

Title	Limit processes for the branching processes with the Perron-Frobenius root. I
Author(s)	Sugitani, Sadao
Citation	Osaka Journal of Mathematics. 1987, 24(3), p. 579–604
Version Type	VoR
URL	https://doi.org/10.18910/8246
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

LIMIT PROCESSES FOR THE BRANCHING PROCESSES WITH THE PERRON-FROBENIUS ROOT I

SADAO SUGITANI

(Received June 12, 1985)

1. Introduction

We consider a multitype branching process whose mean matrix has the Perron-Frobenius root 1. The purpose of this paper is to show that the normalized sequence of such processes converges to some diffusion process when the initial population size goes to infinity.

Let $X(n) = (X_a(n))_{1 \le a \le d}$ be the *d*-type branching process and $M = (m_b^a)_{1 \le a, b \le d}$ its mean matrix. For later convenience we denote by $u = (u^a)_{1 \le a \le d}$ (resp. $v = (v_a)_{1 \le a \le d}$) the column vector (resp. row vector). Type *b* is said to be accessible from type *a* if the (a, b) component of M^a is positive for some $n \ge 0$. This relation is written as $a \rightarrow b$. If $a \rightarrow b$ and $b \rightarrow a$ then *a* and *b* are said to communicate with each other and this relation is written as $a \leftrightarrow b$. Since \leftrightarrow is an equivalence relation we can decompose the set of types $\{1, 2, \dots, d\}$ into the equivalence classes C_1, C_2, \dots, C_N . Set $M^a_\beta = (m^a_b)_{a \in Ca, b \in C\beta}$. Then we can write $M = (M^a_\beta)_{1 \le a, \beta \le N}$ and, by definition, each M^a_a is irreducible.

Hereafter we shall assume the following conditions:

(A.1)
$$M_{\alpha+1}^{\alpha} \neq Q$$
 for any α and $M_{\beta}^{\alpha} = O$ if $\beta < \alpha$,

and

(A.2) M^{α}_{α} is aperiodic and has the Perron-Frobenius root 1 for any α .

The first assumption means that if $a \in C_{\alpha}$, $b \in C_{\beta}$ and $\alpha < \beta$ then $a \rightarrow b$ but $b \Rightarrow a$. The second assumption means that each C_{α} is a final class or a critical class, where we say that C_{α} is a final class (resp. critical class) if the generating functions $F^{a}(s)$, $a \in C_{\alpha}$, are linear with respect to s^{a} , $a \in C_{\alpha}$ (resp. otherwise). Let $e^{a} = (\delta^{a}_{b})_{1 \leq b \leq d}$ where δ^{a}_{b} is the Kronecker's delta. The final assumption is

(A.3)
$$E_{e^a}[X_b(1)^4] < \infty$$
 for any a and b ,

in which P_{e^a} is the measure of the process X(n) starting at e^a . This assumption is needed to prove the tightness of the processes considered later.

We define a sequence of processes $\{X^n(t)\}_{n\geq 1}$ by

(1.1)
$$X^{n}(t) = X([nt]) + (nt - [nt]) (X([nt] + 1) - X([nt]))$$

where [t] denotes the largest integer not exceeding t. Then the following results are known (see. [3] and [5]):

(T.1) If $X^{n}(0) = ne^{a}$, $a \in C_{1}$ and t > 0, then the distribution of $(n^{-\alpha} X^{n}_{\alpha}(t))_{1 \le \alpha \le N}$ converges to some distribution as $n \to \infty$.

(T.2) If C_1 is a final class, $X^n(0) = e^a$, $a \in C_1$ and t > 0, then the distribution of $(n^{1-\omega} X^n_{\omega}(t))_{1 \le \omega \le N}$ converges to some distribution as $n \to \infty$.

The above results suggest us that these processes converge to some process. The meaning of convergence is as follows. Let C be the set of all continuous functions from $[0, \infty)$ to \mathbb{R}^d endowed with the topology of uniform convergence on each finite interval. Then the sequence of processes $\{(X_n(t), P_n)\}_{n\geq 1}$ is said to be convergent to the process (X(t), P) if Q_n converges to Q weakly where Q_n (resp. Q) is the probability measure on C induced from P_n (resp. P) by X_n (resp. X).

For the centered process the following result is known (see. [1: p. 192]):

(T.3) If N = 1, C_1 is a critical class, $\mathbf{X}^n(0) = n\mathbf{e}^a$ and t > 0 then the distribution of $n^{-1/2}(\mathbf{X}^n(t)-(\mathbf{X}^n(t)\mathbf{u})\mathbf{v})$ converges to some distribution as $n \to \infty$ for some suitably chosen vectors \mathbf{u} and \mathbf{v} .

Unfortunately the author could not prove the convergence of the above process. Instead we can prove that the process $\int_0^t (X([ns]) - (X([ns]) u) v) ds$ converges.

We shall study the combined processes $\{(X^{n}(t), Y^{n}(t))\}_{n\geq 1}$ where $Y^{n}(t)$ is defined by

(1.2)

$$Y_{\alpha}^{n}(t) = n \int_{0}^{t} (X_{\alpha}([ns]) - (X_{\alpha}([ns]) u^{\alpha}) v_{\alpha}) ds$$

$$= \sum_{k=0}^{[nt]-1} (X_{\alpha}(k) - (X_{\alpha}(k) u^{\alpha}) v_{\alpha})$$

$$+ (nt - [nt]) (X_{\alpha}([nt]) - (X_{\alpha}([nt]) u^{\alpha}) v_{\alpha}), \quad 1 \leq \alpha \leq N,$$

in which $u^{\sigma} = (u^{\sigma})_{a \in C_{\sigma}}$ and $v_{\sigma} = (v_{\sigma})_{a \in C_{\sigma}}$ are determined by

(1.3)
$$M^{\alpha}_{\alpha} u^{\alpha} = u^{\alpha}, v_{\alpha} M^{\alpha}_{\alpha} = v_{\alpha} \text{ and } v_{\alpha} u^{\alpha} = v_{\alpha} \mathbf{1}^{\alpha} = 1,$$

where $\mathbf{1}^{\boldsymbol{\omega}} = (1)_{a \in C_{\boldsymbol{\omega}}}$.

The purpose of this paper is to show the following two theorems.

Theorem A. If $\lim_{n\to\infty} n^{-\alpha} X^n_{\alpha}(0) = x_{\alpha} v_{\alpha}, 1 \le \alpha \le N$, then the sequence of processes $(n^{-\alpha} X^n_{\alpha}(t), n^{-\alpha} Y^n_{\alpha}(t))_{1 \le \alpha \le N}$ converges to some diffusion process.

Theorem B. Let C_1 be a final class. If $X_1^n(0)=1$, $X_a^n(0)=0$, $a \in C_1$, $a \neq 1$ and $\lim_{n \to \infty} n^{1-\sigma} X_{\sigma}^n(0) = x_{\sigma} v_{\sigma}$, $\alpha \ge 2$, then the sequence of processes $(n^{1-\sigma} X_{\sigma}^n(t), n^{1-\sigma} Y_{\sigma}^n(t))_{2 \le \sigma \le N}$ converges to some diffusion process.

The precise forms of these theorems are given in Theorem 2 in section 5 and Theorem 3 in section 6.

To prove our main theorems we must show the tightness and the convergence of any finite dimensional distributions. In general the tightness for a sequence of continuous processes $\{(X_n(t), P_n)\}_{n\geq 1}$ follows from

(1.4)
$$\sup E_n[X_n(0)^2] < \infty,$$

and the existence of C > 0 such that

(1.5)
$$E_n[(X_n(t)-X_n(s))^4] \leq C(t-s)^2 \text{ for any } n.$$

But in our case (1.4) is trivial by the definition of our process. To show (1.5) we shall estimate several moments of X(n) in section 2. Then we shall show the tightness in section 3. To prove the convergence of finite dimensional distributions we prepare a limit theorem in section 4. Applying these results we shall show our main theorems in sections 5 and 6. We shall give some comments for the limit processes in section 7. An example is given in section 8.

The author wishes to express his thanks to Prof. N. Ikeda and Prof. T. Watanabe who suggested him to extend the results in [5].

2. Preliminary results

In this section we shall estimate several moments for the process X(n). Before stating our results we prepare some notations. Set

(2.1)
$$P_{\beta}^{\sigma} = \begin{cases} \boldsymbol{u}^{\sigma} \otimes \boldsymbol{v}_{\sigma} & \text{if } \beta = \alpha , \\ \frac{1}{(\beta - \alpha)!} \prod_{\gamma = \sigma}^{\beta - 1} (\boldsymbol{v}_{\gamma} M_{\gamma + 1}^{\gamma} \boldsymbol{u}^{\gamma + 1}) (\boldsymbol{u}^{\sigma} \otimes \boldsymbol{v}_{\beta}) & \text{if } \beta \ge \alpha + 1 , \end{cases}$$

where $\boldsymbol{u}^{\boldsymbol{\omega}} \otimes \boldsymbol{v}_{\boldsymbol{\beta}} = (u^a v_b)_{a \in C_{\boldsymbol{\omega}}, b \in C_{\boldsymbol{\beta}}},$

(2.2)
$$Q_{\beta}^{\sigma} = \begin{cases} (I - M_{\alpha}^{\sigma} + P_{\alpha}^{\sigma})^{-1} (I - P_{\alpha}^{\sigma}) & \text{if } \beta = \alpha, \\ (\beta - \alpha)^{-1} P_{\beta-1}^{\sigma} M_{\beta}^{\beta-1} (I - M_{\beta}^{\beta} + P_{\beta}^{\beta})^{-1} (I - P_{\beta}^{\beta}) & \text{if } \beta \ge \alpha + 1, \end{cases}$$
(2.3)
$$\bar{\lambda}^{\sigma} = (I - P_{\alpha}^{\sigma}) \lambda^{\sigma} = \lambda^{\sigma} - (v_{\sigma} \lambda^{\sigma}) u^{\sigma}.$$

Then $v_{\alpha} \bar{\lambda}^{\alpha} = 0$ and

(2.4)
$$Y^{n}_{\omega}(t) \lambda^{\omega} = \sum_{k=0}^{\lfloor nt \rfloor - 1} X_{\omega}(k) \bar{\lambda}^{\omega} + (nt - \lfloor nt \rfloor) X_{\omega}(\lfloor nt \rfloor) \bar{\lambda}^{\omega}.$$

Set $M^n = ((M^n)^{\alpha}_{\beta})_{1 \leq \alpha, \beta \leq N}$. By (A.1) and (A.2) we have

(2.5)
$$\begin{cases} (M^{n})_{\beta}^{\alpha} = 0 \quad \text{if} \quad \beta < \alpha ,\\ (M^{n})_{\alpha}^{\alpha} = (M_{\alpha}^{\alpha})^{n} ,\\ (M^{n})_{\beta}^{\alpha} = \sum_{k=0}^{n-1} \sum_{\alpha < \gamma \le \beta} (M_{\alpha}^{\alpha})^{k} M_{\gamma}^{\alpha} (M^{n-k-1})_{\beta}^{\gamma} \quad \text{if} \quad \beta \ge \alpha + 1 . \end{cases}$$

By (A.2) it follows that $(M_{\alpha}^{\alpha} - P_{\alpha}^{\alpha})^{n} = O(\rho^{n})$ for some $0 < \rho < 1$. Then the following result is easily seen by the induction argument with respect to $\beta - \alpha$ (cf. [5: section 3]).

Lemma 2.1. Let (A.1) and (A.2) be satisfied. Then we have

(2.6)
$$(E_{e^{a}}[X_{\beta}(n) \lambda^{\beta}])_{a \in C_{\alpha}} = (M^{n})_{\beta}^{\alpha} \lambda^{\beta}$$
$$= \begin{cases} P_{\alpha}^{\alpha} \lambda^{\alpha} + (M_{\alpha}^{\alpha} - P_{\alpha}^{\alpha})^{n} \lambda^{\alpha} & \text{if } \beta = \alpha, \\ n^{\beta - \alpha} P_{\beta}^{\alpha} \lambda^{\beta} + O(n^{\beta - \alpha - 1}) & \text{if } \beta \ge \alpha + 1, \end{cases}$$

(2.7)
$$\begin{array}{l} (E_{e^{d}}[X_{\beta}(n)\,\bar{\boldsymbol{\lambda}}^{\beta}])_{a\in C_{\mathfrak{G}}} = (M^{n})^{\alpha}_{\beta}\,\bar{\boldsymbol{\lambda}}^{\beta} \\ = \begin{cases} (M^{\alpha}_{\sigma} - P^{\alpha}_{\sigma})^{n}\,\bar{\boldsymbol{\lambda}}^{\sigma} & \text{if } \beta = \alpha , \\ (\beta - \alpha)\,n^{\beta - \sigma - 1}\,Q^{\alpha}_{\beta}\,\bar{\boldsymbol{\lambda}}^{\beta} + O(n^{\beta - \sigma - 2}) & \text{if } \beta \geq \alpha + 1 , \end{cases}$$

and hence

(2.8)

$$(E_{e^{\sigma}}[\sum_{k=0}^{n-1} X_{\beta}(k) \bar{\lambda}^{\beta}])_{a \in C_{\sigma}} = \sum_{k=0}^{n-1} (M^{k})_{\beta}^{\sigma} \bar{\lambda}^{\beta}$$

$$= \begin{cases} Q_{\alpha}^{\sigma} \lambda^{\sigma} - Q_{\sigma}^{\sigma} (M_{\alpha}^{\sigma} - P_{\alpha}^{\sigma})^{n} \lambda^{\sigma} & \text{if } \beta = \alpha , \\ n^{\beta - \sigma} Q_{\beta}^{\sigma} \lambda^{\beta} + O(n^{\beta - \sigma - 1}) & \text{if } \beta \ge \alpha + 1 . \end{cases}$$

Next we shall estimate the higher order moments. We define $|\lambda^{\alpha}| = \sum_{\alpha \equiv \sigma_{\alpha}} |\lambda^{\alpha}|$.

Lemma 2.2. Let (A.1)–(A.3) be satisfied and $a \in C_{\alpha}$, $\beta \ge \alpha$ be fixed. Then there exists C>0 satisfying the following relations for $n \ge 1$ and p=2, 3, 4,

(2.9)
$$|E_{e^a}[(\boldsymbol{X}_{\boldsymbol{\beta}}(n) \boldsymbol{\lambda}^{\boldsymbol{\beta}})^p]| \leq C n^{p(\boldsymbol{\beta}-\boldsymbol{\alpha}+1)-1} |\boldsymbol{\lambda}^{\boldsymbol{\beta}}|^p,$$

(2.10)
$$|E_{e^{\alpha}}[(X_{\beta}(n)\bar{\lambda}^{\beta})^{p}]| \leq \begin{cases} C |\lambda^{\alpha}|^{p} & \text{if } \beta = \alpha \text{ and } p \leq 3, \\ Cn |\lambda^{\alpha}|^{4} & \text{if } \beta = \alpha \text{ and } p = 4, \\ Cn^{p(\beta-\alpha)-1} |\lambda^{\beta}|^{p} & \text{if } \beta \geq \alpha+1. \end{cases}$$

If C_{α} is a final class then we have the better estimates

(2.11)
$$|E_{e^{a}}[(X_{\beta}(n)\lambda^{\beta})^{p}]| \leq Cn^{p(\beta-\alpha)}|\lambda^{\beta}|^{p},$$

LIMIT PROCESSES FOR THE BRANCHING PROCESSES

$$(2.12) |E_{e^{\sigma}}[(X_{\beta}(n)\,\bar{\boldsymbol{\lambda}}^{\beta})^{p}]| \leq \begin{cases} C\,|\boldsymbol{\lambda}^{\omega}|^{p} & \text{if } \beta = \alpha ,\\ Cn\,|\boldsymbol{\lambda}^{\omega+1}|^{p} & \text{if } \beta = \alpha+1 \text{ and } p \leq 3 ,\\ Cn^{2}\,|\boldsymbol{\lambda}^{\omega+1}|^{4} & \text{if } \beta = \alpha+1 \text{ and } p = 4 ,\\ Cn^{p(\beta-\omega-1)}\,|\boldsymbol{\lambda}^{\beta}|^{p} & \text{if } \beta \geq \alpha+2 . \end{cases}$$

Outline of the proof. We first consider in the case p=2. Set

$$egin{aligned} &m{B}(n;m{\lambda},m{\lambda})=(E_{e^a}[(m{X}(n)\ m{\lambda})^2])_{1\leq a\leq d}\ ,\ &m{B}^a(m{\lambda}^m{eta},m{\lambda}^\gamma)=(\sum\limits_{b\in \mathcal{O}_m{eta}}\sum\limits_{c\in \mathcal{O}_m{\gamma}}D_b\ D_c\ F^a(m{1})\ m{\lambda}^bm{\lambda}^c)_{a\in C_a}\ ,\ &m{B}(m{\lambda},m{\lambda})=(\sum\limits_{m{eta},m{eta}^a(m{\lambda}^m{eta},m{\lambda}^\gamma))_{1\leq a\leq N}\ ,\ \end{aligned}$$

where D_a denotes the partial differentiation with respect to s^a . Then we have

$$\boldsymbol{B}(n;\boldsymbol{\lambda},\boldsymbol{\lambda}) = \boldsymbol{B}(M^{n-1}\boldsymbol{\lambda},M^{n-1}\boldsymbol{\lambda}) + M\boldsymbol{B}(n-1;\boldsymbol{\lambda},\boldsymbol{\lambda}),$$

and hence

(2.13)
$$\boldsymbol{B}(n;\boldsymbol{\lambda},\boldsymbol{\lambda}) = \sum_{k=0}^{n-1} M^{n-k-1} \boldsymbol{B}(M^k \boldsymbol{\lambda}, M^k \boldsymbol{\lambda}) + M^n \boldsymbol{B}(0;\boldsymbol{\lambda},\boldsymbol{\lambda}).$$

This means that

(2.14)
$$(E_{e^{a}}[(X_{\beta}(n) \lambda^{\beta})^{2}])_{a \in C_{a}} = \sum_{k=0}^{n-1} \sum_{\gamma=a}^{\beta} \sum_{\delta=\gamma}^{\beta} \sum_{\epsilon=\gamma}^{\beta} (M^{n-k-1})^{\alpha}_{\gamma} B^{\gamma}((M^{k})^{\delta}_{\beta} \lambda^{\beta}, (M^{k})^{\epsilon}_{\beta} \lambda^{\beta}) + \sum_{\gamma=a}^{\beta} (M^{n})^{\alpha}_{\gamma} B^{\gamma}(0; \lambda^{\beta}, \lambda^{\beta}).$$

Then (2.9) and (2.10) follow from Lemma 2.1. If C_{σ} is a final class then B^{σ} $(\lambda^{\sigma}, \lambda^{\sigma})=0$ and so (2.11) and (2.12) can be shown by the same method.

The other cases can be treated similarly. For the convenience to check the above results we only remark the forms of moments. Set

$$\begin{split} \boldsymbol{C}(n;\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) &= (E_{e^{d}}[(\boldsymbol{X}(n)\,\boldsymbol{\lambda})^{3}])_{1\leq a\leq d},\\ \boldsymbol{D}(n;\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) &= (E_{e^{d}}[(\boldsymbol{X}(n)\,\boldsymbol{\lambda})^{4}])_{1\leq a\leq d},\\ \boldsymbol{C}(\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) &= (\sum_{b,c,f} D_{b}\,D_{c}\,D_{f}\,F^{a}(1)\,\lambda^{b}\,\lambda^{c}\,\lambda^{f})_{1\leq a\leq d},\\ \boldsymbol{D}(\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) &= (\sum_{b,c,f,g} D_{b}\,D_{c}\,D_{f}\,D_{g}\,F^{a}(1)\,\lambda^{b}\,\lambda^{c}\,\lambda^{f}\,\lambda^{g})_{1\leq a\leq d}. \end{split}$$

Then we have

(2.15)
$$C(n; \lambda, \lambda, \lambda) = \sum_{k=0}^{n-1} M^{n-k-1} C(M^k \lambda, M^k \lambda, M^k \lambda) + 3 \sum_{k=0}^{n-1} M^{n-k-1} B(M^k \lambda, B(k; \lambda, \lambda)) + M^n C(0; \lambda, \lambda, \lambda),$$

$$D(n: \lambda, \lambda, \lambda, \lambda) = \sum_{k=0}^{n-1} M^{n-k-1} D(M^k \lambda, M^k \lambda, M^k \lambda, M^k \lambda)$$

+ $6 \sum_{k=0}^{n-1} M^{n-k+1} C(M^k \lambda, M^k \lambda, B(k: \lambda, \lambda))$
(2.16) + $4 \sum_{k=1}^{n-1} M^{n-k-1} B(M^k \lambda, C(k: \lambda, \lambda, \lambda))$
+ $3 \sum_{k=0}^{n-1} M^{n-k-1} B(B(k: \lambda, \lambda), B(k: \lambda, \lambda))$
+ $M^n D(0: \lambda, \lambda, \lambda, \lambda)$.

Next we shall estimate the moments for $Y^{n}(t)$.

Lemma 2.3. Let (A.1)–(A.3) be satisfied and $a \in C_{\alpha}$, $\beta \ge \alpha$ be fixed. Then there exists C > 0 satisfying the following relations for $n \ge 1$,

(2.17)
$$E_{e^{\sigma}}\left[\left(\sum_{k=0}^{n-1} X_{\beta}(k) \, \bar{\boldsymbol{\lambda}}^{\beta}\right)^{2}\right] \leq \begin{cases} Cn \, |\boldsymbol{\lambda}^{\sigma}|^{2} & \text{if } \beta = \alpha ,\\ Cn^{2(\beta-\alpha)} \, |\boldsymbol{\lambda}^{\beta}|^{2} & \text{if } \beta \geq \alpha+1 , \end{cases}$$

(2.18)
$$E_{e^{\mathfrak{a}}}[(\sum_{k=1}^{n-2} X_{\beta}(k) \bar{\boldsymbol{\lambda}}^{\beta})^{4}] \leq C n^{4(\beta-\mathfrak{a}+1)-1} |\boldsymbol{\lambda}^{\beta}|^{4}.$$

If C_{α} is a final class then we obtain the better estimate

(2.19)
$$E_{e^{\alpha}}[(\sum_{k=0}^{n-1} X_{\beta}(k) \bar{\boldsymbol{\lambda}}^{\beta})^{4}] \leq \begin{cases} Cn^{2} |\boldsymbol{\lambda}^{\alpha}|^{4} & \text{if } \beta = \alpha, \\ Cn^{4(\beta-\alpha)} |\boldsymbol{\lambda}^{\beta}|^{4} & \text{if } \beta \geq \alpha+1. \end{cases}$$

Proof. Since

$$\begin{split} E_{e^{d}} & \left[\left(\sum_{k=0}^{n-1} X_{\beta}(k) \lambda^{\beta} \right)^{2} \right] \\ & \leq 2 \sum_{k=0}^{n-1} E_{e^{d}} \left[X_{\beta}(k) \lambda^{\beta} \cdot \sum_{l=k}^{n-1} X_{\beta}(l) \lambda^{\beta} \right] \\ & = 2 \sum_{k=0}^{n-1} \sum_{\gamma=\alpha}^{\beta} E_{e^{d}} \left[X_{\beta}(k) \lambda^{\beta} \cdot X_{\gamma}(k) \left(\sum_{l=0}^{n-k-1} M^{l} \right)^{\gamma}_{\beta} \lambda^{\beta} \right] \\ & \leq 2 \sum_{k=0}^{n-1} \sum_{\gamma=\alpha}^{\beta} \sqrt{E_{e^{d}} \left[(X_{\beta}(k) \lambda^{\beta})^{2} \right] E_{e^{d}} \left[(X_{\gamma}(k) \left(\sum_{l=0}^{n-k-1} M^{k} \right)^{\gamma}_{\beta} \lambda^{\beta} \right)^{2} \right]} \,, \end{split}$$

(2.17) follows from Lemma 2.1 and Lemma 2.2. Combining Lemma 2.2 with

$$E_{e^{\mathbf{d}}}[(\sum_{k=0}^{n-1} X_{\beta}(k) \boldsymbol{\lambda}^{\beta})^{4}] \leq n^{3} \sum_{k=0}^{n-1} E_{e^{\mathbf{d}}}[(X_{\beta}(k) \boldsymbol{\lambda}^{\beta})^{4}],$$

we obtain (2.18) for $\beta \ge \alpha + 1$ and (2.19) for $\beta \ge \alpha + 2$. If C_{α} is a final class then the process $(X_{\alpha}(n), \sum_{a \in \mathcal{O}_{\alpha}} v_a P_{e^a})$ is a stationary Markov chain having the mixing property and so (2.19) holds if $\beta = \alpha$. To show the rest cases, (2.18) for $\beta = \alpha$ and (2.19) for $\beta = \alpha + 1$, we first expand (2.18) as follows. LIMIT PROCESSES FOR THE BRANCHING PROCESSES

(2.20)
$$E_{e^{a}}[(\sum_{k=0}^{n-1} X(k) \lambda)^{4}] = \sum_{k=0}^{n-1} I_{k}^{n} + 4 \sum_{k=0}^{n-1} \sum_{l=k+1}^{n-1} I_{k,l}^{n} + 6 \sum_{k=0}^{n-1} \sum_{l=k+1}^{n-1} II_{k,l}^{n} + 12 \sum_{k=0}^{n-1} \sum_{l=k+1}^{n-1} \sum_{m=l+1}^{n-1} \sum_{k=0}^{n-1} I_{k,l,m}^{n},$$

where

(2.21)

$$\begin{cases}
I_{k}^{n} = E_{e^{d}}[(\mathbf{X}(k) \boldsymbol{\lambda})^{3} \cdot (\mathbf{X}(k) \boldsymbol{\lambda} + 4\sum_{l=m+1}^{n-1} \mathbf{X}(l) \boldsymbol{\lambda})], \\
I_{k,l}^{n} = E_{e^{d}}[\mathbf{X}(k) \boldsymbol{\lambda} \cdot (\mathbf{X}(l) \boldsymbol{\lambda})^{2} \cdot (\mathbf{X}(l) \boldsymbol{\lambda} + 3\sum_{m=l+1}^{n-1} \mathbf{X}(m) \boldsymbol{\lambda})], \\
II_{k,l}^{n} = E_{e^{d}}[(\mathbf{X}(k) \boldsymbol{\lambda})^{2} \cdot \mathbf{X}(l) \boldsymbol{\lambda} \cdot (\mathbf{X}(l) \boldsymbol{\lambda} + 2\sum_{j=m+1}^{n-1} \mathbf{X}(m) \boldsymbol{\lambda})], \\
I_{k,l,m}^{n} = E_{e^{d}}[\mathbf{X}(k) \boldsymbol{\lambda} \cdot \mathbf{X}(l) \boldsymbol{\lambda} \cdot \mathbf{X}(m) \boldsymbol{\lambda} \cdot (\mathbf{X}(m) \boldsymbol{\lambda} + 2\sum_{j=m+1}^{n-1} \mathbf{X}(j) \boldsymbol{\lambda})].
\end{cases}$$

Set
$$\lambda(p, n) = \lambda + p \sum_{j=1}^{n-1} M^j \lambda$$
 and use the Markov property, then we have
(2.22)
$$\begin{cases}
I_k^n = E_{e^a}[(X(k)\lambda)^3 \cdot X(k)\lambda(4, n-k)], \\
I_{k,l}^n = E_{e^a}[X(k)\lambda \cdot (X(l)\lambda)^2 \cdot X(l)\lambda(3, n-l)], \\
II_{k,l}^n = E_{e^a}[(X(k)\lambda)^2 \cdot X(l)\lambda \cdot X(l)\lambda(2, n-l)], \\
I_{k,l,m}^n = E_{e^a}[X(k)\lambda \cdot X(l)\lambda \cdot X(m)\lambda(2, n-m)].
\end{cases}$$

Let $\beta = \alpha$ or $\beta = \alpha + 1$ and set $\lambda = \overline{\lambda}^{\beta}$. Then it follows from Lemma 2.1 that if $\beta = \alpha$ then

(2.23)
$$\boldsymbol{v}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}}(\boldsymbol{p}, \boldsymbol{k}) = 0, \ |\boldsymbol{\lambda}^{\boldsymbol{\omega}}(\boldsymbol{p}, \boldsymbol{k})| \leq C \ |\boldsymbol{\lambda}^{\boldsymbol{\omega}}|,$$

and if $\beta = \alpha + 1$ then

(2.24)
$$\begin{cases} |\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\boldsymbol{p},\boldsymbol{k})| \leq Ck |\boldsymbol{\lambda}^{\boldsymbol{\alpha}+1}|, \, \boldsymbol{v}_{\boldsymbol{\alpha}+1} \, \boldsymbol{\lambda}^{\boldsymbol{\alpha}+1}(\boldsymbol{p},\boldsymbol{k}) = 0, \\ |\boldsymbol{\lambda}^{\boldsymbol{\alpha}+1}(\boldsymbol{p},\boldsymbol{k})| \leq C |\boldsymbol{\lambda}^{\boldsymbol{\alpha}+1}|. \end{cases}$$

Combining these estimates with Lemma 2.2 we obtain

(2.25)
$$|I_{k}^{n}|, |I_{k,l}^{n}|, |II_{k,l}^{n}| \\ \leq \begin{cases} C|\boldsymbol{\lambda}^{\boldsymbol{\omega}}|^{4} & \text{if } \boldsymbol{\beta} = \boldsymbol{\alpha}, \\ Cn|\boldsymbol{\lambda}^{\boldsymbol{\omega}+1}|^{4} & \text{if } C_{\boldsymbol{\omega}} \text{ is a final class and } \boldsymbol{\beta} = \boldsymbol{\alpha} + 1. \end{cases}$$

Hence the proof is completed if we can show

$$\sum_{0 \leq k < l < m \leq n-1} I_{k,k,m}^n$$

$$\leq \begin{cases} Cn^3 |\boldsymbol{\lambda}^{\boldsymbol{\sigma}}|^4 & \text{ if } \boldsymbol{\beta} = \boldsymbol{\alpha}, \\ Cn^4 |\boldsymbol{\lambda}^{\boldsymbol{\sigma}+1}|^4 & \text{ if } C_{\boldsymbol{\sigma}} \text{ is a final class and } \boldsymbol{\beta} = \boldsymbol{\alpha}+1. \end{cases}$$

Set

$$\begin{split} \boldsymbol{A}(\boldsymbol{n}:\boldsymbol{\lambda}(1),\boldsymbol{\lambda}(2)) \\ &= (E_{e^{d}}[(\boldsymbol{X}(\boldsymbol{n})-\boldsymbol{e}^{a}\boldsymbol{M}^{n})\boldsymbol{\lambda}(1)\boldsymbol{\cdot}(\boldsymbol{X}(\boldsymbol{n})-\boldsymbol{e}^{a}\boldsymbol{M}^{n})\boldsymbol{\lambda}(2)])_{1\leq n\leq d} \\ &= \boldsymbol{B}(\boldsymbol{n}:\boldsymbol{\lambda}(1),\boldsymbol{\lambda}(2))-(\boldsymbol{e}^{a}\boldsymbol{M}^{n}\boldsymbol{\lambda}(1)\boldsymbol{\cdot}\boldsymbol{e}^{a}\boldsymbol{M}^{n}\boldsymbol{\lambda}(2))_{1\leq a\leq d} \,. \end{split}$$

Then we have

$$\begin{split} I_{k,l,m}^{n} &= E_{e^{a}}[X(k) \boldsymbol{\lambda} \cdot X(l) \boldsymbol{\lambda} \cdot X(l) \ M^{m-l} \boldsymbol{\lambda} \cdot X(l) \ M^{m-l} \boldsymbol{\lambda}(2, n-m)] \\ &+ E_{e^{a}}[X(k) \boldsymbol{\lambda} \cdot X(l) \boldsymbol{\lambda} \cdot X(l) \ A(m-l; \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m))] \\ &= E_{e^{a}}[X(k) \boldsymbol{\lambda} \cdot X(l) \boldsymbol{\lambda} \cdot X(l) \ M^{m-l} \boldsymbol{\lambda} \cdot X(l) \ M^{m-l} \boldsymbol{\lambda}(2, n-m)] \\ &+ E_{e^{a}}[X(k) \boldsymbol{\lambda} \cdot X(k) \ M^{m-l} \boldsymbol{\lambda} \cdot X(k) \ M^{l-k} \ A(m-l; \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m))] \\ &+ E_{e^{a}}[X(k) \boldsymbol{\lambda} \cdot X(k) \ A(l-k; \boldsymbol{\lambda}, A(m-l; \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))] \\ &= \sum_{p=1}^{3} I_{k,l,m}^{u}(p) \,. \end{split}$$

Therefore it suffices to show that

(2.26)
$$\sum_{\substack{0 \le k < l < m \le n-1}} I^n_{k,l,m}(p) \\ \le \begin{cases} Cn^3 |\boldsymbol{\lambda}^{\boldsymbol{\omega}}|^4 & \text{if } \boldsymbol{\beta} = \boldsymbol{\alpha}, \\ Cn^4 |\boldsymbol{\lambda}^{\boldsymbol{\omega}+1}|^4 & \text{if } C_{\boldsymbol{\omega}} \text{ is a final class and } \boldsymbol{\beta} = \boldsymbol{\alpha}+1, \end{cases}$$

holds for p=1, 2, 3. If p=1 then by Lemma 2.1 and Lemma 2.2 we have

$$(2.27) \begin{array}{l} I_{k,l,m}^{u}(1) \\ \leq \begin{cases} Cn \ \rho^{m-l} | \boldsymbol{\lambda}^{\omega} |^{4} & \text{if } \beta = \alpha , \\ C(n+n^{2} \ \rho^{m-l}) | \boldsymbol{\lambda}^{\omega+1} |^{4} & \text{if } C_{\omega} \text{ is a final class and } \beta = \alpha+1 , \end{cases}$$

for some $0 < \rho < 1$ and hence (2.26) holds. Then we shall consider the rest cases. By (2.13) it follows that if $\beta = \alpha$ or $\alpha + 1$ then

$$|A^{\beta}(m-l:\lambda,\lambda(2,n-m))| \leq C |\lambda^{\beta}|^{2},$$

$$|A^{\beta}(l-k:\lambda,A(m-l:\lambda,\lambda(2,n-m)))| \leq C |\lambda^{\beta}|^{3},$$

and if C_{σ} is a final class then

$$|A^{\alpha}(m-l:\boldsymbol{\lambda},\boldsymbol{\lambda}(2,n-m)))| \leq Cn |\boldsymbol{\lambda}^{\alpha+1}|^2,$$

$$|A^{\alpha}(l-k:\boldsymbol{\lambda},\boldsymbol{A}(m-l:\boldsymbol{\lambda},\boldsymbol{\lambda}(2,n-m)))| \leq Cn |\boldsymbol{\lambda}^{\alpha+1}|^3.$$

Since

$$I_{k,l,m}^{n}(2) = C^{a}(k; \lambda, M^{l-m} \lambda, M^{l-k} A(m-l; \lambda, \lambda(2, n-m))),$$

$$I_{k,l,m}^{n}(3) = B^{a}(k; \lambda, A(l-k; \lambda, A(m-l; \lambda, \lambda(2, n-m))),$$

we obtain the same estimates in (2.27) by (2.13), (2.15) and the preceding estimates. Thus we have completed the proof.

We sometimes assume that λ^{α} is a *d*-dimensional vector such that $(\lambda^{\alpha})^{a} = \lambda^{a}$ if $a \in C_{\alpha}$ and $(\lambda^{\alpha})^{a} = 0$ if $a \notin C_{\alpha}$.

Lemma 2.4. Let (A.1)-(A.3) be satisfied, $a \in C_{\alpha}$, $\beta \geq \alpha$ and T > 0 be fixed and p=2 or 4. Then there exists C(T)>0 such that the following relations hold for $n\geq 1$ and $m\leq nT$,

(2.28)

$$E_{e^{a}}[(\boldsymbol{X}(m) (M^{n}-I) \boldsymbol{\lambda}^{\beta})^{p}] \leq \begin{cases} C(T) |\boldsymbol{\lambda}^{\alpha}|^{b} & \text{if } \beta = \alpha, \\ C(T) n^{p(\beta-\alpha+1)-1} |\boldsymbol{\lambda}^{\beta}|^{b} & \text{if } \beta \ge \alpha+1, \\ E_{e^{a}}[(\boldsymbol{X}(m) \sum_{k=0}^{n-1} M^{k} \boldsymbol{\bar{\lambda}}^{\beta})^{p}] \end{cases}$$

(2.29)
$$\leq \begin{cases} C(T) |\boldsymbol{\lambda}^{\boldsymbol{\alpha}}|^{\boldsymbol{\beta}} & \text{if } \boldsymbol{\beta} = \boldsymbol{\alpha}, \\ C(T) n^{\boldsymbol{p}(\boldsymbol{\beta} - \boldsymbol{\alpha} + 1) - 1} |\boldsymbol{\lambda}^{\boldsymbol{\beta}}|^{\boldsymbol{p}} & \text{if } \boldsymbol{\beta} \ge \boldsymbol{\alpha} + 1 \end{cases}$$

If C_{α} is a final class then

(2.30)
$$E_{e^{a}}[(\boldsymbol{X}(m) (\boldsymbol{M}^{n} - I) \boldsymbol{\lambda}^{\beta})^{p}] \leq C(T) n^{p(\beta - \alpha)} |\boldsymbol{\lambda}^{\beta}|^{p},$$

(2.31)
$$E_{e^{\mathfrak{s}}}[(\boldsymbol{X}(m)\sum_{k=0}^{n-1}M^{k}\,\bar{\boldsymbol{\lambda}}^{\beta})^{p}] \leq C(T)\,n^{p(\beta-\omega)}|\boldsymbol{\lambda}^{\beta}|^{p}\,.$$

Proof. Since p is even we have

$$\begin{split} E_{e^{d}}[(\boldsymbol{X}(m) \left(\boldsymbol{M}^{n}-\boldsymbol{I}\right)\boldsymbol{\lambda}^{\beta})^{p}] &= E_{e^{d}}[\left(\sum_{\boldsymbol{\gamma}=\boldsymbol{\omega}}^{\beta}\boldsymbol{X}_{\boldsymbol{\gamma}}(m) \left(\boldsymbol{M}^{n}-\boldsymbol{I}\right)^{\boldsymbol{\gamma}}_{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\beta})^{p}\right] \\ &\leq N^{p-1}\sum_{\boldsymbol{\gamma}=\boldsymbol{\omega}}^{\beta}E_{e^{d}}[(\boldsymbol{X}_{\boldsymbol{\gamma}}(m) \left(\boldsymbol{M}^{n}-\boldsymbol{I}\right)^{\boldsymbol{\gamma}}_{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\beta})^{p}] \,. \end{split}$$

Then (2.28) and (2.30) follow from $v_{\beta}(M^{n}-I)^{\beta}_{\beta} \lambda^{\beta} = 0$ and the preceding lemmas. Since the rest cases can be treated similarly we omit the proof.

We shall end this section by showing two lemmas which will be used to prove the tightness. Let P_x denote the measure of the process X(n) starting at x.

Lemma 2.5. Let (A.1)–(A.3) be satisfied and $m \ge l+1$. Then we have

(2.32)
$$E_{\mathbf{x}}[(\mathbf{X}(m)\boldsymbol{\lambda} - \mathbf{x}M^{m}\boldsymbol{\lambda} - \mathbf{X}(l)\boldsymbol{\lambda} + \mathbf{x}M^{l}\boldsymbol{\lambda})^{2}] \\ \leq \sum_{a=1}^{d} \sum_{b=1}^{d} x_{a} E_{e^{a}}[X_{b}(l)] E_{e^{b}}[(\mathbf{X}(m-l)\boldsymbol{\lambda})^{2}]$$

$$(2.33) + \sum_{a=1}^{d} x_{a} E_{e^{a}} [(\boldsymbol{X}(l) (M^{m-l}-I)\boldsymbol{\lambda})^{2}],$$

$$E_{\boldsymbol{x}} [(\boldsymbol{X}(m)\boldsymbol{\lambda} - \boldsymbol{x} M^{m}\boldsymbol{\lambda} - \boldsymbol{X}(l)\boldsymbol{\lambda} + \boldsymbol{x} M^{l}\boldsymbol{\lambda})^{4}] \leq 3E_{\boldsymbol{x}} [(\boldsymbol{X}(m)\boldsymbol{\lambda} - \boldsymbol{x} M^{m}\boldsymbol{\lambda} - \boldsymbol{X}(l)\boldsymbol{\lambda} + \boldsymbol{x} M^{l}\boldsymbol{\lambda})^{2}]^{2} + 24d \sum_{a=1}^{d} \sum_{b=1}^{d} x_{a} E_{e^{a}} [X_{b}(l)^{2}] E_{e^{b}} [(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^{2}]^{2} + 128 \sum_{a=1}^{d} \sum_{b=1}^{d} x_{a} E_{e^{a}} [X_{b}(l)] E_{e^{b}} [(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^{4}] + 128 \sum_{a=1}^{d} x_{a} E_{e^{a}} [(\boldsymbol{X}(l) (M^{m-l}-I)\boldsymbol{\lambda})^{4}].$$

Proof. By the branching property of X(n) it follows that

(2.34)

$$E_{\mathbf{x}}[(\mathbf{X}(m) \boldsymbol{\lambda} - \mathbf{x} M^{m} \boldsymbol{\lambda} - \mathbf{X}(l) \boldsymbol{\lambda} + \mathbf{x} M^{l} \boldsymbol{\lambda})^{2}]$$

$$= \sum_{a=1}^{d} x_{a} E_{e^{a}}[(\mathbf{X}(m) \boldsymbol{\lambda} - e^{a} M^{m} \boldsymbol{\lambda} - \mathbf{X}(l) \boldsymbol{\lambda} + e^{a} M^{l} \boldsymbol{\lambda})^{2}]$$

$$= \sum_{a=1}^{d} x_{a} I_{a},$$

$$E_{\mathbf{x}}[(\mathbf{X}(m) \boldsymbol{\lambda} - \mathbf{x} M^{m} \boldsymbol{\lambda} - \mathbf{X}(l) \boldsymbol{\lambda} + \mathbf{x} M^{l} \boldsymbol{\lambda})^{4}]$$

$$\leq 3E_{\mathbf{x}}[(\mathbf{X}(m) \boldsymbol{\lambda} - \mathbf{x} M^{m} \boldsymbol{\lambda} - \mathbf{X}(l) \boldsymbol{\lambda} + \mathbf{x} M^{l} \boldsymbol{\lambda})^{2}]^{2}$$

$$(2.35)$$

$$= \sum_{a=1}^{d} x_{a} E_{\mathbf{x}}[(\mathbf{X}(m) \boldsymbol{\lambda} - \mathbf{x} M^{m} \boldsymbol{\lambda} - \mathbf{X}(l) \boldsymbol{\lambda} + \mathbf{x} M^{l} \boldsymbol{\lambda})^{2}]^{2}$$

$$+ \sum_{a=1}^{d} x_a E_{e^a} [(\boldsymbol{X}(m) \boldsymbol{\lambda} - \boldsymbol{e}^a M^m \boldsymbol{\lambda} - \boldsymbol{X}(l) \boldsymbol{\lambda} + \boldsymbol{e}^a M^l \boldsymbol{\lambda})^4]$$
$$= 3(\sum_{a=1}^{d} x_a I_a)^2 + \sum_{a=1}^{d} x_a II_a.$$

First remark that

$$\begin{split} I_{a} &= E_{e^{a}}[((X(m) \lambda - X(l) M^{m-l} \lambda) + (X(l) \lambda - e^{a} M^{l}) (M^{m-l} - I) \lambda)^{2}] \\ &= E_{e^{a}}[(X(m) \lambda - X(l) M^{m-l} \lambda)^{2}] \\ &+ E_{e^{a}}[((X(l) - e^{a} M^{l}) (M^{m-l} - I) \lambda)^{2}] \\ &\leq I_{a}^{1} + E_{e^{a}}[(X(l) (M^{m-l} - I) \lambda)^{2}] . \end{split}$$

Then by the Markov property and (2.34) we obtain

$$\begin{split} I_a^1 &= E_{e^a}[E_{X(l)}[(\boldsymbol{X}(m-l)\,\boldsymbol{\lambda} - \boldsymbol{X}(0)\,M^{m-l}\,\boldsymbol{\lambda})^2]] \\ &= \sum_{b=1}^d E_{e^a}[X_b(l)]\,E_{e^b}[(\boldsymbol{X}(m-l)\,\boldsymbol{\lambda} - e^bM^{m-l}\,\boldsymbol{\lambda})^2] \\ &\leq \sum_{b=1}^d E_{e^a}[X_b(l)]\,E_{e^b}[(\boldsymbol{X}(m-l)\,\boldsymbol{\lambda})^2]\,, \end{split}$$

and (2.32) follows. Next we shall show (2.33).

$$II_{a} = E_{e^{a}}[((X(m) \lambda - X(l) M^{m-l} \lambda) + (X(l) - e^{a} M^{l}) (M^{m-l} - I) \lambda)^{4}]$$

$$\leq 8E_{e^a}[(\boldsymbol{X}(m) \boldsymbol{\lambda} - \boldsymbol{X}(l) M^{m-l} \boldsymbol{\lambda})^4] \\ + 8E_{e^a}[((\boldsymbol{X}(l) - e^a M^l) (M^{m-l} - I) \boldsymbol{\lambda})^4] \\ \leq 8II_a^1 + 128E_{e^a}[(\boldsymbol{X}(l) (M^{m-l} - I) \boldsymbol{\lambda})^4].$$

Then by the Markov property and (2.35) we have

$$\begin{split} II_{a}^{1} &= E_{e^{a}}[E_{X(l)}[(X(m-l) \lambda - X(0) M^{m-l} \lambda)^{4}]] \\ &\leq 3E_{e^{a}}[(\sum_{b=1}^{d} X_{b}(l) E_{e^{b}}[(X(m-l) \lambda - e^{b} M^{m-l} \lambda)^{2}])^{2}] \\ &+ E_{e^{a}}[\sum_{b=1}^{d} X_{b}(l) E_{e^{b}}[(X(m-l) \lambda - e^{b} M^{m-l} \lambda)^{4}]] \\ &\leq 3d \sum_{b=1}^{d} E_{e^{a}}[X_{b}(l)^{2}] E_{e^{b}}[(X(m-l) \lambda)^{2}]^{2} \\ &+ 16 \sum_{b=1}^{d} E_{e^{a}}[X_{b}(l)] E_{e^{b}}[(X(m-l) \lambda)^{4}] \,, \end{split}$$

and the proof is completed.

We can show the following lemma by the same method and the proof is omitted.

Lemma 2.6. Let (A.1)–(A.3) be satisfied and $m \ge l+1$. Then we have

(2.36)
$$E_{\mathbf{x}}[(\sum_{k=l}^{m-1} (\mathbf{X}(k) \boldsymbol{\lambda} - \mathbf{x} M^{k} \boldsymbol{\lambda}))^{2}] \\ \leq \sum_{a=1}^{d} \sum_{b=1}^{d} x_{a} E_{e^{a}}[X_{b}(l)] E_{e^{b}}[(\sum_{k=l}^{m-1} \mathbf{X}(k) \boldsymbol{\lambda})^{2}] \\ + \sum_{a=1}^{d} x_{a} E_{e^{a}}[(\mathbf{X}(l) \sum_{k=0}^{m-l-1} M^{k} \boldsymbol{\lambda})^{2}],$$

$$E_{\mathbf{x}}\left[\left(\sum_{k=l}^{m-1} (\mathbf{X}(k) \,\mathbf{\lambda} - \mathbf{x} \, M^{k} \,\mathbf{\lambda})\right)^{4}\right]$$

$$\leq 3E_{\mathbf{x}}\left[\left(\sum_{k=l}^{m-1} (\mathbf{X}(k) \,\mathbf{\lambda} - \mathbf{x} \, M^{k} \,\mathbf{\lambda})\right)^{2}\right]^{2}$$

$$+24d \sum_{a=1}^{d} \sum_{b=1}^{d} x_{a} \, E_{e^{a}}\left[X_{b}(l)^{2}\right] E_{e^{b}}\left[\left(\sum_{k=0}^{m-l-1} \mathbf{X}(k) \,\mathbf{\lambda}\right)^{2}\right]^{2}$$

$$+128 \sum_{a=1}^{d} \sum_{b=1}^{d} x_{a} \, E_{e^{a}}\left[X_{b}(l)\right] E_{e^{b}}\left[\left(\sum_{k=0}^{m-l-1} \mathbf{X}(k) \,\mathbf{\lambda}\right)^{4}\right]$$

$$+128 \sum_{a=1}^{d} x_{a} \, E_{e^{a}}\left[\left(\mathbf{X}(k) \sum_{k=0}^{m-l-1} M^{k} \,\mathbf{\lambda}\right)^{4}\right].$$

3. Proof of the tightness

We shall show the tightness part in Theorem A at first. Let $\{x^n\}_{n\geq 1}$ be a sequence of nonnegative integer valued vectors satisfying

(3.1)
$$\lim_{n\to\infty} n^{-\alpha} \mathbf{x}_{\alpha}^{n} = x_{\alpha} \mathbf{v}_{\alpha}, 1 \leq \alpha \leq N,$$

for some nonnegative numbers x_1, x_2, \dots, x_N . Let β nad λ^{β} be fixed arbitrarily. Set

(3.2)
$$\begin{cases} \phi_{n}(t) = E_{x^{n}}[n^{-\beta} X_{\beta}^{n}(t) \lambda^{\beta}] \\ = n^{-\beta} x^{n} M^{[nt]}(I + (nt - [nt]) (M - I)) \lambda^{\beta}, \\ \psi_{n}(t) = E_{x^{n}}[n^{-\beta} Y_{\beta}^{n}(t) \lambda^{\beta}] \\ = n^{-\beta} x^{n} (\sum_{k=0}^{[nt]-1} M^{k} + (nt - [nt]) M^{[nt]}) \overline{\lambda}^{\beta}. \end{cases}$$

By Lemma 2.1 we have

$$\begin{split} \lim_{n \to \infty} \phi_n(t) &= \sum_{\alpha=1}^{\beta} x_{\alpha} \, \boldsymbol{v}_{\sigma} \, P_{\beta}^{\alpha} \, \boldsymbol{\lambda}^{\beta} \, t^{\beta-\alpha} \, ,\\ \lim_{n \to \infty} \psi_n(t) &= \sum_{\alpha=1}^{\beta} x_{\alpha} \, \boldsymbol{v}_{\alpha} \, Q_{\beta}^{\alpha} \, \boldsymbol{\lambda}^{\beta} \, t^{\beta-\alpha} \, , \end{split}$$

uniformly on each finite interval. Set

$$U_n(t) = n^{-\beta} X^n_{\beta}(t) \lambda^{\beta} - \phi_n(t), V_n(t) = n^{-\beta} Y^n_{\beta}(t) \lambda^{\beta} - \psi_n(t).$$

Then it suffices to show the following lemma.

Lemma 3.1. Let (A.1)–(A.3) be satisfied. Then for each fixed T>0 there exists C(T)>0 satisfying

(3.3)
$$E_{x^n}[(U_n(t) - U_n(s))^4 + (V_n(t) - V_n(s))^4] \leq C(T) (t-s)^2,$$

 $n \geq 1, \ 0 \leq s \leq t \leq T.$

Proof. Set l=[ns] and m=[nt]. Then by (2.1) and (2.4) we have

(3.4)
$$n^{\beta}(U_{n}(t)-U_{n}(s)) = \begin{cases} n(t-s) \left(X(l+1)-x^{n}M^{l+1}-X(l)+x^{n}M^{l} \right) \lambda^{\beta} & \text{if } m = l, \\ \left(X(m)-x^{n}M^{m}-X(l)+x^{n}M^{l} \right) \lambda^{\beta} & \\ +(nt-m) \left(X(m+1)-x^{n}M^{m+1}-X(m)+x^{n}M^{m} \right) \lambda^{\beta} & \\ +(ns-l) \left(X(l+1)-x^{n}M^{l+1}-X(l)+x^{n}M^{l} \right) \lambda^{\beta} & \text{if } m \ge l+1, \end{cases}$$

(3.5)
$$= \begin{cases} n(t-s) \left(\mathbf{X}(l) - \mathbf{x}^{n} M^{l} \right) \bar{\mathbf{\lambda}}^{\beta} & \text{if } m = l, \\ \sum_{k=l}^{m-1} \left(\mathbf{X}(k) - \mathbf{x}^{n} M^{k} \right) \bar{\mathbf{\lambda}}^{\beta} + (nt-m) \left(\mathbf{X}(m) - \mathbf{x}^{n} M^{m} \right) \bar{\mathbf{\lambda}}^{\beta} \\ -(ns-l) \left(\mathbf{X}(l) - \mathbf{x}^{n} M^{l} \right) \bar{\mathbf{\lambda}}^{\beta} & \text{if } m \ge l+1. \end{cases}$$

Hence it suffices to show that

(3.6)
$$E_{\mathbf{x}^{\mathbf{n}}}[(\mathbf{X}_{\boldsymbol{\beta}}(m) \boldsymbol{\lambda}^{\boldsymbol{\beta}} - \mathbf{x}^{\mathbf{n}} M^{m} \boldsymbol{\lambda}^{\boldsymbol{\beta}} - \mathbf{X}_{\boldsymbol{\beta}}(l) \boldsymbol{\lambda}^{\boldsymbol{\beta}} + \mathbf{x}^{\mathbf{n}} M^{l} \boldsymbol{\lambda}^{\boldsymbol{\beta}})^{4}] \\ \leq C(T) (m-l)^{2} n^{4\beta-2} |\boldsymbol{\lambda}^{\boldsymbol{\beta}}|^{4},$$

(3.7)
$$E_{\boldsymbol{x}^{n}}[(\sum_{k=l}^{m-1} (\boldsymbol{X}_{\boldsymbol{\beta}}(k) \, \bar{\boldsymbol{\lambda}}^{\boldsymbol{\beta}} - \boldsymbol{x}^{n} M^{k} \, \bar{\boldsymbol{\lambda}}^{\boldsymbol{\beta}}))^{4}] \\ \leq C(T) \, (m-l)^{2} \, n^{4\beta-2} |\boldsymbol{\lambda}^{\boldsymbol{\beta}}|^{4} \, ,$$

hold for any $n \ge 1$ and $l+1 \le m \le nT$.

We shall show (3.6) only, since (3.7) is shown by the same method. Let $\mathbf{x} = \mathbf{x}^n, \mathbf{\lambda} = \mathbf{\lambda}^{\beta}$ in Lemma 2.5. Since $\mathbf{x}_{\alpha}^n = O(n^{\alpha})$, we have

$$E_{\mathbf{x}^{n}}[(\mathbf{X}(m) \boldsymbol{\lambda}^{\beta} - \mathbf{x}^{n} M^{m} \boldsymbol{\lambda}^{\beta} - \mathbf{X}(l) \boldsymbol{\lambda}^{\beta} + \mathbf{x}^{n} M^{l} \boldsymbol{\lambda}^{\beta})^{2}] \\ \leq C(T) n^{\beta} (m-l)^{2\beta-1} |\boldsymbol{\lambda}^{\beta}|^{2},$$

by (2.6), (2.9), (2.28) for $l \leq (m-1)T$ and (2.32). Then (3.6) follows from (2.6), (2.9), (2.28) for $l \leq (m-1)T$ and (2.33).

We can show the tightness part in Theorem B by the same method and the proof is omitted.

4. An auxiliary limit theorem

In this section we shall show a theorem which will be used to prove the convergence of finite dimensional distributions.

Theorem 1. Let (.A.1)–(A.3) be satisfied, α and t>0 be fixed. Then we have

(4.1)
$$\lim_{n \to \infty} n^{\beta - \alpha + 1} (E_{e^{\alpha}} [\exp(i \sum_{\gamma = \alpha}^{N} n^{\alpha - \gamma - 1} (X_{\gamma}^{n}(t) \lambda^{\gamma} + Y_{\gamma}^{n}(t) \mu^{\gamma}))] - 1)_{a \in C_{\beta}} = \left\{ \begin{array}{ll} \psi_{\alpha}(t; \lambda, \mu) u^{\alpha} + i Q_{\alpha}^{\alpha} \mu^{\alpha} & \text{if } \beta = \alpha , \\ i \sum_{\gamma = \beta}^{N} (P_{\gamma}^{\beta} \lambda^{\gamma} + Q_{\gamma}^{\beta} \mu^{\gamma}) t^{\gamma - \beta} & \text{if } \beta \geq \alpha + 1 , \end{array} \right.$$

where $\psi_{a}(t) = \psi_{a}(t; \boldsymbol{\lambda}, \boldsymbol{\mu})$ is the solution of

(4.2)
$$\begin{cases} \psi_{a}(0) = i v_{a} \lambda^{\alpha}, \\ \frac{d}{dt} \psi_{a}(t) \\ = \frac{1}{2} \sum_{a \in \mathcal{O}_{\alpha}} v_{a} E_{e^{a}} [((X_{a}(1) - X_{a}(0) M_{\alpha}^{\alpha}) (\psi_{a}(t) u^{\alpha} + i Q_{\alpha}^{\alpha} \mu^{\alpha}))^{2}] \\ + i \sum_{\beta = \alpha + 1}^{N} (\beta - \alpha) (v_{a} P_{\beta}^{\alpha} \lambda^{\beta} + v_{a} Q_{\beta}^{\alpha} \mu^{\beta}) t^{\beta - \alpha - 1}. \end{cases}$$

Especially if C_{α} is a final class then

(4.3)
$$\psi_{\mathfrak{a}}(t; \boldsymbol{\lambda}, \boldsymbol{\mu}) = -\frac{1}{2} \sum_{a \in \mathcal{O}_{\mathfrak{a}}} v_{a} \, \overline{\mu}^{a} (\overline{\mu}^{a} + 2(M_{\mathfrak{a}}^{\alpha} Q_{\mathfrak{a}}^{\alpha} \boldsymbol{\mu}^{\beta})^{a}) + i \sum_{\beta=\alpha}^{N} v_{\mathfrak{a}} (P_{\beta}^{\alpha} \boldsymbol{\lambda}^{\beta} + Q_{\beta}^{\alpha} \boldsymbol{\mu}^{\beta}) \, t^{\beta-\alpha} \, .$$

Proof. Let $\alpha \leq \beta$, $b \in C_{\beta}$ and t > 0 be fixed and set

$$U_n(t) = \sum_{\gamma=1}^N n^{\alpha-\gamma-1}(X_{\gamma}^n(t) \lambda^{\gamma} + Y_{\gamma}^n(t) \mu^{\gamma}).$$

Then

$$U_{\mathbf{n}}\left(\frac{[nt]}{n}\right) = \sum_{\gamma=1}^{N} n^{\alpha-\gamma-1}(X_{\gamma}([nt]) \,\mathbf{\lambda}^{\gamma} + \sum_{j=1}^{[nt]^{-1}} X_{\gamma}(k) \,\mu_{\gamma}) \,.$$

By Lemma 2.2 and Lemma 2.3 we obtain

(4.4)

$$|E_{e^{b}}[\exp(iU_{n}\left(\frac{[nt]}{n}\right))] - 1 - E_{e^{b}}[iU_{n}\left(\frac{[nt]}{n}\right)]|$$

$$\leq E_{e^{b}}[U_{n}\left(\frac{[nt]}{n}\right)^{2}]$$

$$\leq 2N \sum_{\gamma=\beta}^{N} n^{2(\vartheta-\gamma-1)} E_{e^{b}}[(X_{\gamma}([nt])\lambda^{\gamma})^{2} + (\sum_{k=0}^{[ntl-1]} X_{\gamma}(k) \overline{\mu}^{\gamma})^{2}]$$

$$= O(n^{-2(\beta-\vartheta)-1}).$$

Hence (4.1) in the case $\beta \ge \alpha + 1$ follows from Lemma 2.1 and (4.4). Set

(4.5)
$$G^{a}(n:\boldsymbol{\lambda},\boldsymbol{\mu}) = E_{\boldsymbol{e}^{a}}[\exp(i\boldsymbol{X}(n)\boldsymbol{\lambda} + i\sum_{k=0}^{n}\boldsymbol{X}(k)\boldsymbol{\mu})],$$

(4.6)
$$I(\boldsymbol{\lambda}) = (\delta_b^a \exp(i\lambda^a))_{1 \leq a, b \leq d} .$$

Then $G(n: \lambda, \mu) = (G^{a}(n: \lambda, \mu))_{1 \leq a \leq d}$ satisfies

(4.7)
$$\begin{cases} \boldsymbol{G}(0;\boldsymbol{\lambda},\boldsymbol{\mu}) = I(\boldsymbol{\lambda}+\boldsymbol{\mu}) \mathbf{1}, \\ \boldsymbol{G}(n;\boldsymbol{\lambda},\boldsymbol{\mu}) = I(\boldsymbol{\mu}) \boldsymbol{F}(\boldsymbol{G}(n-1;\boldsymbol{\lambda},\boldsymbol{\mu})) \quad n \geq 1, \end{cases}$$

where F(s) is the vector of generating functions of X(1). To treat this excursion formula we expand the generating functions as follows,

(4.8)
$$F^{a}(s)-1 = \sum_{b=1}^{d} m_{b}^{a}(s^{b}-1) + \frac{1}{2} \sum_{b,c=1}^{d} D_{b} D_{c} F^{a}(s+\theta_{a}(1-s))(s^{b}-1)(s^{c}-1),$$

$$0 < \theta_{a} < 1.$$

Set

(4.9)
$$\boldsymbol{B}(\boldsymbol{s}:\boldsymbol{\lambda},\boldsymbol{\lambda}) = (\sum_{b,c=1}^{d} D_b D_c F^a(\boldsymbol{s} + \theta_a(\boldsymbol{1} - \boldsymbol{s})) \lambda^b \lambda^c)_{1 \leq a \leq d}.$$

Then (4.8) becomes

(4.10)
$$F(s)-1 = M(s-1) + \frac{1}{2} B(s:s-1,s-1).$$

Combining this with (4.7) we obtain

(4.11)
$$\begin{aligned} G(n:\boldsymbol{\lambda},\boldsymbol{\mu}) - \mathbf{1} \\ &= M(G(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}) - \mathbf{1}) + (I(\boldsymbol{\mu}) - I) F(G(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})) \\ &+ \frac{1}{2} B(G(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}): G(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}) - \mathbf{1}, G(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}) - \mathbf{1}), \end{aligned}$$

and hence

(4.12)
$$\begin{aligned} \boldsymbol{G}(\boldsymbol{n}:\boldsymbol{\lambda},\boldsymbol{\mu}) &- \boldsymbol{1} \\ &= M^{\boldsymbol{n}}(I(\boldsymbol{\lambda}+\boldsymbol{\mu})-I) \, \boldsymbol{1} + \sum_{k=0}^{n-1} M^{\boldsymbol{n}-k-1}(I(\boldsymbol{\mu})-I) \, \boldsymbol{F}(\boldsymbol{G}(\boldsymbol{k}:\boldsymbol{\lambda},\boldsymbol{\mu})) \\ &+ \frac{1}{2} \, \sum_{k=0}^{n-1} M^{\boldsymbol{n}-k-1} \, \boldsymbol{B}(\boldsymbol{G}(\boldsymbol{k}:\boldsymbol{\lambda},\boldsymbol{\mu}):\boldsymbol{G}(\boldsymbol{k}:\boldsymbol{\lambda},\boldsymbol{\mu}) - \boldsymbol{1}, \, \boldsymbol{G}(\boldsymbol{k}:\boldsymbol{\lambda},\boldsymbol{\mu}) - \boldsymbol{1}) \, . \end{aligned}$$

Set

(4.13)
$$\boldsymbol{\lambda}_n = (n^{\boldsymbol{\alpha}-\boldsymbol{\beta}-1} \boldsymbol{\lambda}^{\boldsymbol{\beta}})_{1 \leq \boldsymbol{\beta} \leq N},$$

$$(4.14) \qquad \boldsymbol{B}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\boldsymbol{s}:\boldsymbol{\lambda}^{\boldsymbol{\beta}},\boldsymbol{\lambda}^{\boldsymbol{\gamma}}) = \left(\sum_{\boldsymbol{b}\in\mathcal{C}_{\boldsymbol{\beta}}}\sum_{\boldsymbol{c}\in\mathcal{C}_{\boldsymbol{\gamma}}}D_{\boldsymbol{b}}\,D_{\boldsymbol{c}}\,F^{\boldsymbol{a}}(\boldsymbol{s}+\boldsymbol{\theta}_{\boldsymbol{a}}(\boldsymbol{1}-\boldsymbol{s}))\,\lambda^{\boldsymbol{b}}\,\lambda^{\boldsymbol{c}}\right)_{\boldsymbol{a}\in\mathcal{C}_{\boldsymbol{\omega}}}.$$

Then we have

$$(E_{e^{a}}[\exp(i\sum_{\beta=1}^{N}n^{\alpha-\beta-1}(\boldsymbol{X}_{\beta}^{n}(\frac{[nt]}{n})\boldsymbol{\lambda}^{\beta}+\boldsymbol{Y}_{\beta}^{n}(\frac{[nt]+1}{n})\boldsymbol{\mu}^{\beta}))]-1)_{a\in C_{\alpha}}$$

$$=\boldsymbol{G}^{\boldsymbol{\alpha}}([nt]:\boldsymbol{\lambda}_{n},\boldsymbol{\bar{\mu}}_{n})-\boldsymbol{1}^{\boldsymbol{\alpha}}$$

$$=\sum_{\beta=\alpha}^{N}(M^{[nt]})_{\beta}^{\boldsymbol{\alpha}}(I_{\beta}^{\beta}(\boldsymbol{\lambda}_{n}+\boldsymbol{\bar{\mu}}_{n})-I)\boldsymbol{1}^{\beta}$$

$$(4.15) \qquad +\sum_{k=0}^{[nt]-1}\sum_{\beta=\alpha}^{N}(M^{[nt]-k-1})_{\beta}^{\boldsymbol{\alpha}}(I_{\beta}^{\beta}(\boldsymbol{\bar{\mu}}_{n})-I)\boldsymbol{F}^{\beta}(\boldsymbol{G}(k:\boldsymbol{\lambda}_{n},\boldsymbol{\bar{\mu}}_{n}))$$

$$+\frac{1}{2}\sum_{k=0}^{[nt]-1}\sum_{\beta=\alpha}^{N}\sum_{\gamma,\delta=\beta}^{N}(M^{[nt]-k-1})_{\beta}^{\boldsymbol{\alpha}}\boldsymbol{B}_{\gamma,\delta}^{\beta}(\boldsymbol{G}(k:\boldsymbol{\lambda}_{n},\boldsymbol{\bar{\mu}}_{n}):\boldsymbol{G}^{\gamma}(k:\boldsymbol{\lambda}_{n},\boldsymbol{\bar{\mu}}_{n})$$

$$-\boldsymbol{1}^{\gamma},\boldsymbol{G}^{\delta}(k:\boldsymbol{\lambda}_{n},\boldsymbol{\bar{\mu}}_{n})-\boldsymbol{1}^{\delta}).$$

We shall estimate the last three terms. We remark at first

(4.16)
$$\begin{cases} I_{\beta}^{\beta}(\boldsymbol{\lambda}_{n}+\bar{\boldsymbol{\mu}}_{n})-I = \operatorname{in}^{\boldsymbol{\omega}-\beta-1}(\delta_{b}^{a}(\boldsymbol{\lambda}^{a}+\bar{\boldsymbol{\mu}}^{a}))_{a,b\in C_{\beta}}+O(n^{2(\boldsymbol{\omega}-\beta-1)}),\\ I_{\beta}^{\beta}(\bar{\boldsymbol{\mu}}_{n})-I = \operatorname{in}^{\boldsymbol{\omega}-\beta-1}(\delta_{b}^{a}\bar{\boldsymbol{\mu}}^{b})_{a,b\in C_{\beta}}-\frac{1}{2}n^{2(\boldsymbol{\omega}-\beta-1)}(\delta_{b}^{a}(\bar{\boldsymbol{\mu}}^{b})^{2})_{a,b\in C_{\beta}}\\ +O(n^{3(\boldsymbol{\omega}-\beta-1)}). \end{cases}$$

Then the first term is

$$in^{-1}\sum_{\beta=\omega}^{N} P^{\alpha}_{\beta} \lambda^{\beta} t^{\beta-\omega} + o(n^{-1})$$

by Lemma 2.1. By (4.4) and Lemma 2.1 we have

(4.17)
$$\boldsymbol{G}^{\boldsymbol{\beta}}(k;\boldsymbol{\lambda}_{n},\boldsymbol{\mu}_{n}) = \boldsymbol{1}^{\boldsymbol{\beta}} + O(n^{\boldsymbol{\omega}-\boldsymbol{\beta}-\boldsymbol{1}}) \quad k \leq nt,$$

and hence

(4.18)
$$F^{\beta}(G(k; \lambda_n, \overline{\mu}_n)) = 1^{\beta} + O(n^{\sigma-\beta-1}) \quad k \leq nt.$$

Then by (4.16) we obtain

$$\sum_{k=0}^{\lfloor n+1 \rfloor-1} \sum_{\beta=\alpha+1}^{N} (M^k)^{\alpha}_{\beta}(I^{\beta}_{\beta}(\overline{\mu}_n)-I) F^{\beta}(G([nt]-k-1:\lambda_n,\overline{\mu}_n))$$
$$= in^{-1} \sum_{\beta=\alpha+1}^{N} Q^{\alpha}_{\beta} \mu^{\beta} t^{\beta-\alpha} + o(n^{-1}).$$

By (4.10) and (4.17) we have

$$\boldsymbol{F}^{\boldsymbol{\omega}}(\boldsymbol{G}(k;\boldsymbol{\lambda}_n,\boldsymbol{\overline{\mu}}_n)) = \boldsymbol{1}^{\boldsymbol{\omega}} + \boldsymbol{M}^{\boldsymbol{\omega}}_{\boldsymbol{\omega}}(\boldsymbol{1}^{\boldsymbol{\omega}} - \boldsymbol{G}^{\boldsymbol{\omega}}(k;\boldsymbol{\lambda}_n,\boldsymbol{\overline{\mu}}_n)) + o(n^{-2}) \, .$$

Hence by (4.16) and Lemma 2.1 we obtain

$$\begin{split} \sum_{k=0}^{\lfloor nt \rfloor^{-1}} (M^k)^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\overline{\mu}_n) - I) \ F^{\alpha}(G([nt] - k - 1; \lambda_n, \overline{\mu}_o)) \\ &= in^{-1} \sum_{k=0}^{\lfloor nt \rfloor^{-1}} (M^k)^{\alpha}_{\alpha}(\delta^a_b \overline{\mu}^b)_{a,b \in C_{\omega}} \mathbf{1}^{\alpha} \\ &- \frac{1}{2} n^{-2} \sum_{k=0}^{\lfloor nt \rfloor^{-1}} (M^k)^{\alpha}_{\alpha}(\delta^a_b (\overline{\mu}^b)^2)_{a,b \in C_{\omega}} \mathbf{1}^{\alpha} \\ &+ \sum_{k=0}^{\lfloor nt \rfloor^{-1}} (M^k)^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\mu_n) - I) \ M^{\alpha}_{\alpha}(G^{\alpha}([nt] - k - 1; \lambda_n, \overline{\mu}_n) - \mathbf{1}^{\alpha}) \\ &+ o(n^{-1}) \\ &= in^{-1} Q^{\alpha}_{\alpha} \mu^{\alpha} - \frac{1}{2} n^{-1} t \sum_{a \in C_{\alpha}} v_a(\overline{\mu}^a)^2 u^{\alpha} \\ &+ \sum_{k=0}^{\lfloor nt \rfloor^{-1}} P^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\overline{\mu}_n) - I) \ M^{\alpha}_{\alpha}(G^{\alpha}(k; \lambda_n, \overline{\mu}_n) - \mathbf{1}^{\alpha}) + o(n^{-1}) \,. \end{split}$$

Then the second term in (4.15) is

$$-\frac{1}{2} n^{-1} t \sum_{a \to \mathcal{O}_{a}} v_{a}(\overline{\mu}^{a})^{2} u^{a} + i n^{-1} \sum_{\beta=a}^{N} Q_{\beta}^{a} \mu^{\beta} t^{\beta-a} \\ + \sum_{k=0}^{\lfloor nt \rfloor - 1} P_{a}^{a}(I_{a}^{a}(\overline{\mu}_{n}) - I) M_{a}^{a}(G^{a}(k; \lambda_{n}, \overline{\mu}_{n}) - \mathbf{1}^{a}) + o(n^{-1}) .$$

By (4.17) and Lemma 2.1 the last term in (4.15) is

$$\frac{1}{2}\sum_{k=0}^{\lfloor n! \rfloor^{-1}} P^{\alpha}_{\alpha} B^{\alpha}_{\alpha,\alpha}(1: G^{\alpha}(k: \lambda_n, \overline{\mu}_n) - 1^{\alpha}, G^{\alpha}(k: \lambda_n, \overline{\mu}_n) - 1^{\alpha}) + o(n^{-1}).$$

Hence it follows that

(4.19) $n(\boldsymbol{G}^{\boldsymbol{\alpha}}([nt]:\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n) - \boldsymbol{1}^{\boldsymbol{\alpha}})$

LIMIT PROCESSES FOR THE BRANCHING PROCESSES

$$=i\sum_{\beta=\omega}^{N} \left(P_{\beta}^{\omega} \boldsymbol{\lambda}^{\beta} + Q_{\beta}^{\omega} \boldsymbol{\mu}^{\beta}\right) t^{\beta-\omega} - \frac{1}{2} t \sum_{a \in \mathcal{O}_{\omega}} v_{a}(\overline{\mu}^{a})^{2} \boldsymbol{u}^{\omega} + \sum_{k=0}^{\lfloor nt \rfloor - 1} P_{\omega}^{\omega}(I_{\omega}^{\omega}(\overline{\mu}_{n}) - I) M_{\omega}^{\omega} \cdot n(G^{\omega}(k; \boldsymbol{\lambda}_{n}, \overline{\mu}_{n}) - \mathbf{1}^{\omega}) + \frac{1}{2n} \sum_{k=0}^{\lfloor nt \rfloor - 1} P_{\omega}^{\omega} B_{\omega,\omega}^{\alpha}(\mathbf{1}: n(G^{\omega}(k; \boldsymbol{\lambda}_{n}, \overline{\mu}_{n}) - \mathbf{1}^{\omega}), n(G^{\omega}(k; \boldsymbol{\lambda}_{n}, \overline{\mu}_{n}) - \mathbf{1}^{\omega})) + o(1).$$

Set

(4.20)
$$\psi_n(t) = n v_a(G^{\alpha}([nt]:\lambda_n, \overline{\mu}_n) - \mathbf{1}^{\alpha}) + n(nt - [nt]) v_a(G^{\alpha}([nt]+1:\lambda_n, \overline{\mu}_n) - G^{\alpha}([nt]:\lambda_n, \overline{\mu}_n)).$$

Then by (2.1), (2.2) and (4.19) we obtain

(4.21)
$$n(\boldsymbol{G}^{\boldsymbol{\alpha}}([nt]:\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n) - \boldsymbol{1}^{\boldsymbol{\alpha}}) = \psi_n\left(\frac{[nt]}{n}\right)\boldsymbol{u}^{\boldsymbol{\alpha}} + i\boldsymbol{Q}_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \boldsymbol{\mu}^{\boldsymbol{\alpha}} + o(1) \, .$$

But by (4.4), (4.11) and (4.13) we have

(4.22)
$$|\psi_n\left(\frac{[nt]+1}{n}\right)-\psi_n\left(\frac{[nt]}{n}\right)|=O(n^{-1})$$

Hence $\{\psi_n(t)\}_{n\geq 1}$ is equicontinuous on each finite interval. Let $\psi(t)$ be any limit of $\{\psi_n(t)\}_{n\geq 1}$. Then by (4.19) and (4.21) ψ must satisfy

(4.23)

$$\psi(t) = i \sum_{\beta=\alpha}^{N} v_{\alpha} (P_{\beta}^{\alpha} \lambda^{\beta} + Q_{\beta}^{\alpha} \mu^{\beta}) t^{\beta-\alpha} - \frac{t}{2} \sum_{a \in \mathcal{O}_{\alpha}} v_{a} (\overline{\mu}^{a})^{2} \\
+ i \int_{0}^{t} v_{\alpha} (\delta_{\beta}^{a} \overline{\mu}^{b})_{a,t \in \mathcal{O}_{\alpha}} M_{\alpha}^{\alpha} (\psi(s) u^{\alpha} + i Q_{\alpha}^{\alpha} \mu^{\alpha}) ds \\
+ \frac{1}{2} \int_{0}^{t} v_{\alpha} B_{\alpha,\alpha}^{\alpha} (1: \psi(s) u^{\alpha} + i Q_{\alpha}^{\alpha} \mu^{\alpha}, \psi(s) u^{\alpha} + i Q_{\alpha}^{\alpha} \mu^{\alpha}) ds .$$

Then it suffices to show the equivalence of (4.2) and (4.23). Since $\psi(0) = i \boldsymbol{v}_{\sigma} (\boldsymbol{P}_{\sigma}^{\sigma} \boldsymbol{\lambda}^{\sigma} + \boldsymbol{Q}_{\sigma}^{\sigma} \boldsymbol{\mu}^{\sigma}) = i \boldsymbol{v}_{\sigma} \boldsymbol{\lambda}^{\sigma}$ we have only to show that ψ satisfies the differential equation in (4.2). Remark that

$$(4.24) v_a(\delta^a_b \ \overline{\mu}^b)_{a,b \in Ca} = (v_a \ \overline{\mu}^a)_{a \in Ca}.$$

By (2.2) and $P^{\alpha}_{\alpha} Q^{\alpha}_{\alpha} = O$ it follows that

- $(4.25) M^{\alpha}_{\alpha} Q^{\alpha}_{\alpha} = Q^{\alpha}_{\alpha} I + P^{\alpha}_{\alpha},$
- $(4.26) M^{\alpha}_{\alpha} Q^{\alpha}_{\alpha} \mu^{\alpha} = Q^{\alpha}_{\alpha} \mu^{\alpha} \overline{\mu}^{\alpha}.$

For any α we have

$$(4.27) \qquad \begin{aligned} \boldsymbol{v}_{\boldsymbol{\alpha}} \boldsymbol{B}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}^{\boldsymbol{\alpha}}(\mathbf{1}:\boldsymbol{\lambda}^{\boldsymbol{\alpha}},\boldsymbol{\lambda}^{\boldsymbol{\alpha}}) \\ &= \sum_{a \in \mathcal{O}_{\boldsymbol{\alpha}}} \sum_{b,c \in \mathcal{O}_{\boldsymbol{\alpha}}} \boldsymbol{v}_{a} E_{e^{a}}[X_{b}(1) \left(X_{c}(1) - \boldsymbol{\delta}_{c}^{b}\right)] \lambda^{b} \lambda^{c} \\ &= \sum_{a \in \mathcal{O}_{\boldsymbol{\alpha}}} \boldsymbol{v}_{a} E_{e^{a}}[(X_{a}(1) \boldsymbol{\lambda}^{\boldsymbol{\alpha}})^{2}] - \sum_{a \in \mathcal{O}_{\boldsymbol{\alpha}}} \boldsymbol{v}_{a}(\lambda^{a})^{2} \\ &= \sum_{a \in \mathcal{O}_{\boldsymbol{\alpha}}} \boldsymbol{v}_{a} E_{e^{a}}[((X_{\boldsymbol{\alpha}}(1) - X_{\boldsymbol{\alpha}}(0) M_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}}) \boldsymbol{\lambda}^{\boldsymbol{\alpha}})^{2}] \\ &+ \sum_{a \in \mathcal{O}_{\boldsymbol{\alpha}}} \boldsymbol{v}_{a}(e^{a} M_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \boldsymbol{\lambda}^{\boldsymbol{\alpha}})^{2} - \sum_{a \in \mathcal{O}_{\boldsymbol{\alpha}}} \boldsymbol{v}_{a}(\lambda^{a})^{2} . \end{aligned}$$

Hence ψ satisfies

$$\frac{d\psi}{dt}(t) = \sum_{a \in \mathcal{O}_{\alpha}} v_a E_{e^a} [((X_{\alpha}(1) - X_{\alpha}(0) M_{\alpha}^{\alpha}) (\psi(t) u^{\alpha} + iQ_{\alpha}^{\alpha} \mu^{\alpha}))^2] \\
+ \sum_{\beta = \alpha + 1}^{N} i(\beta - \alpha) (P_{\beta}^{\alpha} \lambda^{\beta} + Q_{\beta}^{\alpha} \mu^{\beta}) t^{\beta - \alpha - 1} - \frac{1}{2} \sum_{a \in \mathcal{O}_{\alpha}} v_a (\overline{\mu}^{a})^2 \\
+ i(v_a \overline{\mu}^{a})_{a \in \mathcal{O}_{\alpha}} M_{\alpha}^{\alpha} (\psi(t) u^{\alpha} + iQ_{\alpha}^{\alpha} \mu^{\alpha}) \\
+ \frac{1}{2} \sum_{a \in \mathcal{O}_{\alpha}} v_a (e^{a} M_{\alpha}^{\alpha} (\psi(t) u^{\alpha} + iQ_{\alpha}^{\alpha} \mu^{\alpha}))^2 \\
- \frac{1}{2} \sum_{a \in \mathcal{O}_{\alpha}} v_a (e^{a} (\psi(t) u^{\alpha} + iQ_{\alpha}^{\alpha} \mu^{\alpha}))^2 \\
= \sum_{k=1}^{6} I_k.$$

By (4.26) and $M^{\alpha}_{\alpha} \boldsymbol{u}^{\alpha} = \boldsymbol{u}^{\alpha}$ we have

$$\begin{split} I_4 &= i \sum_{a \in \mathcal{O}_{\alpha}} v_a \, \overline{\mu}^a u^a \psi(t) - \sum_{a \in \mathcal{O}_{\alpha}} v_a \, \overline{\mu}^a (e^a \, Q^a_{\alpha} \, \mu^{\alpha}) + \sum_{a \in \mathcal{O}_{\alpha}} v_a (\overline{\mu}^a)^2 \,, \\ I_5 + I_6 &= -i \sum_{a \in \mathcal{O}_{\alpha}} v_a \, u^a \overline{\mu}^a \psi(t) + \frac{1}{2} \sum_{a \neq \mathcal{O}_{\alpha}} v_a (2e^a \, Q^a_{\alpha} \, \mu^{\alpha} - (\overline{\mu}^a)^2) \,. \end{split}$$

Then it is easy to see that $\sum_{k=3}^{6} I_k = 0$ and we have shown the equivalence of (4.2) and (4.23).

5. Proof of Theorem A

Before proceeding to the proof we state Theorem A more precisely.

Theorem 2. Let (A.1)–(A.3) be satisfied, $\mathbf{x} = (x_{\alpha})_{1 \leq \alpha \leq N}$ be a nonnegative vector and \mathbf{y} be a d-dimensional vector. Assume that $\{\mathbf{x}^n\}_{n\geq 1}$ is a sequence of nonnegative integer valued vectors satisfying $\lim_{n\to\infty} n^{-\alpha} \mathbf{x}_{\alpha}^n = x_{\alpha} \mathbf{v}_{\alpha}, 1 \leq \alpha \leq N$. Then the sequence of processes $\{((n^{-\alpha} \mathbf{X}_{\alpha}^n(t), \mathbf{y}_{\alpha} + n^{-\alpha} \mathbf{Y}_{\alpha}^n(t))_{1\leq \alpha \leq N}, P_{\mathbf{x}^n})\}_{n\geq 1}$ converges to some diffusion process $((\mathbf{X}_{\alpha}(t), \mathbf{Y}_{\alpha}(t))_{1\leq \alpha \leq N}, P_{(\mathbf{x},\mathbf{y})})$ and

(5.1)
$$E_{(\boldsymbol{x},\boldsymbol{y})}[\exp(i\sum_{\alpha=1}^{N} (\boldsymbol{X}_{\boldsymbol{\sigma}}(t) \boldsymbol{\lambda}^{\boldsymbol{\alpha}} + \boldsymbol{Y}_{\boldsymbol{\sigma}}(t) \boldsymbol{\mu}^{\boldsymbol{\sigma}}))] = \exp(i\sum_{\alpha=1}^{N} \boldsymbol{y}_{\boldsymbol{\sigma}} \boldsymbol{\mu}^{\boldsymbol{\alpha}} + x_{1} \psi_{1}(t; \boldsymbol{\lambda}, \boldsymbol{\mu}) + i\sum_{\alpha=2}^{N} \sum_{\beta=\alpha}^{N} x_{\boldsymbol{\sigma}} \boldsymbol{v}_{\boldsymbol{\sigma}}(P_{\beta}^{\boldsymbol{\alpha}} \boldsymbol{\lambda}^{\beta} + Q_{\beta}^{\boldsymbol{\alpha}} \boldsymbol{\mu}^{\beta}) t^{\beta-\boldsymbol{\alpha}}).$$

Proof. First remark that (5.1) is clear if t=0 and if t>0 then (5.1) follows from Theorem 1 and the branching property. If we can show the convergence of any finite dimensional distributions then it is easy to see that the limit process is a diffusion process by (5.1) (cf. section 7). Hence it suffices to show that for any $p \ge 2$ and $0 < t_1 < \cdots < t_p$.

(5.2)
$$E_{\mathbf{x}^{n}}\left[\exp\left(i\sum_{q=1}^{p}\sum_{\alpha=1}^{N}n^{-\alpha}(\mathbf{X}_{\alpha}([nt_{q}])\boldsymbol{\lambda}^{\alpha}(q)+\sum_{k=0}^{[nt_{q}]}\mathbf{X}_{\alpha}(k)\,\overline{\boldsymbol{\mu}}^{\alpha}(q))\right)\right]$$

converges to a continuous function of $(\lambda(1), \dots, \lambda(p), \mu(1), \dots, \mu(p))$. Set

(5.3)
$$\begin{array}{l}
G_{n,q}^{a}(t_{1}, \cdots, t_{p}; \boldsymbol{\lambda}(1), \cdots, \boldsymbol{\mu}(p)) \\
= E_{e^{a}}\left[\exp\left(i\sum_{q=1}^{p}\sum_{\alpha=1}^{N}n^{-\alpha}(\boldsymbol{X}_{\alpha}([nt_{q}])\boldsymbol{\lambda}^{\alpha}(q) + \sum_{k=0}^{[nt_{q}]}\boldsymbol{X}_{\alpha}(k)\,\overline{\boldsymbol{\mu}}^{\alpha}(q))\right)\right].
\end{array}$$

Then, by the branching property, (5.2) becomes

(5.4)
$$\prod_{\alpha=1}^{N} \prod_{a \in \mathcal{C}_{\alpha}} G^{a}_{n,p}(t_{1}, \cdots, t_{p}; \boldsymbol{\lambda}(1), \cdots, \boldsymbol{\mu}(p))^{\boldsymbol{z}^{n}_{\alpha}}.$$

Hence it suffices to show that there exists a vector of continuous functions $(\psi_{\boldsymbol{p}}^{\boldsymbol{\alpha}}(t_1, \dots, t_p; \boldsymbol{\lambda}(1), \dots, \boldsymbol{\mu}(\boldsymbol{p})))_{1 \leq \boldsymbol{\alpha} \leq N}$ such that

(5.5)
$$\lim_{n \to \infty} n^{\boldsymbol{\alpha}}(G^{\boldsymbol{\alpha}}_{n,p}(t_1, \cdots, t_p; \boldsymbol{\lambda}(1), \cdots, \boldsymbol{\mu}(p)) - \mathbf{1})_{\boldsymbol{\alpha} \in C_{\boldsymbol{\alpha}}} = \psi^{\boldsymbol{\alpha}}_p(t_1, \cdots, t_p; \boldsymbol{\lambda}(1), \cdots, \boldsymbol{\mu}(p)), 1 \leq \boldsymbol{\alpha} \leq N.$$

But by Markov property and the branching property we have

(5.6)
$$G^{a}_{n,p}(t_{1},\cdots,t_{p};\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p)) = G^{a}_{n,p-1}(t_{1},\cdots,t_{p-1};\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\lambda}(p-2),\boldsymbol{\lambda}_{n}(p-1),\boldsymbol{\mu}(1),\cdots,\boldsymbol{\mu}(p-1)),$$

in which

(5.7)

$$\lambda_{n}^{a}(p-1) = \lambda^{a}(p-1) + n^{\omega} \log E_{e^{a}}[\exp(i\sum_{\beta=1}^{N} n^{-\beta}(X_{\beta}([nt_{p}]-[nt_{p-1}])\lambda^{\beta}(p) + \sum_{k=0}^{(at_{p})^{-}(at_{p-1})} X_{\beta}(k) \overline{\mu}^{\beta}(p)))], a \in C_{\omega}.$$

Combining this with Theorem 1, (5.5) is easily seen by the induction argument

with respect to p.

6. Proof of Theorem B

We state Theorem B more precisely.

Theorem 3. Let (A.1)-(A.3) be satisfied, C_1 be a final class, $\mathbf{x} = (x_a)_{2 \le a \le N}$ be a nonnegative vector and $\mathbf{y} = (\mathbf{y}_a)_{2 \le a \le N}$ be a real vector. Assume that $\{\mathbf{x}^n\}_{n \ge 1}$ is a sequence of nonnegative integer valued vectors satisfying $x_i^n = 1$, $x_a^n = 0$ for $a \in C_1$, $a \ne 1$ and $\lim_{n \to \infty} n^{1-\alpha} \mathbf{x}_a^n = x_a \mathbf{v}_a$ for $\alpha \ge 2$. Then the sequence of processes $\{((n^{1-\alpha} \mathbf{X}_a^n(t), \mathbf{y}_a + n^{1-\alpha} \mathbf{Y}_a^n(t))_{2 \le a \le N}, P_{\mathbf{x}^n})\}_{n \ge 1}$ converges to some diffusion processes $\{((\hat{\mathbf{X}}_a(t), \hat{\mathbf{Y}}_a(t))_{2 \le a \le N} P_{(\mathbf{x},\mathbf{y})})$ and

$$\begin{aligned}
\hat{E}_{(\boldsymbol{x},\boldsymbol{y})}[\exp\left(i\sum_{\boldsymbol{\alpha}=2}^{N}\left(\hat{\boldsymbol{X}}_{\boldsymbol{\alpha}}(t)\boldsymbol{\lambda}^{\boldsymbol{\alpha}}+\hat{\boldsymbol{Y}}_{\boldsymbol{\alpha}}(t)\boldsymbol{\mu}^{\boldsymbol{\alpha}}\right)\right)]\\ &=\exp\left(i\sum_{\boldsymbol{\alpha}=2}^{N}\boldsymbol{y}_{\boldsymbol{\alpha}}\boldsymbol{\mu}^{\boldsymbol{\alpha}}+\boldsymbol{v}_{1}M_{2}^{1}\boldsymbol{u}^{2}\int_{0}^{t}\boldsymbol{\psi}_{2}(s;\boldsymbol{\lambda},\boldsymbol{\mu})\,ds+it\,\boldsymbol{v}_{1}M_{2}^{1}\,Q_{2}^{2}\,\boldsymbol{\mu}^{2}\right)\\ &\cdot\exp\left(x_{2}\,\boldsymbol{\psi}_{2}(t;\boldsymbol{\lambda},\boldsymbol{\mu})+i\sum_{\boldsymbol{\alpha}=3}^{N}\sum_{\boldsymbol{\beta}=\boldsymbol{\alpha}}^{N}x_{\boldsymbol{\alpha}}\boldsymbol{v}_{\boldsymbol{\alpha}}(\boldsymbol{P}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}+\boldsymbol{Q}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}}\boldsymbol{\mu}^{\boldsymbol{\beta}})\,t^{\boldsymbol{\beta}-\boldsymbol{\alpha}}\right).
\end{aligned}$$

Proof. As the proof of Theorem 2 it suffices to show (6.1) and the convergence of finite dimensional distributions. To show the second part we proceed as follows. Let $p \ge 1$ and $0 < t_1 < \cdots < t_p$ be fixed arbitrarily. Then it suffices to show that

(6.2)
$$E_{\mathbf{x}^{n}}\left[\exp\left(i\sum_{q=1}^{p}\sum_{\boldsymbol{\omega}=2}^{N}n^{1-\boldsymbol{\omega}}(\boldsymbol{X}_{\boldsymbol{\omega}}([nt_{q}])\boldsymbol{\lambda}^{\boldsymbol{\omega}}(q)+\sum_{k=0}^{[nt_{q}]}\boldsymbol{X}_{\boldsymbol{\omega}}(k)\,\bar{\boldsymbol{\mu}}^{\boldsymbol{\omega}}(q))\right)\right]$$

converges to a continuous function of $(\lambda(1), \dots, \mu(p))$. Set

(6.3)
$$\begin{aligned} H^{\alpha}_{n,p}(t_1, \cdots, t_p; \boldsymbol{\lambda}(1), \cdots, \boldsymbol{\mu}(p)) \\ &= E_{e^{\boldsymbol{\alpha}}}[\exp(i\sum_{q=1}^{p}\sum_{\boldsymbol{\alpha}=2}^{N}n^{1-\boldsymbol{\alpha}}(\boldsymbol{X}_{\boldsymbol{\alpha}}([nt_q])\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(q) + \sum_{k=0}^{[nt_q]}\boldsymbol{X}_{\boldsymbol{\alpha}}(k) \, \overline{\boldsymbol{\mu}}^{\boldsymbol{\alpha}}(q)))] \,. \end{aligned}$$

Then by the branching property (6.2) becomes

(6.4)
$$H_{n,p}^{1'}(t_1,\cdots,t_p;\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))\prod_{\alpha=2}^{N}\prod_{a\in\mathcal{O}_{\boldsymbol{\omega}}}H_{n,p}^{a}(t_1,\cdots,t_p;\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))^{x_b^n}.$$

In section 5 we have already shown that the second term converges to a continuous function. Transform the first term as in (5.6). Then applying the following lemma the convergence of the first term is easily seen by the induction argument with respect to p. Also (6.1) follows from this lemma and Theorem 1 applied to (6.4).

Lemma 6.1. Let (A.1)–(A.3) be satisfied, C_1 be a final class and t>0 be

fixed. Then for any $a \in C_1$ we have

(6.5)
$$\lim_{n\to\infty} E_{e^{\alpha}} [\exp\left(i\sum_{\alpha=2}^{N} n^{1-\alpha} (\boldsymbol{X}_{\alpha}^{n}(t) \boldsymbol{\lambda}^{\alpha} + \boldsymbol{Y}_{\alpha}^{n}(t) \boldsymbol{\mu}^{\alpha})\right)] \\ = \exp\left(it \, \boldsymbol{v}_{1} \, M_{2}^{1} \, Q_{2}^{2} \, \boldsymbol{\mu}^{2} + \boldsymbol{v}_{1} \, M_{2}^{1} \, \boldsymbol{u}^{2} \int_{0}^{t} \phi_{2}(s; \boldsymbol{\lambda}, \boldsymbol{\mu}) \, ds\right)$$

Proof. We define $G(n: \lambda, \mu)$ by (4.9). Let $\lambda^1 = \mu^1 = 0$ and set

(6.6)
$$\boldsymbol{\lambda}_n = (n^{1-\omega} \, \boldsymbol{\lambda}^{\omega})_{1 \leq \omega \leq N} \, .$$

Then it is necessary to estimate $G^{1}([nt]: \lambda_{n}, \overline{\mu}_{n})$. To this end we expand the generating functions as follows.

Set $s^{[2,N]} = (s^{\alpha})_{2 \le \alpha \le N_1}$. Since C_1 is a final class we have

(6.7)
$$F_1(s) = M_1^1(s^{[2,N]}) s^1$$

for some $M_1^1(s^{[2,N]}) = (m_b^a(s^{[2,N]}))_{a,b \in C_1}$. Since $m_b^a = m_b^a(1)$ we obtain

(6.8)
$$m_b^a(s^{[2,N]}) = m_b^a + l_b^a(s^{[2,N]}),$$

where

(6.9)
$$l_{b}^{a}(s^{[2,N]}) = \sum_{\alpha=2}^{N} \sum_{c \in \mathcal{O}_{\alpha}} D_{c} m_{b}^{a}(s^{[2,N]} + \theta_{a,b}(1 - s^{[2,N]})) (s^{c} - 1),$$
$$0 < \theta_{a,b} < 1.$$

Set $L_1^1(s^{[2,N]}) = (l_b^a(s^{[2,N]}))_{a,b \in C_1}$. Then we have

(6.10) $M_1^1(s^{[2,N]}) = M_1^1 + L_1^1(s^{[2,N]})$

and hence

(6.11)
$$\begin{cases} \boldsymbol{G}^{1}(0;\boldsymbol{\lambda},\boldsymbol{\mu}) = \boldsymbol{1}^{1}, \\ \boldsymbol{G}^{1}(n;\boldsymbol{\lambda},\boldsymbol{\mu}) = \boldsymbol{M}^{1}(\boldsymbol{G}^{[\boldsymbol{z},N]}(n-1;\boldsymbol{\lambda},\boldsymbol{\mu})) \boldsymbol{G}^{1}(n-1;\boldsymbol{\lambda},\boldsymbol{\mu}), n \geq 1. \end{cases}$$

Combining this with (6.10) we obtain

(6.12)
$$G^{1}(n:\boldsymbol{\lambda},\boldsymbol{\mu}) = M_{1}^{1} G^{1}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}) + L_{1}^{1}(G^{[2,N]}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})) G^{1}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})$$

and it follows that

(6.13)
$$G^{1}(n; \lambda, \mu) = \mathbf{1}^{1} + \sum_{k=0}^{n-1} (M_{1}^{1})^{n-k-1} L_{1}^{1}(G^{[2,N]}(k; \lambda, \mu)) G^{1}(k; \lambda, \mu).$$

Since $(M_1^1)^n - P_1^1 = O(\rho^n)$ for some $0 < \rho < 1$, by (6.6), (6.13) and Theorem 1 we have

(6.14)
$$\boldsymbol{G}^{1}([nt];\boldsymbol{\lambda}_{n},\overline{\boldsymbol{\mu}}_{n})=(\boldsymbol{v}_{1}\,\boldsymbol{G}^{1}([nt];\boldsymbol{\lambda}_{n},\overline{\boldsymbol{\mu}}_{n}))\,\boldsymbol{1}^{1}+O(n^{-1})\,.$$

Hence it is sufficient to estimate $v_1 G^1([nt]: \lambda_n, \overline{\mu}_n)$. Set

(6.15)
$$\psi_n(t) = v_1 \mathbf{G}^1([nt]: \boldsymbol{\lambda}_n, \overline{\mu}_n) + (nt - [nt]) v_1(\mathbf{G}^1([nt] + 1: \boldsymbol{\lambda}_n, \overline{\mu}_n) - \mathbf{G}^1([nt]: \boldsymbol{\lambda}_n, \overline{\mu}_n)).$$

Then, by (6.12) and Theorem 1, it follows that

(6.16)
$$|\psi_n\left(\frac{[nt]+1}{n}\right)-\psi_n\left(\frac{[nt]}{n}\right)|=O(n^{-1}),$$

and $\{\psi_n(t)\}_{n\geq 1}$ is equicontinuous on each finite interval. Let $\psi(t)$ be any limit. Then by Theorem 1 and (6.14) we have

(6.17)
$$\lim_{n \to \infty} n L_1^1(\boldsymbol{G}^{[2,N]}([nt]:\boldsymbol{\lambda}_n, \boldsymbol{\overline{\mu}}_n)) \boldsymbol{G}^1([nt]:\boldsymbol{\lambda}_n, \boldsymbol{\overline{\mu}}_n) \\ = (\sum_{c \in \mathcal{O}_2} \sum_{b \in \mathcal{O}_1} D_c \, m_b^a(\mathbf{1}) \, (\psi_2(t:\boldsymbol{\lambda}, \boldsymbol{\mu}) \, \boldsymbol{u}^c + i(Q_2^2 \, \boldsymbol{\mu}^2)^c))_{a \in \mathcal{O}_1} \, \psi(t) \, .$$

Remark that $\sum_{b \in \mathcal{O}_1} D_c m_b^a(1) = D_c F^a(1) = m_c^a$. Then by (6.13) ψ satisfies

(6.18)
$$\psi(t) = \int_0^t v_1 M_2^1(\psi_2(s; \lambda, \mu) u^2 + i Q_2^2 \mu^2) \psi(s) ds + 1,$$

i.e., $\psi(t)$ is given by (6.5).

7. Some remarks

Set

(7.1)
$$X_{\boldsymbol{\sigma}}(t) = \boldsymbol{X}_{\boldsymbol{\sigma}}(t) \boldsymbol{u}^{\boldsymbol{\sigma}}, \, \hat{X}_{\boldsymbol{\sigma}}(t) = \hat{\boldsymbol{X}}_{\boldsymbol{\sigma}}(t) \boldsymbol{u}^{\boldsymbol{\sigma}}.$$

Then by the preceding three theorems it is easy to see that

(7.2) $P_{(x,y)}(X_{\alpha}(t) = X_{\alpha}(t) v_{\alpha}, t \ge 0) = \hat{P}_{(x,y)}(\hat{X}_{\alpha}(t) = \hat{X}_{\alpha}(t) v_{\alpha}, t \ge 0) = 1$. Set $X(t) = (X_{\alpha}(t))_{1 \le \alpha \le N}$, $\hat{X}(t) = (\hat{X}_{\alpha}(t))_{2 \le \alpha \le N}$ and $B^{\alpha} = (B^{\alpha}_{a,b})_{a,b \in C_{\alpha} \cup \{0\}}$ be the symmetric and non-negative definite matrix defined by

(7.3)
$$\sum_{a,b\in\mathcal{O}_{a}\cup \{0\}} B^{a}_{a,b} \lambda^{a} \lambda^{b} = \sum_{a\in\mathcal{O}_{a}} v_{a} E_{e^{a}} [((\boldsymbol{X}_{a}(1) - \boldsymbol{X}_{a}(0) M^{a}_{a}) (\lambda^{0} \boldsymbol{u}^{a} + Q^{a}_{a} \boldsymbol{\lambda}^{a}))^{2}].$$

Set $\lambda^{\alpha} = \lambda^{\alpha} u^{\alpha}$ for $1 \leq \alpha \leq N$. Then by (7.2) we have

$$\sum_{\alpha=1}^{N} X_{\alpha}(t) \lambda^{\alpha} = \sum_{\alpha=1}^{N} X_{\alpha}(t) \lambda^{\alpha}.$$

Then (5.1) becomes

(7.4)
$$E_{(\boldsymbol{x},\boldsymbol{y})}[\exp\left(i\sum_{\boldsymbol{\alpha}=1}^{N}\left(X_{\boldsymbol{\alpha}}(t)\,\boldsymbol{\lambda}^{\boldsymbol{\alpha}}+\boldsymbol{Y}_{\boldsymbol{\alpha}}(t)\,\boldsymbol{\mu}^{\boldsymbol{\alpha}}\right)\right]$$

LIMIT PROCESSES FOR THE BRANCHING PROCESSES

$$= \exp(i\sum_{\alpha=1}^{N} \boldsymbol{y}_{\alpha} \boldsymbol{\mu}^{\alpha} + x_{1} \psi_{1}(t; \boldsymbol{\lambda}, \boldsymbol{\mu}) + i\sum_{\alpha=2}^{N} \sum_{\beta=\alpha}^{N} x_{\alpha}(\boldsymbol{\lambda}^{\beta} \boldsymbol{v}_{\alpha} P_{\beta}^{\alpha} \boldsymbol{u}^{\beta} + \boldsymbol{v}_{\alpha} Q_{\beta}^{\alpha} \boldsymbol{\mu}^{\beta}) t^{\beta-\alpha}).$$

By (4.2) we obtain

$$\begin{split} \frac{d}{dt} \log E_{(\mathbf{x},\mathbf{y})} [\exp(i\sum_{\alpha=1}^{N} (X_{\alpha}(t) \lambda^{\alpha} + Y_{\alpha}(t) \mu^{\alpha}))]|_{t=0} \\ &= -\frac{1}{2} x_1 (B_{0,0}^1(\lambda^1)^2 + 2\lambda^1 \sum_{\alpha \in \mathcal{O}_1} B_{0,\alpha}^1 \mu^{\alpha} + \sum_{\alpha, b \in \mathcal{O}_1} B_{\alpha,b}^1 \mu^{\alpha} \mu^{b}) \\ &+ i \sum_{\alpha=1}^{N} x_{\alpha} (\lambda^{\alpha+1} v_{\alpha} P_{\alpha+1}^{\alpha} u^{\alpha+1} + v_{\alpha} Q_{\alpha+1}^{\alpha} \mu^{\alpha+1}) \,. \end{split}$$

Then $((X(t), Y(t)), P_{(x,y)})$ is a diffusion process on the state space $[0, \infty)^N \times \mathbb{R}^d$ and the generator is given by

(7.5)
$$\begin{aligned} Af(\mathbf{x}, \mathbf{y}) \\ = \frac{1}{2} x_1 (B_{0,0}^1 D_{x_1}^2 + 2 \sum_{a \neq o_1} B_{0,a}^1 D_{x_1} D_{y_a} + \sum_{a, b \in o_1} B_{a,b}^1 D_{y_a} D_{y_b}) f(\mathbf{x}, \mathbf{y}) \\ + \sum_{\alpha=1}^{N-1} x_{\alpha} (\mathbf{v}_{\alpha} P_{\alpha+1}^{\alpha} \mathbf{u}^{\alpha+1} D_{x_{\alpha+1}} + \sum_{a \in o_{\alpha+1}} (\mathbf{v}_{\alpha} Q_{\alpha+1}^{\alpha})_a D_{y_a}) f(\mathbf{x}, \mathbf{y}) \end{aligned}$$

where D_x denotes the partial differentiation with respect to x.

By the same method it follows that $((\hat{X}(t), \hat{Y}(t)), \hat{P}_{(x,y)})$ is a diffusion process on the state space $[0, \infty)^{N-1} \times \mathbb{R}^{d-d_1}$ (d_1 is the number of elements in C_1) and the generator is given by

(7.6)
$$\hat{A}f(\mathbf{x}, \mathbf{y}) = \mathbf{v}_{1} M_{2}^{1} \mathbf{u}^{2} D_{x_{2}} f(\mathbf{x}, \mathbf{y}) + \frac{1}{2} x_{2} (B_{0,0}^{2} D_{x_{2}}^{2} + 2 \sum_{a \in \mathcal{O}_{2}} B_{0,a}^{2} D_{x_{2}} D_{y_{a}} + \sum_{a,b \neq \mathcal{O}_{2}} B_{a,b}^{2} D_{y_{a}} D_{y_{b}}) f(\mathbf{x}, \mathbf{y}) + \sum_{a \in \mathcal{O}_{2}}^{N-1} x_{a} (\mathbf{v}_{a} P_{a+1}^{a} \mathbf{u}^{a+1} D_{x_{a+1}} + \sum_{a \in \mathcal{O}_{a+1}} (\mathbf{v}_{a} Q_{a+1}^{a})_{a} D_{y_{a}}) f(\mathbf{x}, \mathbf{y}).$$

8. An example

Let $p>0, 0 < q \le 1$ be fixed and set m=[p]+1. In this section we study the 4-type branching process (X(n)) whose generating functions are given by

(8.1)
$$\begin{cases} F^{1}(s) = \frac{1}{m} s^{1} (m - p + p(s^{2})^{m}), \\ F^{2}(s) = \frac{1}{2} (s^{2} s^{3} (s^{4})^{2} + 1), \\ F^{3}(s) = qs^{2} + (1 - q) s^{3}, \\ F^{4}(s) = s^{4}. \end{cases}$$

Then the mean matrix is

(8.2)
$$M = \frac{1}{2} \begin{pmatrix} 2 & 2p & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2q & 2 - 2q & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hence $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$. Remark that C_1 and C_3 are final classes. By elementary calculations we have

(8.3)
$$v_2 = \frac{1}{2q+1} (2q, 1), \ u^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

(8.4)
$$\begin{cases} P_2^2 = \frac{1}{2q+1} \begin{pmatrix} 2q & 1 \\ 2q & 1 \end{pmatrix}, Q_2^2 = \frac{2}{(2q+1)^2} \begin{pmatrix} 1 & -1 \\ -2q & 2q \end{pmatrix}, \\ P_2^1 = \frac{p}{2q+1} (2q, 1), Q_2^1 = \frac{2p}{(2q+1)^2} (1, -1), \\ P_3^2 = \frac{2q}{2q+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q_3^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, P_3^1 = \frac{pq}{2q+1}, Q_3^1 = 0. \end{cases}$$

Since

(8.5)
$$\begin{cases} E_{e^2}[(X_2(1) \lambda^2 - X_2(0) M_2^2 \lambda^2)^2] = \frac{1}{4} (\lambda^2 + \lambda^3)^2, \\ E_{e^3}[(X_2(1) \lambda^2 - X_2(0) M_2^2 \lambda^2)^2] = q(1-q) (\lambda^2 - \lambda^3)^2, \end{cases}$$

the bilinear form (7.3) is

(8.6)
$$\sum_{a=2}^{3} v_{a} E_{e^{a}} [((X_{2}(1) - X_{2}(0) M_{2}^{2}) (\lambda^{0} u^{2} + Q_{2}^{2} \lambda^{2}))^{2}] = B_{0}(q) (\lambda^{0})^{2} + 2B_{1}(q) \lambda^{0} (\lambda^{2} - \lambda^{3}) + B_{2}(q) (\lambda^{2} - \lambda^{3})^{2},$$

where

(8.7)
$$B_0(q) = \frac{2q}{2q+1}, B_1(q) = \frac{2q(1-2q)}{(2q+1)^3}, B_2(q) = \frac{2q(3+2q+4q^2-8q^3)}{(2q+1)^5}.$$

By (4.2) $\psi_2 = \psi_2(t; \boldsymbol{\lambda}, \boldsymbol{\mu})$ is the solution of

(8.8)
$$\begin{cases} \psi_2(0; \boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{i}{2q+1} \left(2q \,\lambda^2 + \lambda^3 \right), \\ \frac{d}{dt} \psi_2 = \frac{1}{2} \left(B_0(q) \,\psi_2^2 + 2iB_1(q) \left(\boldsymbol{\mu}^2 - \boldsymbol{\mu}^3 \right) \,\psi_2 - B_2(q) \left(\boldsymbol{\mu}^2 - \boldsymbol{\mu}^3 \right)^2 \right) + \frac{2iq}{2q+1} \,\lambda^4. \end{cases}$$

Let $x_0 \ge 0$, $x_4 \ge 0$, y_2 , y_3 be fixed arbitrarily and set $x_2 = \frac{2q}{2q+1} x_0$, $x_3 = \frac{1}{2q+1} x_0$, x_2

= $(x_2, x_3), y_2$ = (y_2, y_3) . Then by Theorem 2, the sequence of processes

$$\{((n^{-1}X_2^n(t), n^{-2}X_4^n(t), y_2 + n^{-2}Y_2^n(t)), P_{(0, [nx_2], [nx_3], [n^2x_4])})\}_{n \ge 1}$$

converges to a diffusion process $((X_2(t), X_4(t), Y_2(t)), P_{(x_0, x_4, y_3)})$ and

(8.9)
$$\begin{array}{c} E_{(\boldsymbol{x}_{5},\boldsymbol{x}_{4},\boldsymbol{y}_{2})}[\exp\left(i(\boldsymbol{X}_{2}(t)\,\boldsymbol{\lambda}^{2}+\boldsymbol{X}_{4}(t)\,\boldsymbol{\lambda}^{4}+\boldsymbol{Y}_{2}(t)\,\boldsymbol{\mu}^{2}\right))]\\ =\exp\left(i\boldsymbol{y}_{2}\,\boldsymbol{\mu}^{2}+\boldsymbol{x}_{0}\,\boldsymbol{\psi}_{2}(t:\boldsymbol{\lambda},\,\boldsymbol{\mu})+i\boldsymbol{x}_{4}\,\boldsymbol{\lambda}^{4}\right). \end{array}$$

By Theorem 3 the sequence of processes

$$\{((n^{-1} X_2^n(t), n^{-2} X_4^n(t), y_2 + n^{-1} Y_2^n(t)), P_{(1, \lceil nx_2 \rceil, \lceil nx_3 \rceil, \lceil n^2 x_4 \rceil)})\}_{n \ge 1}$$

converges to a diffusion process
$$((X_2(t), \dot{X}_4(t), Y_2(t)), \dot{P}_{(x_0, x_4, y_2)})$$
 and
 $\hat{E}_{(x_5, x_4, y_2)}[\exp(i(\hat{X}_2(t) \lambda^2 + \hat{X}_4(t) \lambda^4 + \hat{Y}_2(t) \mu^2))]$
(8.10) $= \exp(iy_2 \mu^2 + p \int_0^t \psi_2(s; \lambda, \mu) \, ds + x_0 \, \psi_2(t; \lambda, \mu) + \frac{2ipt}{(2q+1)^2} (\mu^2 - \mu^3) + ix_4 \, \lambda^4).$

We shall clarify these limit processes applying the remarks in section 7. Set $X(t) = X_2(t) u^2 = X_2(t) + X_3(t)$. Then $X_2(t) = \frac{X(t)}{2q+1}$ (2q, 1). By (8.8) and (8.9) it follows that $Y_2(t) + Y_3(t) = y_2 + y_3$. Hence $Y_2(t)$ is determined by $Y(t) = Y_2(t)$ and y_3 . For the convenience we ste $Z(t) = X_4(t)$, $x = x_0$, $z = x_4$. Let $P_{(x,y,z)}$ be the probability measure induced from $P_{(x,z,y,0)}$ by the diffusion process (X(t), Y(t), Z(t)). Then by (8.9) it follows that

(8.11)
$$E_{(\mathbf{x},\mathbf{y},\mathbf{z})}[\exp\left(i(X(t)\,\lambda+Y(t)\,\mu+Z(t)\,\nu)\right)] \\ = \exp\left(iy\mu+x\phi(t:\lambda,\mu,\nu)+iz\nu\right),$$

where $\phi = \phi(t; \lambda, \mu, \nu)$ is the solution of

(8.12)
$$\begin{cases} \phi(0; \lambda, \mu, \nu) = i\lambda, \\ \frac{d\phi}{dt}(t; \lambda, \mu, \nu) = \frac{1}{2} (B_0(q) \phi^2 + 2iB_1(q) \mu\phi - B_2(q) \mu^2) + \frac{2iq}{2q+1}\nu. \end{cases}$$

By the same method we can define the process $(\hat{X}(t), \hat{Y}(t), \hat{Z}(t))$ and let $\hat{P}_{(x,y,t)}$ be the measure induced from $\hat{P}_{(x,z,y,0)}$ by this process. Then by (8.10) we have

$$(8.13) \qquad \begin{array}{l} \hat{E}_{(x,x,z)}[\exp\left(i(\hat{X}(t)\,\lambda+\hat{Y}(t)\,\mu+\hat{Z}(t)\,\nu\right)]\\ =\exp\left(iy\mu+p\int_{0}^{t}\phi(s;\lambda,\,\mu,\,\nu)\,ds+x\phi\left(t;\lambda,\,\mu,\,\nu\right)+\frac{2ipt}{(2q+1)^{2}}\,\mu+iz\nu\right)\,. \end{array}$$

Hence the generator A of the first process is

(8.14)
$$Af(x, y, z) = \frac{1}{2} x(B_0(q) D_x^2 + 2B_1(q) D_x D_y + B_2(q) D_y^2) f(x, y, z) + \frac{2q}{2q+1} xD_z f(x, y, z),$$

and the generator \hat{A} of the second process is

(8.15)
$$\hat{A}f(x, y, z) = Af(x, y, z) + pD_{z}f(x, y, z) + \frac{2p}{(2q+1)^{2}}D_{z}f(x, y, z)$$

We shall end this section by giving the forms of characteristic functions for Y(t) and $\hat{Y}(t)$ in some special cases. (The forms of Laplace transforms for X(t), $\hat{X}(t)$, Z(t) and $\hat{Z}(t)$ are given in [5]). If q=1 then $\phi(t:0, \mu, 0)=\frac{-t\mu^2}{9(27+it\mu)}$. Hence we have

(8.16)
$$E_{(x,0,0)}[\exp(iY(t)\mu)] = \exp\left(\frac{-xt\mu^2}{9(27+it\mu)}\right)$$

(8.17)
$$\hat{E}_{(s,0,0)}[\exp(i\hat{Y}(t)\mu)] = \left(1 + \frac{it}{27}\mu\right)^{-3p} \exp\left(\frac{i}{9}pt\mu - \frac{xt\mu^2}{9(27+it\mu)}\right).$$

If
$$q = \frac{1}{2}$$
 then $\phi(t; 0, \mu, 0) = -\frac{\mu}{2} \tanh\left(\frac{\mu}{8}t\right)$. Hence we ha have

(8.18)
$$E_{(z,0,0)}[\exp(iY(t)\mu)] = \exp\left(-\frac{x}{2}\mu\tanh\left(\frac{\mu}{8}t\right)\right),$$

(8.19)
$$\hat{E}_{(x,0,0)}[\exp(i\hat{Y}(t)\mu)] = (\cosh\left(\frac{\mu}{8}t\right))^{-4p}\exp\left(-\frac{x}{2}\mu\tanh\left(\frac{\mu}{8}t\right) + \frac{i}{2}pt\mu\right).$$

References

- [1] K. Athreya and P. Ney: Branching processes, Springer, Berlin, 1972.
- [2] P. Billingsley: Convergence of probability measures, John Wiley & Sons, New York, 1968.
- [3] J. Foster and P. Ney: Limit laws for decomposable critical branching processes, Z. Wahrsch. Verb. Gebiete 46 (1978), 15-43.
- [4] N. Ikeda and S. Watanabe: Stochastic differential equations and diffusion processes, North-Holland, Amsterdam, 1981.
- [5] S. Sugitani: On the limit distributions of decomposable Galton-Watson processes with the Perron-Frobenius root 1, Osaka J. Math. 18 (1981), 175-224.

Department of Mathematics Faculty of Education Fukui University Bunkyo, Fukui 910 Japan