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THE CENTRAL LIMIT THEOREM FOR CURRENT-VALUED PROCESSES INDUCED BY GEODESIC FLOWS

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0. Introduction

In this article, we study the central limit theorem for current-valued processes induced by the geodesic flows over a compact Riemannian manifold. Recently several works concerning the central limit theorems for current-valued processes have been done. N. Ikeda [7] (see also Ikeda-Ochi [8]) discussed several limit theorems for a class of current-valued processes induced by various stochastic processes. In this direction, Ochi [13] proved the central limit theorem for a current-valued process induced by diffusion process on manifold. In this paper, we establish a similar result for geodesic flows over compact Riemannian manifolds of negative curvature. Since the first half of our results can be treated for transitive Anosov K-flows, we formulate the problem in the framework of Anosov flows. Let M be a compact, connected Riemannian manifold. Let $\{T^t\}$ be a transitive Anosov K-flow on M. By using Markov partition, we can construct a special representation $\{S^t\}$ of $\{T^t\}$. For invariant measure μ of $\{T^t\}$, we consider one which is constructed from invariant measure $\tilde{\nu}$ of $\{S^t\}$ (see Section 1). Let $\Lambda_1(M)$ be the space of all smooth 1-forms on M. For $\alpha\in\Lambda_1(M)$, we consider the following line integral

$$Y_t(\alpha) = Y_t(\alpha; \xi) = \int_{T([0,t],\xi)} \alpha\,, \qquad t\geq 0,\ \xi\in M$$

where $T([0, t], \xi)=\{T^s\xi;\ 0\leq s\leq t\}$. We consider $\{Y_t\}$ as a random process on the probability space (M, μ). Then we can regard $\{Y_t\}$ is a $\Lambda_1(M)'$-valued process in the sense of K.Ito ([10]). If we denote by X the vector field on M which generates the flow $\{T^t\}$, $Y_t(\alpha)$ can be expressed as follows:

$$Y_t(\alpha; \xi) = \int_0^t \langle\alpha, X\rangle(T^s\xi)ds\,.$$

We consider a family of $\Lambda_1(M)'$-valued processes $Y^{(\lambda)}=\{Y_t^{(\lambda)}\}$ defined by

$$Y_t^{(\lambda)}(\alpha; \xi) = \frac{1}{\sqrt{\lambda}}(Y_{\lambda t}(\alpha; \xi)-\lambda t\overline{f[\alpha]})\,.$$

We set $\tilde{f}[\alpha](\xi)=f[\alpha](\xi)-\overline{f[\alpha]}$, where $f[\alpha](\xi)=\langle\alpha, X\rangle(\xi)$ and $\overline{f[\alpha]}=\int_M f[\alpha](\xi)\mu(d\xi)$.

The first result of this paper is the following theorem.

Theorem 1. *The family of $\Lambda_1(M)'$-valued processes $\{Y_t^{(\lambda)}(\alpha)\}$ converges to a $\Lambda_1(M)'$-valued Wiener process with mean functional* 0 *and covariance functional $(t\wedge s)\rho(\alpha, \beta)$. The continuous bilinear functional $\rho(\alpha, \beta)$ is determined by*

$$\lim_{\lambda\to\infty}\int_M Y_t^{(\lambda)}(\alpha)Y_t^{(\lambda)}(\beta)d\mu(\xi)=t\rho(\alpha, \beta)\,, \qquad \alpha, \beta\in\Lambda_1(M)$$

(*See Proposition* 2.1. *for this formula and for continuity of ρ*).

Our next concern is the nondegeneracy of ρ. We confine ourselves to the case that the flow $\{T^t\}$ is the geodesic flow $\{G^t\}$ over a d-dimensional compact, connected Riemannian manifold V of negative curvature. In this case, M is the unit tangent bundle of V: $M=\{\xi=(x, v)\,|\,x\in V,\ v\in T_xV,\ \|v\|=1\}$. We consider $\{Y_t\}$ the $\Lambda_1(V)'$-valued process. From the definition, ρ is a nonnegative definite bilinear functional on $\Lambda_1(V)$, but may be degenerate. In fact, for any exact form $\alpha=dh$, we have $\rho(dh, dh)=0$. Therefore it is an interesting problem to show that ρ is nondegenerate on $\Lambda_1(V)$-{exact forms}.

Before stating our result, we mention several results concerning the nondegeneracy. In the case of usual central limit theorems, there are few researches which proved the nondegeneracy of the limit distribution. Among them, in the paper concerning the homological position of geodesic flows, Gelfand-Pyateckii-Shapiro [5] considered the integral $\int_{G([0,t],\xi)}\alpha$ (α: harmonic form) for the geodesic flow over the Riemann surface of constant negative curvature. They showed $\rho(\alpha, \alpha)>0$ by using the theory of unitary representations. Sinai [16] gave a condition for nondegeneracy of limit distribution, but it seems difficult to verify this condition.

We consider the nondegeneracy problem under more general situations. Recall that V is called a-pinched if for any $x\in V$, there exists a positive constant A such that

$$\left|\frac{K}{A}+1\right|>a\,,$$

where K is the sectional curvature of V.
We note that if V has constant negative curvature, then V is $1/d$-pinched.

Our result is the following

Theorem 2. *Let V be $1/d$-pinched. Then ρ is nondegenerate on $\Lambda_1(V)$-{exact forms}.*

From this we see that the limit process in Theorem 1 is in fact infinite-dimensional.

If we drop the pinching condition, we do not succeed yet to prove the statement in Theorem 2. But we can show the following proposition. Before stating the proposition, we prepare notations. Let $E^s(E^u)$ be the stable (unstable) subbundle and Z be the one-dimensional subbundle along the trajectories of $\{G^t\}$:

$$TM = E^s + Z + E^u .$$

For $\alpha \in \Lambda_1(V)$, we define a one form $\tilde{\alpha}$ on M (see Guillemin-Kazhdan [6. Appendix]) by

$$\langle \tilde{\alpha}_\xi, X \rangle = f[\alpha](\xi) ,$$

$$\langle \tilde{\alpha}_\xi, X^+ \rangle = -\int_0^\infty (G^t_* X^+)_\xi (f[\alpha]) dt ,$$

$$\langle \tilde{\alpha}_\xi, X^- \rangle = \int_\infty^0 (G^t_* X^-)_\xi (f[\alpha]) dt ,$$

where $\xi \in M$.

Proposition 1. *If $\rho(\alpha, \alpha)=0$, then $\tilde{\alpha}$ is exact.*

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1. Preliminaries

In this section, we collect several facts concerning Markov partitions and symbolic representations of geodesic flow, which will be used later sections. It is well-known that any transitive Anosov flow admits a Markov partition. This partition determines a matrix $\pi=(\pi_{ij})_{1\le i,\, j\le r}$, $\pi_{ij}=0$, 1 of order r, such that for some integer $m>0$, the elements of the matrix π^m are positive. Using this matrix, one can construct the space $\Omega=\Omega_\pi \subset \{1, 2, \cdots, r\}^Z$ of sequences $\omega=\{\omega_i\}_{i=-\infty}^{\infty}$, $\pi_{\omega_i,\omega_{i+1}}=1$. The metric d of Ω is defined by $d(\omega, \omega')=\sum_{i=-\infty}^{\infty} 2^{-|i|}(1-\delta_{\omega_i,\omega_i'})$, where $\delta_{a,b}$ is the Kronecker delta. Let ϕ be the shift on Ω: $(\phi\omega)_i=\omega_{i+1}$, for any i. One can define a special flow $\{S^t\}$ acting in the space $\tilde{\Omega}=\{(\omega, \tau);\ \omega\in\Omega,\ 0\le\tau<l(\omega),\ (\omega, l(\omega))=(\phi\omega, 0)\}$, where l is a Hölder continuous positive function on Ω. The special flow $S^t(\omega, \tau)$ is defined as follows:

$$(1.1)\quad S^t(\omega, \tau) = \begin{cases} (\omega, \tau+t), & -\tau \leq t < -\tau + l(\omega), \\ (\phi^n\omega, \tau+t-l_n(\omega)), & -\tau+l_n(\omega) \leq t < -\tau+l_{n+1}(\omega),\ n \geq 1, \\ (\phi^{-n}\omega, \tau+t+l_{-n}(\omega)), & -\tau-l_{-n}(\omega) \leq t < -\tau-l_{-n+1}(\omega),\ n \geq 1, \end{cases}$$

where

$$(1.2)\qquad l_n(\omega) = \begin{cases} \sum_{k=0}^{n-1} l(\phi^k\omega), & n > 0, \\ 0, & n = 0, \\ \sum_{k=n}^{-1} l(\phi^k\omega), & n < 0. \end{cases}$$

It is known that there exists a continuous mapping $\psi: \tilde{\Omega} \to M$ such that $\psi S^t = T^t\psi$. For an $\{S^t\}$-invariant measure $\tilde{\nu}$ on $\tilde{\Omega}$, we define a probability measure μ on M by $\mu(A) = \tilde{\nu}(\psi^{-1}A)$. The flow $\{S^t\}$ in $(\tilde{\Omega}, \tilde{\nu})$ is isomorphic to $\{T^t\}$ in (M, μ). The measure $\tilde{\nu}$ on $\tilde{\Omega}$ induces a ϕ-invariant measure ν on Ω such that $d\tilde{\nu} = (d\nu \times dt)\frac{1}{\langle l \rangle}$, where $\langle l \rangle = \int_\Omega l(\omega)d\nu(\omega)$. (Ω, ν, ϕ) has the following mixing property (see Bowen [2]): There exist positive constants C_1, C_2 such that

$$(1.3)\qquad \sup_{A \in \mathfrak{M}_{-\infty}^a,\ B \in \mathfrak{M}_{a+b}^\infty} |\nu(A \cap B) - \nu(A)\nu(B)| \leq C_1\nu(A)\nu(B)e^{-C_2 b},$$

where $\mathfrak{M}_a^b$ is the σ-field generated by ω_i, $a \leq i \leq b$.

Throughout this paper, we shall use the following notations. For $f \in C^\infty(M)$, we set $f^*(\tilde{\omega}) = f \circ \psi(\tilde{\omega})$ and $F(\omega) = \int_0^{l(\omega)} f^*(S^u(\omega, 0))du$.

2. Convergence of finite dimensional distributions

In this section, we show the convergence of finite dimensional distributions of $\{Y_t\}$. Throughout sections 2 and 3, we always assume that $\{T^t\}$ is a transitive Anosov flow. It is known (Denker-Philipp [3]) that the following limit exists:

$$(2.1)\qquad \rho(\alpha) = \lim_{\lambda \to \infty} \frac{1}{\lambda} \int_M d\mu(\xi) \left(\int_0^\lambda \tilde{f}[\alpha](T^s\xi)ds \right)^2.$$

We define $\rho(\alpha, \beta)$ by

$$(2.2)\qquad \rho(\alpha, \beta) = \frac{1}{4}[\rho(\alpha+\beta) - \rho(\alpha-\beta)].$$

With this notation, we can show the following

Proposition 2.1. *Let n be an arbitrary positive integer. For any $t_1 < \cdots < t_n$ and any elements $\alpha^{(1)}, \cdots, \alpha^{(n)}$ of Λ_1, the n-dimensional random variable*

$(Y_{t_1}^{(\lambda)}(\alpha^{(1)}),\cdots,Y_{t_n}^{(\lambda)}(\alpha^{(n)}))$ converges as $\lambda\to\infty$ to n-dimensional Gaussian distribution whose mean is 0 and covariance matrix (possibly degenerate) is $((t_j\wedge t_k)\rho(\alpha^{(j)},\alpha^{(k)}))$.

Proof. Let $\varphi_{t_1\cdots t_n}^{(\lambda)}$ be the characteristic function of $(Y_{t_1}^{(\lambda)}(\alpha^{(1)}),\cdots,Y_{t_n}^{(\lambda)}(\alpha^{(n)}))$:

$$\varphi_{t_1\cdots t_n}^{(\lambda)}(z_1,\cdots,z_n)=\int_M d\mu(\xi)\exp\,[i\sum_{j=1}^n z_jY_{t_j}^{(\lambda)}(\alpha^{(j)})]\,,$$

where $z_1,\cdots,z_n\in\boldsymbol{C}$. We want to show that

$$\lim_{\lambda\to\infty}\varphi_{t_1\cdots t_n}^{(\lambda)}(z_1,\cdots,z_n)=\exp\left[-\frac{1}{2}(t_j\wedge t_k)\rho(\alpha^{(j)},\alpha^{(k)})z_jz_k\right]. \tag{2.3}$$

We first remark that if $t_1=\cdots=t_n$, then by noting the linearity of $Y_t^{(\lambda)}$: $\sum_{j=1}^n z_jY_1(\alpha^{(j)})=Y_t(\sum_{j=1}^n z_j\alpha^{(j)})$, (2.3) follows from Theorem 1 of Sinai [17]. In the following, we consider only the case $n=2$, since the general case can be treated similarly. By the linearity, we can write

$$\begin{aligned}&\int_M d\mu(\xi)\exp\,[iz_1Y_{t_1}^{(\lambda)}(\alpha^{(1)})+iz_2Y_{t_2}^{(\lambda)}(\alpha^{(2)})]\\&=\int_M d\mu(\xi)\exp\left[\frac{i}{\sqrt{\lambda}}\int_0^{\lambda t_1}\tilde{f}[z_1\alpha^{(1)}+z_2\alpha^{(2)}](T^s\xi)ds\right.\\&\qquad\left.+\frac{i}{\sqrt{\lambda}}\int_{\lambda t_1}^{\lambda t_2}\tilde{f}[z_2\alpha^{(2)}](T^s\xi)ds\right].\end{aligned}$$

Define $\tilde{f}_1(\xi)=\tilde{f}[z_1\alpha^{(1)}+z_2\alpha^{(2)}](\xi)$, $\tilde{f}_2(\xi)=\tilde{f}[z_2\alpha^{(2)}](\xi)$ and $\tilde{f}_j^*(\tilde{\omega})=\tilde{f}_j(\psi(\tilde{\omega}))$, $j=1,2$. Then we have

$$\begin{aligned}&\int_M d\mu(\xi)\exp\,[iz_1Y_{t_1}^{(\lambda)}(\alpha^{(1)})+iz_2\,Y_{t_2}^{(\lambda)}(\alpha^{(2)})]\\&=\int_{\tilde{\Omega}}d\tilde{\nu}(\tilde{\omega})\exp\left[\frac{i}{\sqrt{\lambda}}\int_0^{\lambda t_1}\tilde{f}_1^*(S^u\tilde{\omega})\,du+\frac{i}{\sqrt{\lambda}}\int_{\lambda t_1}^{\lambda t_2}\tilde{f}_2^*(S^u\tilde{\omega})du\right].\end{aligned}$$

Using the following inequality

$$\sup_{t,\omega}\sup_{0\le\tau\le l(\omega)}\left|\int_0^t\tilde{f}^*(S^u(\omega,\tau))-\int_0^t\tilde{f}^*(S^u(\omega,0))du\right|\le2\|f\|_\infty\|l\|_\infty\,, \tag{2.4}$$

we have only to consider the integral

$$I(\lambda)=\int_{\tilde{\Omega}}d\tilde{\nu}(\tilde{\omega})\exp\left[\frac{i}{\sqrt{\lambda}}\int_0^{\lambda t_1}\tilde{f}_1^*(S^u(\omega,0))\,du+\frac{i}{\sqrt{\lambda}}\int_{\lambda t_1}^{\lambda t_2}\tilde{f}_2^*(S^u(\omega,0))\,du\right].$$

which can be written as follows

$$\frac{1}{\langle l\rangle}\int_\Omega d\nu(\omega)\,l(\omega)\exp\left[\frac{i}{\sqrt{\lambda}}\int_0^{\lambda t_1}\tilde{f}_1^*(S^u(\omega,0))\,du+\frac{i}{\sqrt{\lambda}}\int_{\lambda t_1}^{\lambda t_2}\tilde{f}_2^*(S^u(\omega,0))du\right].$$

Let $N(t, \omega)$ be the integer N such that

$$\sum_{j=0}^{N} l(\phi^j\omega) < t \leq \sum_{j=0}^{N+1} l(\phi^j \omega) . \tag{2.5}$$

Since

$$\int_0^{\lambda t_j} \tilde{f}_j^*(S^u(\omega, 0))du = \sum_{j=0}^{N(\lambda t_j^-,\omega)} F_j(\phi^j \omega) + \int_0^{\lambda_j t - l_{N(\lambda t_j,\omega)+1}(\omega)} \tilde{f}_j^*(\phi^{N(\lambda t_j,\omega)}\omega, u)\, du ,$$

$j=1, 2$, using the following inequality ([3], Lemma 2),

$$\nu(\sup | \int_0^t f^*(S^u(\omega, \tau))du - \sum_{j=0}^{N(t,\omega)} F(\phi^j\omega) | \leq C_3) = 1 , \tag{2.6}$$

we have

$$I(\lambda) = \frac{1}{\langle l \rangle} \int_\Omega d\nu(\omega) l(\omega) \exp \left[\frac{i}{\sqrt{\lambda}} \sum_{j=0}^{N(\lambda t_1,\omega)} F_1(\phi^j\omega) \right.$$
$$\left. + \frac{i}{\sqrt{\lambda}} \sum_{j=N(\lambda t_1,\omega)+1}^{N(\lambda t_2,\omega)} F_2(\phi^j\omega) + o(1) \right] .$$

Denote by A_1, A_2 the differences

$$A_1 = \frac{1}{\sqrt{\lambda}} \Big(\sum_{j=0}^{N(\lambda t_1,\omega)} F_1(\phi^j\omega) - \sum_{j=0}^{[\lambda t_1/\langle l\rangle]} F_1(\phi^j\omega) \Big)$$

$$A_2 = \frac{1}{\sqrt{\lambda}} \Big(\sum_{j=N(\lambda t_1,\omega)+1}^{N(\lambda t_2,\omega)} F_2(\phi^j\omega) - \sum_{j=[\lambda t_1/\langle l\rangle]+1}^{[\lambda t_2/\langle l\rangle]} F_2(\phi^j\omega) \Big) .$$

We have $A_j \to 0$ in probability as $\lambda \to \infty$, $j=1, 2$. In fact, for any $\varepsilon > 0$, we have

$$\begin{aligned} \nu(|A_1|>\varepsilon) &= \nu(|A_1>|\varepsilon, \ |\sum_{j=1}^{[\lambda t_1/\langle l\rangle]} l(\phi^j\omega) - \langle l\rangle[\lambda t_1/\langle l\rangle]| \leq \lambda^{2/3}) \\ &\quad + \nu(|A_1|>\varepsilon, \ |''| > \lambda^{2/3}) \\ &\leq \nu(|A_1|>\varepsilon, \ |N(\lambda t_1, \omega) - [\lambda t_1/\langle l\rangle]| \leq \text{const. } \lambda^{2/3}) \\ &\quad + \nu(|\sum_{j=0}^{[\lambda t_1/\langle l\rangle]} l(\phi^j\omega) - \langle l\rangle[\lambda t_1/\langle l\rangle]| > \lambda^{2/3}) . \end{aligned}$$

The second term tends to zero as $\lambda \to \infty$. The first term can be estimated as follows.

$$\begin{aligned} &\nu(|A_1|>\varepsilon, \ |N(t\lambda_1, \omega) - [t\lambda_1/\langle l\rangle]| \leq \text{const. } \lambda^{2/3}) \\ &\leq \nu(\sup_{|h-[\lambda t_1/\langle l\rangle]| \leq \text{const. } \lambda^{2/3}} | \sum_{j=[\lambda t_1/\langle l\rangle]}^{h} F_1(\phi^j\omega) | > \varepsilon\sqrt{\lambda}) \\ &\leq \frac{\text{const.}}{\varepsilon^4\lambda^2} \lambda^{4/3} \to 0 \quad \text{as} \quad \lambda \to \infty . \end{aligned}$$

Here we used the stationarity and the following fact which follows from Lemma 3.2 of Ratner [14] and Theorem B of Serfling [15].

$$\int \max_{m\leq n} |\sum_{i=0}^{m} F(\phi^i\omega)|^4 d\nu \leq C_4 \|F\|_\infty^4 n^2, \tag{2.7}$$

for any Hölder continuous function on Ω with $\int_\Omega F d\nu = 0$. The statement for A_2 can be proven entirely in the same way. Thus we have only to estimate

$$I_1(\lambda) = \frac{1}{\langle l\rangle}\int_\Omega d\nu(\omega) l(\omega) \exp\left[\frac{i}{\sqrt{\lambda}} \sum_{j=0}^{[\lambda t_1/\langle l\rangle]} F_1(\phi^j\omega) + \frac{i}{\sqrt{\lambda}} \sum_{j=[\lambda t_1/\langle l\rangle]+[(\lambda t_1/\langle l\rangle)^a]}^{[\lambda t_2/\langle l\rangle]} F_2(\phi^j\omega)\right],$$

where $0<a<1/2$. Approximating F_j by an $\mathfrak{M}_{-[\lambda t_1/\langle l\rangle]}^{[\lambda t_1/\langle l\rangle]}$-measurable function $F_{j,[\lambda t_1/\langle l\rangle]}$ ($j=1, 2$) and using the inequality

$$|\int HK d\nu - \int H d\nu \int K d\nu| \leq g(n) \|H\|_p \|K\|_q, \qquad (g(n)\to 0, \text{ as } n\to\infty)$$

for H: $\mathfrak{M}_{-\infty}^{a}$-measurable and K: $\mathfrak{M}_{a+n}^{\infty}$-measurable $\left(H\in L^p, K\in L^q, \frac{1}{p}+\frac{1}{q}=1\right)$, we have

$$I_1(\lambda) = \frac{1}{\langle l\rangle}\int_\Omega d\nu(\omega) l_{[\lambda t_1/\langle l\rangle]}(\omega) \exp\left[\frac{1}{\sqrt{\lambda}} \sum_{j=0}^{[\lambda t_1/\langle l\rangle]} F_{1,[\lambda t_1/\langle l\rangle]}(\phi^j\omega)\right]$$
$$\times \int_\Omega d\nu(\omega) \exp\left[\frac{1}{\sqrt{\lambda}} \sum_{j=[\lambda t_1/\langle l\rangle]+[(\lambda t_1/\langle l\rangle)^a]}^{[\lambda t_2/\langle l\rangle]} F_{2,[\lambda t_1/\langle l\rangle]}(\phi^j\omega)\right.$$
$$+ \text{negligible term}.$$

Now by the central limit theorem, we have

$$I_1(\lambda) \to \exp\left[-\frac{1}{2}\rho(z_1\alpha^{(1)}+z_2\alpha^{(2)})t_1\right] \exp\left[-\frac{1}{2}\rho(z_2\alpha^{(2)})(t_2-t_1)\right]$$
$$= \exp\left[-\frac{1}{2}\sum_{j,k=1}^{2} (t_j\wedge t_k)\rho(\alpha^{(j)}, \alpha^{(k)}) z_j z_k\right],$$

which proves Proposition 2.1 for $n=2$.

3. Tightness of $\{Y^{(\lambda)}\}$

In this section, we shall show that the tightness of $\{Y_t^{(\lambda)}\}$. Recall the topology of Λ_1 and Λ_1' (see Ochi [13]). Let $\{U_n\}_{n=1}^{n_0}$ be a finite open covering of M satisfying

(i) for each n, U_n is a coordinate neighborhood,

(ii) for each n, U_n is homeomorphic to an open cube $I_n \subset \boldsymbol{R}^d$. Let $\{\varphi_n\}_{n=1}^{n_0}$ be a partition of unity subordinate to $\{U_n\}_{n=1}^{n_0}$. We define $||\alpha||_p$ by

$$(3.1) \qquad ||\alpha||_p^2 = \sum_{n=1}^{n_0} \sum_{i=1}^{d} ||(-\Delta)^p(\varphi_n \alpha_i)(x)||_{L^2(I_n)}^2 ,$$

where Δ is the Laplacian on $\boldsymbol{R}^d$ and $\alpha = \sum_{i=1}^{d} \alpha_i(x) dx^i$ on U_n. The topology determined by $\{|| \ ||_p\}_{p \in \boldsymbol{R}}$ is consistent with Schwartz topology and for $p, q \in \boldsymbol{R}$ with $q > p + \frac{d}{4}$, $|| \ ||_p <_{HS} || \ ||_q$, that is, for an orthonormal basis $\{e_k\}_{k \in N}$ on $(\Lambda_1, || \ ||_q)$, it holds that

$$\sum_{k=1}^{\infty} ||e_k||_p^2 < \infty .$$

We denote by $(\Lambda_1)_p$ the completion of $\Lambda_1/\mathrm{Ker}|| \ ||_p$. We use the same notation $|| \ ||_p$ for the induced norm on $(\Lambda_1)_p$. Let $((\Lambda_1)_p', || \ ||_p')$ be the dual space of $((\Lambda_1)_p, || \ ||_p)$, where $||A||_p' = \sup_{\alpha \in (\Lambda_1)_p, ||\alpha||_p = 1} |A(\alpha)|$.

Proposition 3.1. *The family $\{Y^{(\lambda)}\}$ is tight in the space $C([0, \infty) \to \Lambda_1')$.*

Proof. It is sufficient to show that there exists a sufficiently large $p > 0$ for which the following holds:
For any $\eta > 0$, $\varepsilon > 0$, there exists a δ with $0 < \delta < 1$ and a $\lambda_0 > 0$ such that for $\lambda > \lambda_0$

$$\frac{1}{\delta} \mu(\{\xi \in M; \sup_{s \leq t \leq s+\delta} ||Y_t^{(\lambda)} - Y_s^{(\lambda)}||_p' > \varepsilon\}) < \eta , \qquad \text{for all} \quad s \in [0, 1] .$$

Let us consider a partition of $[0, 1]$:

$$0 = t_0 < t_1 < \cdots < t_N = 1 ,$$

such that

$$\frac{C_1}{2} \lambda^{-7/12} < t_j - t_{j-1} \leq C_2 \lambda^{-7/12} , \qquad \text{for} \quad j = 1, \cdots, N .$$

Set $\tilde{f}^*[\alpha](\omega, \tau) = \tilde{f}[\alpha] \circ \psi(\omega, \tau)$, then $Y_t^{(\lambda)}(\alpha)$ can be written as

$$Y_t^{(\lambda)}(\alpha) = \frac{1}{\sqrt{\lambda}} \int_0^{\lambda t} \tilde{f}^*[\alpha](S^u(\omega, \tau))\, du.$$

By the triangular inequality,

$$||Y_t^{(\lambda)} - Y_s^{(\lambda)}||_p' \leq ||Y_t^{(\lambda)} - Y_{t_k}^{(\lambda)}||_p' + ||Y_{t_k}^{(\lambda)} - Y_{t_j}^{(\lambda)}||_p' + ||Y_{t_j}^{(\lambda)} - Y_s^{(\lambda)}||_p' ,$$

where t_k is the nearest point of t with $t_k \leq t$ and t_j is the nearest point of s with $t_j \geq s$. Using Sobolev's lemma, we have

$$||Y_t^{(\lambda)}-Y_{t_k}^{(\lambda)}||_p' \leq \sup_{\alpha\in\Lambda_1, ||\alpha||_p=1} [||\tilde{f}[\alpha]||_\infty \lambda^{-1/12}] \leq C_0\lambda^{-1/12},$$

where

$$\sup|\tilde{f}^*[\alpha]| \leq ||\tilde{f}[\alpha]||_\infty \leq C_0||\alpha||_p .$$

Similarly,

$$||Y_t^{(\lambda)}-Y_s^{(\lambda)}||_p' \leq C_0\lambda^{-1/12} .$$

Therefore it suffices to estimate

$$\mu(\{\xi\in M; \sup_{k\,:\,t_j\leq t_k\leq t_j+\delta} ||Y_{t_j}^{(\lambda)}-Y_{t_k}^{(\lambda)}||_p' > \varepsilon\}) .$$

We write

$$F_\alpha(\omega) = \int_0^{l(\omega)} \tilde{f}^*[\alpha](S^u(\omega, 0))du , \quad \tau_j = [\lambda t_j/\langle l\rangle] .$$

Set

$$Z_{\alpha,j}^{(\lambda)}(\omega) = Y_{t_j}^{(\lambda)}(\alpha) - \frac{1}{\sqrt{\lambda}}\sum_{i=0}^{\tau_j-1} F_\alpha(\phi^i\omega) .$$

Then we have

$$|Y_{t_k}^{(\lambda)}(\omega)-Y_{t_j}^{(\lambda)}(\alpha)| \leq \frac{1}{\sqrt{\lambda}}|\sum_{i=\tau_j}^{\tau_k-1} F_\alpha(\phi^i\omega)| + |Z_{\alpha,j}^{(\lambda)}(\omega)| + |Z_{\alpha,k}^{(\lambda)}(\omega)| .$$

Let $\{\alpha^{(m)}\}_{m=1}^\infty$ be an orthonormal basis of $(\Lambda_1)_p$, then

$$\{||Y_{t_k}^{(\lambda)}-Y_{t_j}^{(\lambda)}||_p'\}^2 \leq \sum_{m=1}^\infty |Y_{t_k}^{(\lambda)}(\alpha^{(m)})-Y_{t_j}^{(\lambda)}(\alpha^{(m)})|^2$$

$$\leq 3\sum_{m=1}^\infty \left\{\frac{1}{\sqrt{\lambda}}|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega)|\right\}^2 + 3\sum_{m=1}^\infty |Z_{m,k}^{(\lambda)}(\tilde{\omega})|^2 + 3\sum_{m=1}^\infty |Z_{m,j}^{(\lambda)}(\tilde{\omega})|^2,$$

where we set $F_m=F_{\alpha^{(m)}}$ and $Z_{m,j}^{(\lambda)}=Z_{\alpha^{(m)},j}^{(\lambda)}$. Thus

$$\mu(\xi\in M; \sup_k ||Y_{t_k}^{(\lambda)}-Y_{t_j}^{(\lambda)}||_p' > \varepsilon)$$

$$\leq \nu\left(\omega\in\Omega; \sup_k \sum_{m=1}^\infty \left\{\frac{1}{\sqrt{\lambda}}|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega)|\right\}^2 > \frac{\varepsilon^2}{9}\right)$$

$$+2\tilde{\nu}\left(\tilde{\omega}\in\tilde{\Omega}; \sup_k \sum_{m=1}^\infty |Z_{m,k}^{(\lambda)}(\tilde{\omega})|^2 > \frac{\varepsilon^2}{9}\right)$$

$$\leq \nu\left(\omega\in\Omega; \sum_{m=1}^\infty \sup_k \left\{\frac{1}{\sqrt{\lambda}}|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega)|\right\}^2 > \frac{\varepsilon^2}{9}\right)$$

$$+2\tilde{\nu}\left(\tilde{\omega}\in\tilde{\Omega}; \sup_k \sum_{m=1}^\infty |Z_{m,k}^{(\lambda)}(\tilde{\omega})|^2 > \frac{\varepsilon^2}{9}\right)$$

$$= \nu(\Omega_1) + 2\tilde{\nu}\left(\tilde{\omega}\in\tilde{\Omega};\ \sup_k \sum_{m=1}^{\infty} |Z^{(\lambda)}_{m,k}(\tilde{\omega})|^2 > \frac{\varepsilon^2}{9}\right)$$
$$= I + II .$$

The term I can be estimated by Chebyshev's inequality.

$$I \le \frac{81}{\varepsilon^4}\int_{\Omega_1} d\nu \left[\sum_{m=1}^{\infty} \sup_k \left\{\frac{1}{\sqrt{\lambda}} \left|\sum_{i=\tau_j}^{\tau_k - i} F_m(\phi^i\omega)\right|\right\}^2\right]^2$$
$$\le \frac{81}{\varepsilon^4}\int d\nu \sum_{m,n} \sup_k \left\{\frac{1}{\sqrt{\lambda}} \left|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega)\right|\right\}^2 \sup_k \left\{\frac{1}{\sqrt{\lambda}} \left|\sum_{i=\tau_j}^{\tau_k-1} F_n(\phi^i\omega)\right|\right\}^2$$
$$\le \frac{81}{\varepsilon^4}\sum_{m,n}\left(\int d\nu \sup_k \left\{\frac{1}{\sqrt{\lambda}} \left|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega\right|\right\}^4\right)^{1/2}$$
$$\times\left(\int d\nu \sup_k \left\{\frac{1}{\sqrt{\lambda}} \left|\sum_{i=\tau_j}^{\tau_k-1} F_n(\phi^i\omega)\right|\right\}^4\right)^{1/2} .$$

To estimate the last sum, we need the following fact:

There exists a sequence $\{a_m\}$ of positive numbers such that

$$\int d\nu \sup_k \left\{\left|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega)\right|^4\right\} \le a_m \left(\frac{\lambda\delta}{\langle l\rangle}\right)^2 \tag{3.2a}$$

and

$$\sum_{m=1}^{\infty} \sqrt{a_m} < \infty . \tag{3.2b}$$

This inequality can be proven as follows. Using stationarity, we have

$$\int d\nu \sup_k \left|\sum_{i=\tau_j}^{\tau_k-1} F_m(\phi^i\omega)\right|^4 = \int d\nu \sup_{0\le\tau\le[\lambda\delta/\langle l\rangle]} \left|\sum_{i=0}^{\tau-1} F_m(\phi^i\omega)\right|^4 .$$

By (2.7),

$$\int d\nu \sup_{0\le\tau\le[\lambda\delta/\langle l\rangle]} \left|\sum_{i=0}^{\tau-1} F_m(\phi^i\omega)\right|^4 \le C_4 \|F_m\|_\infty^4 \left(\frac{\lambda\delta}{\langle l\rangle}\right)^2 .$$

Since $\|F_m\|_\infty \le C_5 \sup_{j=1,\cdots,n_0} \sup_{x\in\bar{U}_j} |\varphi_j \alpha_i^{(m)}(x)| \le C_0 \|\alpha^{(m)}\|_q$, for some $q+\frac{1}{4}<p$, we have

$$\sum_m \|F_m\|_\infty^2 \le C_6^2 \sum_m \|\alpha^{(m)}\|_q^2 < \infty , \quad \text{which proves (3.2).}$$

By virtue of (3.2), we now obtain

$$I \le \frac{81}{\varepsilon^4}\sum_{m,n} \sqrt{a_m\left(\frac{\delta}{\langle l\rangle}\right)^2}\sqrt{a_n\left(\frac{\delta}{\langle l\rangle}\right)^2} = \frac{81}{\varepsilon^4}\left(\frac{\delta}{\langle l\rangle}\right)^2 \left(\sum_m \sqrt{a_m}\right)^2$$

We therefore obtain $I \leqq \delta\eta/3$, if we choose δ so small that

$$(\sum_m \sqrt{a_m}/\langle l\rangle)^2 \delta\varepsilon^{-4} \leq \eta/3 .$$

Next we estimate *II*.

$$Z_{m,k}^{(\lambda)}(\tilde\omega) = \frac{1}{\sqrt{\lambda}}\int_0^{\lambda t_k} \tilde f_m^*(S^u\tilde\omega)du - \frac{1}{\sqrt{\lambda}}\int_0^{\lambda t_k}\tilde f_m^*(S^u(\omega, 0))du$$

$$+\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t_k}\tilde f_m^*(S^u(\tilde\omega, 0))du - \frac{1}{\sqrt{\lambda}}\sum_{i=0}^{\tau_k-1} F(\phi^i\omega) = z_1(m)+z_2(m) ,$$

where $\tilde\omega=(\omega, \tau)$. By (2.4), we have

$$|z_1(m)| < C_7/\sqrt{\lambda} .$$

To estimate $z_2(m)$, we divide $z_2(m)$ into two terms.

$$z_2(m) = \Big(\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t_k}\tilde f_m^*(S^u(\omega, 0))du - \frac{1}{\sqrt{\lambda}}\int_0^{l_{N(\lambda t_k,\omega)}}\tilde f_m^*(S^u(\omega, 0))du\Big)$$

$$+\Big(\frac{1}{\sqrt{\lambda}}\int_0^{l_{N(\lambda t_k,\omega)}}\tilde f_m^*(S^u(\omega, 0))du - \frac{1}{\sqrt{\lambda}}\sum_{i=0}^{\tau_k-1} F_m(\phi^i\omega)\Big)$$

$$= \frac{1}{\sqrt{\lambda}}J_1^{(m)} + \frac{1}{\sqrt{\lambda}}J_2^{(m)} .$$

First we estimate $\frac{1}{\sqrt{\lambda}}J_1^{(m)}$. Noting that $\|\tilde f_m^*\|_\infty^2 \leqq \text{const.}\ \|\alpha^{(m)}\|_q^2$, for some $q+\frac{1}{4}<p$, we have $\sum_m \|\tilde f_m^*\|_\infty^2 \leq \text{const.} \sum \|\alpha^{(m)}\|_q^2 < \infty$. Using Chebyshev's inequality, we have

$$\tilde\nu\Big(\sup_{k\,;\,t_j\leq t_k\leq t_j+\delta}\sum_m |J_1^{(m)}|^2 > \frac{\varepsilon^2}{144}\lambda\Big) \leq \sum_{k=1}^{[\lambda^{7/12}]} \tilde\nu\Big(\sum_m |J_1^{(m)}|^2 > \frac{\varepsilon^2}{144}\lambda\Big)$$

$$\leq C_8 \frac{\lambda^{7/12}}{\varepsilon^2\lambda} = C_8\frac{1}{\varepsilon^2\lambda^{5/12}} ,$$

where we used the fact that the number of k which satisfies the requirement does not exceed $\delta\lambda^{7/12} < \lambda^{7/12}$ for $\delta < \frac{1}{2}$. For the estimate of $\frac{1}{\sqrt{\lambda}}J_2^{(m)}$, note that $J_2^{(m)}$ is of the form

$$J_2^{(m)} = \sum_{i=0}^{N(\lambda t_k,\omega)-1} F_m(\phi^i\omega) - \sum_{i=0}^{\tau_k-1} F_m(\phi^i\omega) .$$

Then

$$\nu\Big(\sup_{k\,:\,t_j\leq t_k\leq t_j+\delta}\sum_m |J_2^{(m)}|^2 > \frac{\varepsilon^2\lambda}{144}\Big) \leq \sum_{1=k}^{[\lambda^{7/12}]} \nu\Big(\sum_m |J_2^{(m)}|^2 > \frac{\varepsilon^2\lambda}{144}\Big)$$

$$= \sum_{k=1}^{[\lambda^{7/12}]} \nu\Big(\sum_m |J_2^{(m)}|^2 > \frac{\varepsilon^2\lambda}{144},\ \Big|\sum_{i=0}^{\tau_k-1} l(\phi^i\omega) - \langle l\rangle\tau_k\Big| \leq \lambda^{2/3}\Big)$$

$$+\sum_{k=1}^{[\lambda^{7/12}]}\nu\Big(\sum_m |J_2^{(m)}|^2>\frac{\varepsilon^2\lambda}{144},\ |\sum_{i=0}^{\tau_k-1} l(\phi^i\omega)-\langle l\rangle\tau_k|>\lambda^{2/3}\Big)$$
$$\leq \sum_{k=1}^{[\lambda^{7/12}]}\nu\Big(\sum_m |J_2^{(m)}|^2>\frac{\varepsilon^2\lambda}{144},\ |N(t\lambda_k,\omega)-\tau_k|\leq C\lambda^{2/3}\Big)$$
$$+\sum_{k=1}^{[\lambda^{7/12}]}\nu(|\sum_{i=0}^{\tau_k-1} l(\phi^i\omega)-\langle l\rangle\tau_k|>\lambda^{2/3})=A+B,\ \text{say}.$$

The second term can be estimated as follows.

$$B\leq \sum_{k=2}^{[\lambda^{7/12}]}\lambda^{-8/3}\int|\sum_{i=0}^{\lambda_k-1} l(\phi^i\omega)-\langle l\rangle\tau_k|^4 d\nu(\omega)$$
$$\leq C\sum_{k=1}^{[\lambda^{7/12}]}\lambda^{-8/3}(\lambda t_k)^2=C\lambda^{-2/3}\sum_{k=1}^{[\lambda^{7/12}]}t_k^2\,.$$

By the definition of the partition, $t_k\sim k\lambda^{-7/12}$. Therefore we have

$$B\leq C\lambda^{-2/3}\lambda^{-14/12}\sum_{k=1}^{[\lambda^{7/12}]}k^2\sim C\lambda^{-8/12-14/12+21/12}=C\lambda^{-1/12}\,.$$

For the first term, we write $A=\sum_{k=1}^{[\lambda^{7/12}]}A_k$.

$$A_k\leq\nu\Big(\sum_m \sup_{[\tau_k-C\lambda^{2/3}]\leq h\leq[\tau_k+C\lambda^{2/3}]}|\sum_{i=\tau_k-1}^{h}F_m(\phi^i\omega)|^2>\frac{\varepsilon^2\lambda}{144}\Big)$$
$$\leq\nu\Big(\sum_m \sup_{[\tau_k-C\lambda^{2/3}]\leq h\leq\tau_k}|\sum_{i=h}^{\tau_k-1}F_m(\phi^i\omega)|^2>\frac{\varepsilon^2\lambda}{144}\Big)$$
$$+\nu\Big(\sum_m \sup_{\tau_k\leq h\leq[\tau_k+C\lambda^{2/3}]}|\sum_{i=\tau_k-1}^{h}F_m(\phi^i\omega)|^2>\frac{\varepsilon^2\lambda}{144}\Big)$$

By (3.2), we have

$$A_k\leq C\frac{1}{\varepsilon^4\lambda^2}\lambda^{4/3}=\frac{C}{\varepsilon^4}\lambda^{-2/3}.$$

This implies $A_k\leq\frac{C}{\varepsilon^4}\lambda^{-1/12}$. So we have $II<\frac{2}{3}\eta\delta$, for sufficiently large λ. Thus we obtain

$$\frac{1}{\delta}\mu(\sup_{s\leq t\leq s+\delta}||Y_t^{(\lambda)}-Y_s^{(\lambda)}||'>\varepsilon)<\eta\,,$$

for sufficiently small $\delta>0$ and $\lambda>\lambda_0$, which completes the proof of tightness. To prove the continuity of $\rho(\alpha,\beta)$, it is sufficient to show that there exists a constant $C>0$ such that $\rho_\lambda(\alpha,\alpha)\leq C||\alpha||_p^2$ for any $\Lambda_1(V)$. But this can be shown by similar argument used in the above proof.

4. Nondegeneracy of $\rho(\alpha,\beta)$

In this section, we consider the nondegeneracy of $\rho(\alpha,\beta)$. Let us recall the setting of Theorem 2. Let V be a d-dimensional compact, connected Riemannian manifold of negative curvature. We assume V is $1/d$-pinched. $\{G^t\}$ is the geodesic flow on the unit tangent bundle M of V. We denote by μ the normalized invariant measure given by $\mu(d\xi)=\text{const.}\, m(dx)\sigma_x(dv)$, where m is the Riemannian volume and σ_x is the uniform measure on $\{v\in T_x(V);\ \|v\|=1\}$. Note that $Y_t^{(\lambda)}(\alpha;\xi)=\frac{1}{\sqrt{\lambda}}\int_{G([0,\lambda t],\xi)}\alpha=\frac{1}{\sqrt{\lambda}}\int_0^t f[\alpha](G^s\xi)ds$, $\alpha\in\Lambda_1(V)$. In fact,

$$\overline{f[\alpha]}=\int_M f[\alpha](\xi)d\mu(\xi)=\int_V dm(x)<\alpha,\int_{\{v\in T_xV\,;\,\|v\|=1\}} vd\sigma_x>=0\,.$$

Proof of Theorem 2*. We make use of Theorem 4 of Guillemin-Kazhdan [7]. Following [7], we denote by $H^k(V)$ the totality of C^∞-functions $f(x,v)$ on M, each of which is a harmonic polynomial as a function of v of degree k for each x. We note that $H^0(V)=C^\infty(V)$. Recall $f[\alpha](\xi)=\langle\pi^*\alpha, X\rangle$ $(\alpha\in\Lambda_1(V))$, where X is the vector field generating the geodesic flow $\{G^t\}$ (in local coordinates, $X_{(x,v)}=(v^i\partial/\partial x^i-\Gamma_{kl}^j v^k v^l\partial/\partial v^j)$. Since $\pi_*X=v^i\partial/\partial x^i$, we see that $f[\alpha]\in H^1(V)$, because $f[\alpha](\xi)=\langle\pi^*\alpha, X\rangle(\xi)=\langle\alpha, v\rangle(x)$. Assume that $\rho(\alpha)=0$. Then by a result (for stationary processes) of Leonov [11], there exists an $L^2(M,\mu)$-function u such that

$$\int_0^t f[\alpha](G^s\xi)ds=u(G^t\xi)-u(\xi)\,. \tag{4.1}$$

Furthermore for transitive Anosov flows, Livsic [12. Theorem 9] showed that the function u in (4.1) can be chosen as a continuous function. Therefore $\int_{G([0,t],\xi)}\pi^*\alpha=\int_0^t f[\alpha](G^t\xi)ds=0$ for any periodic geodesic. By virtue of Theorem 4 of Guillemin-Kazhdan [7], there exists an $h\in H^\infty(V)=C^\infty(V)$ such that $Xh=f[\alpha]$. It follows that $f[\alpha]=Xh=\langle d\pi^*h, X\rangle=\langle\pi^*dh, X\rangle$. Since $f[\alpha]=\langle\pi^*\alpha, X\rangle=\langle\alpha, v\rangle$ and $\langle\pi^*dh, X\rangle=\langle dh, v\rangle$, we have $\langle\alpha-dh, v\rangle=0$ for any $v\in T_xV$ with $\|v\|=1$, which implies $\alpha=dh$. This proves Theorem 2.

REMARK 4.1. We do not know whether Theorem 2 holds without the pinching condition. But for closed 1-forms this holds: Let α be closed. Then $\rho(\alpha,\alpha)=0$ if and only if α is exact. The following proof is due to T. Sunada. By the Theorem H of Fried [4], the closed orbits of the geodesic flow generates $H_1(V,\boldsymbol{R})$. Therefore, in view of (4.1),

$$\int_A\alpha=0\,,\qquad\text{for any cycle } A \text{ in } V\,.$$

By de Rham's theorem, this implies that α is exact.

* We thank to T. Sunada, who kindly informed us the literature [7].

Proof of Proposition 1. By (4.1), $\rho(\alpha)=0$ implies the assumption of Theorem A.3 of Guillemin and Kazhdan [6], Proposition 1 is a special case of their result.

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