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SCATTERING THEORY FOR TIME-DEPENDENT HARTREE-FOCK TYPE EQUATION

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1. Introduction

In this paper we consider the scattering problem for the following system of non-linear Schrödinger equations with nonlocal interaction

$$(1) \quad i \frac{\partial}{\partial t} u_j = -\frac{1}{2} \Delta u_j + f_j(\vec{u}), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n,$$

$$(2) \quad u_j(0, x) = \phi_j(x), \quad j = 1, \dots, N.$$

Here Δ denotes the Laplacian in x ,

$$f_j(\vec{u}) = \sum_{k=1}^N (V * |u_k|^2) u_j - \sum_{k=1}^N [V * (u_j \bar{u}_k)] u_k,$$

and $*$ denotes the convolution in \mathbf{R}^n . In this paper we treat the case $n \geq 2$ and $V(x) = |x|^{-\gamma}$ with $0 < \gamma < n$.

The system (1)-(2) appears in the quantum mechanics as an approximation to a fermionic N-body system and is called the time-dependent Hartree-Fock type equation.

Throughout the paper we use the following notation:

$\mathbf{N} = \{1, 2, 3, \dots\}$, $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$, $U(t) = \exp(it\Delta/2)$, $M(t) = \exp(i|x|^2/2t)$, $J = U(t)xU(-t) = M(t)(it\nabla)M(-t)$. For $1 \leq p \leq \infty$, $p' = p/(p-1)$, $\delta(p) = n/2 - n/p$. $\|\cdot\|_p$ denotes the norm of $L^p(\mathbf{R}^n)$ (if $p = 2$, we write $\|\cdot\|_2 = \|\cdot\|$). For $1 \leq q, r \leq \infty$ and for the interval $I \subset \mathbf{R}$, $\|\cdot\|_{q,r,I}$ denotes the norm of $L^r(I; L^q(\mathbf{R}^n))$, namely, $\|u\|_{q,r,I} = \left[\int_I \left(\int_{\mathbf{R}^n} |u(t, x)|^q dx \right)^{r/q} dt \right]^{1/r}$. For positive

integers l and m , $\Sigma^{l,m}$ denotes the Hilbert space defined as

$$\Sigma^{l,m} = \left\{ \psi \in L^2(\mathbf{R}^n); \|\psi\|_{\Sigma^{l,m}} = \left(\sum_{|\alpha| \leq l} \|\nabla^\alpha \psi\|^2 + \sum_{|\beta| \leq m} \|x^\beta \psi\|^2 \right)^{1/2} < \infty \right\}.$$

When we use N 'th direct sums of various function spaces, we denote them by the same symbols and denote these elements by writing arrow over the letter, like \vec{f} .

Now we state our main theorem.

Theorem 1.1. (i) *Suppose that $1 < \gamma < \min(4, n)$, and $l, m \in \mathbf{N}$. Then for any $\vec{\phi}^{(+)} \in \Sigma^{l,m}$, there exists a unique $\vec{\phi} \in \Sigma^{l,m}$ such that*

$$(3) \quad \lim_{t \rightarrow +\infty} \|\vec{\phi}^{(+)} - U(-t)\vec{u}(t)\|_{\Sigma^{l,m}} = 0,$$

where $\vec{u}(t)$ is the solution of (1)-(2) with $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$. For any $\vec{\phi}^{(-)} \in \Sigma^{l,m}$, the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (3).

(ii) *Suppose that $4/3 < \gamma < \min(4, n)$, and $l, m \in \mathbf{N}$. And if $\gamma \leq \sqrt{2}$, suppose, in addition, that $m \geq 2$. Then for any $\vec{\phi} \in \Sigma^{l,m}$, there exist $\vec{\phi}^{(\pm)} \in \Sigma^{l,m}$ such that the solution of (1)-(2) with $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$ satisfies*

$$(4) \quad \lim_{t \rightarrow \pm\infty} \|\vec{\phi}^{(\pm)} - U(-t)\vec{u}(t)\|_{\Sigma^{l,m}} = 0.$$

By Theorem 1.1 (i), if $1 < \gamma < \min(4, n)$, we can define the operator W_+ in $\Sigma^{l,m}$ as

$$W_+ : \vec{\phi}^{(+)} \mapsto \vec{\phi},$$

which is called the wave operator. The operator W_- is defined similarly. Under the condition of $4/3 < \gamma < \min(4, n)$ ($m \geq 2$ if $\gamma \leq \sqrt{2}$), Theorem 1.1 (ii) implies the completeness of W_{\pm} , namely, $\text{Range}W_{\pm} = \Sigma^{l,m}$.

There are many papers for the following equation

$$(5) \quad i \frac{\partial u}{\partial t} = -\frac{1}{2} \Delta u + f(u), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n,$$

$$(6) \quad u(0, x) = \phi(x),$$

where

$$f(u) = [V * |u|^2]u = \int_{\mathbf{R}^n} |x - y|^{-\gamma} |u(t, y)|^2 dy u(t, x)$$

(see, for example, [5, 7, 8, 9, 12]). The equation (5)-(6) is called the Hartree type equation. For the scattering problem for (5)-(6), the following results are known (see [9]).

[A] *Suppose that $1 < \gamma < \min(4, n)$, and $l, m \in \mathbf{N}$. Then, for any $\phi^{(+)} \in \Sigma^{l,m}$, there exists a unique $\phi \in \Sigma^{l,m}$ such that*

$$(7) \quad \lim_{t \rightarrow +\infty} \|\phi^{(+)} - U(-t)u(t)\|_{\Sigma^{l,m}} = 0,$$

where $u(t)$ is the solution of (5)-(6) with $U(-t)u(t) \in C(\mathbf{R}; \Sigma^{l,m})$. For any $\phi^{(-)} \in \Sigma^{l,m}$, the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (7).

[B] Suppose that $4/3 < \gamma < \min(4, n)$, and $l, m \in \mathbf{N}$. Then, for any $\phi \in \Sigma^{l,m}$, there exist unique $\phi^{(\pm)} \in \Sigma^{l,m}$ such that the solution $u(t)$ of (5)-(6) with $U(-t)u(t) \in C(\mathbf{R}; \Sigma^{l,m})$ satisfies

$$(8) \quad \lim_{t \rightarrow \pm\infty} \|\phi^{(\pm)} - U(-t)u(t)\|_{\Sigma^{l,m}} = 0.$$

Our main Theorem is the analogous results to [A], [B].

Since $U(t)$ is unitary in H^l , (4) implies that the asymptotic profiles of $\vec{u}(t)$ as $t \rightarrow \pm\infty$ are $U(t)\vec{\phi}^{(\pm)}$; and by the estimates

$$\|U(t)\vec{\phi}^{(\pm)}\|_p \leq (2\pi|t|)^{-\delta(p)} \|\vec{\phi}^{(\pm)}\|_{p'}, \quad 2 \leq p \leq \infty,$$

it is expected that

$$(9) \quad \|\vec{u}(t)\|_p = O(|t|^{-\delta(p)})$$

as $t \rightarrow \pm\infty$. Indeed, in Corollary 4.1, we shall prove (9) for $p = \infty$ under the suitable condition for $\vec{\phi}$.

Conversely, if (9) holds for some p sufficiently large, We can prove Theorem 1.1(ii). Actually, in Propositions 3.1 and 3.2, we prove (9) for some $p > 2$. This decay estimate is the key point of our proof of the main theorem.

The proof of Theorems [B] is much more simple than our proof of Theorem 1.1 (ii). But we cannot apply the method in [9] for (5)-(6) to prove Theorem 1.1 (ii). So we shall use the method in our work [15] to prove the main theorem.

2. Preliminaries

First, we collect various inequalities which will be used in later sections.

Lemma 2.1. (The Gagliardo-Nirenberg inequality) *Let $1 \leq q, r \leq \infty$ and j, m be any integers satisfying $0 \leq j < m$. If u is any function in $W^{m,q}(\mathbf{R}^n) \cap L^r(\mathbf{R}^n)$, then*

$$(10) \quad \sum_{|\alpha|=j} \|\nabla^\alpha u\|_p \leq C \left(\sum_{|\beta|=m} \|\nabla^\beta u\|_q \right)^a \|u\|_r^{1-a}$$

where

$$\frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{q} - \frac{m}{n}\right) + (1-a)\frac{1}{r}$$

for all a in the interval $j/m \leq a \leq 1$, where the constant C is independent of u , with the following exception: if $m - j - (n/q)$ is a nonnegative integer, then (10) is asserted for $j/m \leq a < 1$.

For the proof of Lemma 2.1, see [3, 14].

Lemma 2.2. *Let $\alpha > 0$. Then*

$$(11) \quad \|(-\Delta)^{\alpha/2} fg\| \leq C(\|(-\Delta)^{\alpha/2} f\| \|g\|_{\infty} + \|f\|_{\infty} \|(-\Delta)^{\alpha/2} g\|).$$

This lemma is essentially due to [4, 6]. The lemma is obtained as in the proof of Lemma 3.4 in [4] and Lemma 3.2 in [6], by using the theory of Besov space (for Besov space, see [1]).

Lemma 2.3. (The Hardy-Littlewood-Sobolev inequality) *Let $0 < \gamma < n, 1 < p, q < \infty$ and $1 + 1/p = \gamma/n + 1/q$. Then*

$$(12) \quad \| |x|^{-\gamma} * \phi \|_p \leq C \|\phi\|_q.$$

For the proof, see [10, 13].

A pair (q, r) of real numbers is called admissible, if it satisfies the condition $0 \leq \delta(q) = 2/r < 1$. Then

Lemma 2.4. *If a pair (q, r) is admissible, then for any $\psi \in L^2(\mathbf{R}^n)$, we have*

$$(13) \quad \|U(t)\psi\|_{q,r,\mathbf{R}} \leq C\|\psi\|.$$

Lemma 2.5. *We put $(Gu)(t) = \int_{t_0}^t U(t-\tau)u(\tau)d\tau$. Let $I \subset \mathbf{R}$ be an interval containing t_0 , and let pairs $(q_j, r_j), j = 1, 2$, be admissible. Then G maps $L^{r'_1}(I; L^{q'_1})$ into $L^{r_2}(I; L^{q_2})$ and satisfies*

$$(14) \quad \|Gu\|_{q_2, r_2, I} \leq C\|u\|_{q'_1, r'_1, I},$$

where C is independent of I .

For the proof of Lemmas 2.4 and 2.5, see [11, 16].

Next, we summarize the results for the Cauchy problem to (1)-(2) . We convert (1)-(2) into the integral equations

$$(15) \quad u_j(t) = U(t)\phi_j - i \int_0^t U(t - \tau)f_j(\vec{u}(\tau))d\tau, \quad j = 1, \dots, N,$$

then

Proposition 2.1. (i) *Suppose that $n \geq 2$, $0 < \gamma < \min(4, n)$, and $l, m \in \mathbf{N}$. Then for any $\vec{\phi} \in H^l$, there exists a unique solution $\vec{u}(t) \in C(\mathbf{R}; H^l)$ of (15). The solution $\vec{u}(t)$ satisfies following equalities.*

$$(16) \quad (u_j(t), u_k(t)) = (\phi_j, \phi_k), \quad j, k = 1, \dots, N,$$

especially,

$$(17) \quad \|u_j(t)\| = \|\phi_j\|, \quad j = 1, \dots, N;$$

and

$$(18) \quad E(\vec{u}(t)) = E(\vec{\phi}),$$

where

$$E(\vec{\psi}) = \sum_{j=1}^N \|\nabla\psi_j\|^2 + P(\vec{\psi}),$$

$$P(\vec{\psi}) = \sum_{j,k=1}^N \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |x - y|^{-\gamma} (|\psi_j(x)|^2 |\psi_k(y)|^2 - \psi_j(x)\bar{\psi}_k(x)\psi_k(y)\bar{\psi}_j(y)) dx dy;$$

(ii) *Furthermore, if $\vec{\phi} \in \Sigma^{l,m}$, then $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$, and the solution $\vec{u}(t)$ satisfies*

$$(19) \quad \sum_{j=1}^N \|xU(-t)u_j(t)\|^2 + t^2 P(\vec{u}(t)) = \sum_{j=1}^N \|x\psi_j\|^2 + (2 - \gamma) \int_0^t \tau P(\vec{u}(\tau)) d\tau.$$

REMARK. (i) By the Cauchy-Schwarz inequality, $P(\vec{\psi}) \geq 0$.

(ii) The equalities (17), (18) and (19) are called the L^2 -norm, the energy, and the pseudo-conformal conservation laws, respectively.

The proof of Propositions 2.1 is similar to that of the corresponding result for (5)-(6), so we shall omit it (see, for example, [8, 9, 12]).

3. Decay estimates for some norm of the solution

In this section we shall estimate the L^p -norm of the solution $\vec{u}(t)$ of (1)-(2) to prove the main theorem. We use the following transform

$$\begin{aligned} v_j(t) &= \mathcal{F}M(t)U(-t)u_j(t) \\ &= (it)^{n/2} \exp(-it|x|^2/2)u_j(t, tx), \end{aligned}$$

where \mathcal{F} is the Fourier transform in \mathbf{R}^n . This transform was introduced by N. Hayashi and T. Ozawa [7]. Then the equations (1) are transformed into the equations

$$(20) \quad i \frac{\partial}{\partial t} v_j = -\frac{1}{2t^2} \Delta v_j + \frac{1}{t^\gamma} f_j(\vec{v}), \quad j = 1, \dots, N,$$

and if $\vec{\phi} \in \Sigma^{1,m}$, then $\vec{v}(t) \in C((0, \infty); \Sigma^{m,1})$. The relations (17) and (19) are equivalent to

$$(21) \quad \frac{d}{dt} \|v_j(t)\| = 0, \quad j = 1, \dots, N$$

and

$$(22) \quad t^{-2} \frac{d}{dt} \sum_{j=1}^N \|\nabla v_j(t)\|^2 + t^{-\gamma} \frac{d}{dt} P(\vec{v}(t)) = 0,$$

respectively. Using the relation (22), we show

Lemma 3.1. *Suppose that $n \geq 2$, $0 < \gamma < \min(4, n)$, and $\vec{\phi} \in \Sigma^{1,1}$. Then, for $t \geq 1$,*

$$(23) \quad \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq \begin{cases} Ct^{2-\gamma} & \text{if } \gamma \leq \sqrt{2}, \\ C & \text{if } \gamma > \sqrt{2}. \end{cases}$$

Here, the constants C depend on $\|\vec{\phi}\|_{\Sigma^{1,1}}$.

Proof. If $\gamma < 2$,

$$\frac{d}{dt} \left(t^{\gamma-2} \sum_{j=1}^N \|\nabla v_j(t)\|^2 + P(\vec{v}(t)) \right) = (\gamma - 2)t^{\gamma-3} \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq 0,$$

and if $\gamma \geq 2$,

$$\frac{d}{dt} \left(\sum_{j=1}^N \|\nabla v_j(t)\|^2 + t^{2-\gamma} P(\vec{v}(t)) \right) = (2 - \gamma)t^{1-\gamma} P(\vec{v}(t)) \leq 0.$$

Hence

$$(24) \quad \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq \begin{cases} Ct^{2-\gamma} & \text{if } \gamma < 2, \\ C & \text{if } \gamma \geq 2. \end{cases}$$

So we shall prove (23) when $\sqrt{2} < \gamma < 2$. We multiply (20) by $\Delta \bar{v}_j$, and integrate the imaginary part over \mathbf{R}^n . Then

$$\frac{1}{2} \frac{d}{dt} \|\nabla v_j(t)\|^2 = t^{-\gamma} \text{Im} \int_{\mathbf{R}^n} f_j(\vec{v}) \Delta \bar{v}_j dx.$$

Since $\text{Im} \int_{\mathbf{R}^n} V * |v_k|^2 |\nabla v_j|^2 dx$ and $\text{Im} \sum_{j,k=1}^N \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) \nabla v_k \cdot \nabla \bar{v}_j dx$ are equal to zero, we have, by Hölder's inequality and Lemma 2.3,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \|\nabla v_j(t)\|^2 \\ &= t^{-\gamma} \text{Im} \sum_{j,k=1}^N \left[\int_{\mathbf{R}^n} v_j \nabla (V * |v_k|^2) \cdot \nabla \bar{v}_j dx + \int_{\mathbf{R}^n} v_k \nabla (V * (v_j \bar{v}_k)) \cdot \nabla \bar{v}_j dx \right] \\ &\leq Ct^{-\gamma} \|\vec{v}(t)\|_{\rho}^2 \sum_{j=1}^N \|\nabla v_j(t)\|^2, \end{aligned}$$

where $\rho = 2n/(n - \gamma)$. By Lemma 2.1 and (24), we have

$$\begin{aligned} \|v_j(t)\|_{\rho} &\leq C \|v_j\|^{1-\gamma/2} \|\nabla v_j\|^{\gamma/2} \\ &\leq Ct^{(2\gamma-\gamma^2)/4}. \end{aligned}$$

Therefore,

$$(25) \quad \frac{d}{dt} \sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq Ct^{-\gamma^2/2} \sum_{j=1}^N \|\nabla v_j(t)\|^2.$$

Since $\gamma^2/2 > 1$ if $\gamma > \sqrt{2}$, (25) and Gronwall's inequality yield (23).

Lemma 3.1 immediately implies

Proposition 3.1. *Suppose that $\sqrt{2} < \gamma < \min(4, n)$, and $\vec{\phi} \in \Sigma^{1,1}$. Then for the number p satisfying $0 \leq \delta(p) \leq 1$ if $n \geq 3$ and $0 \leq \delta(p) < 1$ if $n = 2$, the solution of (1)-(2) has the estimate*

$$(26) \quad \|\vec{u}(t)\|_p \leq C(1 + |t|)^{-\delta(p)}.$$

Proof. Since $\|\vec{u}(t)\|_p = t^{-\delta(p)}\|\vec{v}(t)\|_p$, Lemma 2.1 and Lemma 3.1 yield (26).

Now we show the L^p decay estimate of the solution in case $1 < \gamma \leq \sqrt{2}$.

Lemma 3.2. *Suppose that $1 < \gamma \leq \sqrt{2}$ and $\vec{\phi} \in \Sigma^{1,2}$. Then we have for $t \geq 1$,*

$$(27) \quad \sum_{j=1}^N \|\Delta v_j(t)\|^2 \leq \begin{cases} Ct^{(\gamma^2-8\gamma+10)/(2-\gamma)} & \text{if } n \geq 3, \\ Ct^{(\gamma^2-8\gamma+10)/(2-\gamma)+\varepsilon} & \text{if } n = 2. \end{cases}$$

Here ε is a positive number which can be chosen arbitrarily small, and the constant C depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$, and ε (the case $n = 2$).

Proof. We apply Δ to the both side of (20) and obtain

$$(28) \quad i \frac{\partial}{\partial t} \Delta v_j = -\frac{1}{2t^2} \Delta^2 v_j + \frac{1}{t^\gamma} \Delta f_j(\vec{v}), \quad j = 1, \dots, N.$$

Multiplying (28) by $\Delta \bar{v}_j$, integrating the imaginary part over \mathbf{R}^n , we have

$$\frac{1}{2} \frac{d}{dt} \|\Delta v_j(t)\|^2 = \frac{1}{t^\gamma} \text{Im} \int_{\mathbf{R}^n} \Delta f_j(\vec{v}) \Delta \bar{v}_j dx.$$

Since $\text{Im} \int_{\mathbf{R}^n} V * |v_k|^2 |\Delta v_j|^2 dx$ and $\text{Im} \sum_{j,k=1}^N \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) \Delta v_k \Delta \bar{v}_j dx$ are equal to zero,

$$(29) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \|\Delta v_j(t)\|^2 \\ &= t^{-\gamma} \text{Im} \sum_{j,k=1}^N \left[\int_{\mathbf{R}^n} \Delta(V * |v_k|^2) v_j \Delta \bar{v}_j dx + 2 \int_{\mathbf{R}^n} \nabla(V * |v_k|^2) \cdot \nabla v_j \Delta \bar{v}_j dx \right. \\ & \quad \left. + \int_{\mathbf{R}^n} \Delta(V * (v_j \bar{v}_k)) v_k \Delta \bar{v}_j dx + 2 \int_{\mathbf{R}^n} \nabla(V * (v_j \bar{v}_k)) \cdot \nabla v_k \Delta \bar{v}_j dx \right]. \end{aligned}$$

(i) Case $n \geq 3$. Hölder's inequality, Lemma 2.1 and Lemma 2.3 imply that the first term in the brackets of the right of (29) is dominated by

$$\begin{aligned} & C \int_{\mathbf{R}^n} |x|^{-\gamma-1} * (|\nabla v_k| |v_k|) |v_j| |\Delta v_j| dx \\ & \leq C \|\nabla v_k\| \|v_k\|_{2n/(n-2\gamma)} \|v_j\|_{2n/(n-2)} \|\Delta v_j\| \\ & \leq C \left(\sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma} \left(\sum_{j=1}^N \|\Delta v_j\|^2 \right)^{\gamma/2}. \end{aligned}$$

The other terms are estimated similarly. Therefore, it follows from (23) that for $t \geq 1$,

$$\begin{aligned} (30) \quad \frac{d}{dt} \sum_{j=1}^N \|\Delta v_j(t)\|^2 & \leq C t^{-\gamma} \left(\sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma} \left(\sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{\gamma/2} \\ & \leq C t^{(8-8\gamma+\gamma^2)/2} \left(\sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{\gamma/2}. \end{aligned}$$

Integrating this differential inequality, we have

$$(31) \quad \left(\sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{1-\gamma/2} \leq C t^{(10-8\gamma+\gamma^2)/2} + \left(\sum_{j=1}^N \|\Delta v_j(1)\|^2 \right)^{1-\gamma/2},$$

which implies (27). Since $\|\Delta v_j(1)\| = \| |x|^2 U(-1)u(1) \| \leq C \|\vec{\phi}\|_{\Sigma^{1,2}}$, the constant C in (23) depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$.

(ii) Case $n = 2$. Since

$$V_* = \frac{2^{n-\gamma} \pi^{n/2} \Gamma(\frac{n-\gamma}{2})}{\Gamma(\frac{\gamma}{2})} (-\Delta)^{(\gamma-n)/2}, \quad 0 < \gamma < n,$$

we have for $n = 2$, $-\Delta V_* = C(-\Delta)^{\gamma/2}$. Hence, by using Hölder's inequality, Lemma 2.1 and Lemma 2.2, we can estimate the first term in the brackets of the right of (29) by

$$\begin{aligned} (32) \quad & C \|(-\Delta)^{\gamma/2} |v_k|^2\| \|v_j\|_{\infty} \|\Delta v_j\| \\ & \leq C \|(-\Delta)^{\gamma/2} v_k\| \|\vec{v}\|_{\infty}^2 \|\Delta v_j\| \\ & \leq C \|\vec{v}\|_{\infty}^2 \left(\sum_{j=1}^N \|\nabla v_j\| \right)^{2-\gamma} \left(\sum_{j=1}^N \|\Delta v_j\|^2 \right)^{\gamma/2}. \end{aligned}$$

Since Lemma 2.1 implies

$$\begin{aligned} \|v_k\|_{\infty} & \leq C \|\Delta v_k\|^{2/(\theta+2)} \|v_k\|_{\theta}^{\theta/(\theta+2)} \\ & \leq C \|v_k\|^{2/(\theta+2)} \|\nabla v_k\|^{(\theta-2)/(\theta+2)} \|\Delta v_k\|^{2/(\theta+2)}, \end{aligned}$$

where $2 \leq \theta < \infty$, the right of (32) is dominated by

$$C \|v\|^a \left(\sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma-2a} \left(\sum_{j=1}^N \|\Delta v_j\|^2 \right)^{(\gamma+a)/2}$$

with $a = 2/(\theta + 2)$. The second term in the brackets of the right of (29) is estimated by

$$\begin{aligned} & \|V * (|\nabla v_k| |v_k|) \|_{n/(\gamma-1)} \|\nabla v_j\|_{2n/(n-\gamma+1)} \|\Delta v_j\| \\ & \leq C \|\vec{v}\|_\infty \left(\sum_{j=1}^N \|\nabla v_j\| \right)^{3-\gamma} \left(\sum_{j=1}^N \|\Delta v_j\|^2 \right)^{\gamma/2} \\ & \leq C \|\vec{v}\|^a \left(\sum_{j=1}^N \|\nabla v_j\| \right)^{4-\gamma-2a} \left(\sum_{j=1}^N \|\Delta v_j\|^2 \right)^{(\gamma+a)/2} \end{aligned}$$

The other terms are estimated similarly. Therefore, we have

$$(33) \quad \frac{d}{dt} \sum_{j=1}^N \|\Delta v_j(t)\|^2 \leq C t^{(8-8\gamma+\gamma^2)/2} \left(\sum_{j=1}^N \|\Delta v_j(t)\|^2 \right)^{(\gamma+a)/2}$$

Since the number a can be chosen arbitrarily small, this differential equation implies (27).

Lemma 3.3. *Suppose that $n \geq 2$, $1 < \gamma \leq \sqrt{2}$ and $\vec{\phi} \in \Sigma^{1,2}$. Then we have for $t \geq 1$,*

$$(34) \quad \|\vec{v}(t)\|_p \leq C.$$

Here, p satisfies $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$, and the constant C depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$.

Proof. For simplicity, we prove the lemma in case $n \geq 3$. We put $\|\vec{v}\|_{p,*} = [\int_{\mathbf{R}^n} (\sum_{l=1}^N |v_l|^2)^{p/2} dx]^{1/p}$, which is equivalent to the norm $\|\vec{v}\|_p = \sum_{l=1}^N \|v_l\|_p$. We multiply the equation (20) by $(\sum_{l=1}^N |v_l|^2)^{(p-2)/2} \bar{v}_j$, integrate their imaginary part over \mathbf{R}^n , and add them. Then we have

$$(35) \quad \frac{1}{p} \frac{d}{dt} \|\vec{v}(t)\|_{p,*}^p = -\frac{1}{2t^2} \text{Im} \sum_{j=1}^N \int_{\mathbf{R}^n} \Delta v_j \left(\sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \bar{v}_j dx,$$

since $\text{Im} \int_{\mathbf{R}^n} V * |v_k|^2 \left(\sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} |v_j|^2 dx$ and $\text{Im} \sum_{j,k=1}^N \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) v_k \bar{v}_j \left(\sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} dx$ are equal to zero. By the integration by parts and Hölder's inequality,

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\vec{v}(t)\|_{p,*}^p &= \frac{1}{2t^2} \text{Im} \sum_{j=1}^N \int_{\mathbf{R}^n} \nabla v_j \cdot \nabla \left(\left(\sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \bar{v}_j \right) dx \\ &\leq Ct^{-2} \sum_{j=1}^N \int_{\mathbf{R}^n} |\nabla v_j|^2 \left(\sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} dx \\ &\leq Ct^{-2} \sum_{j=1}^N \|\nabla v_j\|_p^2 \|\vec{v}(t)\|_{p,*}^{(p-2)}. \end{aligned}$$

We note that when $1 < \gamma \leq \sqrt{2}$, we have $0 < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma) < 1$, and so $2 < p < 2n/(n - 2)$. Then, Lemma 2.1, Lemma 3.1 and Lemma 3.2 yield

$$\begin{aligned} \|\nabla v_j\|_p &\leq C \|\nabla v_j\|^{1-\delta(p)} \|\Delta v_j\|^{\delta(p)} \\ &\leq Ct^\eta. \end{aligned}$$

Here

$$\eta = 2 - \gamma + \frac{6 - 4\gamma}{2 - \gamma} \delta(p),$$

and the constant C depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$. Therefore,

$$(36) \quad \frac{d}{dt} \|\vec{v}(t)\|_{p,*}^p \leq Ct^{-2+\eta} \|\vec{v}(t)\|_{p,*}^{(p-2)}.$$

Since $\eta < 1$ for p satisfying $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$, the estimate (34) follows by integrating the differential inequality (36).

By this lemma, we have

Proposition 3.2. *Suppose that $n \geq 2$, $1 < \gamma \leq \sqrt{2}$ and $\vec{\phi} \in \Sigma^{1,2}$. Then the solution of (1)-(2) has the following estimate*

$$(37) \quad \|\vec{u}(t)\|_p \leq C(1 + |t|)^{-\delta(p)},$$

where p satisfies $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$.

4. Proof of the main theorem

In this section, we shall prove Theorem 1.1. Since we can prove part (i) of the Theorem similar to the Theorem [A] for Hartree type equation, we omit the proof. Throughout this section, we put $q = 4n/(2n - \gamma)$ and $r = 8/\gamma$. Then the pair (q, r) is admissible. To prove part (ii), we introduce the following Banach space:

$$X^{l,m}(I) = \{u \in C(I; H^l); \|u\|_{X^{l,m}(I)} < \infty\},$$

where

$$\|u\|_{X^{l,m}(I)} = \sum_{|\alpha| \leq l} (\|\nabla^\alpha u\|_{2,\infty,I} + \|\nabla^\alpha u\|_{q,r,I}) + \sum_{|\beta| \leq m} (\|J^\beta u\|_{2,\infty,I} + \|J^\beta u\|_{q,r,I}).$$

Let $I = [T, \infty)$, where T will be defined later. Using Hölder’s inequality, Lemma 2.1 and Lemma 2.3, we have

$$(38) \quad \sum_{|\alpha|=l} \|\nabla^\alpha f_j(\vec{u})\|_{q'} \leq C \|\vec{u}\|_q^2 \sum_{k=1}^N \sum_{|\alpha|=l} \|\nabla^\alpha u_k\|_q$$

and

$$(39) \quad \sum_{|\beta|=m} \|J^\beta f_j(\vec{u})\|_{q'} \leq C \|\vec{u}\|_q^2 \sum_{k=1}^N \sum_{|\beta|=m} \|J^\beta u_k\|_q.$$

So we have, by Lemma 2.1 and Lemma 2.5,

$$(40) \quad \sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{2,\infty,I} \leq \sum_{|\alpha| \leq l} \|\nabla^\alpha U(-T)u_j(T)\| + C \sum_{|\alpha| \leq l} \|\nabla^\alpha f_j(\vec{u})\|_{q',r',I}.$$

Under the assumption of the theorem, Proposition 3.1 or Proposition 3.2 implies $\|\vec{u}(t)\|_q \leq Ct^{-\gamma/4}$. Therefore, by using (38) and Hölder’s inequality, the second term in the right of (40) is dominated by

$$(41) \quad \begin{aligned} & C \sum_{k=1}^N \sum_{|\alpha| \leq l} \left[\int_T^\infty (\|\vec{u}(\tau)\|_q^2 \|\nabla^\alpha u_k(\tau)\|_q)^{r'} d\tau \right]^{1/r'} \\ & \leq C \sum_{k=1}^N \sum_{|\alpha| \leq l} \left[\int_T^\infty (\tau^{-\gamma/2} \|\nabla^\alpha u_k(\tau)\|_q)^{r'} d\tau \right]^{1/r'} \\ & \leq C \left(\int_T^\infty \tau^{-2\gamma/(4-\gamma)} d\tau \right)^{(4-\gamma)/4} \sum_{k=1}^N \sum_{|\alpha| \leq l} \|\nabla^\alpha u_k\|_{q,r,I}. \end{aligned}$$

If $\gamma > 4/3$, the integral in the right of (41) converges. Hence,

$$(42) \quad \sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{2,\infty,I} \leq \|U(-T)u_j(T)\|_{\Sigma^{l,m}} + CT^{(4-3\gamma)/4} \|\vec{u}\|_{X^{l,m}(I)}.$$

We can estimate

$$\sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{q,r,I}, \sum_{|\beta| \leq m} \|J^\beta u_j\|_{2,\infty,I}, \text{ and } \sum_{|\beta| \leq m} \|J^\beta u_j\|_{q,r,I}$$

similarly. Therefore,

$$(43) \quad \|\vec{u}\|_{X^{l,m}(I)} \leq C\|U(-T)\vec{u}(T)\|_{\Sigma^{l,m}} + CT^{(4-3\gamma)/4} \|\vec{u}\|_{X^{l,m}(I)}.$$

If we choose T sufficiently large so that $CT^{(4-3\gamma)/4} \leq 1/2$, (43) implies

$$\|\vec{u}\|_{X^{l,m}(I)} \leq C\|U(-T)\vec{u}(T)\|_{\Sigma^{l,m}}.$$

Therefore, $\|\vec{u}\|_{X^{l,m}(\mathbf{R})}$ is finite. Once this has been proved, by the similar argument, for $t > s > 0$, we have

$$(44) \quad \begin{aligned} \|U(-t)\vec{u}(t) - U(-s)\vec{u}(s)\|_{\Sigma^{l,m}} &\leq C \left(\int_s^t \tau^{-2\gamma/(4-\gamma)} d\tau \right)^{(4-\gamma)/4} \|\vec{u}\|_{X^{l,m}(\mathbf{R})} \\ &\leq C \left(t^{(4-3\gamma)/4} - s^{(4-3\gamma)/4} \right). \end{aligned}$$

The right of (44) tends to zero as s, t tend to infinity. Thus the theorem has been proved.

Corollary 4.1. *Suppose that $4/3 < \gamma < \min(4, n)$, and $l, m \geq 1 + [n/2]$. Then for any $\vec{\phi} \in \Sigma^{l,m}$, the solution $\vec{u}(t)$ of (1)-(2) satisfies*

$$(45) \quad \|\vec{u}(t)\|_\infty \leq C(1 + |t|)^{-n/2}.$$

Proof. By the relation $J^\beta \vec{u}(t) = M(t)x^\beta M(-t)\vec{u}(t)$ and Lemma 2.1.

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