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## SCATTERING THEORY FOR TIME-DEPENDENT HARTREE-FOCK TYPE EQUATION

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### 1. Introduction

In this paper we consider the scattering problem for the following system of non-linear Schrödinger equations with nonlocal interaction

$$(1) \quad i \frac{\partial}{\partial t} u_j = -\frac{1}{2} \Delta u_j + f_j(\vec{u}), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n,$$

$$(2) \quad u_j(0, x) = \phi_j(x), \quad j = 1, \dots, N.$$

Here  $\Delta$  denotes the Laplacian in  $x$ ,

$$f_j(\vec{u}) = \sum_{k=1}^N (V * |u_k|^2) u_j - \sum_{k=1}^N [V * (u_j \bar{u}_k)] u_k,$$

and  $*$  denotes the convolution in  $\mathbf{R}^n$ . In this paper we treat the case  $n \geq 2$  and  $V(x) = |x|^{-\gamma}$  with  $0 < \gamma < n$ .

The system (1)-(2) appears in the quantum mechanics as an approximation to a fermionic N-body system and is called the time-dependent Hartree-Fock type equation.

Throughout the paper we use the following notation:

$\mathbf{N} = \{1, 2, 3, \dots\}$ ,  $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ ,  $U(t) = \exp(it\Delta/2)$ ,  $M(t) = \exp(i|x|^2/2t)$ ,  $J = U(t)xU(-t) = M(t)(it\nabla)M(-t)$ . For  $1 \leq p \leq \infty$ ,  $p' = p/(p-1)$ ,  $\delta(p) = n/2 - n/p$ .  $\|\cdot\|_p$  denotes the norm of  $L^p(\mathbf{R}^n)$  (if  $p = 2$ , we write  $\|\cdot\|_2 = \|\cdot\|$ ). For  $1 \leq q, r \leq \infty$  and for the interval  $I \subset \mathbf{R}$ ,  $\|\cdot\|_{q,r,I}$  denotes the norm of  $L^r(I; L^q(\mathbf{R}^n))$ , namely,  $\|u\|_{q,r,I} = \left[ \int_I \left( \int_{\mathbf{R}^n} |u(t, x)|^q dx \right)^{r/q} dt \right]^{1/r}$ . For positive integers  $l$  and  $m$ ,  $\Sigma^{l,m}$  denotes the Hilbert space defined as

$$\Sigma^{l,m} = \left\{ \psi \in L^2(\mathbf{R}^n); \|\psi\|_{\Sigma^{l,m}} = \left( \sum_{|\alpha| \leq l} \|\nabla^\alpha \psi\|^2 + \sum_{|\beta| \leq m} \|x^\beta \psi\|^2 \right)^{1/2} < \infty \right\}.$$

When we use  $N$ 'th direct sums of various function spaces, we denote them by the same symbols and denote these elements by writing arrow over the letter, like  $\vec{f}$ .

Now we state our main theorem.

**Theorem 1.1.** (i) *Suppose that  $1 < \gamma < \min(4, n)$ , and  $l, m \in \mathbf{N}$ . Then for any  $\vec{\phi}^{(+)} \in \Sigma^{l,m}$ , there exists a unique  $\vec{\phi} \in \Sigma^{l,m}$  such that*

$$(3) \quad \lim_{t \rightarrow +\infty} \|\vec{\phi}^{(+)} - U(-t)\vec{u}(t)\|_{\Sigma^{l,m}} = 0,$$

where  $\vec{u}(t)$  is the solution of (1)-(2) with  $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$ . For any  $\vec{\phi}^{(-)} \in \Sigma^{l,m}$ , the same result as above holds valid with  $+\infty$  replaced by  $-\infty$  in (3).

(ii) *Suppose that  $4/3 < \gamma < \min(4, n)$ , and  $l, m \in \mathbf{N}$ . And if  $\gamma \leq \sqrt{2}$ , suppose, in addition, that  $m \geq 2$ . Then for any  $\vec{\phi} \in \Sigma^{l,m}$ , there exist  $\vec{\phi}^{(\pm)} \in \Sigma^{l,m}$  such that the solution of (1)-(2) with  $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$  satisfies*

$$(4) \quad \lim_{t \rightarrow \pm\infty} \|\vec{\phi}^{(\pm)} - U(-t)\vec{u}(t)\|_{\Sigma^{l,m}} = 0.$$

By Theorem 1.1 (i), if  $1 < \gamma < \min(4, n)$ , we can define the operator  $W_+$  in  $\Sigma^{l,m}$  as

$$W_+ : \vec{\phi}^{(+)} \longmapsto \vec{\phi},$$

which is called the wave operator. The operator  $W_-$  is defined similarly. Under the condition of  $4/3 < \gamma < \min(4, n)$  ( $m \geq 2$  if  $\gamma \leq \sqrt{2}$ ), Theorem 1.1 (ii) implies the completeness of  $W_{\pm}$ , namely,  $\text{Range}W_{\pm} = \Sigma^{l,m}$ .

There are many papers for the following equation

$$(5) \quad i \frac{\partial u}{\partial t} = -\frac{1}{2} \Delta u + f(u), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n,$$

$$(6) \quad u(0, x) = \phi(x),$$

where

$$f(u) = [V * |u|^2]u = \int_{\mathbf{R}^n} |x - y|^{-\gamma} |u(t, y)|^2 dy u(t, x)$$

(see, for example, [5, 7, 8, 9, 12]). The equation (5)-(6) is called the Hartree type equation. For the scattering problem for (5)-(6), the following results are known (see [9]).

[A] *Suppose that  $1 < \gamma < \min(4, n)$ , and  $l, m \in \mathbf{N}$ . Then, for any  $\phi^{(+)} \in \Sigma^{l,m}$ , there exists a unique  $\phi \in \Sigma^{l,m}$  such that*

$$(7) \quad \lim_{t \rightarrow +\infty} \|\phi^{(+)} - U(-t)u(t)\|_{\Sigma^{l,m}} = 0,$$

where  $u(t)$  is the solution of (5)-(6) with  $U(-t)u(t) \in C(\mathbf{R}; \Sigma^{l,m})$ . For any  $\phi^{(-)} \in \Sigma^{l,m}$ , the same result as above holds valid with  $+\infty$  replaced by  $-\infty$  in (7).

[B] Suppose that  $4/3 < \gamma < \min(4, n)$ , and  $l, m \in \mathbf{N}$ . Then, for any  $\phi \in \Sigma^{l,m}$ , there exist unique  $\phi^{(\pm)} \in \Sigma^{l,m}$  such that the solution  $u(t)$  of (5)-(6) with  $U(-t)u(t) \in C(\mathbf{R}; \Sigma^{l,m})$  satisfies

$$(8) \quad \lim_{t \rightarrow \pm\infty} \|\phi^{(\pm)} - U(-t)u(t)\|_{\Sigma^{l,m}} = 0.$$

Our main Theorem is the analogous results to [A], [B].

Since  $U(t)$  is unitary in  $H^l$ , (4) implies that the asymptotic profiles of  $\vec{u}(t)$  as  $t \rightarrow \pm\infty$  are  $U(t)\vec{\phi}^{(\pm)}$ ; and by the estimates

$$\|U(t)\vec{\phi}^{(\pm)}\|_p \leq (2\pi|t|)^{-\delta(p)} \|\vec{\phi}^{(\pm)}\|_{p'}, \quad 2 \leq p \leq \infty,$$

it is expected that

$$(9) \quad \|\vec{u}(t)\|_p = O(|t|^{-\delta(p)})$$

as  $t \rightarrow \pm\infty$ . Indeed, in Corollary 4.1, we shall prove (9) for  $p = \infty$  under the suitable condition for  $\vec{\phi}$ .

Conversely, if (9) holds for some  $p$  sufficiently large, We can prove Theorem 1.1(ii). Actually, in Propositions 3.1 and 3.2, we prove (9) for some  $p > 2$ . This decay estimate is the key point of our proof of the main theorem.

The proof of Theorems [B] is much more simple than our proof of Theorem 1.1 (ii). But we cannot apply the method in [9] for (5)-(6) to prove Theorem 1.1 (ii). So we shall use the method in our work [15] to prove the main theorem.

## 2. Preliminaries

First, we collect various inequalities which will be used in later sections.

**Lemma 2.1.** (The Gagliardo-Nirenberg inequality) *Let  $1 \leq q, r \leq \infty$  and  $j, m$  be any integers satisfying  $0 \leq j < m$ . If  $u$  is any function in  $W^{m,q}(\mathbf{R}^n) \cap L^r(\mathbf{R}^n)$ , then*

$$(10) \quad \sum_{|\alpha|=j} \|\nabla^\alpha u\|_p \leq C \left( \sum_{|\beta|=m} \|\nabla^\beta u\|_q \right)^a \|u\|_r^{1-a}$$

where

$$\frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{q} - \frac{m}{n}\right) + (1 - a)\frac{1}{r}$$

for all  $a$  in the interval  $j/m \leq a \leq 1$ , where the constant  $C$  is independent of  $u$ , with the following exception: if  $m - j - (n/q)$  is a nonnegative integer, then (10) is asserted for  $j/m \leq a < 1$ .

For the proof of Lemma 2.1, see [3, 14].

**Lemma 2.2.** *Let  $\alpha > 0$ . Then*

$$(11) \quad \|(-\Delta)^{\alpha/2} fg\| \leq C(\|(-\Delta)^{\alpha/2} f\| \|g\|_{\infty} + \|f\|_{\infty} \|(-\Delta)^{\alpha/2} g\|).$$

This lemma is essentially due to [4, 6]. The lemma is obtained as in the proof of Lemma 3.4 in [4] and Lemma 3.2 in [6], by using the theory of Besov space (for Besov space, see [1]).

**Lemma 2.3.** (The Hardy-Littlewood-Sobolev inequality) *Let  $0 < \gamma < n, 1 < p, q < \infty$  and  $1 + 1/p = \gamma/n + 1/q$ . Then*

$$(12) \quad \| |x|^{-\gamma} * \phi \|_p \leq C \|\phi\|_q.$$

For the proof, see [10, 13].

A pair  $(q, r)$  of real numbers is called admissible, if it satisfies the condition  $0 \leq \delta(q) = 2/r < 1$ . Then

**Lemma 2.4.** *If a pair  $(q, r)$  is admissible, then for any  $\psi \in L^2(\mathbf{R}^n)$ , we have*

$$(13) \quad \|U(t)\psi\|_{q,r,\mathbf{R}} \leq C\|\psi\|.$$

**Lemma 2.5.** *We put  $(Gu)(t) = \int_{t_0}^t U(t - \tau)u(\tau)d\tau$ . Let  $I \subset \mathbf{R}$  be an interval containing  $t_0$ , and let pairs  $(q_j, r_j), j = 1, 2$ , be admissible. Then  $G$  maps  $L^{r'_1}(I; L^{q'_1})$  into  $L^{r_2}(I; L^{q_2})$  and satisfies*

$$(14) \quad \|Gu\|_{q_2,r_2,I} \leq C\|u\|_{q'_1,r'_1,I},$$

where  $C$  is independent of  $I$ .

For the proof of Lemmas 2.4 and 2.5, see [11, 16].

Next, we summarize the results for the Cauchy problem to (1)-(2) . We convert (1)-(2) into the integral equations

$$(15) \quad u_j(t) = U(t)\phi_j - i \int_0^t U(t - \tau)f_j(\vec{u}(\tau))d\tau, \quad j = 1, \dots, N,$$

then

**Proposition 2.1.** (i) *Suppose that  $n \geq 2$ ,  $0 < \gamma < \min(4, n)$ , and  $l, m \in \mathbf{N}$ . Then for any  $\vec{\phi} \in H^l$ , there exists a unique solution  $\vec{u}(t) \in C(\mathbf{R}; H^l)$  of (15). The solution  $\vec{u}(t)$  satisfies following equalities.*

$$(16) \quad (u_j(t), u_k(t)) = (\phi_j, \phi_k), \quad j, k = 1, \dots, N,$$

especially,

$$(17) \quad \|u_j(t)\| = \|\phi_j\|, \quad j = 1, \dots, N;$$

and

$$(18) \quad E(\vec{u}(t)) = E(\vec{\phi}),$$

where

$$E(\vec{\psi}) = \sum_{j=1}^N \|\nabla\psi_j\|^2 + P(\vec{\psi}),$$

$$P(\vec{\psi}) = \sum_{j,k=1}^N \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |x - y|^{-\gamma} (|\psi_j(x)|^2 |\psi_k(y)|^2 - \psi_j(x)\bar{\psi}_k(x)\psi_k(y)\bar{\psi}_j(y)) dx dy;$$

(ii) *Furthermore, if  $\vec{\phi} \in \Sigma^{l,m}$ , then  $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$ , and the solution  $\vec{u}(t)$  satisfies*

$$(19) \quad \sum_{j=1}^N \|xU(-t)u_j(t)\|^2 + t^2 P(\vec{u}(t)) = \sum_{j=1}^N \|x\psi_j\|^2 + (2 - \gamma) \int_0^t \tau P(\vec{u}(\tau)) d\tau.$$

REMARK. (i) By the Cauchy-Schwarz inequality,  $P(\vec{\psi}) \geq 0$ .

(ii) The equalities (17), (18) and (19) are called the  $L^2$ -norm, the energy, and the pseudo-conformal conservation laws, respectively.

The proof of Propositions 2.1 is similar to that of the corresponding result for (5)-(6), so we shall omit it ( see, for example, [8, 9, 12]).

**3. Decay estimates for some norm of the solution**

In this section we shall estimate the  $L^p$ -norm of the solution  $\vec{u}(t)$  of (1)-(2) to prove the main theorem. We use the following transform

$$\begin{aligned} v_j(t) &= \mathcal{F}M(t)U(-t)u_j(t) \\ &= (it)^{n/2} \exp(-it|x|^2/2)u_j(t, tx), \end{aligned}$$

where  $\mathcal{F}$  is the Fourier transform in  $\mathbf{R}^n$ . This transform was introduced by N. Hayashi and T. Ozawa [7]. Then the equations (1) are transformed into the equations

$$(20) \quad i \frac{\partial}{\partial t} v_j = -\frac{1}{2t^2} \Delta v_j + \frac{1}{t^\gamma} f_j(\vec{v}), \quad j = 1, \dots, N,$$

and if  $\vec{\phi} \in \Sigma^{1,m}$ , then  $\vec{v}(t) \in C((0, \infty); \Sigma^{m,1})$ . The relations (17) and (19) are equivalent to

$$(21) \quad \frac{d}{dt} \|v_j(t)\| = 0, \quad j = 1, \dots, N$$

and

$$(22) \quad t^{-2} \frac{d}{dt} \sum_{j=1}^N \|\nabla v_j(t)\|^2 + t^{-\gamma} \frac{d}{dt} P(\vec{v}(t)) = 0,$$

respectively. Using the relation (22), we show

**Lemma 3.1.** *Suppose that  $n \geq 2$ ,  $0 < \gamma < \min(4, n)$ , and  $\vec{\phi} \in \Sigma^{1,1}$ . Then, for  $t \geq 1$ ,*

$$(23) \quad \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq \begin{cases} Ct^{2-\gamma} & \text{if } \gamma \leq \sqrt{2}, \\ C & \text{if } \gamma > \sqrt{2}. \end{cases}$$

Here, the constants  $C$  depend on  $\|\vec{\phi}\|_{\Sigma^{1,1}}$ .

**Proof.** If  $\gamma < 2$ ,

$$\frac{d}{dt} \left( t^{\gamma-2} \sum_{j=1}^N \|\nabla v_j(t)\|^2 + P(\vec{v}(t)) \right) = (\gamma - 2)t^{\gamma-3} \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq 0,$$

and if  $\gamma \geq 2$ ,

$$\frac{d}{dt} \left( \sum_{j=1}^N \|\nabla v_j(t)\|^2 + t^{2-\gamma} P(\vec{v}(t)) \right) = (2 - \gamma)t^{1-\gamma} P(\vec{v}(t)) \leq 0.$$

Hence

$$(24) \quad \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq \begin{cases} Ct^{2-\gamma} & \text{if } \gamma < 2, \\ C & \text{if } \gamma \geq 2. \end{cases}$$

So we shall prove (23) when  $\sqrt{2} < \gamma < 2$ . We multiply (20) by  $\Delta \bar{v}_j$ , and integrate the imaginary part over  $\mathbf{R}^n$ . Then

$$\frac{1}{2} \frac{d}{dt} \|\nabla v_j(t)\|^2 = t^{-\gamma} \text{Im} \int_{\mathbf{R}^n} f_j(\vec{v}) \Delta \bar{v}_j dx.$$

Since  $\text{Im} \int_{\mathbf{R}^n} V * |v_k|^2 |\nabla v_j|^2 dx$  and  $\text{Im} \sum_{j,k=1}^N \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) \nabla v_k \cdot \nabla \bar{v}_j dx$  are equal to zero, we have, by Hölder's inequality and Lemma 2.3,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \|\nabla v_j(t)\|^2 \\ &= t^{-\gamma} \text{Im} \sum_{j,k=1}^N \left[ \int_{\mathbf{R}^n} v_j \nabla (V * |v_k|^2) \cdot \nabla \bar{v}_j dx + \int_{\mathbf{R}^n} v_k \nabla (V * (v_j \bar{v}_k)) \cdot \nabla \bar{v}_j dx \right] \\ &\leq Ct^{-\gamma} \|\vec{v}(t)\|_{\rho}^2 \sum_{j=1}^N \|\nabla v_j(t)\|^2, \end{aligned}$$

where  $\rho = 2n/(n - \gamma)$ . By Lemma 2.1 and (24), we have

$$\begin{aligned} \|v_j(t)\|_{\rho} &\leq C \|v_j\|^{1-\gamma/2} \|\nabla v_j\|^{\gamma/2} \\ &\leq Ct^{(2\gamma-\gamma^2)/4}. \end{aligned}$$

Therefore,

$$(25) \quad \frac{d}{dt} \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq Ct^{-\gamma^2/2} \sum_{j=1}^N \|\nabla v_j(t)\|^2.$$

Since  $\gamma^2/2 > 1$  if  $\gamma > \sqrt{2}$ , (25) and Gronwall's inequality yield (23).

Lemma 3.1 immediately implies



**Proposition 3.1.** *Suppose that  $\sqrt{2} < \gamma < \min(4, n)$ , and  $\vec{\phi} \in \Sigma^{1,1}$ . Then for the number  $p$  satisfying  $0 \leq \delta(p) \leq 1$  if  $n \geq 3$  and  $0 \leq \delta(p) < 1$  if  $n = 2$ , the solution of (1)-(2) has the estimate*

$$(26) \quad \|\vec{u}(t)\|_p \leq C(1 + |t|)^{-\delta(p)}.$$

Proof. Since  $\|\vec{u}(t)\|_p = t^{-\delta(p)}\|\vec{v}(t)\|_p$ , Lemma 2.1 and Lemma 3.1 yield (26).

Now we show the  $L^p$  decay estimate of the solution in case  $1 < \gamma \leq \sqrt{2}$ .

**Lemma 3.2.** *Suppose that  $1 < \gamma \leq \sqrt{2}$  and  $\vec{\phi} \in \Sigma^{1,2}$ . Then we have for  $t \geq 1$ ,*

$$(27) \quad \sum_{j=1}^N \|\Delta v_j(t)\|^2 \leq \begin{cases} Ct^{(\gamma^2-8\gamma+10)/(2-\gamma)} & \text{if } n \geq 3, \\ Ct^{(\gamma^2-8\gamma+10)/(2-\gamma)+\varepsilon} & \text{if } n = 2. \end{cases}$$

Here  $\varepsilon$  is a positive number which can be chosen arbitrarily small, and the constant  $C$  depends on  $\|\vec{\phi}\|_{\Sigma^{1,2}}$ , and  $\varepsilon$  ( the case  $n = 2$  ).

Proof. We apply  $\Delta$  to the both side of (20) and obtain

$$(28) \quad i \frac{\partial}{\partial t} \Delta v_j = -\frac{1}{2t^2} \Delta^2 v_j + \frac{1}{t^\gamma} \Delta f_j(\vec{v}), \quad j = 1, \dots, N.$$

Multiplying (28) by  $\Delta \bar{v}_j$ , integrating the imaginary part over  $\mathbf{R}^n$ , we have

$$\frac{1}{2} \frac{d}{dt} \|\Delta v_j(t)\|^2 = \frac{1}{t^\gamma} \text{Im} \int_{\mathbf{R}^n} \Delta f_j(\vec{v}) \Delta \bar{v}_j dx.$$

Since  $\text{Im} \int_{\mathbf{R}^n} V * |v_k|^2 |\Delta v_j|^2 dx$  and  $\text{Im} \sum_{j,k=1}^N \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) \Delta v_k \Delta \bar{v}_j dx$  are equal to zero,

$$(29) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \|\Delta v_j(t)\|^2 \\ &= t^{-\gamma} \text{Im} \sum_{j,k=1}^N \left[ \int_{\mathbf{R}^n} \Delta(V * |v_k|^2) v_j \Delta \bar{v}_j dx + 2 \int_{\mathbf{R}^n} \nabla(V * |v_k|^2) \cdot \nabla v_j \Delta \bar{v}_j dx \right. \\ & \quad \left. + \int_{\mathbf{R}^n} \Delta(V * (v_j \bar{v}_k)) v_k \Delta \bar{v}_j dx + 2 \int_{\mathbf{R}^n} \nabla(V * (v_j \bar{v}_k)) \cdot \nabla v_k \Delta \bar{v}_j dx \right]. \end{aligned}$$

(i) Case  $n \geq 3$ . Hölder's inequality, Lemma 2.1 and Lemma 2.3 imply that the first term in the brackets of the right of (29) is dominated by

$$\begin{aligned} & C \int_{\mathbf{R}^n} |x|^{-\gamma-1} * (|\nabla v_k| |v_k|) |v_j| |\Delta v_j| dx \\ & \leq C \|\nabla v_k\| \|v_k\|_{2n/(n-2\gamma)} \|v_j\|_{2n/(n-2)} \|\Delta v_j\| \\ & \leq C \left( \sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma} \left( \sum_{j=1}^N \|\Delta v_j\|^2 \right)^{\gamma/2}. \end{aligned}$$

The other terms are estimated similarly. Therefore, it follows from (23) that for  $t \geq 1$ ,

$$\begin{aligned} (30) \quad \frac{d}{dt} \sum_{j=1}^N \|\Delta v_j(t)\|^2 & \leq C t^{-\gamma} \left( \sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma} \left( \sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{\gamma/2} \\ & \leq C t^{(8-8\gamma+\gamma^2)/2} \left( \sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{\gamma/2}. \end{aligned}$$

Integrating this differential inequality, we have

$$(31) \quad \left( \sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{1-\gamma/2} \leq C t^{(10-8\gamma+\gamma^2)/2} + \left( \sum_{j=1}^N \|\Delta v_j(1)\|^2 \right)^{1-\gamma/2},$$

which implies (27). Since  $\|\Delta v_j(1)\| = \| |x|^2 U(-1)u(1) \| \leq C \|\vec{\phi}\|_{\Sigma^{1,2}}$ , the constant  $C$  in (23) depends on  $\|\vec{\phi}\|_{\Sigma^{1,2}}$ .

(ii) Case  $n = 2$ . Since

$$V_* = \frac{2^{n-\gamma} \pi^{n/2} \Gamma(\frac{n-\gamma}{2})}{\Gamma(\frac{\gamma}{2})} (-\Delta)^{(\gamma-n)/2}, \quad 0 < \gamma < n,$$

we have for  $n = 2$ ,  $-\Delta V_* = C(-\Delta)^{\gamma/2}$ . Hence, by using Hölder's inequality, Lemma 2.1 and Lemma 2.2, we can estimate the first term in the brackets of the right of (29) by

$$\begin{aligned} (32) \quad & C \|(-\Delta)^{\gamma/2} |v_k|^2\| \|v_j\|_{\infty} \|\Delta v_j\| \\ & \leq C \|(-\Delta)^{\gamma/2} v_k\| \|\vec{v}\|_{\infty}^2 \|\Delta v_j\| \\ & \leq C \|\vec{v}\|_{\infty}^2 \left( \sum_{j=1}^N \|\nabla v_j\| \right)^{2-\gamma} \left( \sum_{j=1}^N \|\Delta v_j\|^2 \right)^{\gamma/2}. \end{aligned}$$

Since Lemma 2.1 implies

$$\begin{aligned} \|v_k\|_{\infty} & \leq C \|\Delta v_k\|^{2/(\theta+2)} \|v_k\|_{\theta}^{\theta/(\theta+2)} \\ & \leq C \|v_k\|^{2/(\theta+2)} \|\nabla v_k\|^{(\theta-2)/(\theta+2)} \|\Delta v_k\|^{2/(\theta+2)}, \end{aligned}$$

where  $2 \leq \theta < \infty$ , the right of (32) is dominated by

$$C \|v\|^a \left( \sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma-2a} \left( \sum_{j=1}^N \|\Delta v_j\|^2 \right)^{(\gamma+a)/2}$$

with  $a = 2/(\theta + 2)$ . The second term in the brackets of the right of (29) is estimated by

$$\begin{aligned} & \|V * (|\nabla v_k| |v_k|) \|_{n/(\gamma-1)} \|\nabla v_j\|_{2n/(n-\gamma+1)} \|\Delta v_j\| \\ & \leq C \|\vec{v}\|_\infty \left( \sum_{j=1}^N \|\nabla v_j\| \right)^{3-\gamma} \left( \sum_{j=1}^N \|\Delta v_j\|^2 \right)^{\gamma/2} \\ & \leq C \|\vec{v}\|^a \left( \sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma-2a} \left( \sum_{j=1}^N \|\Delta v_j\|^2 \right)^{(\gamma+a)/2} \end{aligned}$$

The other terms are estimated similarly. Therefore, we have

$$(33) \quad \frac{d}{dt} \sum_{j=1}^N \|\Delta v_j(t)\|^2 \leq C t^{(8-8\gamma+\gamma^2)/2} \left( \sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{(\gamma+a)/2}$$

Since the number  $a$  can be chosen arbitrarily small, this differential equation implies (27).

**Lemma 3.3.** *Suppose that  $n \geq 2$ ,  $1 < \gamma \leq \sqrt{2}$  and  $\vec{\phi} \in \Sigma^{1,2}$ . Then we have for  $t \geq 1$ ,*

$$(34) \quad \|\vec{v}(t)\|_p \leq C.$$

Here,  $p$  satisfies  $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$ , and the constant  $C$  depends on  $\|\vec{\phi}\|_{\Sigma^{1,2}}$ .

**Proof.** For simplicity, we prove the lemma in case  $n \geq 3$ . We put  $\|\vec{v}\|_{p,*} = [\int_{\mathbf{R}^n} (\sum_{l=1}^N |v_l|^2)^{p/2} dx]^{1/p}$ , which is equivalent to the norm  $\|\vec{v}\|_p = \sum_{l=1}^N \|v_l\|_p$ . We multiply the equation (20) by  $(\sum_{l=1}^N |v_l|^2)^{(p-2)/2} \bar{v}_j$ , integrate their imaginary part over  $\mathbf{R}^n$ , and add them. Then we have

$$(35) \quad \frac{1}{p} \frac{d}{dt} \|\vec{v}(t)\|_{p,*}^p = -\frac{1}{2t^2} \text{Im} \sum_{j=1}^N \int_{\mathbf{R}^n} \Delta v_j \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \bar{v}_j dx,$$

since  $\text{Im} \int_{\mathbf{R}^n} V * |v_k|^2 \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} |v_j|^2 dx$  and  $\text{Im} \sum_{j,k=1}^N \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) v_k \bar{v}_j \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} dx$  are equal to zero. By the integration by parts and Hölder's inequality,

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\vec{v}(t)\|_{p,*}^p &= \frac{1}{2t^2} \text{Im} \sum_{j=1}^N \int_{\mathbf{R}^n} \nabla v_j \cdot \nabla \left( \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \bar{v}_j \right) dx \\ &\leq Ct^{-2} \sum_{j=1}^N \int_{\mathbf{R}^n} |\nabla v_j|^2 \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} dx \\ &\leq Ct^{-2} \sum_{j=1}^N \|\nabla v_j\|_p^2 \|\vec{v}(t)\|_{p,*}^{(p-2)}. \end{aligned}$$

We note that when  $1 < \gamma \leq \sqrt{2}$ , we have  $0 < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma) < 1$ , and so  $2 < p < 2n/(n - 2)$ . Then, Lemma 2.1, Lemma 3.1 and Lemma 3.2 yield

$$\begin{aligned} \|\nabla v_j\|_p &\leq C \|\nabla v_j\|^{1-\delta(p)} \|\Delta v_j\|^{\delta(p)} \\ &\leq Ct^\eta. \end{aligned}$$

Here

$$\eta = 2 - \gamma + \frac{6 - 4\gamma}{2 - \gamma} \delta(p),$$

and the constant  $C$  depends on  $\|\vec{\phi}\|_{\Sigma^{1,2}}$ . Therefore,

$$(36) \quad \frac{d}{dt} \|\vec{v}(t)\|_{p,*}^p \leq Ct^{-2+\eta} \|\vec{v}(t)\|_{p,*}^{(p-2)}.$$

Since  $\eta < 1$  for  $p$  satisfying  $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$ , the estimate (34) follows by integrating the differential inequality (36).

By this lemma, we have

**Proposition 3.2.** *Suppose that  $n \geq 2$ ,  $1 < \gamma \leq \sqrt{2}$  and  $\vec{\phi} \in \Sigma^{1,2}$ . Then the solution of (1)-(2) has the following estimate*

$$(37) \quad \|\vec{u}(t)\|_p \leq C(1 + |t|)^{-\delta(p)},$$

where  $p$  satisfies  $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$ .

**4. Proof of the main theorem**

In this section, we shall prove Theorem 1.1. Since we can prove part (i) of the Theorem similar to the Theorem [A] for Hartree type equation, we omit the proof. Throughout this section, we put  $q = 4n/(2n - \gamma)$  and  $r = 8/\gamma$ . Then the pair  $(q, r)$  is admissible. To prove part (ii), we introduce the following Banach space:

$$X^{l,m}(I) = \{u \in C(I; H^l); \|u\|_{X^{l,m}(I)} < \infty\},$$

where

$$\|u\|_{X^{l,m}(I)} = \sum_{|\alpha| \leq l} (\|\nabla^\alpha u\|_{2,\infty,I} + \|\nabla^\alpha u\|_{q,r,I}) + \sum_{|\beta| \leq m} (\|J^\beta u\|_{2,\infty,I} + \|J^\beta u\|_{q,r,I}).$$

Let  $I = [T, \infty)$ , where  $T$  will be defined later. Using Hölder’s inequality, Lemma 2.1 and Lemma 2.3, we have

$$(38) \quad \sum_{|\alpha|=l} \|\nabla^\alpha f_j(\vec{u})\|_{q'} \leq C \|\vec{u}\|_q^2 \sum_{k=1}^N \sum_{|\alpha|=l} \|\nabla^\alpha u_k\|_q$$

and

$$(39) \quad \sum_{|\beta|=m} \|J^\beta f_j(\vec{u})\|_{q'} \leq C \|\vec{u}\|_q^2 \sum_{k=1}^N \sum_{|\beta|=m} \|J^\beta u_k\|_q.$$

So we have, by Lemma 2.1 and Lemma 2.5,

$$(40) \quad \sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{2,\infty,I} \leq \sum_{|\alpha| \leq l} \|\nabla^\alpha U(-T)u_j(T)\| + C \sum_{|\alpha| \leq l} \|\nabla^\alpha f_j(\vec{u})\|_{q',r',I}.$$

Under the assumption of the theorem, Proposition 3.1 or Proposition 3.2 implies  $\|\vec{u}(t)\|_q \leq Ct^{-\gamma/4}$ . Therefore, by using (38) and Hölder’s inequality, the second term in the right of (40) is dominated by

$$(41) \quad \begin{aligned} & C \sum_{k=1}^N \sum_{|\alpha| \leq l} \left[ \int_T^\infty (\|\vec{u}(\tau)\|_q^2 \|\nabla^\alpha u_k(\tau)\|_q)^{r'} d\tau \right]^{1/r'} \\ & \leq C \sum_{k=1}^N \sum_{|\alpha| \leq l} \left[ \int_T^\infty (\tau^{-\gamma/2} \|\nabla^\alpha u_k(\tau)\|_q)^{r'} d\tau \right]^{1/r'} \\ & \leq C \left( \int_T^\infty \tau^{-2\gamma/(4-\gamma)} d\tau \right)^{(4-\gamma)/4} \sum_{k=1}^N \sum_{|\alpha| \leq l} \|\nabla^\alpha u_k\|_{q,r,I}. \end{aligned}$$

If  $\gamma > 4/3$ , the integral in the right of (41) converges. Hence,

$$(42) \quad \sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{2,\infty,I} \leq \|U(-T)u_j(T)\|_{\Sigma^{l,m}} + CT^{(4-3\gamma)/4} \|\vec{u}\|_{X^{l,m}(I)}.$$

We can estimate

$$\sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{q,r,I}, \sum_{|\beta| \leq m} \|J^\beta u_j\|_{2,\infty,I}, \text{ and } \sum_{|\beta| \leq m} \|J^\beta u_j\|_{q,r,I}$$

similarly. Therefore,

$$(43) \quad \|\vec{u}\|_{X^{l,m}(I)} \leq C\|U(-T)\vec{u}(T)\|_{\Sigma^{l,m}} + CT^{(4-3\gamma)/4} \|\vec{u}\|_{X^{l,m}(I)}.$$

If we choose  $T$  sufficiently large so that  $CT^{(4-3\gamma)/4} \leq 1/2$ , (43) implies

$$\|\vec{u}\|_{X^{l,m}(I)} \leq C\|U(-T)\vec{u}(T)\|_{\Sigma^{l,m}}.$$

Therefore,  $\|\vec{u}\|_{X^{l,m}(\mathbf{R})}$  is finite. Once this has been proved, by the similar argument, for  $t > s > 0$ , we have

$$(44) \quad \begin{aligned} \|U(-t)\vec{u}(t) - U(-s)\vec{u}(s)\|_{\Sigma^{l,m}} &\leq C \left( \int_s^t \tau^{-2\gamma/(4-\gamma)} d\tau \right)^{(4-\gamma)/4} \|\vec{u}\|_{X^{l,m}(\mathbf{R})} \\ &\leq C \left( t^{(4-3\gamma)/4} - s^{(4-3\gamma)/4} \right). \end{aligned}$$

The right of (44) tends to zero as  $s, t$  tend to infinity. Thus the theorem has been proved.

**Corollary 4.1.** *Suppose that  $4/3 < \gamma < \min(4, n)$ , and  $l, m \geq 1 + [n/2]$ . Then for any  $\vec{\phi} \in \Sigma^{l,m}$ , the solution  $\vec{u}(t)$  of (1)-(2) satisfies*

$$(45) \quad \|\vec{u}(t)\|_\infty \leq C(1 + |t|)^{-n/2}.$$

**Proof.** By the relation  $J^\beta \vec{u}(t) = M(t)x^\beta M(-t)\vec{u}(t)$  and Lemma 2.1.

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