because ${\rm diag}^{-1}(\widehat{\boldsymbol{W}}\boldsymbol{e})\widehat{\boldsymbol{W}}\boldsymbol{e} = \boldsymbol{e}$, we have

$$\begin{aligned} \|\boldsymbol{\varpi} - \boldsymbol{\varpi}^{\rm approx}\|_1 &= \Big\|\frac{\boldsymbol{\pi}{\rm diag}(\widehat{\boldsymbol{W}}\boldsymbol{e})}{\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}}{\rm diag}^{-1}(\widehat{\boldsymbol{W}}\boldsymbol{e})\widehat{\boldsymbol{W}} - \frac{\boldsymbol{\pi}^{\rm approx}{\rm diag}(\widehat{\boldsymbol{W}}\boldsymbol{e})}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}{\rm diag}^{-1}(\widehat{\boldsymbol{W}}\boldsymbol{e})\widehat{\boldsymbol{W}}\Big\|_1 \\ &\le \Big\|\frac{\boldsymbol{\pi}{\rm diag}(\widehat{\boldsymbol{W}}\boldsymbol{e})}{\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}} - \frac{\boldsymbol{\pi}^{\rm approx}{\rm diag}(\widehat{\boldsymbol{W}}\boldsymbol{e})}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\Big\|_1 \\ &\le \Big\|\frac{(\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}){\rm diag}(\widehat{\boldsymbol{W}}\boldsymbol{e})}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\Big\|_1 + \Big|\frac{1}{\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}} - \frac{1}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\Big|\|\boldsymbol{\pi}{\rm diag}(\widehat{\boldsymbol{W}}\boldsymbol{e})\|_1 \\ &\le \frac{T_{\max}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1 + \Big|\frac{(\boldsymbol{\pi}^{\rm approx} - \boldsymbol{\pi})\widehat{\boldsymbol{W}}\boldsymbol{e}}{\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}\cdot\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\Big|\cdot T_{\max} \\ &\le \frac{T_{\max}}{\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1 + \frac{T_{\max}^2}{\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}\cdot\boldsymbol{\pi}^{\rm approx}\widehat{\boldsymbol{W}}\boldsymbol{e}}\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\rm approx}\|_1. \end{aligned} \tag{5.56}$$

Note here that $T_{\max}$ can be bounded from above.

$$T_{\max} = \max_{(k,i)\in\mathbb{Z}^+\times\mathcal{M}}{\rm E}[\tau_{n+1} - \tau_n \mid L_n = k, S_n = i] \le {\rm E}[H] + \max_{i\in\mathcal{M}}[(-\boldsymbol{C}_0)^{-1}\boldsymbol{e}]_i. \tag{5.57}$$

Note also that $\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e}$ can be bounded from below.

$$\boldsymbol{\pi}\widehat{\boldsymbol{W}}\boldsymbol{e} = \boldsymbol{\pi}^{\mathrm{approx}}\widehat{\boldsymbol{W}}\boldsymbol{e} + (\boldsymbol{\pi} - \boldsymbol{\pi}^{\mathrm{approx}})\widehat{\boldsymbol{W}}\boldsymbol{e} \geq \boldsymbol{\pi}^{\mathrm{approx}}\widehat{\boldsymbol{W}}\boldsymbol{e} - \|\boldsymbol{\pi} - \boldsymbol{\pi}^{\mathrm{approx}}\|_1 \cdot T_{\max}. \quad (5.58)$$

Theorem 5.4 now follows from (5.56), (5.57), and (5.58). □

We can evaluate the error bound (5.55) for $\boldsymbol{\varpi}^{\mathrm{approx}}$ if the upper bound of $\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\mathrm{approx}}\|_1$ is available. Recall that the upper bound of $\|\boldsymbol{\pi} - \boldsymbol{\pi}^{\mathrm{approx}}\|_1$ is given by $2\xi_{\mathrm{lv}}(K) + \epsilon_{\mathrm{lv}}(K)$ (cf. (5.15)). While the upper bound $\varepsilon_{\mathrm{lv}}(K)$ of $\epsilon_{\mathrm{lv}}(K)$ is given by (5.54), the tail probability $\xi_{\mathrm{lv}}(K)$ is hard to evaluate analytically. We will discuss a rough estimation of $\xi_{\mathrm{lv}}(K)$ using numerical examples in the next section.

5.8 Numerical examples and discussions

The purpose of this section is two-fold: one is to demonstrate the soundness of our computational procedure and the other is to discuss how to obtain a qualitatively decent approximation. For these purposes, we consider a single-server queue with balking and level-dependent disasters, where customers are served on a FIFO basis with deterministic service times with $\mathrm{E}[H] = 1.0$. We assume that customers arrive according to a Markov-modulated Poisson process (MMPP), whose underlying Markov chain $\{S(t)\}_{t\geq 0}$ is assumed to be a two-state birth-and-death process with identical transition rate $\alpha = 0.01$. Note that the stationary distribution $\boldsymbol{\varpi}_{\mathrm{U}}$ of the underlying Markov chain $\{S(t)\}_{t\geq 0}$ is given by $\boldsymbol{\varpi}_{\mathrm{U}} = (0.5 \quad 0.5)$. We assume that if $S(t) = 1$ (resp. $S(t) = 2$), customers arrive according to a Poisson process with rate $\lambda_1 = 1.2$ (resp. $\lambda_2 = 1.8$). If a customer finds k ($k \in \mathbb{Z}^+$) customers on arrival, he/she joins the queue with probability $\beta_k = (2.0 + 0.99^k)/3$. Let $L(t)$ ($t \geq 0$) denote the number of customers in the system at time t. We assume that when $L(t) = k$ ($k \in \mathbb{Z}^+$), a disaster occurs at rate $r_k = \max(0, (k - 200)/100)$.

The *net* arrival/disaster process is characterized by $\{(\boldsymbol{C}_k, \boldsymbol{D}_k, \boldsymbol{\Gamma}_k);\ k \in \mathbb{Z}^+\}$, where

$$\boldsymbol{C}_k = \begin{pmatrix} c_{k,1} & \alpha \\ \alpha & c_{k,2} \end{pmatrix}, \quad \boldsymbol{D}_k = \beta_k \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \boldsymbol{\Gamma}_k = r_k \boldsymbol{I}, \quad k = 0, 1, \ldots,$$

where $c_{k,i} = -\alpha - \beta_k \lambda_i - r_k$ ($k \in \mathbb{Z}^+$, $i = 1, 2$). It then follows that $\mathcal{J}_{\mathrm{lv}}(K) = \{0, K\} \times \{1, 2\}$ and θ_m is given by

$$\begin{aligned} \theta_m &= \max_{k \in \mathbb{Z}_1^m} \Big(0.01 + \frac{2 + 0.99^k}{3} \cdot 1.8 + \max\big(0, \frac{k - 200}{100}\big) \Big) \\ &= \begin{cases} 1.804, & m = 1, 2, \ldots, 254, \\ 1.21 + 0.6 \cdot 0.99^k + \dfrac{k - 200}{100}, & m = 255, 256, \ldots. \end{cases} \end{aligned}$$

It is easy to see that $\{(L(t), S(t))\}_{t\geq 0}$ is non-explosive, $\exp[\boldsymbol{U}t]$ is non-uniformizable, and Assumption 5.1 (i) holds with $K_1^\dagger = 208$. By definition, $\gamma_n(m_{\mathrm{P}})$ and $\eta_n(m_{\mathrm{W}})$ ($n = 0, 1, \ldots$) are given by

$$\gamma_n(m_{\mathrm{P}}) = \frac{(\theta_{m_{\mathrm{P}}} \mathrm{E}[H])^n}{n!} e^{-\theta_{m_{\mathrm{P}}} \mathrm{E}[H]}, \qquad \eta_n(m_{\mathrm{W}}) = \frac{1}{\theta_{m_{\mathrm{W}}} \mathrm{E}[H]} (1 - \sum_{i=0}^{n} \gamma_i(m_{\mathrm{W}})).$$

In what follows, we set $\epsilon_{\rm P} = \epsilon_{\rm W} = 10^{-14}$.

In Step 1, we conduct the procedure in Figure 5.2 (a) to compute $\boldsymbol{P}(K)$. Figure 5.5 shows the error bound $\varepsilon_{\rm P}(K)$ for $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})$'s in $\boldsymbol{P}(K)$ as a function of K ($K > K_1^\dagger = 208$), where

$$\begin{aligned}\varepsilon_{\rm P}(K) &= \max_{\substack{k=0,1,\ldots,K\\ j\in\mathcal{M}}} \Big[\boldsymbol{e} - \sum_{\ell=0}^{m_{\rm P}-1} \boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\boldsymbol{e}\Big]_j \\ &\geq \max_{\substack{k=0,1,\ldots,K\\ j\in\mathcal{M}}} \Big[\sum_{\ell=0}^{K} \big(\boldsymbol{P}_{k,\ell} - \boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})\big)\boldsymbol{e}\Big]_j.\end{aligned}$$

Note that the inequality in the above comes from (5.30). We observe that the computational procedure for $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})$'s works as designed because we set $\epsilon_{\rm P} = 10^{-14}$. We make a comment on the ripple-like shape of $\varepsilon_{\rm P}(K)$. As shown in Theorem 5.1, there are two sources of errors in computing an approximation to $\{\boldsymbol{P}_{k,\ell}\}$: the truncation of $\boldsymbol{U}$ (i.e., $m_{\rm P}$ in (5.34)) and the truncation of infinite sums (i.e., $n_{\rm P}$ in (5.34)). The selection of $m_{\rm P}$ is conservative as shown in the proof of Theorem 5.2, so that the numerical error due to the truncation of $\boldsymbol{U}$ seems to be negligible. On the other hand, the selection of $n_{\rm P}$ is rigid and for a fixed $n_{\rm P}$, the second term on the right-hand side of (5.34) increases with K because $\theta_{m_{\rm P}}$ increases with K. This causes a ripple-like shape of $\varepsilon_{\rm P}(K)$, which is inherent in the non-uniformizable case.

Next, we consider the ATA in Step 2. According to the procedure in Figure 5.4, we compute an approximation $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ to the conditional stationary distribution $\boldsymbol{\pi}_{\rm lv}(K)$. Figure 5.6 shows the error bound $\varepsilon_{\rm lv}(K)$ of $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ in (5.54) as a function of K ($K > K_1^\dagger = 208$). We observe that (i) $\varepsilon_{\rm lv}(K)$ takes a large value when k is close to $K_1^\dagger$, (ii) it decreases as K increases, and then, (iii) it takes a ripple-like shape as $\varepsilon_{\rm P}(K)$ in Figure 5.5.

Note that the conditional stationary distribution in any ergodic Markov chain with the same northwest corner submatrix $\boldsymbol{P}(K)$ and the same $\mathcal{J}_{\rm lv}(K)$ defined in (5.51) is given by a mixture of four probability distributions $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\ \boldsymbol{e}_{(k,i)}^T)$ $(k = 0, K;\ i = 1, 2)$ with the positive weights, as stated in Theorem 4.2. Therefore, the degree of freedom (i.e., the extent to which the transition structure in level $K+1$ or higher affects the conditional stationary distribution $\boldsymbol{\pi}_{\rm lv}(K)$) decreases as K increases, which leads to the above-mentioned observations (i) and (ii). Let $\boldsymbol{P}^{\rm comp}(K)$ denote the computed $\boldsymbol{P}$ with $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(K)$'s. When K is large enough to suppress the degree of freedom, $\|\boldsymbol{P}(K) - \boldsymbol{P}^{\rm comp}(K)\|$ becomes the dominant source of errors which leads to the observation (iii).

Figure 5.7 shows approximations to the conditional stationary distribution $\boldsymbol{\pi}_k(K)\boldsymbol{e}$ $(k \geq 200)$ for $K = 220, 240, \ldots, 320$. Recall that the drift is negative if $L(t) > K_1^\dagger = 208$. Therefore, we expect that $\boldsymbol{\pi}_k(K)\boldsymbol{e}$ $(k \geq 208)$ is a decreasing function of k, whereas $\boldsymbol{\pi}_k^{\rm approx}(K)\boldsymbol{e}$ takes the minimum value at some $k = u_{\rm min}(K)$ close to K and it turns upward. This pathological phenomenon is caused by our selection of equal weights in (5.53) for $\boldsymbol{v}_{k,i}(K) = \boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\ \boldsymbol{e}_{(k,i)}^T)$ $(k = 0, K;\ i = 1, 2)$. While $\boldsymbol{v}_{0,i}(K)$ $(i = 1, 2)$ has a strictly decreasing tail, $\boldsymbol{v}_{K,i}(K)$ $(i = 1, 2)$ takes the minimum value at some k close to K and it turns upward. The equally-weighted mixture of those distributions results in this pathological phenomenon. Note that similar phenomena can also occur in standard augmentation strategies such as last-column/last-column-block augmentations. When

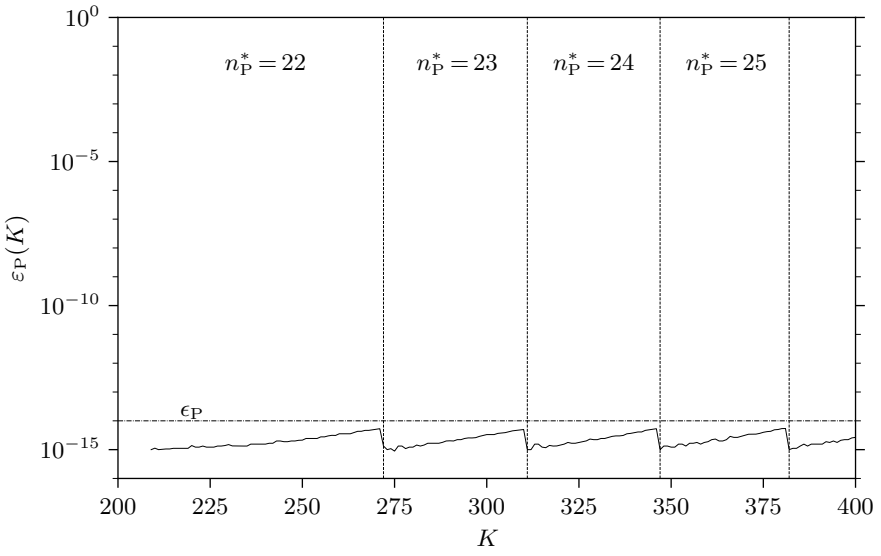


Figure 5.5: Error bounds $\varepsilon_{\rm P}(K)$ in $\boldsymbol{P}_{k,\ell}^{{\rm comp},n_{\rm P}}(m_{\rm P})$'s.

K is large enough (i.e., $K \geq 260$), however, $\boldsymbol{\pi}_K^{\rm approx}(K)\boldsymbol{e}$ is less than $\epsilon_{\rm P}(K) = 10^{-14}$, which suggests that such a K is large enough to obtain a reliable result in finite precision computation. Furthermore, in such a case, $\boldsymbol{\pi}_k^{\rm approx}(K)\boldsymbol{e}$ for k less than and not close to $u_{\min}(K)$ seems to have a decent accuracy as they are indistinguishable from $\boldsymbol{\pi}_k^{\rm approx}(K)\boldsymbol{e}$ for $K = 320$. In what follows, we restrict our attention to $K = 300$.

We attempt to eliminate an inaccurate tail in $\{\boldsymbol{\pi}_k^{\rm approx}(K);\, k \in \mathbb{Z}_0^K\}$ by utilizing the results in Section 4.4 and the last 2 paragraph of Section 5.2, i.e., we set N ($N < K = 300$) and consider an approximation $\boldsymbol{\pi}_k^{\rm approx}(K;\ N)$ to the conditional stationary distribution $\boldsymbol{\pi}_{\rm lv}(N)$, based on $\boldsymbol{P}(K)$. Although $\boldsymbol{\pi}_k^{\rm approx}(K)\boldsymbol{e} \simeq \epsilon_{\rm P} = 10^{-14}$ for $k = 257$ can be a guideline for it, Figure 5.7 suggests that it would be too conservative because $u_{\min}(K) = 286$ for $K = 300$. We thus set $N = 280$ and obtain an approximation $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N) = (\boldsymbol{\pi}_0^{\rm approx}(K;\, N)\ \boldsymbol{\pi}_1^{\rm approx}(K;\, N)\ \cdots\ \boldsymbol{\pi}_N^{\rm approx}(K;\, N))$, where

$$\boldsymbol{\pi}_k^{\rm approx}(K;\, N) = \boldsymbol{\pi}_k^{\rm approx}(K) \Big/ \sum_{n=0}^{N} \boldsymbol{\pi}_n^{\rm approx}(K)\boldsymbol{e}, \quad k \in \mathbb{Z}_0^N.$$

We adopt $\boldsymbol{\pi}^{{\rm approx},N} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N)\ \ \boldsymbol{0})$ instead of $\boldsymbol{\pi}^{\rm approx} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)\ \ \boldsymbol{0})$. Figure 5.8 shows $\boldsymbol{\pi}_k^{\rm approx}(K)\boldsymbol{e}$ for $K = 300$ and $\boldsymbol{\pi}_k^{\rm approx}(K;\, N)\boldsymbol{e}$ for $(K, N) = (300, 280)$.

We then consider the error bound for $\boldsymbol{\pi}^{{\rm approx},N}$. We can show that the truncated approximation $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N)$ lies in the minimum convex polytope shown in Section 4.4. Let $L_\ell := (N+1)M$ ($\ell \in \mathbb{Z}^+$) denote the number of states in levels ℓ or lower. Recall that $\boldsymbol{P}(K)$ is the $L_K \times L_K$ northwest corner submatrix of $\boldsymbol{P}$ and $\boldsymbol{\pi}_{\rm lv}(N) := \boldsymbol{\pi}(L_N - 1)$ is the $1 \times L_N$ probability vector given by normalizing the first L_N elements of $\boldsymbol{\pi}$. In our model, it is readily shown that $\mathcal{D}(L_K - 1, L_N - 1)$ defined in Procedure 4.1 is given by $\mathcal{D}(L_K - 1, L_N - 1) = \mathcal{J}^*(L_K - 1) = \mathcal{J}(L_K - 1) = \mathcal{J}_{\rm lv}(K)$, so that $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N)$ lies in the minimum convex polytope whose vertex $\boldsymbol{\gamma}(k, i)$ ($(k, i) \in \mathcal{J}_{\rm lv}(K)$) is given by the

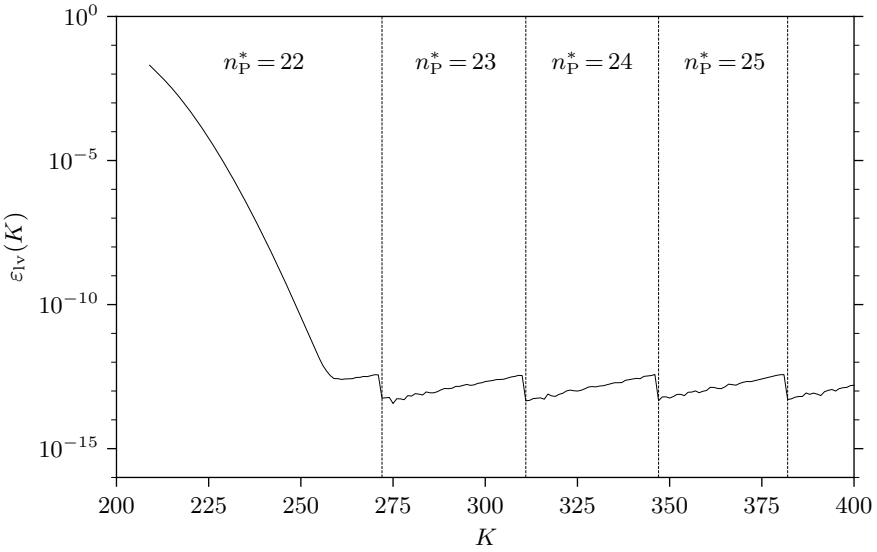


Figure 5.6: Error bounds $\varepsilon_{\rm lv}(K)$ in $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$.

normalized first L_N elements of $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)$.

$$\boldsymbol{\gamma}(k,i) = \frac{(\boldsymbol{\pi}_0^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)\ \boldsymbol{\pi}_1^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)\ \cdots\ \boldsymbol{\pi}_N^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T))}{(\boldsymbol{\pi}_0^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)\ \boldsymbol{\pi}_1^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T)\ \cdots\ \boldsymbol{\pi}_N^{\rm approx}(K;\, \boldsymbol{e}_{(k,i)}^T))\boldsymbol{e}},$$
$$(k,i) \in \mathcal{J}_{\rm lv}(K).$$

Furthermore, the upper bound of the error $\|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N) - \boldsymbol{\pi}_{\rm lv}(N)\|_1$ is obtained by (5.8).

$$\|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N) - \boldsymbol{\pi}_{\rm lv}(N)\|_1 \le \max_{(k,i)\in\mathcal{J}_{\rm lv}(K)} \|\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N) - \boldsymbol{\gamma}(k,i)\|_1 =: \varepsilon_{\rm lv}(K;\, N).$$

As a result, we have $\varepsilon_{\rm lv}(K;\, N) \simeq 2.1084 \cdot 10^{-13}$ for $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N)$, while $\varepsilon_{\rm lv}(K) \simeq 2.1270 \cdot 10^{-13}$ for $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$. Therefore, in terms of 1-norm, the difference between two approximations $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N)$ and $\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K)$ is negligible.

Next we consider a rough estimation of the upper bound of the tail probability $\xi_{\rm lv}(N)$ in (5.15). Because the magnitude of the negative drift increases with k ($k \ge 208$), we expect that the tail probability decreases more rapidly than any exponential function, as suggested in Figure 5.8. Noting $\boldsymbol{\pi}_{257}^{\rm approx}(K;\, N)\boldsymbol{e} \simeq \epsilon_{\rm P} = 10^{-14}$, we assume that

$$\boldsymbol{\pi}_k \boldsymbol{e} \le \boldsymbol{\pi}_N^{\rm approx}(K;\, N)\boldsymbol{e} \cdot r^{k-N}, \quad k = N+1, N+2, \ldots,$$

where $r^{N-257} = \boldsymbol{\pi}_N^{\rm approx}(K;\, N)\boldsymbol{e} / \boldsymbol{\pi}_{257}^{\rm approx}(K;\, N)\boldsymbol{e} \simeq 1.2338 \cdot 10^{-8}$. We thus have $r \simeq 0.4530$ and $\xi_{\rm lv}(N) \le \boldsymbol{\pi}_N^{\rm approx}(K;\, N)\boldsymbol{e} \cdot r/(1-r) \simeq 7.9031 \cdot 10^{-23}$, which indicates that $N = 280$ is large enough to ignore the error due to the truncation at N.

Let $\boldsymbol{\pi}^{{\rm approx},N} = (\boldsymbol{\pi}_{\rm lv}^{\rm approx}(K;\, N)\ \ \boldsymbol{0})$. We then have $\|\boldsymbol{\pi}^{{\rm approx},N} - \boldsymbol{\pi}\|_1 \le 2\xi_{\rm lv}(K) + \varepsilon_{\rm lv}(K;\, N) \le 2.1084 \cdot 10^{-13}$. In a similar way, we also obtain $\|\boldsymbol{\pi}^{\rm approx} - \boldsymbol{\pi}\|_1 \le 3.2362 \cdot 10^{-13}$. Moreover, $\|\boldsymbol{\pi}^{\rm approx} - \boldsymbol{\pi}^{{\rm approx},N}\|_1 \simeq 2.2151 \cdot 10^{-15}$. We thus conclude that in terms of 1-norm, the difference between $\boldsymbol{\pi}^{{\rm approx},N}$ and $\boldsymbol{\pi}^{\rm approx}$ is not significant.

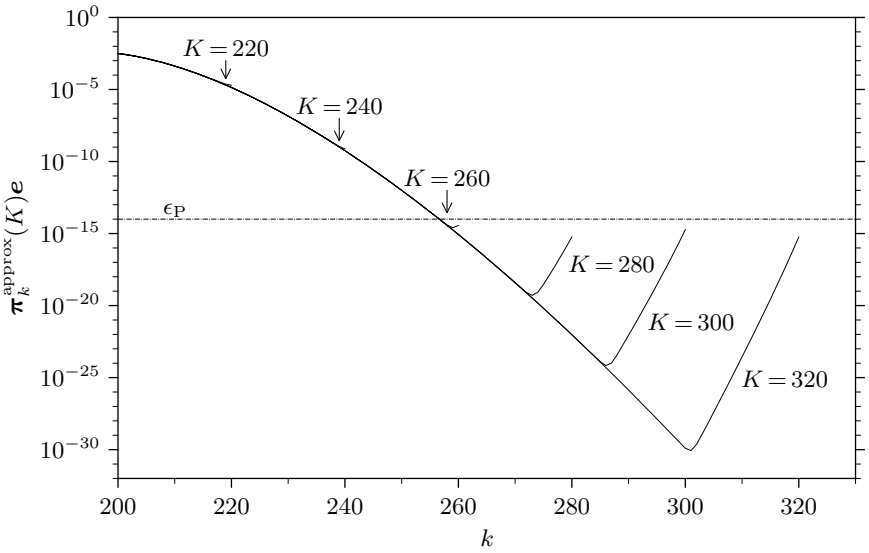


Figure 5.7: Examples of $\boldsymbol{\pi}_k^{\mathrm{approx}}(K)\boldsymbol{e}$'s.

Finally, in Step 3, we compute an approximation to the stationary distribution $\boldsymbol{\varpi}$. We first compute $\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}}(m_{\mathrm{W}})$'s according to the procedure in Figure 5.3 (a), where for K is replaced by $N = 280$ and we obtain $m_{\mathrm{W}}^* = 332$, and $n_{\mathrm{W}}^* = 21$. We then obtain the error bound in (5.50) for $N = 280$.

$$\max_{\substack{k=1,2,\ldots,N\\ j\in\mathcal{M}}}\Big[\sum_{\ell=k}^{\infty}(\boldsymbol{W}_{k,\ell}-\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}^*}(m_{\mathrm{W}}^*))\boldsymbol{e}\Big]_j \le \max_{\substack{k=1,2,\ldots,N\\ j\in\mathcal{M}}}\Big[\mathrm{E}[H]\boldsymbol{e}-\sum_{\ell=0}^{m_{\mathrm{W}}^*}\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}^*}(m_{\mathrm{W}}^*)\boldsymbol{e}\Big]_j \simeq 4.2188\cdot 10^{-15}.$$

This result shows that the procedure for $\boldsymbol{W}_{k,\ell}^{\mathrm{comp},n_{\mathrm{W}}^*}(m_{\mathrm{W}}^*)$'s works as designed because $\epsilon_{\mathrm{W}} = 10^{-14}$.

We then compute $\{\boldsymbol{\varpi}_k^{\mathrm{approx}}(K;\,N);\,k\in\mathbb{Z}^+\}$ and construct the stationary distribution of the number of customers.

$$\boldsymbol{\varpi}^{\mathrm{approx},N} = (\boldsymbol{\varpi}_0^{\mathrm{approx}}(K;\,N)\ \boldsymbol{\varpi}_1^{\mathrm{approx}}(K;\,N)\ \cdots \boldsymbol{\varpi}_{301}^{\mathrm{approx}}(K;\,N)\ \ \mathbf{0}\ \ \mathbf{0}\ \cdots),$$

where we use $\min(m_{\mathrm{W}}^*, N+n_{\mathrm{W}}^*) = 301$ for $N = 280$. Figure 5.9 shows $\boldsymbol{\varpi}^{\mathrm{approx},N}$, where the corresponding result for $\boldsymbol{\varpi}^{\mathrm{approx}}$ based on $\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx}}(K)$ for $K = 300$ is also plotted for reference. The error bound (5.55) for $\boldsymbol{\varpi}^{\mathrm{approx},N}$ is given by $1.092\cdot 10^{-12}$, where we use $T_{\max}^{\mathrm{upper}} \simeq 1.8311$ and $\boldsymbol{\pi}^{\mathrm{approx},N}\widehat{\boldsymbol{W}}\boldsymbol{e} \simeq 1.0004$. It thus seems that $\boldsymbol{\varpi}^{\mathrm{approx},N}$ is not only a qualitatively but also quantitatively decent approximation.

5.9 Conclusion

In this chapter, we first considered implications of the results in Chapter 4 for the augmented truncation approximation (ATA). Specifically, we showed that the linear ATA solution has the same degree of freedom as the general ATA solution does. We

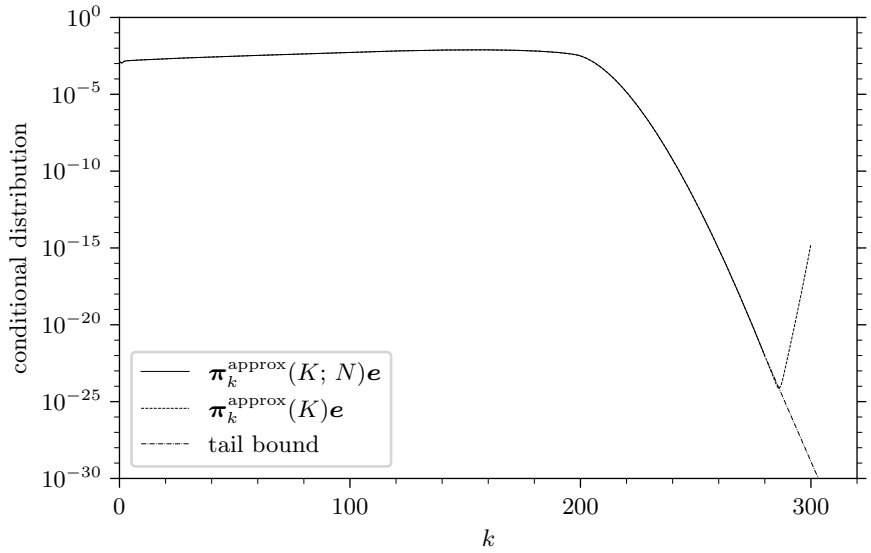


Figure 5.8: The conditional stationary distribution in the imbedded Markov chain.

also showed that the error bound for an arbitrary approximation to the conditional stationary distribution can be evaluated using the probability vectors that constitute the ATA solutions.

Next we considered applications of the implementations to single-server queues with level-dependent arrivals and disasters and generally distributed service times. We first discussed how to compute the truncated transition probability matrix $\boldsymbol{P}(K)$ of the imbedded Markov chain. If the continuous-time Markov chain representing arrivals and disasters is not uniformizable, the computation of $\boldsymbol{P}(K)$ is not straightforward. In this chapter, we developed numerical procedures for computing $\boldsymbol{P}(K)$ that satisfies a predefined error bound, under some stability conditions. We also briefly discussed how to construct the augmentation matrix in the ATA and how to obtain the error bound of the approximate solution to the stationary queue length distribution. Numerical examples demonstrated that our procedure works well as designed.

We also demonstrated numerically that the ATA solution with the sufficiently large truncation point K may require the further truncation of the solution. We showed that the truncated ATA solution also lies in the minimum convex polytope defined in Section 4.4. The numerical examples indicated that the (K, N)-skip-free sets are useful for obtaining a qualitatively good approximation.

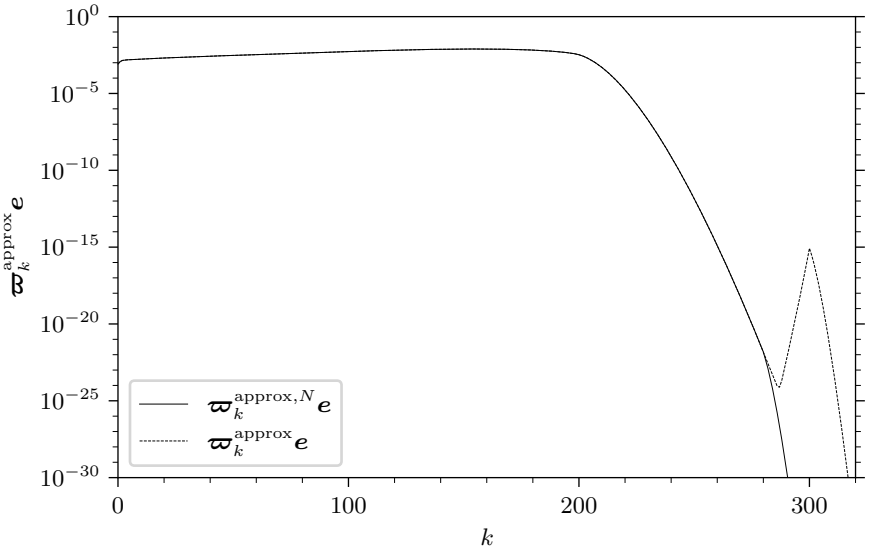


Figure 5.9: The stationary distribution of the number of customers.

6 Conclusion

In this dissertation, we studied the conditional stationary distribution in ergodic, time-homogeneous, continuous-time Markov chains on a countably infinite state space. In almost all the existing works, the (conditional) stationary distribution has been studied based on systems of linear equations because it is given by the unique solution of the global balance equation. We took different approaches from those and characterized the conditional stationary distribution via systems of linear inequalities. Note that such a system of linear inequalities specifies a convex cone on the first orthant and the conditional stationary distribution lies in a convex region given by the intersection of such a convex cone and the hyperplane containing all probability vectors. Furthermore, we studied the limit of those convex regions and based on it, we developed numerical algorithms for computing the conditional stationary distribution. The main results in this dissertation are summarized as follows.

In Chapter 2, we first obtained a convex polytope $\mathcal{P}^{+}_{k,\ell}$ that contains the conditional stationary distribution $\overline{\boldsymbol{\pi}}_{\ell}$ in a general bivariate Markov chain, where $\mathcal{P}^{+}_{k,\ell}$ is given in terms of the reverse-directional R-matrix $\boldsymbol{N}_{k,\ell}$. We then showed the limiting formula that each vertex spanning $\mathcal{P}^{+}_{k,\ell}$ converges to $\overline{\boldsymbol{\pi}}_{\ell}$ as k goes to infinity. Note that this limiting formula holds in any ergodic Markov chain and reveals the essential property of reverse-directional R-matrices with respect to the conditional stationary distribution. In addition, we showed the inclusion property $\mathcal{P}^{+}_{k+1,\ell} \subseteq \mathcal{P}^{+}_{k,\ell}$ for Markov chains of LD-M/G/1-type.

In Chapter 3, we considered computation of the conditional stationary distribution in Markov chains of LD-M/G/1-type, based on reverse-directional R-matrices. The main body of the numerical algorithm is identical to that in [Taki16]. Note, however, that owing to the results in Chapter 2, we eliminated the assumption that downward transition rate matrices $\boldsymbol{Q}_{\ell,\ell-1}$ are nonsingular, which is put in [Taki16]. We developed the stopping criterion in terms of the error bound of the output of the algorithm (i.e., the conditional stationary distribution), which is a notable feature of our algorithm. Through numerical examples, we confirmed that our algorithm works well as designed.

In Chapter 4, we considered the conditional stationary distribution $\boldsymbol{\pi}(N)$ in an ergodic Markov chain based on northwest corner submatrices of the infinitesimal generator. We first derived a system of linear inequalities that $\boldsymbol{\pi}(N)$ satisfies from the $(N+1) \times (N+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(N)$. This result was refined by taking account of the structural information $\mathcal{J}(N)$. Next, we considered $\boldsymbol{\pi}(N)$ using the $(K+1) \times (K+1)$ northwest corner submatrix $\boldsymbol{Q}^{(1,1)}(K)$, where $K > N$. We then identified convex polytopes containing $\boldsymbol{\pi}(N)$. We further discussed the inclusion and limiting properties of those convex polytopes and identified the minimum convex polytope by introducing a new structural property called (K, N)-skip-free sets.

In Chapter 5, we considered an application of the ATA to a single-server queue with level-dependent arrivals and disasters, where service times are assumed to be generally distributed. We first provided some implications of the results in Chapter 4 for the ATA. Next, we considered the above-mentioned queueing model by the following standard approach: (i) The construction of the imbedded Markov chain, (ii) the computation of the stationary distribution in the imbedded Markov chain, and (iii) the computation of the stationary distribution in the original model. In Step (i), we developed an error-controllable computational method for transition probabilities in the imbedded Markov chain. In Step (ii), we applied the ATA to the computation of the stationary distribution in the imbedded Markov chain, taking account of the above-mentioned implications. In Step (iii), we discussed how to adjust the inaccurate tail distribution, along with numerical examples.

In this dissertation, we characterized the conditional stationary distribution in ergodic Markov chains via systems of linear inequalities, which determine convex regions that the conditional stationary distribution lies in. The convexity is useful in numerical computation, i.e., we can obtain approximations to the conditional stationary distribution, together with their error bounds. We also showed that the sequences of convex regions converge to the singleton containing the conditional stationary distribution. These results guarantee that in principle, we can obtain a good approximation to the stationary distribution in compensation for computational cost. We hope that the results in this dissertation will be a basis for the future development of efficient and accurate numerical algorithm for the stationary distribution in Markov chains.

Appendices

A Proof of Eq. (2.24)

Let $\boldsymbol{F}_{\ell,\ell}$ ($\ell \in \mathbb{Z}^+$) denote an $M_\ell \times M_\ell$ matrix whose (i,j)th ($i,j \in \mathcal{M}_\ell$) element represents the conditional probability that the recurrence time to level ℓ ends at state (ℓ, j), given that it starts from state (ℓ, i). Since the imbedded Markov chain $\{(L_n, J_n)\}_{n=0,1,\ldots}$ is irreducible and positive-recurrent, $\boldsymbol{F}_{\ell,\ell}$ is an irreducible stochastic matrix with an invariant probability vector $\overline{\boldsymbol{\rho}}_\ell$ determined uniquely by

$$\overline{\boldsymbol{\rho}}_\ell = \overline{\boldsymbol{\rho}}_\ell \boldsymbol{F}_{\ell,\ell}, \qquad \overline{\boldsymbol{\rho}}_\ell \boldsymbol{e} = 1. \tag{A.1}$$

It then follows from (2.20) and (A.1) that

$$\begin{aligned}(_k\hat{\boldsymbol{\rho}}_\ell(i) - \overline{\boldsymbol{\rho}}_\ell)\,(\boldsymbol{I} - \boldsymbol{F}_{\ell,\ell} + \boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell) &= {}_k\hat{\boldsymbol{\rho}}_\ell(i)\,(\boldsymbol{I} - \boldsymbol{F}_{\ell,\ell}) + \overline{\boldsymbol{\rho}}_\ell - \overline{\boldsymbol{\rho}}_\ell \\ &= {}_k\hat{\boldsymbol{\rho}}_\ell(i)(_k\hat{\boldsymbol{F}}_\ell(i) - \boldsymbol{F}_{\ell,\ell}).\end{aligned}$$

Since $\boldsymbol{F}_{\ell,\ell}$ is irreducible, $\boldsymbol{I} - \boldsymbol{F}_{\ell,\ell} + \boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell$ is nonsingular. We thus have

$$_k\hat{\boldsymbol{\rho}}_\ell(i) = \overline{\boldsymbol{\rho}}_\ell + {}_k\hat{\boldsymbol{\rho}}_\ell(i)(_k\hat{\boldsymbol{F}}_\ell(i) - \boldsymbol{F}_{\ell,\ell})(\boldsymbol{I} - \boldsymbol{F}_{\ell,\ell} + \boldsymbol{e}\overline{\boldsymbol{\rho}}_\ell)^{-1}. \tag{A.2}$$

For an arbitrarily fixed ℓ ($\ell \in \mathbb{Z}^+$), if

$$\lim_{k\to\infty} {}_k\boldsymbol{F}_{\ell,\ell} = \boldsymbol{F}_{\ell,\ell}, \tag{A.3}$$

we have $\lim_{k\to\infty}(\boldsymbol{F}_{\ell,\ell} - {}_k\boldsymbol{F}_{\ell,\ell})\boldsymbol{e} = \lim_{k\to\infty}(\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})\boldsymbol{e} = \boldsymbol{0}$, and therefore it follows from (2.19) and (A.3) that for $i \in \mathcal{M}_\ell$,

$$\lim_{k\to\infty} {}_k\hat{\boldsymbol{F}}_\ell(i) = \lim_{k\to\infty} {}_k\boldsymbol{F}_{\ell,\ell} + \lim_{k\to\infty} (\boldsymbol{I} - {}_k\boldsymbol{F}_{\ell,\ell})\boldsymbol{e}\boldsymbol{e}_i^T = \lim_{k\to\infty} {}_k\boldsymbol{F}_{\ell,\ell} = \boldsymbol{F}_{\ell,\ell}. \tag{A.4}$$

Therefore, if (A.3) holds, taking the limit $k \to \infty$ on both sides of (A.2) and using (A.4) yield (2.24). In what follows, we will show (A.3).

We define τ_ℓ ($\ell \in \mathbb{Z}^+$) as the first passage time to level ℓ.

$$\tau_\ell = \inf(n \in \mathbb{N};\ L_n = \ell).$$

We also define events $\Omega_\ell(i)$, ${}_k\Omega_\ell(i)$ and ${}_{k+}\Omega_\ell(i)$ ($\ell \in \mathbb{Z}^+$, $k \in \mathbb{Z}_{\ell+1}^\infty$) as

$$\begin{aligned}\Omega_\ell(i) &= \{L_0 = \ell, J_0 = i, \tau_\ell < \infty\}, \\ {}_k\Omega_\ell(i) &= \{L_0 = \ell, J_0 = i, \tau_\ell < \infty, L_s \neq k\ (s = 1, 2, \ldots, \tau_\ell - 1)\}, \\ {}_{k+}\Omega_\ell(i) &= \{L_0 = \ell, J_0 = i, \tau_\ell < \infty, L_s < k\ (s = 1, 2, \ldots, \tau_\ell - 1)\}.\end{aligned}$$

Note that

$$[{}_k\boldsymbol{F}_{\ell,\ell}]_{i,j} = \Pr({}_k\Omega_\ell(i) \cap \{J_{\tau_\ell} = j\} \mid L_0 = \ell, J_0 = i),$$
$$[\boldsymbol{F}_{\ell,\ell}]_{i,j} = \Pr(\Omega_\ell(i) \cap \{J_{\tau_\ell} = j\} \mid L_0 = \ell, J_0 = i).$$

Let ${}_{k+}\boldsymbol{F}_{\ell,\ell}$ ($\ell \in \mathbb{Z}^+$, $k \in \mathbb{Z}^\infty_{\ell+1}$) denote an $M_\ell \times M_\ell$ matrix whose (i,j)th ($i,j \in \mathcal{M}_\ell$) element is given by

$$[{}_{k+}\boldsymbol{F}_{\ell,\ell}]_{i,j} = \Pr({}_{k+}\Omega_\ell(i) \cap \{J_{\tau_\ell} = j\} \mid L_0 = \ell, J_0 = i).$$

By definition, ${}_{k+}\Omega_\ell(i) \subset {}_k\Omega_\ell(i) \subset \Omega_\ell(i)$ ($k \in \mathbb{Z}^\infty_{\ell+1}$). It then follows that

$${}_{k+}\boldsymbol{F}_{\ell,\ell} \le {}_k\boldsymbol{F}_{\ell,\ell} \le \boldsymbol{F}_{\ell,\ell}, \quad k \in \mathbb{Z}^\infty_{\ell+1}. \tag{A.5}$$

Furthermore, because ${}_{k+}\Omega_\ell(i) \subset {}_{(k+1)+}\Omega_\ell(i)$ ($k \in \mathbb{Z}^\infty_{\ell+1}$) and $\lim_{k\to\infty} {}_{k+}\Omega_\ell(i) = \Omega_\ell(i)$, we have $\lim_{k\to\infty} {}_{k+}\boldsymbol{F}_{\ell,\ell} = \boldsymbol{F}_{\ell,\ell}$. (A.3) now follows from (A.5), which completes the proof.

B The main error source of the algorithm in Figure 3.1 [Taki16, Appendix 3]

The main source of numerical errors in the algorithm in Figure 3.1 is the computation of the inverse of $\boldsymbol{T}$ ($= \boldsymbol{T}_k$). By definition, $\boldsymbol{T}_k$ is given in terms of $\boldsymbol{Z}_\ell$'s ($\ell \in \mathbb{Z}_0^{k-1}$) being composed of ($-\boldsymbol{T}_\ell$)'s. In other word, values (including errors) of $(-\boldsymbol{T}_k)^{-1}$ tend to oscillate since the inverse matrix is calculated recursively. We make an adjustment to suppress the oscillation as follows. As stated in [Taki16], $\boldsymbol{T}_k$ can be interpreted as a part of an infinitesimal generator, and since the row some of the infinitesimal generator is equal to zero, it satisfies

$$\boldsymbol{T}_k\boldsymbol{e} + \sum_{n=1}^{\infty} \boldsymbol{Q}_{k,n}\boldsymbol{e} + \sum_{n=0}^{k-1} \boldsymbol{Z}_{k-1}\boldsymbol{Z}_{k-2}\cdots\boldsymbol{Z}_n \sum_{\ell=k+1-n}^{\infty} \boldsymbol{Q}_{n,n+\ell}\boldsymbol{e} = \boldsymbol{0}.$$

We can rewrite it to be

$$(-\boldsymbol{T}_k)^{-1}\Big(\sum_{n=1}^{\infty} \boldsymbol{Q}_{k,n}\boldsymbol{e} + \sum_{n=0}^{k-1} \boldsymbol{Z}_{k-1}\boldsymbol{Z}_{k-2}\cdots\boldsymbol{Z}_n \sum_{\ell=k+1-n}^{\infty} \boldsymbol{Q}_{n,n+\ell}\boldsymbol{e}\Big) = \boldsymbol{e}. \tag{A.6}$$

Therefore, we normalize every row of $(-\boldsymbol{T}_k)^{-1}$ in such a way that (A.6) holds. It is claimed in [Taki16] that this adjustment worked to the stability of large-scale computation, and its effect is confirmed in our examples.

C Proof of Lemma 3.2

We first consider (3.22). For $\ell \in \mathbb{Z}_0^{N^*}$, we have

$$\begin{aligned}
\boldsymbol{\pi}_\ell^{\mathrm{approx},N+1}(N) &= \frac{\overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \cdot \boldsymbol{N}_{N,\ell}}{\overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} = \frac{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N}}{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N}\boldsymbol{e}} \cdot \frac{\boldsymbol{N}_{N,\ell}}{\overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \\
&= \frac{\boldsymbol{e}^T \boldsymbol{N}_{N+1,\ell}}{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N}\boldsymbol{e} \cdot \overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} = \frac{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N^*}\boldsymbol{N}_{N^*,\ell}}{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N}\boldsymbol{e} \cdot \overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \\
&= \frac{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N^*}\boldsymbol{e}}{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N}\boldsymbol{e} \cdot \overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \widetilde{\boldsymbol{N}}_N \boldsymbol{e}} \cdot \overline{\overline{\boldsymbol{\nu}}}_{N+1,N^*} \boldsymbol{N}_{N^*,\ell},
\end{aligned}$$

and therefore

$$\boldsymbol{\pi}_{\ell}^{\mathrm{approx},N+1}(N) = c_{N,N^*} \cdot \boldsymbol{\pi}_{\ell}^{\mathrm{approx},N+1}(N^*), \quad n = 0, 1, \ldots, N^*,$$

where c_{N,N^*} is a constant depending only on N and N^*.

$$c_{N,N^*} = \frac{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N^*} \boldsymbol{e}}{\boldsymbol{e}^T \boldsymbol{N}_{N+1,N} \boldsymbol{e}} \cdot \frac{\overline{\boldsymbol{\nu}}_{N+1,N^*} \widetilde{\boldsymbol{N}}_{N^*} \boldsymbol{e}}{\overline{\overline{\boldsymbol{\nu}}}_{N+1,N} \widetilde{\boldsymbol{N}}_{N} \boldsymbol{e}}.$$

We thus have

$$\overline{\boldsymbol{\pi}}_{-}^{\mathrm{approx},N+1}(N; N^*) = \boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},N+1}(N^*).$$

It then follows that

$$\begin{aligned}
&\|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},N+1}(N) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \\
&\quad \le \|(\boldsymbol{\pi}_{-}^{\mathrm{approx},N+1}(N; N^*)\ \mathbf{0}\ \mathbf{0}\ \cdots\ \mathbf{0}) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 + \|\boldsymbol{\pi}_{+}^{\mathrm{approx},N+1}(N; N^*)\|_1 \\
&\quad = \|(1-\varepsilon_2) \cdot (\overline{\boldsymbol{\pi}}_{-}^{\mathrm{approx},N+1}(N; N^*)\ \mathbf{0}\ \mathbf{0}\ \cdots\ \mathbf{0}) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 + \varepsilon_2 \\
&\quad \le \|(\overline{\boldsymbol{\pi}}_{-}^{\mathrm{approx},N+1}(N; N^*)\ \mathbf{0}\ \mathbf{0}\ \cdots\ \mathbf{0}) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \\
&\qquad + \varepsilon_2 \|\overline{\boldsymbol{\pi}}_{-}^{\mathrm{approx},N+1}(N; N^*)\|_1 + \varepsilon_2 \\
&\quad = \|(\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},N+1}(N^*)\ \mathbf{0}\ \mathbf{0}\ \cdots\ \mathbf{0}) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 + 2\varepsilon_2, \qquad \text{(A.7)}
\end{aligned}$$

and an adaptation of (3.21) (i.e., (1.25)) yields

$$\|(\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},N+1}(N^*)\ \mathbf{0}\ \mathbf{0}\ \cdots\ \mathbf{0}) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 \le \varepsilon_1 + 2\Pr(L(0) > N^* \mid L(0) \le N). \quad \text{(A.8)}$$

(3.22) now follows from (A.7) and (A.8).

Next we consider (3.23). It follows from Theorem 3.1 that

$$\begin{aligned}
&\Pr(L(0) > N^* \mid L(0) \le N) \\
&\le |\boldsymbol{\pi}_{+}^{\mathrm{approx},k}(N; N^*)\boldsymbol{e} - \Pr(L(0) > N^* \mid L(0) \le N)| + \boldsymbol{\pi}_{+}^{\mathrm{approx},k}(N; N^*)\boldsymbol{e} \\
&\le \|\boldsymbol{\pi}_{+}^{\mathrm{approx},k}(N; N^*) - (\boldsymbol{\pi}_{N^*+1}(N)\ \boldsymbol{\pi}_{N^*+2}(N)\ \cdots\ \boldsymbol{\pi}_{N}(N))\|_1 + \boldsymbol{\pi}_{+}^{\mathrm{approx},k}(N; N^*)\boldsymbol{e} \\
&\le \|\boldsymbol{\pi}_{\mathrm{lv}}^{\mathrm{approx},k}(N) - \boldsymbol{\pi}_{\mathrm{lv}}(N)\|_1 + \boldsymbol{\pi}_{+}^{\mathrm{approx},k}(N; N^*)\boldsymbol{e} \\
&\le \frac{e_N(k,N)}{\overline{\overline{\boldsymbol{\nu}}}_{k,N}\, \boldsymbol{w}_N} + \boldsymbol{\pi}_{+}^{\mathrm{approx},k}(N; N^*)\boldsymbol{e},
\end{aligned}$$

from which (3.23) follows.

D Proof of Corollary 4.2

Substituting (4.14) and (4.15) into (4.11) yields (4.16). We thus consider (4.14) and (4.15) below. Because

$$\boldsymbol{H}(N) = (-\boldsymbol{Q}^{(1,1)}(N))^{-1} = \int_0^{\infty} \exp[\boldsymbol{Q}^{(1,1)}(N)t]\mathrm{d}t,$$

we have

$$[\boldsymbol{H}(N)]_{i,j} = \mathrm{E}\Big[\int_{T_n^{(1)}}^{T_n^{(2)}} \mathbb{1}(X(t) = j)\mathrm{d}t \mid X(T_n^{(1)}) = i\Big], \quad i, j \in \mathbb{Z}_0^N,$$

and therefore

$$[\overline{\boldsymbol{H}}(N)]_{i,j} = \frac{1}{\mathrm{E}\big[T_n^{(2)} - T_n^{(1)} \mid X(T_n^{(1)}) = i\big]} \mathrm{E}\Big[\int_{T_n^{(1)}}^{T_n^{(2)}} \mathbb{1}(X(t) = j)\mathrm{d}t \mid X(T_n^{(1)}) = i\Big],$$

from which (4.14) follows.

We also have $\boldsymbol{\pi}^{(1)}(N) = \boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\boldsymbol{H}(N)$ from (4.3) and (4.8), so that

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} = \boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}.$$

It then follows from (4.9) and (4.11) that

$$\boldsymbol{\alpha}^*(N) = \frac{\boldsymbol{\pi}^{(2)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}} \cdot \boldsymbol{Q}^{(2,1)}(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}) = \frac{\boldsymbol{\zeta}^*(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})}{\boldsymbol{\zeta}^*(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}},$$

where

$$\boldsymbol{\zeta}^*(N) = \frac{\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)}{\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\boldsymbol{e}}. \tag{A.9}$$

It is readily seen that

$$[\boldsymbol{\zeta}^*(N)]_i = \Pr(X(T_n^{(1)}) = i), \quad i \in \mathbb{Z}_0^N.$$

We thus have

$$\begin{aligned}
[\boldsymbol{\alpha}^*(N)]_i &= \frac{1}{\mathrm{E}\big[T_n^{(2)} - T_n^{(1)}\big]} \mathrm{E}\Big[\int_{T_n^{(1)}}^{T_n^{(2)}} \mathbb{1}(X(T_n^{(1)}) = i)\mathrm{d}t\Big], \\
&= \frac{1}{\mathrm{E}\big[T_n^{(2)} - T_n^{(1)}\big]} \mathrm{E}\Big[\int_{T_n^{(1)}}^{T_n^{(2)}} \mathbb{1}(I(t) = i)\mathrm{d}t\Big], \quad i \in \mathbb{Z}_0^N,
\end{aligned}$$

from which (4.15) follows.

E Equivalence between (4.21) and (4.22) for $\mathcal{J}(N) \neq \mathbb{Z}_0^N$

Suppose $\mathcal{J}(N) \neq \mathbb{Z}_0^N$. We partition $\overline{\boldsymbol{H}}(N)$ into two matrices:

$$\overline{\boldsymbol{H}}(N) = \begin{matrix} \\ \mathcal{J}(N) \\ \mathbb{Z}_0^N \backslash \mathcal{J}(N) \end{matrix} \begin{matrix} \mathbb{Z}_0^N \\ \begin{pmatrix} \overline{\boldsymbol{H}}_+(N) \\ \overline{\boldsymbol{H}}_0(N) \end{pmatrix} \end{matrix}.$$

Note that (4.22) for $\mathcal{J}(N) \neq \mathbb{Z}_0^N$ is equivalent to

$$\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}_+(N),\ \boldsymbol{\alpha} \geq \boldsymbol{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1\}.$$

We now rewrite (4.21) to be

$$\begin{aligned}
\mathcal{P}^+(N) &= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N)) \geq \boldsymbol{0},\ \boldsymbol{x}(-\boldsymbol{Q}_0^{(1,1)}(N)) = \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N)) = \boldsymbol{y},\ \boldsymbol{x}(-\boldsymbol{Q}_0^{(1,1)}(N)) = \boldsymbol{0},\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N) \ \ -\boldsymbol{Q}_0^{(1,1)}(N)) = (\boldsymbol{y}\ \ \boldsymbol{0}),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) = (\boldsymbol{y}\ \ \boldsymbol{0}),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = (\boldsymbol{y}\ \ \boldsymbol{0})\boldsymbol{H}(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = (\boldsymbol{y}\ \ \boldsymbol{0})\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = (\boldsymbol{y}\mathrm{diag}(\boldsymbol{H}_+(N)\boldsymbol{e})\ \ \boldsymbol{0})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}_+(N)\boldsymbol{e})\overline{\boldsymbol{H}}_+(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}_+(N),\ \boldsymbol{\alpha} \geq \boldsymbol{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1\},
\end{aligned}$$

where the last equality follows from the same reasoning as in (4.10).

F Proof of Corollary 4.4

.

We first prove (4.45) by showing $\boldsymbol{\beta} \in \Gamma(N) \Rightarrow \boldsymbol{\alpha} \in \Gamma(N)$ in (4.43), i.e.,

$$\boldsymbol{\beta} \geq \boldsymbol{0},\ \boldsymbol{\beta}\boldsymbol{e} = 1 \quad \Rightarrow \quad \boldsymbol{\alpha} \geq \boldsymbol{0},\ \boldsymbol{\alpha}\boldsymbol{e} = \boldsymbol{\beta}\boldsymbol{e}\ (= 1).$$

For this purpose, we consider (4.34). By definition, we have for $i, j \in \mathbb{Z}_0^N$,

$$[\boldsymbol{R}(K,N)]_{i,j} = \Pr(X(T_{n+1}^{(1)}) = j,\ X(t) \in \mathbb{Z}_{N+1}^K\,(T_n^{(2)} \leq t < T_{n+1}^{(1)}) \mid X(T_n^{(1)}) = i),$$

and therefore $\boldsymbol{R}(K,N)$ is sub-stochastic, i.e.,

$$\boldsymbol{R}(K,N) \geq \boldsymbol{O}, \qquad \boldsymbol{R}(K,N)\boldsymbol{e} \leq \boldsymbol{e}, \qquad \boldsymbol{R}(K,N)\boldsymbol{e} \neq \boldsymbol{e}.$$

It follows that

$$(\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1} = \sum_{n=0}^{\infty} \big(\boldsymbol{R}(K,N)\big)^n \geq \boldsymbol{O}.$$

Recall that $\boldsymbol{H}(N) \geq \boldsymbol{O}$ and $\widetilde{\boldsymbol{H}}(K,N) \geq \boldsymbol{O}$. We thus conclude $\boldsymbol{\beta} \geq \boldsymbol{0} \ \Rightarrow\ \boldsymbol{\alpha} \geq \boldsymbol{0}$ from (4.43). Furthermore, we have

$$\begin{aligned}
\boldsymbol{\alpha}\boldsymbol{e} &= \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}\boldsymbol{H}(N)\boldsymbol{e} \\
&= \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e} \\
&= \boldsymbol{\beta}\boldsymbol{e}.
\end{aligned}$$

Next we show (4.46). For this purpose, we consider the censored process on $\mathbb{Z}_0^N \cup \mathbb{Z}_{K_1+1}^{\infty}$ whose generator is given by $\widetilde{\boldsymbol{Q}}(K_1,N)$. We then apply Theorem 4.1 with $N = K = K_1$ and obtain $\widetilde{\Gamma}(K_1,N)$. Moreover, we apply Theorem 4.4 to the above censored process on $\mathbb{Z}_0^N \cup \mathbb{Z}_{K_1+1}^{\infty}$ with $K = K_2$ and obtain $\widetilde{\Gamma}(K_2,N)$. (4.46) now follows from (4.45), where $\widetilde{\Gamma}(K_1,N)$ and $\widetilde{\Gamma}(K_2,N)$ correspond to $\Gamma(N)$ and $\widetilde{\Gamma}(K,N)$ in (4.45).

G Proof of Corollary 4.5

We first consider (4.58). If $\mathcal{J}(N) = \mathbb{Z}_0^N$, (4.58) is immediate because

$$\widetilde{\Gamma}^+(K,N) \subseteq \widetilde{\Gamma}(K,N) \subseteq \Gamma(N) = \Gamma^+(N).$$

We thus assume $\mathcal{J}(N) \neq \mathbb{Z}_0^N$. Noting (4.49), we partition $\mathbb{Z}_0^N$ into three subsets $\widetilde{\mathcal{J}}(K,N)$, $\mathcal{J}(N) \setminus \widetilde{\mathcal{J}}(K,N)$, and $\mathbb{Z}_0^N \setminus \mathcal{J}(N)$.

$$\boldsymbol{Q}^{(2,1)}(K,N) = \begin{array}{c} \overset{\widetilde{\mathcal{J}}(K,N)}{} \quad \overset{\mathcal{J}(N)\setminus\widetilde{\mathcal{J}}(K,N)}{} \quad \overset{\mathbb{Z}_0^N\setminus\mathcal{J}(N)}{} \\ \Big(\boldsymbol{Q}_{+}^{(2,1)}(K,N) \quad \boldsymbol{Q}_{++}^{(2,1)}(K,N) \quad \boldsymbol{O} \Big) \end{array}. \tag{A.10}$$

Note here that $\mathcal{J}(N) \setminus \widetilde{\mathcal{J}}(K,N)$ may be empty and if this is the case, we simply ignore the corresponding terms. It follows from (4.34) that

$$(\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1} = \boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}\boldsymbol{R}(K,N) = \boldsymbol{I} + \boldsymbol{B}\boldsymbol{Q}^{(2,1)}(K,N), \tag{A.11}$$

where

$$\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}(-\boldsymbol{Q}^{(1,1)}(K,N))^{-1}\boldsymbol{Q}^{(1,2)}(K,N)(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1} \geq \boldsymbol{O}.$$

Suppose $\boldsymbol{\alpha} \in \widetilde{\Gamma}^+(K,N)$, i.e., there exists nonnegative vector $\boldsymbol{\beta}$ such that

$$\boldsymbol{\beta} = \begin{array}{c} \overset{\widetilde{\mathcal{J}}(K,N)}{} \quad \overset{\mathcal{J}(N)\setminus\widetilde{\mathcal{J}}(K,N)}{} \quad \overset{\mathbb{Z}_0^N\setminus\mathcal{J}(N)}{} \\ (\quad \boldsymbol{\beta}_1 \quad \boldsymbol{0} \quad \boldsymbol{0} \quad) \end{array}, \quad \boldsymbol{\beta}_1 \geq \boldsymbol{0}, \quad \boldsymbol{\beta}_1\boldsymbol{e} = 1,$$

and

$$\begin{aligned} \boldsymbol{\alpha} &= \begin{array}{c} \overset{\widetilde{\mathcal{J}}(K,N)}{} \quad \overset{\mathcal{J}(N)\setminus\widetilde{\mathcal{J}}(K,N)}{} \quad \overset{\mathbb{Z}_0^N\setminus\mathcal{J}(N)}{} \\ (\quad \boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \boldsymbol{\alpha}_3 \quad) \end{array} \\ &= (\boldsymbol{\beta}_1 \;\; \boldsymbol{0} \;\; \boldsymbol{0})\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}). \end{aligned}$$

It follows from (4.45) and (4.56) that $\widetilde{\Gamma}^+(K,N) \subseteq \Gamma(N)$, so that $\boldsymbol{\alpha}\boldsymbol{e} = 1$ and $\boldsymbol{\alpha} \geq \boldsymbol{0}$. Moreover, using (A.10) and (A.11), we rewrite $\boldsymbol{\alpha}$ to be

$$\begin{aligned} \boldsymbol{\alpha} &= (\boldsymbol{\beta}_1 \;\; \boldsymbol{0} \;\; \boldsymbol{0})\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}) \\ &\quad + \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})\boldsymbol{B}\big(\boldsymbol{Q}_{+}^{(2,1)}(K,N) \;\; \boldsymbol{Q}_{++}^{(2,1)}(K,N) \;\; \boldsymbol{O}\big)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}), \end{aligned}$$

and therefore $\boldsymbol{\alpha}_3 = \boldsymbol{0}$, which completes the proof of (4.58).

We omit the proof of (4.59) because the proof is almost the same as that of (4.46) in Corollary 4.4, where we use Theorem 4.2, Theorem 4.5, and (4.58), instead of Theorem 4.1, Theorem 4.4, and (4.45), respectively.

H Proof of Theorem 4.6

Without loss of generality, we assume $\mathbb{Z}_0^N \cup \mathcal{X}^* = \mathbb{Z}_0^M$, where $M = |\mathbb{Z}_0^N \cup \mathcal{X}^*| - 1$. It then follows from Theorem 4.4 with $N := M$ that

$$\boldsymbol{\pi}(M) \in \widetilde{\mathcal{P}}(K,M),$$

where from (4.40) and (4.44),

$$\widetilde{\mathcal{P}}(K,M)=\widehat{\mathcal{P}}(K,M)=\big\{\boldsymbol{x}\in\mathbb{R}^{M+1};\ \boldsymbol{x}=\boldsymbol{\beta}_M\overline{\widetilde{\boldsymbol{H}}}(K,M),\ \boldsymbol{\beta}_M\in\Gamma(M)\big\}.$$

Note here that if $X(0)\in\mathbb{Z}_{K+1}^{\infty}$, the first passage time to $\mathbb{Z}_0^M$ ends on $\mathcal{X}^*$. Therefore, following a discussion similar to the proof of Theorem 4.5, we obtain

$$\boldsymbol{\pi}(M)\in\widehat{\mathcal{P}}^{+}(K,M;\,\mathcal{X}^*)\quad\text{for}\quad\mathcal{X}\in S(K,N),\tag{A.12}$$

where

$$\widehat{\mathcal{P}}^{+}(K,M;\,\mathcal{X}^*)=\big\{\boldsymbol{x}\in\mathbb{R}^{M+1};\ \boldsymbol{x}=\boldsymbol{\beta}_M\overline{\widetilde{\boldsymbol{H}}}(K,M),\ \boldsymbol{\beta}_M\in\Gamma(M),[\boldsymbol{\beta}_M]_i=0\ (i\in\mathbb{Z}_0^M\setminus\mathcal{X}^*)\big\}.\tag{A.13}$$

For a proper $\mathcal{X}^*$, we have $\mathcal{X}^*\cap\mathcal{N}_0(K,N)=\emptyset$, i.e., the probability that the first passage time from $\mathbb{Z}_{K+1}^{\infty}$ to $\mathbb{Z}_0^M$ ends on state i is strictly positive for all $i\in\mathcal{X}^*$. We thus have $\mathcal{X}^*=\widetilde{\mathcal{J}}(K,M)$ for $\mathcal{X}^*\in\mathcal{S}^*(K,N)$ and therefore from Theorem 4.5,

$$\boldsymbol{\pi}(M)\in\mathrm{ri}\,\widetilde{\mathcal{P}}^{+}(K,M;\,\mathcal{X}^*)\quad\text{for}\quad\mathcal{X}^*\in S^*(K,N).$$

Since $\mathbb{Z}_0^N\cup\mathcal{X}^*=\mathbb{Z}_0^M$, $\boldsymbol{\pi}(N)$ can be expressed in terms of $\boldsymbol{\pi}(M)$. Specifically, we first partition $\boldsymbol{\pi}(M)$ and $\widetilde{\boldsymbol{H}}(K,M)$ as follows.

$$\boldsymbol{\pi}(M)=\begin{array}{c}\overset{\mathbb{Z}_0^N}{}\\(\ \boldsymbol{\pi}^{(1)}(M;\,N)\end{array}\quad\begin{array}{c}\overset{\mathbb{Z}_{N+1}^M}{}\\\boldsymbol{\pi}^{(2)}(M;\,N)\ )\end{array},$$

$$\widetilde{\boldsymbol{H}}(K,M)=\begin{array}{c}\overset{\mathbb{Z}_0^N}{}\\\Big(\ \widetilde{\boldsymbol{H}}^{(*,1)}(K,M;\,N)\end{array}\quad\begin{array}{c}\overset{\mathbb{Z}_{N+1}^M}{}\\\widetilde{\boldsymbol{H}}^{(*,2)}(K,M;\,N)\ \Big)\end{array}.$$

It then follows from (A.12) and (A.13) that $\boldsymbol{\pi}^{(1)}(M;\,N)\in\widehat{\mathcal{P}}_M^{+(1)}(K,M;\,\mathcal{X}^*)$, where

$$\widehat{\mathcal{P}}_M^{+(1)}(K,M;\,\mathcal{X}^*)=\big\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}=\boldsymbol{\beta}_M\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,M)\boldsymbol{e})\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;\,N),\ \boldsymbol{\beta}_M\in\Gamma(M),\ [\boldsymbol{\beta}_M]_i=0\ (i\in\mathbb{Z}_0^M\setminus\mathcal{X}^*)\big\}.\tag{A.14}$$

By definition, $\boldsymbol{\pi}(N)=\boldsymbol{\pi}^{(1)}(M;\,N)/\boldsymbol{\pi}^{(1)}(M;\,N)\boldsymbol{e}$. It then follows from (A.14) that $\boldsymbol{\pi}^{(1)}(M;\,N)$ is given by a weighted sum of row vectors of $\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;\,N)$. On the other hand, it follows from Remark 4.2 that if $\mathbb{Z}_0^N\cup\mathcal{X}^*=\mathbb{Z}_0^M$, $\widetilde{\boldsymbol{H}}(K,M)$ is identical to the $(M+1)\times(M+1)$ northwest corner block of $\boldsymbol{H}(K)$. Therefore, the row vectors corresponding to state $i\in\mathcal{X}^*$ in $\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;\,N)$ and in $\boldsymbol{H}^{(*,1)}(K;\,N)$ are identical. Therefore, replacing $\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;\,N)$ by $\boldsymbol{H}^{(*,1)}(K;\,N)$ and normalizing $\boldsymbol{\pi}^{(1)}(M;\,N)$, we obtain

$$\boldsymbol{\pi}(N)\in\widehat{\mathcal{P}}^{+}(K,N;\,\mathcal{X}^*)\ \text{ for }\ \mathcal{X}\in\mathcal{S}(K,N),$$

where

$$\widehat{\mathcal{P}}^{+}(K,N;\,\mathcal{X}^*)=\big\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}=\boldsymbol{\beta}\mathrm{diag}^{*}(\boldsymbol{H}^{(*,1)}(K;\,N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K;\,N),\ \boldsymbol{\beta}\in\Gamma(K),\ [\boldsymbol{\beta}]_i=0\ (i\in\mathbb{Z}_0^K\setminus\mathcal{X}^*)\big\}.\tag{A.15}$$

In particular, for a proper $\mathcal{X}^*$, we have

$$\boldsymbol{\pi}(N) \in \mathrm{ri}\, \widehat{\mathcal{P}}^+(K, N;\, \mathcal{X}) \;\; \text{for} \;\; \mathcal{X}^* \in \mathcal{S}^*(K, N).$$

Note that $\widehat{\mathcal{P}}^+(K, N;\, \mathcal{X}^*)$ in (A.15) is a convex polytope on the first orthant of $\mathbb{R}^{N+1}$.

The proof of (4.65) will complete if

$$\widehat{\mathcal{P}}^+(K, N;\, \mathcal{X}^*) = \mathcal{P}^+(K, N;\, \mathcal{X}^*), \tag{A.16}$$

which will be shown below. Note that $\boldsymbol{H}^{(*,1)}(K;\, N)$ are given in terms of $\widetilde{\boldsymbol{H}}(K, N)$ in (4.36).

$$\boldsymbol{H}^{(*,1)}(K;\, N) = \begin{matrix} \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^K \end{matrix} \begin{pmatrix} \boldsymbol{I} \\ (-\boldsymbol{Q}^{(2,2)}(K, N))^{-1}\boldsymbol{Q}^{(2,1)}(K, N) \end{pmatrix} \widetilde{\boldsymbol{H}}(K, N). \tag{A.17}$$

It then follows from (4.41) and (A.17) that

$$\begin{aligned} &\boldsymbol{\beta}\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;\, N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K;\, N) \\ &\quad = \boldsymbol{\beta}\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;\, N)\boldsymbol{e}) \begin{pmatrix} \boldsymbol{I} \\ (-\boldsymbol{Q}^{(2,2)}(K, N))^{-1}\boldsymbol{Q}^{(2,1)}(K, N) \end{pmatrix} \widetilde{\boldsymbol{H}}(K, N) \\ &\quad = \boldsymbol{\beta}\boldsymbol{U}(K, N)\overline{\boldsymbol{H}}(N), \end{aligned}$$

so that $\widehat{\mathcal{P}}^+(K, N;\, \mathcal{X}^*)$ is identical to $\mathcal{P}^+(K, N;\, \mathcal{X}^*)$ in (4.66). The equivalence between (4.65) and (4.66) can be shown in the same way as in the proof of Theorem 4.2, so that we omit it.

I Proof of Lemma 4.2

If $\mathcal{X}_A^* = \mathcal{X}_B^*$, Lemma 4.2 holds. We thus assume $\mathcal{X}_A^* \neq \mathcal{X}_B^*$. Note that for $\mathcal{X} \in S(K, N)$, $\mathcal{P}^+(K, N;\, \mathcal{X}^*)$ is equivalent to $\widehat{\mathcal{P}}^+(K, N;\, \mathcal{X}^*)$ in (A.15), i.e.,

$$\begin{aligned} \widehat{\mathcal{P}}^+(K, N;\, \mathcal{X}^*) = \big\{ \boldsymbol{x} \in \mathbb{R}^{N+1};\; \boldsymbol{x} = \boldsymbol{\beta}\overline{\boldsymbol{H}}^{(*,1)}(K;\, N), \\ \boldsymbol{\beta} \in \Gamma(K),\; [\boldsymbol{\beta}]_i = 0 \;(i \in \mathbb{Z}_0^K \setminus \mathcal{X}^*) \big\}, \end{aligned} \tag{A.18}$$

where

$$\overline{\boldsymbol{H}}^{(*,1)}(K;\, N) = \mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;\, N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K;\, N).$$

Let $\boldsymbol{\eta}_j^{(*,1)}(K;\, N)$ and $\overline{\boldsymbol{\eta}}_j^{(*,1)}(K;\, N)$ $(j \in \mathbb{Z}_0^K)$ denote the jth row vectors of $\boldsymbol{H}^{(*,1)}(K;\, N)$ and $\overline{\boldsymbol{H}}^{(*,1)}(K;\, N)$. Note that the lemma is proven if

$$\overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N) \in \widehat{\mathcal{P}}(K, N;\, \mathcal{X}_B^*), \quad i \in \mathcal{X}_A^*. \tag{A.19}$$

It is clear that (A.19) holds for $i \in \mathcal{X}_A^* \cap \mathcal{X}_B^*$. We thus show (A.19) for $i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*$ below.

For a fixed $i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*$, we consider a censored Markov chain $\{\widetilde{X}(t)\}_{t\geq 0}$ by observing $\{X(t)\}_{t\geq 0}$ only when $X(t) \in \mathbb{Z}_0^N \cup \mathcal{X}_B^* \cup \{i\} \cup \mathbb{Z}_{K+1}^\infty$. The infinitesimal generator $\widetilde{\boldsymbol{Q}} := \widetilde{\boldsymbol{Q}}(K, N, \mathcal{X}_B^*, i)$ of the censored Markov chain $\{\widetilde{X}(t)\}_{t\geq 0}$ takes the following form:

$$\widetilde{\boldsymbol{Q}} = \begin{matrix} \\ \mathbb{Z}_0^N \cup \mathcal{X}_B^* \cup \{i\} \\ \mathbb{Z}_{K+1}^\infty \end{matrix} \begin{pmatrix} \overset{\mathbb{Z}_0^N \cup \mathcal{X}_B^* \cup \{i\}}{\widetilde{\boldsymbol{Q}}^{(1,1)}} & \overset{\mathbb{Z}_{K+1}^\infty}{\widetilde{\boldsymbol{Q}}^{(1,2)}} \\ \widetilde{\boldsymbol{Q}}^{(2,1)} & \widetilde{\boldsymbol{Q}}^{(2,2)} \end{pmatrix},$$

where, with $\mathbb{Z}_B = \mathbb{Z}_0^N \cup \mathcal{X}_B^*$,

$$\widetilde{\boldsymbol{Q}}^{(1,1)} = \begin{matrix} \\ \mathbb{Z}_B \\ \{i\} \end{matrix} \begin{pmatrix} \overset{\mathbb{Z}_B}{\widetilde{\boldsymbol{Q}}^{(1,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}} & \overset{\{i\}}{\widetilde{\boldsymbol{q}}^{(1,1)}_{\mathbb{Z}_B,i}} \\ \widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B} & -\widetilde{q}^{(1,1)}_{i,i} \end{pmatrix}.$$

Note here that

$$\widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B} = \begin{pmatrix} \overset{\mathbb{Z}_0^N \setminus \mathcal{X}_B^*}{\mathbf{0}} & \overset{\mathcal{X}_B^*}{\widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B,+}} \end{pmatrix}, \tag{A.20}$$

since $i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*$ and (4.67) holds sample path-wise. We define $\widetilde{\boldsymbol{H}} := \widetilde{\boldsymbol{H}}(K, N, \mathcal{X}_B^*, i)$ as

$$\begin{aligned} \widetilde{\boldsymbol{H}} = \big(-\widetilde{\boldsymbol{Q}}^{(1,1)}\big)^{-1} &= \begin{matrix} \\ \mathbb{Z}_B \\ \{i\} \end{matrix} \begin{pmatrix} \overset{\mathbb{Z}_B}{\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B}} & \overset{\{i\}}{\widetilde{\boldsymbol{h}}_{\mathbb{Z}_B,i}} \\ \widetilde{\boldsymbol{\eta}}_{i,\mathbb{Z}_B} & \widetilde{h}_{i,i} \end{pmatrix} \\ &= \begin{matrix} \\ \mathbb{Z}_B \\ \{i\} \end{matrix} \begin{pmatrix} \overset{\mathbb{Z}_B}{\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B}} & \overset{\{i\}}{\widetilde{\boldsymbol{h}}_{\mathbb{Z}_B,i}} \\ (\widetilde{q}^{(1,1)}_{i,i})^{-1}\widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B}\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{h}_{i,i} \end{pmatrix}. \end{aligned}$$

Note here that if we partition $\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B}$ as

$$\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} = \begin{matrix} \\ \mathbb{Z}_0^N \setminus \mathcal{X}_B^* \\ \mathcal{X}_B^* \end{matrix} \begin{pmatrix} \overset{\mathbb{Z}_0^N}{\widetilde{\boldsymbol{H}}^{(1,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}} & \overset{\mathbb{Z}_B \setminus \mathbb{Z}_0^N}{\widetilde{\boldsymbol{H}}^{(1,2)}_{\mathbb{Z}_B,\mathbb{Z}_B}} \\ \widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{\boldsymbol{H}}^{(2,2)}_{\mathbb{Z}_B,\mathbb{Z}_B} \end{pmatrix},$$

the row vector of $\widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}$, corresponding to $j \in \mathcal{X}_B^*$, is identical to $\boldsymbol{\eta}_j^{(*,1)}(K;\, N)$. Furthermore, noting (A.20), we have

$$\begin{aligned} \widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B}\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} &= \left(\widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B}\begin{pmatrix}\widetilde{\boldsymbol{H}}^{(1,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} \\ \widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}\end{pmatrix} \quad \widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B}\begin{pmatrix}\widetilde{\boldsymbol{H}}^{(1,2)}_{\mathbb{Z}_B,\mathbb{Z}_B} \\ \widetilde{\boldsymbol{H}}^{(2,2)}_{\mathbb{Z}_B,\mathbb{Z}_B}\end{pmatrix}\right) \\ &= \big(\widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B,+}\widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} \quad \widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B,+}\widetilde{\boldsymbol{H}}^{(2,2)}_{\mathbb{Z}_B,\mathbb{Z}_B}\big), \end{aligned}$$

and

$$\boldsymbol{\eta}_i^{(*,1)}(K;\, N) = (\widetilde{q}^{(1,1)}_{i,i})^{-1}\widetilde{\boldsymbol{\phi}}^{(1,1)}_{i,\mathbb{Z}_B,+}\widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}.$$

Therefore, $\boldsymbol{\eta}_i^{(*,1)}(K;\, N)$ $(i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*)$ is given by a linear combination of $\boldsymbol{\eta}_j^{(*,1)}(K;\, N)$ $(j \in \mathcal{X}_B^*)$ with nonnegative weights $\alpha_{i,j}$.

$$\boldsymbol{\eta}_i^{(*,1)}(K;\, N) = \sum_{j \in \mathcal{X}_B^*} \alpha_{i,j} \boldsymbol{\eta}_j^{(*,1)}(K;\, N).$$

We thus conclude that $\overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N)$ $(i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*)$ is given by a convex combination of $\overline{\boldsymbol{\eta}}_j^{(*,1)}(K;\, N)$ $(j \in \mathcal{X}_B^*)$, i.e.,

$$\overline{\boldsymbol{\eta}}_i^{(*,1)}(K;\, N) = \sum_{j \in \mathcal{X}_B^*} \beta_{i,j} \overline{\boldsymbol{\eta}}_j^{(*,1)}(K;\, N),$$

where

$$\beta_{i,j} = \frac{\alpha_{i,j} \boldsymbol{\eta}_j^{(*,1)}(K;\, N)\boldsymbol{e}}{\boldsymbol{\eta}_i^{(*,1)}(K;\, N)\boldsymbol{e}}, \quad j \in \mathcal{X}_B^*.$$

Note that $\beta_{i,j} \geq 0$ and $\sum_{j \in \mathcal{X}_B^*} \beta_{i,j} = 1$, which completes the proof.

J Proof of Lemma 4.3

We first prove that $\mathcal{D}(K, N)$ is an (s, t)-node cut in $G(K, N)$ and then we show its minimality. To prove the former, we show that the graph $G(K, N;\, -\mathcal{D}(K, N))$, which is obtained from $G(K, N)$ by removing all nodes in $\mathcal{D}(K, N)$, has no directed paths from s to t. To show this fact, we assume that $G(K, N;\, -\mathcal{D}(K, N))$ has a directed path P from s to t and we show contradiction. Since $\mathcal{J}(K)$ is the set of all neighboring nodes of s, P must contain node $v \in \mathcal{J}(K) \setminus \mathcal{D}(K, N)$, where we assume that v is the closest to t if P contains more than one node in $\mathcal{J}(K) \setminus \mathcal{D}(K, N)$. Because the directed path from v to t has no edges incoming to any nodes in $\mathcal{J}(K)$, $G^*(K, N)$ should contain the sub-path $P_{v,t}$ of P from v to t. This, however, contradicts that $\mathcal{D}(K, N)$ contains all nodes in $\mathcal{J}(K)$ from which t is reachable in $G^*(K, N)$.

Next we show the minimality of $\mathcal{D}(K, N)$. For each node $v \in \mathcal{D}(K, N)$, $G^*(K, N)$ contains a directed path $P_{v,t}$ from v that visits t without passing through any other nodes in $\mathcal{J}(K)$. As a result, edge $(s, v) \in E(K, N)$ and $P_{v,t}$ form a directed path from s to t in $G(K, N)$ and thereby $v \in \mathcal{D}(K, N)$ must be included in any (s, t)-node cut $\mathcal{X} \subseteq \mathcal{J}(K)$ in $G(K, N)$. We thus conclude that $\mathcal{D}(K, N) \subseteq \mathcal{X}$ for any $\mathcal{X}$ such that $\mathcal{X} \subseteq \mathcal{J}(K)$ and $\mathcal{X} \in S(K, N)$, which completes the proof.

K Proof of Theorem 4.7

We have $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^+(K, N;\, \mathcal{J}^*(K))$ from Lemma 4.4 and $\mathcal{P}^+(K, N;\, \mathcal{J}^*(K)) = \widehat{\mathcal{P}}^+(K, N;\, \mathcal{J}^*(K))$ from (A.16), where $\widehat{\mathcal{P}}^+(K, N;\, \mathcal{J}^*(K))$ is given in (A.15). We thus have $\boldsymbol{x} \in \operatorname{ri} \widehat{\mathcal{P}}^+(K, N;\, \mathcal{J}^*(K))$ and therefore, there exists $\boldsymbol{\beta} \in \Gamma(K)$ such that $[\boldsymbol{\beta}]_i > 0$ $(i \in \mathcal{J}^*(K))$, $[\boldsymbol{\beta}]_i = 0$ $(i \in \mathbb{Z}_0^K \setminus \mathcal{J}^*(K))$, and

$$\boldsymbol{x} = \boldsymbol{\beta} \operatorname{diag}^*(\boldsymbol{H}^{(*,1)}(K;\, N)\boldsymbol{e}) \boldsymbol{H}^{(*,1)}(K;\, N).$$

On the other hand, it follows from (4.68) that $\mathcal{J}(K) \setminus \mathcal{J}^*(K) = \mathcal{J}(K) \cap \mathcal{N}_0^{(3)}(K, N)$. Therefore, we have from (4.63),

$$\boldsymbol{\eta}_i^{(*,1)}(K;\, N) = \mathbf{0}, \quad i \in \mathcal{J}(K) \setminus \mathcal{J}^*(K),$$

where $\boldsymbol{\eta}_i^{(*,1)}(K;\, N)$ $(i \in \mathbb{Z}_0^K)$ denotes the ith row vector of $\boldsymbol{H}^{(*,1)}(K;\, N)$. Let $\boldsymbol{\gamma} \in \mathbb{R}^{K+1}$ denote

$$\boldsymbol{\gamma} = \boldsymbol{\beta}\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;\, N)\boldsymbol{e}) + \sum_{i \in \mathcal{J}(K) \setminus \mathcal{J}^*(K)} \boldsymbol{e}_i,$$

where $\boldsymbol{e}_i \in \Gamma(K)$ $(i \in \mathbb{Z}_0^K)$ denotes the unit vector whose ith element is equal to one. Note here that

$$[\boldsymbol{\gamma}]_i > 0. \quad i \in \mathcal{J}(K), \tag{A.21}$$

$$[\boldsymbol{\gamma}]_i = 0, \quad i \in \mathbb{Z}_0^K \setminus \mathcal{J}(K). \tag{A.22}$$

We thus have

$$\boldsymbol{\gamma}\boldsymbol{H}^{(*,1)}(K;\, N) = \boldsymbol{x} + \sum_{i \in \mathcal{J}(K) \setminus \mathcal{J}^*(K)} \boldsymbol{\eta}_i^{(*,1)}(K;\, N) = \boldsymbol{x}.$$

Let $\boldsymbol{y} = \boldsymbol{\gamma}\boldsymbol{H}^{(*,2)}(K;\, N)$. We then have

$$(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K)) = \boldsymbol{\gamma}.$$

It then follows from (A.21) and (A.22) that

$$[(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{Q}^{(1,1)}(K)]_i > 0, \quad i \in \mathcal{J}(K),$$
$$[(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{Q}^{(1,1)}(K)]_i = 0, \quad i \in \mathbb{Z}_0^K \setminus \mathcal{J}(K).$$

Furthermore, $(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{e} = \boldsymbol{\gamma}\boldsymbol{H}(K)\boldsymbol{e} > \mathbf{0}$ since $\boldsymbol{H}(K)\boldsymbol{e} > \mathbf{0}$. We thus have $(\boldsymbol{x}\ \boldsymbol{y})/((\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{e}) \in \mathrm{ri}\,\mathcal{P}^+(K)$ and therefore,

$$(\boldsymbol{x}\ \boldsymbol{y}) > \mathbf{0},$$

from Lemma 4.1. We then define $\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y})$ as

$$\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y}) = \frac{(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))}{(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\gamma}\boldsymbol{e}}.$$

Note that $\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y}) \in \mathrm{ri}\,\Gamma^+(K)$, where

$$\Gamma^+(K) = \left\{\boldsymbol{\alpha} \in \mathbb{R}^{K+1};\ \boldsymbol{\alpha} \geq \mathbf{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1,\ [\boldsymbol{\alpha}]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{J}(K))\right\}.$$

We now consider a Markov chain whose infinitesimal generator $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{matrix} \\ \mathbb{Z}_0^K \\ \mathbb{Z}_{K+1}^\infty \end{matrix} \begin{matrix} \begin{matrix} \mathbb{Z}_0^K & \mathbb{Z}_{K+1}^\infty \end{matrix} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(K) & (-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}\boldsymbol{z} \\ \boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y}) & -\boldsymbol{I}, \end{pmatrix} \end{matrix}. \tag{A.23}$$

where $\boldsymbol{z}$ denotes a $1 \times \infty$ positive probability vector. Note here that the Markov chain with $\boldsymbol{Q}$ in (A.23) is a member of $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ if it is ergodic, i.e., the global

balance equation $\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0}$ has a unique, positive solution $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0}$ is written to be

$$\begin{aligned}\boldsymbol{\pi}^{(1)}(K)\boldsymbol{Q}^{(1,1)}(K) + \boldsymbol{\pi}^{(2)}(K)\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x},\boldsymbol{y}) &= \boldsymbol{0},\\ \boldsymbol{\pi}^{(1)}(N)(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}\boldsymbol{z} - \boldsymbol{\pi}^{(2)}(K) &= \boldsymbol{0}.\end{aligned}$$

Solving the above with $\boldsymbol{\pi}^{(1)}(K)\boldsymbol{e} + \boldsymbol{\pi}^{(2)}(K)\boldsymbol{e} = 1$, we obtain

$$\begin{aligned}\boldsymbol{\pi} &= \begin{pmatrix}\boldsymbol{\pi}^{(1)}(K) & \boldsymbol{\pi}^{(2)}(K)\end{pmatrix}\\ &= \frac{1}{(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{e} + (\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}} \begin{pmatrix}(\boldsymbol{x}\ \boldsymbol{y}) & (\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}\cdot\boldsymbol{z}\end{pmatrix}\\ &= \frac{1}{1 + \boldsymbol{y}\boldsymbol{e} + \boldsymbol{\gamma}\boldsymbol{e}} \begin{pmatrix}(\boldsymbol{x}\ \boldsymbol{y}) & \boldsymbol{\gamma}\boldsymbol{e}\cdot\boldsymbol{z}\end{pmatrix} > \boldsymbol{0},\end{aligned}$$

especially,

$$\boldsymbol{\pi}^{(1)}(N) = \boldsymbol{\pi}^{(1)}(K,N) = \frac{1}{1 + \boldsymbol{y}\boldsymbol{e} + \boldsymbol{\gamma}\boldsymbol{e}} \cdot \boldsymbol{x}.$$

We thus conclude that the Markov chain with $\boldsymbol{Q}$ in (A.23) is a member of $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ and it has $\boldsymbol{\pi}(N) = \boldsymbol{x}$.

L Comparison of the results in Chapter 2 and Chapter 4 in Markov chains of LD-M/G/1-type

We first compare results in Chapter 2 and those in Chapter 4. To this end, we apply the results in Chapter 4 to an ergodic, bivariate Markov chain $\{(L(t), J(t))\}_{t\geq 0}$. We define $\mathcal{L}_k^{\ell}$ $(k, \ell \in \mathbb{Z}^+, k < \ell)$ and $\mathcal{L}_k^{\infty}$ $(k \in \mathbb{Z}^+)$ as

$$\mathcal{L}_k^{\ell} = \bigcup_{m=k}^{\ell} \mathcal{L}_m, \quad \mathcal{L}_k^{\infty} = \bigcup_{m=k}^{\infty} \mathcal{L}_m.$$

We then partition the infinitesimal generator $\boldsymbol{Q}$ as follows.

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathcal{L}_0^K \\ \mathcal{L}_{K+1}^{\infty} \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{L}_0^K & \mathcal{L}_{K+1}^{\infty} \end{array} \\ \begin{pmatrix} \boldsymbol{Q}_{\mathrm{lv}}^{(1,1)}(K) & \boldsymbol{Q}_{\mathrm{lv}}^{(1,2)}(K) \\ \boldsymbol{Q}_{\mathrm{lv}}^{(2,1)}(K) & \boldsymbol{Q}_{\mathrm{lv}}^{(2,2)}(K) \end{pmatrix} \end{array}.$$

We define $\boldsymbol{H}_{\mathrm{lv}}(K)$ and partition it as follows.

$$\boldsymbol{H}_{\mathrm{lv}}(K) = (-\boldsymbol{Q}_{\mathrm{lv}}^{(1,1)}(K))^{-1} = \begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_K \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{L}_0 & \mathcal{L}_1 & \cdots & \mathcal{L}_K \end{array} \\ \begin{pmatrix} \boldsymbol{H}_{0,0}(K) & \boldsymbol{H}_{0,1}(K) & \cdots & \boldsymbol{H}_{0,K}(K) \\ \boldsymbol{H}_{1,0}(K) & \boldsymbol{H}_{1,1}(K) & \cdots & \boldsymbol{H}_{1,K}(K) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{H}_{K,0}(K) & \boldsymbol{H}_{K,1}(K) & \cdots & \boldsymbol{H}_{K,K}(K) \end{pmatrix} \end{array}. \quad \text{(A.24)}$$

Note that $\boldsymbol{Q}_{\mathrm{lv}}^{(1,1)}(K) = \boldsymbol{Q}^{(1,1)}(L_K - 1)$ and $\boldsymbol{H}_{\mathrm{lv}}(K) = \boldsymbol{H}(L_K - 1)$ because of $|\mathcal{L}_0^K| = L_K$. Theorem 4.4 implies that for $N < K$,

$$\overline{\boldsymbol{\pi}}_N \in \{\boldsymbol{x} \in \mathbb{R}^{M_N};\ \boldsymbol{x} = \sum_{k=0}^{N} \boldsymbol{\alpha}_k \overline{\boldsymbol{H}}_{k,N}(K),\ \sum_{k=0}^{N} \boldsymbol{\alpha}_k \boldsymbol{e} = 1,\ \boldsymbol{\alpha}_k \geq \boldsymbol{0}\ (k \in \mathbb{Z}_0^N)\}. \quad \text{(A.25)}$$

Recall that $\overline{\boldsymbol{H}}_{k,N}(K) = \mathrm{diag}^{-1}(\boldsymbol{H}_{k,N}(K)\boldsymbol{e})\boldsymbol{H}_{k,N}(K)$.

We define $\mathcal{J}_{\mathrm{lv}}(K) := \mathcal{J}(L_K - 1)$.

$$\mathcal{J}_{\mathrm{lv}}(K) = \{(\ell, j) \in \mathcal{L}_0^K;\ [\boldsymbol{Q}_{k,\ell}]_{i,j} > 0 \text{ for some } (k,i) \in \mathcal{L}_{K+1}^{\infty}\}.$$

If $\{(L(t), J(t))\}_{t\geq 0}$ is of LD-M/G/1-type, it is clear that $\mathcal{J}_{\mathrm{lv}}(K) \subseteq \mathcal{L}_K$.

$$\mathcal{J}_{\mathrm{lv}}(K) = \{(K, j) \in \mathcal{L}_K;\ [\boldsymbol{Q}_{K+1,K}]_{i,j} > 0 \text{ for some } i \in \mathcal{M}_{K+1}\}.$$

In the rest of this section, we assume that $\{(L(t), J(t))\}_{t\geq 0}$ is of LD-M/G/1-type. We define $\mathcal{M}_{\mathrm{J}}(K)$ as follows.

$$\begin{aligned}\mathcal{M}_{\mathrm{J}}(K) &= \{j \in \mathcal{M}_K;\ (K,j) \in \mathcal{J}_{\mathrm{lv}}(K)\} \\ &= \{j \in \mathcal{M}_K;\ [\boldsymbol{Q}_{K+1,K}]_{i,j} > 0 \text{ for some } i \in \mathcal{M}_{K+1}\}. \end{aligned} \tag{A.26}$$

Let $\overline{\boldsymbol{\eta}}_{(K,i),N}(K)$ $(i \in \mathcal{M}_K)$ denote the ith row vector of $\overline{\boldsymbol{H}}_{K,N}(K)$. By applying Theorem 4.6, (A.25) is refined as follows.

$$\overline{\boldsymbol{\pi}}_N \in \mathcal{P}^{+}_{\mathrm{H},K,N},$$

where $\mathcal{P}^{+}_{\mathrm{H},K,N}$ denotes a convex polytope spanned by $\overline{\boldsymbol{\eta}}_{(K,i),N}(K)$ $(i \in \mathcal{M}_{\mathrm{J}}(K))$.

$$\begin{aligned}\mathcal{P}^{+}_{\mathrm{H},K,N} = \{\boldsymbol{x} \in \mathbb{R}^{M_N};\ \boldsymbol{x} = &\sum_{i\in\mathcal{M}_{\mathrm{J}}(K)} \alpha(i)\overline{\boldsymbol{\eta}}_{(K,i),N}(K), \\ &\sum_{i\in\mathcal{M}_{\mathrm{J}}(K)} \alpha(i) = 1,\ \alpha(i) \geq 0\ (i \in \mathcal{M}_{\mathrm{J}}(K))\}.\end{aligned} \tag{A.27}$$

Corollaries 4.6 and 4.7 imply that

$$\lim_{K\to\infty} \mathcal{P}^{+}_{\mathrm{H},K,N} = \{\overline{\boldsymbol{\pi}}_N\}. \tag{A.28}$$

By definition, we can show stochastically that

$$\boldsymbol{N}_{K+1,N} = \boldsymbol{Q}_{K+1,K}\boldsymbol{H}_{K,N}(K). \tag{A.29}$$

Because of $\boldsymbol{Q}_{K+1,K} \geq \boldsymbol{O}$, (A.26), and (A.29), we have

$$\mathcal{P}^{+}_{K+1,N} \subseteq \mathcal{P}^{+}_{\mathrm{H},K,N},$$

which implies that Theorem 2.3 can be shown by Corollary 4.6. On the other hand, we can show algebraically that

$$\boldsymbol{H}_{K,N}(K) = (-\boldsymbol{T}_k)^{-1}\boldsymbol{N}_{K,N},$$

where $\boldsymbol{T}_k$ is given in (3.3). Because of $(-\boldsymbol{T}_k)^{-1} \geq \boldsymbol{O}$, we have

$$\mathcal{P}^{+}_{\mathrm{H},K,N} \subseteq \mathcal{P}^{+}_{K,N},$$

which implies that Corollary 4.6 can be shown by Theorem 2.3. Therefore, *the limit formulas* in Chapter 2 and those in Chapter 4 are essentially identical in Markov chains of LD-M/G/1-type.

We next compare numerical computation based on Chapter 2, i.e., the algorithm in Figure 3.1, and that based on Chapter 4. The numerical computation based on Chapter 2 is stabler than that based on Chapter 4 because the former utilizes the stochastic property of levels. Specifically, we can obtain the relation $\boldsymbol{\pi}_n(N) = \boldsymbol{\pi}_k(N)\boldsymbol{N}_{k,n}$ $(N \in \mathbb{Z}^+,\ n < k \leq N)$ between $\boldsymbol{\pi}_n(N)$ and $\boldsymbol{\pi}_k(N)$ exactly. Therefore, the algorithm in Figure 3.1 rarely output an inaccurate tail like Figure 5.7. We thus recommend the algorithm based on Chapter 2 for Markov chains in LD-M/G/1-type.

M Proof of Lemma 5.1

Since $\mathcal{A}_{\mathrm{L}}(N) \subseteq \mathcal{A}(N)$, we have $\mathcal{T}_{\mathrm{L}}(N) \subseteq \mathcal{T}(N)$. We thus show $\mathcal{T}(N) \subseteq \mathcal{P}(N)$ and $\mathcal{P}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$, from which the lemma follows.

We first prove $\mathcal{T}(N) \subseteq \mathcal{P}(N)$. Suppose $\boldsymbol{x} \in \mathcal{T}(N)$, i.e., $\boldsymbol{x}[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)] = \boldsymbol{0}$ for some $\boldsymbol{Q}_{\mathrm{A}}(N) \in \mathcal{A}(N)$. Post-multiplying both sides of this equation by $\boldsymbol{H}(N) = (-\boldsymbol{Q}^{(1,1)}(N))^{-1}$ and rearranging terms, we obtain

$$\boldsymbol{x} = \boldsymbol{x}\boldsymbol{Q}_{\mathrm{A}}(N)\boldsymbol{H}(N) = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),$$

where $\boldsymbol{\alpha} = \boldsymbol{x}\boldsymbol{Q}_{\mathrm{A}}(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})$. Note here that $\boldsymbol{\alpha} \in \Gamma(N)$ because $\boldsymbol{\alpha} \geq \boldsymbol{0}$, $\boldsymbol{x}\boldsymbol{e} = 1$, and $\boldsymbol{x}\boldsymbol{e} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N)\boldsymbol{e} = \boldsymbol{\alpha}\boldsymbol{e}$. We thus obtain $\boldsymbol{x} \in \mathcal{P}(N)$ from (4.7), so that $\mathcal{T}(N) \subseteq \mathcal{P}(N)$.

Next, we prove $\mathcal{P}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$. For an arbitrarily fixed $\boldsymbol{x} \in \mathcal{P}(N)$, let $\boldsymbol{y} = \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \boldsymbol{0}$ (cf. (4.6)). Since $\boldsymbol{x}\boldsymbol{e} = 1$ and $\boldsymbol{Q}^{(1,1)}(N)$ is nonsingular, we have $\boldsymbol{y} \neq \boldsymbol{0}$. We then consider a linear augmentation matrix $\boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N)$ given by

$$\boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{\boldsymbol{y}}{\boldsymbol{y}\boldsymbol{e}} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{\boldsymbol{y}}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}.$$

It is clear that $\boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N) \in \mathcal{A}_{\mathrm{L}}(N)$ and

$$\boldsymbol{x}[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N)] = -\boldsymbol{y} + \boldsymbol{x} \cdot \frac{(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \cdot \boldsymbol{y} = \boldsymbol{0},$$

which implies $\boldsymbol{x} \in \mathcal{T}_{\mathrm{L}}(N)$, so that $\mathcal{P}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$.

N Proof of Lemma 5.2

If $\mathcal{J}(N) = \mathbb{Z}_0^N$, Lemma 5.2 immediately follows from Lemma 5.1. We thus assume $\mathcal{J}(N) \neq \mathbb{Z}_0^N$. Implication 5.2 with $\boldsymbol{\zeta} \in \Gamma^{+}(N)$ implies $\mathcal{T}^{+}_{\mathrm{L}}(N) \subseteq \mathcal{P}^{+}(N)$. We thus show $\mathcal{P}^{+}(N) \subseteq \mathcal{T}^{+}_{\mathrm{L}}(N)$ below. For an arbitrarily fixed $\boldsymbol{x} \in \mathcal{P}^{+}(N)$, let $\boldsymbol{y} = \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))$. Since $\boldsymbol{x}\boldsymbol{e} = 1$ and $\boldsymbol{Q}^{(1,1)}(N)$ is nonsingular, $\boldsymbol{y} \neq \boldsymbol{0}$. Furthermore, by definition, $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}_{+}(N)) \geq \boldsymbol{0}$ and $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}_{0}(N)) = \boldsymbol{0}$, where $\boldsymbol{Q}^{(1,1)}_{+}(N)$ and $\boldsymbol{Q}^{(1,1)}_{0}(N)$ are given in (4.17). We then partition $\boldsymbol{y}$ into two parts.

$$\boldsymbol{y} = \begin{pmatrix} \overset{\mathcal{J}(N)}{\boldsymbol{y}_{+}} & \overset{\mathbb{Z}_0^N \setminus \mathcal{J}(N)}{\boldsymbol{0}} \end{pmatrix},$$

where $\boldsymbol{y}+ \geq \boldsymbol{0}$ and $\boldsymbol{y}_{+}\boldsymbol{e} > 0$. We now consider a linear augmentation matrix $\boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N)$ given by

$$\boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{\boldsymbol{y}}{\boldsymbol{y}\boldsymbol{e}} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{(\boldsymbol{y}_{+} \;\; \boldsymbol{0})}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}.$$

It is clear that

$$\frac{(\boldsymbol{y}_{+} \;\; \boldsymbol{0})}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \in \Gamma^{+}(N),$$

and

$$\boldsymbol{x}[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}^{\dagger}_{\mathrm{A}}(N)] = -\boldsymbol{y} + \boldsymbol{x} \cdot \frac{(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \cdot \boldsymbol{y} = \boldsymbol{0},$$

which implies $\boldsymbol{x} \in \mathcal{T}^{+}_{\mathrm{L}}(N)$, so that $\mathcal{P}^{+}(N) \subseteq \mathcal{T}^{+}_{\mathrm{L}}(N)$.

O Proof of Theorem 5.3

Suppose that a service of length H starts at time 0. Let T_1 denote the first occurrence time of arrivals after time 0 and we define T_n $(n = 2, 3, \dots)$ as the inter-arrival time of the $(n-1)$st and nth arrivals after time 0. We also define Γ_1 as the first occurrence time of disasters after time 0. We then have

$$b^+_{\mathrm{P},(K,i)}(m_{\mathrm{P}}) = \Pr\bigl(T_1 + T_2 + \cdots + T_{m_{\mathrm{P}}-K+1} \le \min(H, \Gamma_1) \mid \bigl(L(0), S(0)\bigr) = (K, i)\bigr), \qquad i \in \mathcal{M}.$$

Associated with the Markov chain $\{(L(t), S(t))\}_{t\ge 0}$, we consider a Markov chain $\{(L^\natural(t),\, S^\natural(t))\}_{t\ge 0}$ characterized by $\{(\boldsymbol{C}^\natural_k, \boldsymbol{D}_k);\ k = 0, 1, \dots\}$, where $\boldsymbol{C}^\natural_k$ is introduced in Lemma 5.3. In what follows, we use the following convention: For any symbol X related to the original Markov chain $\{(L(t), S(t))\}_{t\ge 0}$, $X^\natural$ denotes the corresponding quantity for the Markov chain $\{(L^\natural(t), S^\natural(t))\}_{t\ge 0}$. It then follows from Lemma 5.3 (i) that

$$\begin{aligned}\boldsymbol{b}^+_{\mathrm{P},K}(m_{\mathrm{P}}) &= \int\cdots\int_{t_1+t_2+\cdots+t_{m_{\mathrm{P}}-K+1}\le y} \exp[\boldsymbol{C}_K t_1]\boldsymbol{D}_K \cdots \exp[\boldsymbol{C}_{m_{\mathrm{P}}} t_{m_{\mathrm{P}}-K+1}]\boldsymbol{D}_{m_{\mathrm{P}}}\boldsymbol{e}\mathrm{d}H(y)\\ &\le \int\cdots\int_{t_1+t_2+\cdots+t_{m_{\mathrm{P}}-K+1}\le y} \exp[\boldsymbol{C}^\natural_K t_1]\boldsymbol{D}_K \cdots \exp[\boldsymbol{C}^\natural_{m_{\mathrm{P}}} t_{m_{\mathrm{P}}-K+1}]\boldsymbol{D}_{m_{\mathrm{P}}}\boldsymbol{e}\mathrm{d}H(y),\end{aligned}$$

so that

$$b^+_{\mathrm{P},(K,i)}(m_{\mathrm{P}}) \le \Pr\bigl(T^\natural_1 + T^\natural_2 + \cdots + T^\natural_{m_{\mathrm{P}}-K+1} \le H \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (K, i)\bigr). \tag{A.30}$$

Moreover, associated with the Markov chain $\{(L^\natural(t), S^\natural(t))\}_{t\ge 0}$, we introduce a Markov chain $\{(L^*(t), S^*(t))\}_{t\ge 0}$ characterized by $\{(\boldsymbol{C}^*_k(\boldsymbol{x}_k), \boldsymbol{D}^*_k(\boldsymbol{x}_k));\ k = 0, 1, \dots\}$, where

$$\boldsymbol{C}^*_k(\boldsymbol{x}_k) = \boldsymbol{C}^\natural_k - \mathrm{diag}(\boldsymbol{x}_k), \qquad \boldsymbol{D}^*_k(\boldsymbol{x}_k) = \boldsymbol{D}_k + \mathrm{diag}(\boldsymbol{x}_k)(-\boldsymbol{C}^\natural_k)^{-1}\boldsymbol{D}_k, \quad k = 1, 2, \dots,$$

for $\boldsymbol{x}_k \ge \boldsymbol{0}$. We use the superscript * to represent quantities related to the Markov chain $\{(L^*(t), S^*(t))\}_{t\ge 0}$. It then follows from Lemma 5.3 (iii) that for $k = 1, 2, \dots$,

$$\begin{aligned}&\Pr\bigl(T^\natural_1 \le \tau, S^\natural(T^\natural_1) = j \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (k, i)\bigr)\\ &\quad= \Bigl[\int_0^\tau \exp[\boldsymbol{C}^\natural_k t]\boldsymbol{D}_k \mathrm{d}t\Bigr]_{i,j} \le \Bigl[\int_0^\tau \exp\bigl[\bigl(\boldsymbol{C}^\natural_k - \mathrm{diag}(\boldsymbol{x}_k)\bigr)t\bigr]\boldsymbol{D}^*_k(\boldsymbol{x}_k)\mathrm{d}t\Bigr]_{i,j}\\ &\quad= \Pr\bigl(T^*_1 \le \tau, S^*(T^*_1) = j \mid \bigl(L^*(0), S^*(0)\bigr) = (k, i)\bigr).\end{aligned}$$

Note here that Lemma 5.3 (ii) implies

$$\Pr\bigl(S^\natural(T^\natural_1) = j \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (k, i)\bigr) = \Pr\bigl(S^*(T^*_1) = j \mid \bigl(L^*(0), S^*(0)\bigr) = (k, i)\bigr). \tag{A.31}$$

It then follows that

$$\begin{aligned}&\Pr\bigl(T^\natural_1 \le \tau \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (k, i), S^\natural(T^\natural_1) = j\bigr)\\ &\qquad\le \Pr\bigl(T^*_1 \le \tau \mid \bigl(L^*(0), S^*(0)\bigr) = (k, i), S^*(T^*_1) = j\bigr),\end{aligned}$$

which implies that $T_1^\natural$ is larger than T_1^* in a sense of the usual stochastic order under the condition that the states of the Markov chains at time zero and the phases immediately after the next arrival are given. Let $S_n^\natural = S^\natural(a_n^\natural)$ and $S_n^* = S_n^*(a_n^*)$ $(n = 1, 2, \ldots)$, where $a_n^\natural = T_1^\natural + T_2^\natural + \cdots + T_n^\natural$ and $a_n^* = T_1^* + T_2^* + \cdots + T_n^*$. Because $T_n^\natural$'s (resp. T_n^*'s) are conditionally independent given $(L^\natural(0), S^\natural(0))$ and $S_n^\natural$'s (resp. $(L^*(0), S^*(0))$ and S_n^*'s), it follows from [Sha07, Theorem 1.A.3 (b)] that for $m = 1, 2, \ldots$,

$$\begin{aligned}&\Pr\bigl(T_1^\natural + T_2^\natural + \cdots + T_m^\natural \le x \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (K, i), S_n^\natural = j_n \ (n = 1, 2, \ldots, m)\bigr)\\ &\quad \le \Pr\bigl(T_1^* + T_2^* + \cdots + T_m^* \le x \mid \bigl(L^*(0), S^*(0)\bigr) = (K, i), S_n^* = j_n \ (n = 1, 2, \ldots, m)\bigr).\end{aligned}$$

Furthermore, it follows from (A.31) that

$$\begin{aligned}&\Pr\bigl(S_n^\natural = j_n \ (n = 1, 2, \ldots, m) \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (K, i)\bigr)\\ &\qquad = \Pr\bigl(S_n^* = j_n \ (n = 1, 2, \ldots, m) \mid \bigl(L^*(0), S^*(0)\bigr) = (K, i)\bigr).\end{aligned}$$

We thus have

$$\begin{aligned}&\Pr\bigl(T_1^\natural + T_2^\natural + \cdots + T_{m_{\mathrm{P}}-K+1}^\natural \le x \mid \bigl(L^\natural(0), S^\natural(0)\bigr) = (K, i)\bigr)\\ &\qquad \le \Pr\bigl(T_1^* + T_2^* + \cdots + T_{m_{\mathrm{P}}-K+1}^* \le x \mid \bigl(L^*(0), S^*(0)\bigr) = (K, i)\bigr).\end{aligned} \tag{A.32}$$

It then follows from (A.30) and (A.32) that

$$b_{(K,i)}^+(m_{\mathrm{P}}) \tag{A.33}$$

$$\begin{aligned}&\le \Pr\bigl(T_1^* + T_2^* + \cdots + T_{m_{\mathrm{P}}-K+1}^* \le H \mid \bigl(L^*(0), S^*(0)\bigr) = (K, i)\bigr)\\ &\le \Pr\bigl(T_{n(K,K_2^\dagger)}^* + T_{n(K,K_2^\dagger)+1}^* + \cdots + T_{m_{\mathrm{P}}-K+1}^* \le H \mid \bigl(L^*(0), S^*(0)\bigr) = (K, i)\bigr),\end{aligned} \tag{A.34}$$

where $n(K, K_2^\dagger) = \max(0, K_2^\dagger - K) + 1 \ge 1$.

We now set $\boldsymbol{x}_k = \boldsymbol{0}$ for $k = 1, 2, \ldots, K_2^\dagger - 1$ and $\boldsymbol{x}_k = \mu\boldsymbol{e} - \boldsymbol{D}_k\boldsymbol{e}$ for $k = K_2^\dagger, K_2^\dagger + 1, \ldots$. Note that $\boldsymbol{x}_k \ge \boldsymbol{0}$ $(k = 1, 2, \ldots)$ under Assumption 5.1 (ii). Furthermore, we have $\boldsymbol{D}_k^*(\boldsymbol{x}_k)\boldsymbol{e} = \mu\boldsymbol{e}$ $(k = K_2^\dagger, K_2^\dagger + 1, \ldots)$, which implies that if $L^*(t) \ge K_2^\dagger$, arrivals after time t occur at constant rate μ, regardless of $S^*(t)$. In other words, when $(L^*(0), S^*(0)) = (K, i)$, T_n^*'s $(n = n(K, K_2^\dagger), n(K, K_2^\dagger) + 1, \ldots)$ are i.i.d. exponential random variables. The theorem now follows from (A.34) and this observation.

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Publication List

A. Journal Papers (refereed)

1. M. Kimura and T. Takine, "Computing the conditional stationary distribution in Markov chains of level-dependent M/G/1-type," *Stochastic Models*, Vol.34, No.2, pp.207–238, 2018 (**the Best Paper Award of the SIG of Queueing Theory, the Operations Research Society of Japan**).

2. M. Kimura and T. Takine, "Characterization of the conditional stationary distribution in Markov chains via systems of linear inequalities," *Advances in Applied Probability*, Vol.52, No.4, pp.1249–1283, 2020.

3. M. Kimura and T. Takine, "Numerical implementation of the augmented truncation approximation to single-server queues with level-dependent arrivals and disasters," to appear in *Journal of the Operations Research Society of Japan,* Vol.64, No.3, July 2021 (26 pages).

B. International Conference (refereed)

1. M. Kimura and Y. Inoue, "The joint distribution of two successive sojourn times of tagged class customers in the multi-class FCFS M/G/1 queue," presented at the 13th International Conference on Queueing Theory and Network Applications (QTNA2018), Tsukuba, Japan, July 25–27, 2018.

C. Domestic Conferences (non-refereed, written in Japanese)

1. M. Kimura and T. Takine, "The computational algorithm for the conditional stationary distribution in Markov chains of level-dependent M/G/1-type," presented at the Queueing Symposium 2016: Stochastic Models and Their Applications, Tokyo, Japan, January 19–21, 2017 (**the Research Encouragement Award, the SIG of Queueing Theory in the Operations Research Society of Japan**).

2. M. Kimura and T. Takine, "On the computation of the conditional stationary distribution in Markov chains of level-dependent M/G/1-type with the unbounded number of phases," presented at the 2017 Autumn National Conference of the Operations Research Society of Japan, Osaka, Japan, September 14–15, 2017.

3. M. Kimura and T. Takine, "The numerical computation of the stationary distribution in Markov chains of level-dependent M/G/1-type and its application,"

presented at the 2017 Young Researchers Conference in Operations Research Kansai, Osaka, Japan, October 28, 2017 (**the Best Young Researcher Award for Presentation, the Kansai Branch of the Operations Research Society of Japan**).

4. M. Kimura and T. Takine, "The numerical computation of the conditional stationary distribution in Markov chains of level-dependent G/G/1-type with the error bounds," presented at the Queueing Symposium 2017: Stochastic Models and Their Applications, Osaka, Japan, January 17–19, 2018 (**the Research Encouragement Award, the SIG of Queueing Theory in the Operations Research Society of Japan**).

5. M. Kimura, "The numerical computation with the error bounds of the conditional stationary distribution in Markov chains on infinite state space," presented at Summer Seminar in Operations Research, Gunma, Japan, August 29–31, 2018 (**the Best Student Award for Presentation, the Headquarter of the Operations Research Society of Japan**).

6. M. Kimura, "The numerical computation and the error evaluation of the conditional stationary distribution in Markov chains with infinitely many states," presented at Summer Seminar in Operations Research Kansai, Nara, Japan, November 2–4, 2018 (**the Best Young Researcher Award for Presentation, the Kansai Branch of the Operations Research Society of Japan**).

7. M. Kimura and T. Takine, "Characterization of the conditional stationary distribution in Markov chains via systems of linear inequalities," presented at the Queueing Symposium 2018: Stochastic Models and Their Applications, Tokyo, Japan, January 23–25, 2019 (**the Research Encouragement Award, the SIG of Queueing Theory in the Operations Research Society of Japan**).

8. M. Kimura and T. Takine, "On the augmented truncation approximation to single-server queues with level-dependent disasters," presented at the Queueing Symposium 2019: Stochastic Models and Their Applications, Kanagawa, Japan, January 22–24, 2020.