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Osaka J. Math.
24 (1987), 579–604

LIMIT PROCESSES FOR THE BRANCHING PROCESSES WITH THE PERRON-FROBENIUS ROOT 1

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(Received June 12, 1985)

1. Introduction

We consider a multitype branching process whose mean matrix has the Perron-Frobenius root 1. The purpose of this paper is to show that the normalized sequence of such processes converges to some diffusion process when the initial population size goes to infinity.

Let $\boldsymbol{X}(n)=(X_a(n))_{1\leqq a\leqq d}$ be the d-type branching process and $M=(m_b^a)_{1\leqq a,b\leqq d}$ its mean matrix. For later convenience we denote by $\boldsymbol{u}=(u^a)_{1\leqq a\leqq d}$ (resp. $\boldsymbol{v}=(v_a)_{1\leqq a\leqq d}$) the column vector (resp. row vector). Type b is said to be accessible from type a if the (a, b) component of M^n is positive for some $n\geqq 0$. This relation is written as $a\rightarrow b$. If $a\rightarrow b$ and $b\rightarrow a$ then a and b are said to communicate with each other and this relation is written as $a\leftrightarrow b$. Since $\leftrightarrow$ is an equivalence relation we can decompose the set of types $\{1, 2, \cdots, d\}$ into the equivalence classes $C_1, C_2, \cdots, C_N$. Set $M_\beta^\alpha=(m_b^a)_{a\in C_\alpha, b\in C_\beta}$. Then we can write $M=(M_\beta^\alpha)_{1\leqq\alpha,\beta\leqq N}$ and, by definition, each M_α^α is irreducible.

Hereafter we shall assume the following conditions:

(A.1) $M_{\alpha+1}^\alpha\neq \mathrm{O}$ for any α and $M_\beta^\alpha=\mathrm{O}$ if $\beta<\alpha$,

and

(A.2) M_α^α is aperiodic and has the Perron-Frobenius root 1 for any α.

The first assumption means that if $a\in C_\alpha$, $b\in C_\beta$ and $\alpha<\beta$ then $a\rightarrow b$ but $b\nrightarrow a$. The second assumption means that each C_α is a final class or a critical class, where we say that C_α is a final class (resp. critical class) if the generating functions $F^a(s)$, $a\in C_\alpha$, are linear with respect to s^a, $a\in C_\alpha$ (resp. otherwise). Let $e^a=(\delta_b^a)_{1\leqq b\leqq d}$ where δ_b^a is the Kronecker's delta. The final assumption is

(A.3) $$E_{e^a}[X_b(1)^4]<\infty \quad \text{for any } a \text{ and } b,$$

in which P_{e^a} is the measure of the process $\boldsymbol{X}(n)$ starting at e^a. This assumption is needed to prove the tightness of the processes considered later.

We define a sequence of processes $\{\boldsymbol{X}^n(t)\}_{n\geqq 1}$ by

$$X^n(t) = X([nt])+(nt-[nt])\,(X([nt]+1)-X([nt]))\,, \tag{1.1}$$

where $[t]$ denotes the largest integer not exceeding t. Then the following results are known (see. [3] and [5]):

(T.1) If $X^n(0) = ne^a$, $a\in C_1$ and $t>0$, then the distribution of $(n^{-\alpha} X^n_\alpha(t))_{1\leq\alpha\leq N}$ converges to some distribution as $n\to\infty$.

(T.2) If C_1 is a final class, $X^n(0) = e^a$, $a\in C_1$ and $t>0$, then the distribution of $(n^{1-\alpha} X^n_\alpha(t))_{1\leq\alpha\leq N}$ converges to some distribution as $n\to\infty$.

The above results suggest us that these processes converge to some process. The meaning of convergence is as follows. Let C be the set of all continuous functions from $[0, \infty)$ to R^d endowed with the topology of uniform convergence on each finite interval. Then the sequence of processes $\{(X_n(t), P_n)\}_{n\geq 1}$ is said to be convergent to the process $(X(t), P)$ if Q_n converges to Q weakly where Q_n (resp. Q) is the probability measure on C induced from P_n (resp. P) by X_n (resp. X).

For the centered process the following result is known (see. [1: p. 192]):

(T.3) If $N= 1$, C_1 is a critical class, $X^n(0)=ne^a$ and $t>0$ then the distribution of $n^{-1/2}(X^n(t)-(X^n(t)\,u)\,v)$ converges to some distribution as $n\to\infty$ for some suitably chosen vectors u and v.

Unfortunately the author could not prove the convergence of the above process. Instead we can prove that the process $\int_0^t (X([ns])-(X([ns])\,u)\,v)\,ds$ converges.

We shall study the combined processes $\{(X^n(t), Y^n(t))\}_{n\geq 1}$ where $Y^n(t)$ is defined by

$$\begin{aligned} Y^n_\alpha(t) &= n\int_0^t (X_\alpha([ns])-(X_\alpha([ns])\,u^\alpha)\,v_\alpha)\,ds \\ &= \sum_{k=0}^{[nt]-1} (X_\alpha(k)-(X_\alpha(k)\,u^\alpha)\,v_\alpha) \\ &\quad +(nt-[nt])\,(X_\alpha([nt])-(X_\alpha([nt])\,u^\alpha)\,v_\alpha)\,, \qquad 1\leq\alpha\leq N\,, \end{aligned} \tag{1.2}$$

in which $u^\alpha=(u^a)_{a\in C_\alpha}$ and $v_\alpha=(v_a)_{a\in C_\alpha}$ are determined by

$$M^\alpha_\alpha u^\alpha = u^\alpha,\ v_\alpha M^\alpha_\alpha = v_\alpha \quad\text{and}\quad v_\alpha u^\alpha = v_\alpha \mathbf{1}^\alpha = 1\,, \tag{1.3}$$

where $\mathbf{1}^\alpha=(1)_{a\in C_\alpha}$.

The purpose of this paper is to show the following two theorems.

Theorem A. *If* $\lim_{n\to\infty} n^{-\alpha} X^n_\alpha(0)=x_\alpha v_\alpha$, $1\leq\alpha\leq N$, *then the sequence of processes* $(n^{-\alpha} X^n_\alpha(t), n^{-\alpha} Y^n_\alpha(t))_{1\leq\alpha\leq N}$ *converges to some diffusion process.*

Theorem B. *Let C_1 be a final class. If $X_1^n(0)=1$, $X_a^n(0)=0$, $a\in C_1$, $a\neq 1$ and $\lim_{n\to\infty} n^{1-\alpha} X_\alpha^n(0)=x_\alpha v_\alpha$, $\alpha\geqq 2$, then the sequence of processes $(n^{1-\alpha} X_\alpha^n(t), n^{1-\alpha} Y_\alpha^n(t))_{2\leqq\alpha\leqq N}$ converges to some diffusion process.*

The precise forms of these theorems are given in Theorem 2 in section 5 and Theorem 3 in section 6.

To prove our main theorems we must show the tightness and the convergence of any finite dimensional distributions. In general the tightness for a sequence of continuous processes $\{(X_n(t), P_n)\}_{n\geqq 1}$ follows from

$$\sup_n E_n[X_n(0)^2]<\infty\,, \tag{1.4}$$

and the existence of $C>0$ such that

$$E_n[(X_n(t)-X_n(s))^4]\leqq C(t-s)^2 \quad \text{for any } n. \tag{1.5}$$

But in our case (1.4) is trivial by the definition of our process. To show (1.5) we shall estimate several moments of $X(n)$ in section 2. Then we shall show the tightness in section 3. To prove the convergence of finite dimensional distributions we prepare a limit theorem in section 4. Applying these results we shall show our main theorems in sections 5 and 6. We shall give some comments for the limit processes in section 7. An example is given in section 8.

The author wishes to express his thanks to Prof. N. Ikeda and Prof. T. Watanabe who suggested him to extend the results in [5].

2. Preliminary results

In this section we shall estimate several moments for the process $X(n)$. Before stating our results we prepare some notations. Set

$$P_\beta^\alpha=\begin{cases} u^\alpha\otimes v_\alpha & \text{if } \beta=\alpha\,,\\ \dfrac{1}{(\beta-\alpha)!}\prod_{\gamma=\alpha}^{\beta-1}(v_\gamma M_{\gamma+1}^\gamma u^{\gamma+1})\,(u^\alpha\otimes v_\beta) & \text{if } \beta\geqq\alpha+1\,,\end{cases} \tag{2.1}$$

where $u^\alpha\otimes v_\beta=(u^a v_b)_{a\in C_\alpha, b\in C_\beta}$,

$$Q_\beta^\alpha=\begin{cases}(I-M_\alpha^\alpha+P_\alpha^\alpha)^{-1}(I-P_\alpha^\alpha) & \text{if } \beta=\alpha\,,\\ (\beta-\alpha)^{-1}P_{\beta-1}^\alpha M_\beta^{\beta-1}(I-M_\beta^\beta+P_\beta^\beta)^{-1}(I-P_\beta^\beta) & \text{if } \beta\geqq\alpha+1\,,\end{cases} \tag{2.2}$$

$$\bar{\lambda}^\alpha=(I-P_\alpha^\alpha)\,\lambda^\alpha=\lambda^\alpha-(v_\alpha\lambda^\alpha)\,u^\alpha\,. \tag{2.3}$$

Then $v_\alpha\bar{\lambda}^\alpha=0$ and

$$Y_\alpha^n(t)\,\lambda^\alpha=\sum_{k=0}^{[nt]-1}X_\alpha(k)\,\bar{\lambda}^\alpha+(nt-[nt])\,X_\alpha([nt])\,\bar{\lambda}^\alpha\,. \tag{2.4}$$

Set $M^n=((M^n)^\alpha_\beta)_{1\leqq\alpha,\beta\leqq N}$. By (A.1) and (A.2) we have

$$(2.5)\qquad \begin{cases} (M^n)^\alpha_\beta = O & \text{if} \quad \beta<\alpha\,, \\ (M^n)^\alpha_\alpha = (M^\alpha_\alpha)^n\,, \\ (M^n)^\alpha_\beta = \sum_{k=0}^{n-1} \sum_{\alpha<\gamma\leqq\beta} (M^\alpha_\alpha)^k M^\alpha_\gamma (M^{n-k-1})^\gamma_\beta & \text{if} \quad \beta\geqq\alpha+1\,. \end{cases}$$

By (A.2) it follows that $(M^\alpha_\alpha-P^\alpha_\alpha)^n=O(\rho^n)$ for some $0<\rho<1$. Then the following result is easily seen by the induction argument with respect to $\beta-\alpha$ (cf. [5: section 3]).

Lemma 2.1. *Let* (A.1) *and* (A.2) *be satisfied. Then we have*

$$(2.6)\qquad \begin{aligned} (E_{e^a}[X_\beta(n)\,\boldsymbol{\lambda}^\beta])_{a\in C_\alpha} &= (M^n)^\alpha_\beta\,\boldsymbol{\lambda}^\beta \\ &= \begin{cases} P^\alpha_\alpha\,\boldsymbol{\lambda}^\alpha+(M^\alpha_\alpha-P^\alpha_\alpha)^n\,\boldsymbol{\lambda}^\alpha & \textit{if} \quad \beta=\alpha\,, \\ n^{\beta-\alpha}P^\alpha_\beta\,\boldsymbol{\lambda}^\beta+O(n^{\beta-\alpha-1}) & \textit{if} \quad \beta\geqq\alpha+1\,, \end{cases} \end{aligned}$$

$$(2.7)\qquad \begin{aligned} (E_{e^a}[X_\beta(n)\,\bar{\boldsymbol{\lambda}}^\beta])_{a\in C_\alpha} &= (M^n)^\alpha_\beta\,\bar{\boldsymbol{\lambda}}^\beta \\ &= \begin{cases} (M^\alpha_\alpha-P^\alpha_\alpha)^n\,\bar{\boldsymbol{\lambda}}^\alpha & \textit{if} \quad \beta=\alpha\,, \\ (\beta-\alpha)\,n^{\beta-\alpha-1}\,Q^\alpha_\beta\,\bar{\boldsymbol{\lambda}}^\beta+O(n^{\beta-\alpha-2}) & \textit{if} \quad \beta\geqq\alpha+1\,, \end{cases} \end{aligned}$$

and hence

$$(2.8)\qquad \begin{aligned} (E_{e^a}[\sum_{k=0}^{n-1} X_\beta(k)\,\bar{\boldsymbol{\lambda}}^\beta])_{a\in C_\alpha} &= \sum_{k=0}^{n-1} (M^k)^\alpha_\beta\,\bar{\boldsymbol{\lambda}}^\beta \\ &= \begin{cases} Q^\alpha_\alpha\,\boldsymbol{\lambda}^\alpha-Q^\alpha_\alpha(M^\alpha_\alpha-P^\alpha_\alpha)^n\,\boldsymbol{\lambda}^\alpha & \textit{if} \quad \beta=\alpha\,, \\ n^{\beta-\alpha}\,Q^\alpha_\beta\,\boldsymbol{\lambda}^\beta+O(n^{\beta-\alpha-1}) & \textit{if} \quad \beta\geqq\alpha+1\,. \end{cases} \end{aligned}$$

Next we shall estimate the higher order moments. We define $|\boldsymbol{\lambda}^\alpha|=\sum_{a\ni C_\alpha} |\lambda^a|$.

Lemma 2.2. *Let* (A.1)–(A.3) *be satisfied and $a\in C_\alpha$, $\beta\geqq\alpha$ be fixed. Then there exists $C>0$ satisfying the following relations for $n\geqq1$ and $p=2, 3, 4$,*

$$(2.9)\qquad |E_{e^a}[(X_\beta(n)\,\boldsymbol{\lambda}^\beta)^p]|\leqq Cn^{p(\beta-\alpha+1)-1}|\boldsymbol{\lambda}^\beta|^p\,,$$

$$(2.10)\qquad |E_{e^a}[(X_\beta(n)\,\bar{\boldsymbol{\lambda}}^\beta)^p]|\leqq \begin{cases} C|\boldsymbol{\lambda}^\alpha|^p & \textit{if} \quad \beta=\alpha \textit{ and } p\leqq3\,, \\ Cn|\boldsymbol{\lambda}^\alpha|^4 & \textit{if} \quad \beta=\alpha \textit{ and } p=4\,, \\ Cn^{p(\beta-\alpha)-1}|\lambda^\beta|^p & \textit{if} \quad \beta\geqq\alpha+1\,. \end{cases}$$

If C_α is a final class then we have the better estimates

$$(2.11)\qquad |E_{e^a}[(X_\beta(n)\,\boldsymbol{\lambda}^\beta)^p]|\leqq Cn^{p(\beta-\alpha)}|\boldsymbol{\lambda}^\beta|^p\,,$$

$$(2.12)\qquad |E_{e^a}[(X_\beta(n)\,\bar{\boldsymbol{\lambda}}^\beta)^p]| \leqq \begin{cases} C|\boldsymbol{\lambda}^\alpha|^p & \textit{if } \beta=\alpha\,, \\ Cn|\boldsymbol{\lambda}^{\alpha+1}|^p & \textit{if } \beta=\alpha+1 \textit{ and } p\leqq 3\,, \\ Cn^2|\boldsymbol{\lambda}^{\alpha+1}|^4 & \textit{if } \beta=\alpha+1 \textit{ and } p=4\,, \\ Cn^{p(\beta-\alpha-1)}|\boldsymbol{\lambda}^\beta|^p & \textit{if } \beta\geqq\alpha+2\,. \end{cases}$$

Outline of the proof. We first consider in the case $p=2$. Set

$$\begin{aligned} &\boldsymbol{B}(n:\boldsymbol{\lambda},\boldsymbol{\lambda}) = (E_{e^a}[(\boldsymbol{X}(n)\,\boldsymbol{\lambda})^2])_{1\leqq a\leqq d}\,, \\ &\boldsymbol{B}^\alpha(\boldsymbol{\lambda}^\beta,\boldsymbol{\lambda}^\gamma) = (\sum_{b\ni C_\beta}\sum_{c\in C\gamma} D_b\,D_c\,F^a(\mathbf{1})\,\lambda^b\lambda^c)_{a\in C_\alpha}\,, \\ &\boldsymbol{B}(\boldsymbol{\lambda},\boldsymbol{\lambda}) = (\sum_{\beta,\gamma}\boldsymbol{B}^\alpha(\boldsymbol{\lambda}^\beta,\boldsymbol{\lambda}^\gamma))_{1\leqq\alpha\leqq N}\,, \end{aligned}$$

where D_a denotes the partial differentiation with respect to s^a. Then we have

$$\boldsymbol{B}(n:\boldsymbol{\lambda},\boldsymbol{\lambda}) = \boldsymbol{B}(M^{n-1}\boldsymbol{\lambda}, M^{n-1}\boldsymbol{\lambda}) + M\boldsymbol{B}(n-1:\boldsymbol{\lambda},\boldsymbol{\lambda})\,,$$

and hence

$$(2.13)\qquad \boldsymbol{B}(n:\boldsymbol{\lambda},\boldsymbol{\lambda}) = \sum_{x=0}^{n-1} M^{n-k-1}\boldsymbol{B}(M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}) + M^n\boldsymbol{B}(0:\boldsymbol{\lambda},\boldsymbol{\lambda})\,.$$

This means that

$$(2.14)\qquad \begin{aligned} &(E_{e^a}[(\boldsymbol{X}_\beta(n)\,\boldsymbol{\lambda}^\beta)^2])_{a\in C_\alpha} \\ &= \sum_{k=0}^{n-1}\sum_{\gamma=\alpha}^{\beta}\sum_{\delta=\gamma}^{\beta}\sum_{\varepsilon=\gamma}^{\beta}(M^{n-k-1})^\alpha_\gamma\,\boldsymbol{B}^\gamma((M^k)^\delta_\beta\,\boldsymbol{\lambda}^\beta, (M^k)^\varepsilon_\beta\,\boldsymbol{\lambda}^\beta) \\ &\quad + \sum_{\gamma=\alpha}^{\beta}(M^n)^\alpha_\gamma\,\boldsymbol{B}^\gamma(0:\boldsymbol{\lambda}^\beta,\boldsymbol{\lambda}^\beta)\,. \end{aligned}$$

Then (2.9) and (2.10) follow from Lemma 2.1. If C_α is a final class then $\boldsymbol{B}^\alpha(\boldsymbol{\lambda}^\alpha,\boldsymbol{\lambda}^\alpha)=\mathbf{0}$ and so (2.11) and (2.12) can be shown by the same method.

The other cases can be treated similarly. For the convenience to check the above results we only remark the forms of moments. Set

$$\begin{aligned} &\boldsymbol{C}(n:\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) = (E_{e^a}[(\boldsymbol{X}(n)\,\boldsymbol{\lambda})^3])_{1\leqq a\leqq d}\,, \\ &\boldsymbol{D}(n:\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) = (E_{e^a}[(\boldsymbol{X}(n)\,\boldsymbol{\lambda})^4])_{1\leqq a\leqq d}\,, \\ &\boldsymbol{C}(\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) = (\sum_{b,c,f} D_b\,D_c\,D_f\,F^a(\mathbf{1})\,\lambda^b\,\lambda^c\,\lambda^f)_{1\leqq a\leqq d}\,, \\ &\boldsymbol{D}(\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) = (\sum_{b,c,f,g} D_b\,D_c\,D_f\,D_g\,F^a(\mathbf{1})\,\lambda^b\,\lambda^c\,\lambda^f\,\lambda^g)_{1\leqq a\leqq d}\,. \end{aligned}$$

Then we have

$$(2.15)\qquad \begin{aligned} \boldsymbol{C}(n:\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda}) &= \sum_{k=0}^{n-1} M^{n-k-1}\boldsymbol{C}(M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}) \\ &\quad + 3\sum_{k=0}^{n-1} M^{n-k-1}\boldsymbol{B}(M^k\boldsymbol{\lambda}, \boldsymbol{B}(k:\boldsymbol{\lambda},\boldsymbol{\lambda})) + M^n\boldsymbol{C}(0:\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{\lambda})\,, \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{D}(n: \boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) = & \sum_{k=0}^{n-1} M^{n-k-1} \boldsymbol{D}(M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}) \\
& +6 \sum_{k=0}^{n-1} M^{n-k+1} \boldsymbol{C}(M^k\boldsymbol{\lambda}, M^k\boldsymbol{\lambda}, \boldsymbol{B}(k: \boldsymbol{\lambda}, \boldsymbol{\lambda})) \\
& +4 \sum_{k=1}^{n-1} M^{n-k-1} \boldsymbol{B}(M^k\boldsymbol{\lambda}, \boldsymbol{C}(k: \boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda})) \\
& +3 \sum_{k=0}^{n-1} M^{n-k-1} \boldsymbol{B}(\boldsymbol{B}(k: \boldsymbol{\lambda}, \boldsymbol{\lambda}), \boldsymbol{B}(k: \boldsymbol{\lambda}, \boldsymbol{\lambda})) \\
& +M^n \boldsymbol{D}(0: \boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) .
\end{aligned} \tag{2.16}
$$

Next we shall estimate the moments for $\boldsymbol{Y}^n(t)$.

Lemma 2.3. *Let* (A.1)–(A.3) *be satisfied and* $a \in C_\alpha$, $\beta \geqq \alpha$ *be fixed. Then there exists* $C>0$ *satisfying the following relations for* $n \geqq 1$,

$$
E_{e^a}[(\sum_{k=0}^{n-1} \boldsymbol{X}_\beta(k)\, \bar{\boldsymbol{\lambda}}^\beta)^2] \leqq \begin{cases} Cn|\boldsymbol{\lambda}^\alpha|^2 & \text{if } \beta=\alpha, \\ Cn^{2(\beta-\alpha)}|\boldsymbol{\lambda}^\beta|^2 & \text{if } \beta \geqq \alpha+1, \end{cases} \tag{2.17}
$$

$$
E_{e^a}[(\sum_{k=1}^{n-2} \boldsymbol{X}_\beta(k)\, \bar{\boldsymbol{\lambda}}^\beta)^4] \leqq Cn^{4(\beta-\alpha+1)-1}|\boldsymbol{\lambda}^\beta|^4 . \tag{2.18}
$$

If C_α *is a final class then we obtain the better estimate*

$$
E_{e^a}[(\sum_{k=0}^{n-1} \boldsymbol{X}_\beta(k)\, \bar{\boldsymbol{\lambda}}^\beta)^4] \leqq \begin{cases} Cn^2|\boldsymbol{\lambda}^\alpha|^4 & \text{if } \beta=\alpha, \\ Cn^{4(\beta-\alpha)}|\boldsymbol{\lambda}^\beta|^4 & \text{if } \beta \geqq \alpha+1 . \end{cases} \tag{2.19}
$$

Proof. Since

$$
\begin{aligned}
& E_{e^a}[(\sum_{k=0}^{n-1} \boldsymbol{X}_\beta(k)\, \boldsymbol{\lambda}^\beta)^2] \\
& \leqq 2 \sum_{k=0}^{n-1} E_{e^a}[\boldsymbol{X}_\beta(k)\, \boldsymbol{\lambda}^\beta \cdot \sum_{l=k}^{n-1} \boldsymbol{X}_\beta(l)\, \boldsymbol{\lambda}^\beta] \\
& = 2 \sum_{k=0}^{n-1} \sum_{\gamma=\alpha}^{\beta} E_{e^a}[\boldsymbol{X}_\beta(k)\, \boldsymbol{\lambda}^\beta \cdot \boldsymbol{X}_\gamma(k) (\sum_{l=0}^{n-k-1} M^l)^\gamma_\beta\, \boldsymbol{\lambda}^\beta] \\
& \leqq 2 \sum_{k=0}^{n-1} \sum_{\gamma=\alpha}^{\beta} \sqrt{E_{e^a}[(\boldsymbol{X}_\beta(k)\, \boldsymbol{\lambda}^\beta)^2]\, E_{e^a}[(\boldsymbol{X}_\gamma(k) (\sum_{l=0}^{n-k-1} M^k)^\gamma_\beta\, \boldsymbol{\lambda}^\beta)^2]} ,
\end{aligned}
$$

(2.17) follows from Lemma 2.1 and Lemma 2.2. Combining Lemma 2.2 with

$$
E_{e^a}[(\sum_{k=0}^{n-1} \boldsymbol{X}_\beta(k)\, \boldsymbol{\lambda}^\beta)^4] \leqq n^3 \sum_{k=0}^{n-1} E_{e^a}[(\boldsymbol{X}_\beta(k)\, \boldsymbol{\lambda}^\beta)^4] ,
$$

we obtain (2.18) for $\beta \geqq \alpha+1$ and (2.19) for $\beta \geqq \alpha+2$. If C_α is a final class then the process $(\boldsymbol{X}_\alpha(n), \sum_{a \in C_\alpha} v_a P_{e^a})$ is a stationary Markov chain having the mixing property and so (2.19) holds if $\beta=\alpha$. To show the rest cases, (2.18) for $\beta=\alpha$ and (2.19) for $\beta=\alpha+1$, we first expand (2.18) as follows.

$$E_{e^a}[(\sum_{k=0}^{n-1} X(k)\boldsymbol{\lambda})^4]$$

$$(2.20)\qquad = \sum_{k=0}^{n-1} I_k^n + 4\sum_{k=0}^{n-1}\sum_{l=k+1}^{n-1} I_{k,l}^n + 6\sum_{k=0}^{n-1}\sum_{l=k+1}^{n-1} II_{k,l}^n + 12\sum_{k=0}^{n-1}\sum_{l=k+1}^{n-1}\sum_{m=l+1}^{n-1} I_{k,l,m}^n,$$

where

$$(2.21)\qquad \begin{cases} I_k^n = E_{e^a}[(X(k)\boldsymbol{\lambda})^3\cdot(X(k)\boldsymbol{\lambda}+4\sum_{l=m+1}^{n-1} X(l)\boldsymbol{\lambda})], \\ I_{k,l}^n = E_{e^a}[X(k)\boldsymbol{\lambda}\cdot(X(l)\boldsymbol{\lambda})^2\cdot(X(l)\boldsymbol{\lambda}+3\sum_{m=l+1}^{n-1} X(m)\boldsymbol{\lambda})], \\ II_{k,l}^n = E_{e^a}[(X(k)\boldsymbol{\lambda})^2\cdot X(l)\boldsymbol{\lambda}\cdot(X(l)\boldsymbol{\lambda}+2\sum_{j=m+1}^{n-1} X(m)\boldsymbol{\lambda})], \\ I_{k,l,m}^n = E_{e^a}[X(k)\boldsymbol{\lambda}\cdot X(l)\boldsymbol{\lambda}\cdot X(m)\boldsymbol{\lambda}\cdot(X(m)\boldsymbol{\lambda}+2\sum_{j=m+1}^{n-1} X(j)\boldsymbol{\lambda})]. \end{cases}$$

Set $\boldsymbol{\lambda}(p, n)=\boldsymbol{\lambda}+p\sum_{j=1}^{n-1} M^j\boldsymbol{\lambda}$ and use the Markov property, then we have

$$(2.22)\qquad \begin{cases} I_k^n = E_{e^a}[(X(k)\boldsymbol{\lambda})^3\cdot X(k)\boldsymbol{\lambda}(4, n-k)], \\ I_{k,l}^n = E_{e^a}[X(k)\boldsymbol{\lambda}\cdot(X(l)\boldsymbol{\lambda})^2\cdot X(l)\boldsymbol{\lambda}(3, n-l)], \\ II_{k,l}^n = E_{e^a}[(X(k)\boldsymbol{\lambda})^2\cdot X(l)\boldsymbol{\lambda}\cdot X(l)\boldsymbol{\lambda}(2, n-l)], \\ I_{k,l,m}^n = E_{e^a}[X(k)\boldsymbol{\lambda}\cdot X(l)\boldsymbol{\lambda}\cdot X(m)\boldsymbol{\lambda}\cdot X(m)\boldsymbol{\lambda}(2, n-m)]. \end{cases}$$

Let $\beta=\alpha$ or $\beta=\alpha+1$ and set $\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}^\beta$. Then it follows from Lemma 2.1 that if $\beta=\alpha$ then

$$(2.23)\qquad v_\alpha\boldsymbol{\lambda}^\alpha(p, k)=0,\ |\boldsymbol{\lambda}^\alpha(p, k)|\leqq C|\boldsymbol{\lambda}^\alpha|,$$

and if $\beta=\alpha+1$ then

$$(2.24)\qquad \begin{cases} |\boldsymbol{\lambda}^\alpha(p, k)|\leqq Ck|\boldsymbol{\lambda}^{\alpha+1}|,\ v_{\alpha+1}\boldsymbol{\lambda}^{\alpha+1}(p, k)=0, \\ |\boldsymbol{\lambda}^{\alpha+1}(p, k)|\leqq C|\boldsymbol{\lambda}^{\alpha+1}|. \end{cases}$$

Combining these estimates with Lemma 2.2 we obtain

$$(2.25)\qquad |I_k^n|,\ |I_{k,l}^n|,\ |II_{k,l}^n| \leqq \begin{cases} C|\boldsymbol{\lambda}^\alpha|^4 & \text{if } \beta=\alpha, \\ Cn|\boldsymbol{\lambda}^{\alpha+1}|^4 & \text{if } C_\alpha \text{ is a final class and } \beta=\alpha+1. \end{cases}$$

Hence the proof is completed if we can show

$$\sum_{0\leqq k<l<m\leqq n-1} I_{k,k,m}^n$$

$$\leqq \begin{cases} Cn^3|\boldsymbol{\lambda}^\alpha|^4 & \text{if } \beta=\alpha, \\ Cn^4|\boldsymbol{\lambda}^{\alpha+1}|^4 & \text{if } C_\alpha \text{ is a final class and } \beta=\alpha+1. \end{cases}$$

Set

$$\begin{aligned} &\boldsymbol{A}(n\colon \boldsymbol{\lambda}(1), \boldsymbol{\lambda}(2)) \\ &\quad = (E_{e^a}[(\boldsymbol{X}(n)-e^aM^n)\boldsymbol{\lambda}(1)\cdot(\boldsymbol{X}(n)-e^aM^n)\boldsymbol{\lambda}(2)])_{1\leqq n\leqq d} \\ &\quad = \boldsymbol{B}(n\colon \boldsymbol{\lambda}(1), \boldsymbol{\lambda}(2))-(e^aM^n\boldsymbol{\lambda}(1)\cdot e^aM^n\boldsymbol{\lambda}(2))_{1\in a\in d}. \end{aligned}$$

Then we have

$$\begin{aligned} &I^n_{k,l,m} \\ &\quad = E_{e^a}[\boldsymbol{X}(k)\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda}(2, n-m)] \\ &\qquad +E_{e^a}[\boldsymbol{X}(k)\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)\boldsymbol{A}(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m))] \\ &\quad = E_{e^a}[\boldsymbol{X}(k)\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda}\cdot\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda}(2, n-m)] \\ &\qquad +E_{e^a}[\boldsymbol{X}(k)\boldsymbol{\lambda}\cdot\boldsymbol{X}(k)M^{m-l}\boldsymbol{\lambda}\cdot\boldsymbol{X}(k)M^{l-k}\boldsymbol{A}(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m))] \\ &\qquad +E_{e^a}[\boldsymbol{X}(k)\boldsymbol{\lambda}\cdot\boldsymbol{X}(k)\boldsymbol{A}(l-k\colon \boldsymbol{\lambda}, \boldsymbol{A}(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))] \\ &\quad = \sum_{p=1}^{3} I^u_{k,l,m}(p). \end{aligned}$$

Therefore it suffices to show that

$$\begin{aligned} &\sum_{0\leqq k<l<m\leqq n-1} I^n_{k,l,m}(p) \\ &\quad \leqq \begin{cases} Cn^3|\boldsymbol{\lambda}^\alpha|^4 & \text{if } \beta=\alpha, \\ Cn^4|\boldsymbol{\lambda}^{\alpha+1}|^4 & \text{if } C_\alpha \text{ is a final class and } \beta=\alpha+1, \end{cases} \end{aligned} \tag{2.26}$$

holds for $p=1, 2, 3$. If $p=1$ then by Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned} &I^u_{k,l,m}(1) \\ &\quad \leqq \begin{cases} Cn\,\rho^{m-l}|\boldsymbol{\lambda}^\alpha|^4 & \text{if } \beta=\alpha, \\ C(n+n^2\rho^{m-l})|\boldsymbol{\lambda}^{\alpha+1}|^4 & \text{if } C_\alpha \text{ is a final class and } \beta=\alpha+1, \end{cases} \end{aligned} \tag{2.27}$$

for some $0<\rho<1$ and hence (2.26) holds. Then we shall consider the rest cases. By (2.13) it follows that if $\beta=\alpha$ or $\alpha+1$ then

$$\begin{aligned} &|\boldsymbol{A}^\beta(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m))|\leqq C|\boldsymbol{\lambda}^\beta|^2, \\ &|\boldsymbol{A}^\beta(l-k\colon \boldsymbol{\lambda}, \boldsymbol{A}(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))|\leqq C|\boldsymbol{\lambda}^\beta|^3, \end{aligned}$$

and if C_α is a final class then

$$\begin{aligned} &|\boldsymbol{A}^\alpha(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))|\leqq Cn|\boldsymbol{\lambda}^{\alpha+1}|^2, \\ &|\boldsymbol{A}^\alpha(l-k\colon \boldsymbol{\lambda}, \boldsymbol{A}(m-l\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))|\leqq Cn|\boldsymbol{\lambda}^{\alpha+1}|^3. \end{aligned}$$

Since

$$I_{k,l,m}^{n}(2) = C^a(k: \boldsymbol{\lambda}, M^{l-m} \boldsymbol{\lambda}, M^{l-k} \boldsymbol{A}(m-l: \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))\,,$$
$$I_{k,l,m}^{n}(3) = B^a(k: \boldsymbol{\lambda}, \boldsymbol{A}(l-k: \boldsymbol{\lambda}, \boldsymbol{A}(m-l: \boldsymbol{\lambda}, \boldsymbol{\lambda}(2, n-m)))\,,$$

we obtain the same estimates in (2.27) by (2.13), (2.15) and the preceding estimates. Thus we have completed the proof.

We sometimes assume that $\boldsymbol{\lambda}^\alpha$ is a d-dimensional vector such that $(\boldsymbol{\lambda}^\alpha)^a = \lambda^a$ if $a \in C_\alpha$ and $(\boldsymbol{\lambda}^\alpha)^a = 0$ if $a \notin C_\alpha$.

Lemma 2.4. *Let* (A.1)–(A.3) *be satisfied,* $a \in C_\alpha$, $\beta \geqq \alpha$ *and* $T > 0$ *be fixed and* $p=2$ *or* 4. *Then there exists* $C(T) > 0$ *such that the following relations hold for* $n \geqq 1$ *and* $m \leqq nT$,

$$\tag{2.28} E_{e^a}[(\boldsymbol{X}(m)(M^n - I)\boldsymbol{\lambda}^\beta)^p] \leqq \begin{cases} C(T)|\boldsymbol{\lambda}^\alpha|^p & \text{if } \beta = \alpha\,, \\ C(T)\, n^{p(\beta-\alpha+1)-1} |\boldsymbol{\lambda}^\beta|^p & \text{if } \beta \geqq \alpha+1\,, \end{cases}$$

$$\tag{2.29} E_{e^a}[(\boldsymbol{X}(m) \sum_{k=0}^{n-1} M^k \bar{\boldsymbol{\lambda}}^\beta)^p] \leqq \begin{cases} C(T)|\boldsymbol{\lambda}^\alpha|^p & \text{if } \beta = \alpha\,, \\ C(T)\, n^{p(\beta-\alpha+1)-1} |\boldsymbol{\lambda}^\beta|^p & \text{if } \beta \geq \alpha+1\,. \end{cases}$$

If C_α *is a final class then*

$$\tag{2.30} E_{e^a}[(\boldsymbol{X}(m)(M^n - I)\boldsymbol{\lambda}^\beta)^p] \leqq C(T)\, n^{p(\beta-\alpha)} |\boldsymbol{\lambda}^\beta|^p\,,$$

$$\tag{2.31} E_{e^a}[(\boldsymbol{X}(m) \sum_{k=0}^{n-1} M^k \bar{\boldsymbol{\lambda}}^\beta)^p] \leqq C(T)\, n^{p(\beta-\alpha)} |\boldsymbol{\lambda}^\beta|^p\,.$$

Proof. Since p is even we have

$$E_{e^a}[(\boldsymbol{X}(m)(M^n - I)\boldsymbol{\lambda}^\beta)^p] = E_{e^a}[(\sum_{\gamma=\alpha}^{\beta} \boldsymbol{X}_\gamma(m)(M^n - I)_\beta^\gamma \boldsymbol{\lambda}^\beta)^p]$$
$$\leqq N^{p-1} \sum_{\gamma=\alpha}^{\beta} E_{e^a}[(\boldsymbol{X}_\gamma(m)(M^n - I)_\beta^\gamma \boldsymbol{\lambda}^\beta)^p]\,.$$

Then (2.28) and (2.30) follow from $\boldsymbol{v}_\beta (M^n - I)_\beta^\beta \boldsymbol{\lambda}^\beta = 0$ and the preceding lemmas. Since the rest cases can be treated similarly we omit the proof.

We shall end this section by showing two lemmas which will be used to prove the tightness. Let P_x denote the measure of the process $\boldsymbol{X}(n)$ starting at $\boldsymbol{x}$.

Lemma 2.5. *Let* (A.1)–(A.3) *be satisfied and* $m \geqq l+1$. *Then we have*

$$\tag{2.32} \begin{aligned} &E_x[(\boldsymbol{X}(m)\boldsymbol{\lambda} - \boldsymbol{x} M^m \boldsymbol{\lambda} - \boldsymbol{X}(l)\boldsymbol{\lambda} + \boldsymbol{x} M^l \boldsymbol{\lambda})^2] \\ &\leqq \sum_{a=1}^{d} \sum_{b=1}^{d} x_a E_{e^a}[X_b(l)]\, E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^2] \end{aligned}$$

$$+\sum_{a=1}^{d} x_a E_{e^a}[(\boldsymbol{X}(l)(M^{m-l}-I)\boldsymbol{\lambda})^2],$$

$$\begin{aligned} &E_{\boldsymbol{x}}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{x}M^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{x}M^l\boldsymbol{\lambda})^4] \\ &\quad\leqq 3E_{\boldsymbol{x}}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{x}M^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{x}M^l\boldsymbol{\lambda})^2]^2 \\ &\qquad+24d\sum_{a=1}^{d}\sum_{b=1}^{d} x_a E_{e^a}[X_b(l)^2]E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^2]^2 \\ &\qquad+128\sum_{a=1}^{d}\sum_{b=1}^{d} x_a E_{e^a}[X_b(l)]E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^4] \\ &\qquad+128\sum_{a=1}^{d} x_a E_{e^a}[(\boldsymbol{X}(l)(M^{m-l}-I)\boldsymbol{\lambda})^4]. \end{aligned} \tag{2.33}$$

Proof. By the branching property of $\boldsymbol{X}(n)$ it follows that

$$\begin{aligned} &E_{\boldsymbol{x}}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{x}M^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{x}M^l\boldsymbol{\lambda})^2] \\ &\quad=\sum_{a=1}^{d} x_a E_{e^a}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{e}^aM^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{e}^aM^l\boldsymbol{\lambda})^2] \\ &\quad=\sum_{a=1}^{d} x_a I_a, \end{aligned} \tag{2.34}$$

$$\begin{aligned} &E_{\boldsymbol{x}}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{x}M^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{x}M^l\boldsymbol{\lambda})^4] \\ &\quad\leqq 3E_{\boldsymbol{x}}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{x}M^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{x}M^l\boldsymbol{\lambda})^2]^2 \\ &\qquad+\sum_{a=1}^{d} x_a E_{e^a}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{e}^aM^m\boldsymbol{\lambda}-\boldsymbol{X}(l)\boldsymbol{\lambda}+\boldsymbol{e}^aM^l\boldsymbol{\lambda})^4] \\ &\quad=3\Big(\sum_{a=1}^{d} x_a I_a\Big)^2+\sum_{a=1}^{d} x_a II_a. \end{aligned} \tag{2.35}$$

First remark that

$$\begin{aligned} I_a &= E_{e^a}[((\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda})+(\boldsymbol{X}(l)\boldsymbol{\lambda}-\boldsymbol{e}^aM^l)(M^{m-l}-I)\boldsymbol{\lambda})^2] \\ &= E_{e^a}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda})^2] \\ &\quad+E_{e^a}[((\boldsymbol{X}(l)-\boldsymbol{e}^aM^l)(M^{m-l}-I)\boldsymbol{\lambda})^2] \\ &\leqq I_a^1+E_{e^a}[(\boldsymbol{X}(l)(M^{m-l}-I)\boldsymbol{\lambda})^2]. \end{aligned}$$

Then by the Markov property and (2.34) we obtain

$$\begin{aligned} I_a^1 &= E_{e^a}[E_{\boldsymbol{X}(l)}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda}-\boldsymbol{X}(0)M^{m-l}\boldsymbol{\lambda})^2]] \\ &= \sum_{b=1}^{d} E_{e^a}[X_b(l)]E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda}-\boldsymbol{e}^bM^{m-l}\boldsymbol{\lambda})^2] \\ &\leqq \sum_{b=1}^{d} E_{e^a}[X_b(l)]E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^2], \end{aligned}$$

and (2.32) follows. Next we shall show (2.33).

$$II_a = E_{e^a}[((\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda})+(\boldsymbol{X}(l)-\boldsymbol{e}^aM^l)(M^{m-l}-I)\boldsymbol{\lambda})^4]$$

$$
\begin{aligned}
&\leqq 8E_{e^a}[(\boldsymbol{X}(m)\boldsymbol{\lambda}-\boldsymbol{X}(l)M^{m-l}\boldsymbol{\lambda})^4] \\
&\quad +8E_{e^a}[((\boldsymbol{X}(l)-\boldsymbol{e}^aM^l)(M^{m-l}-I)\boldsymbol{\lambda})^4] \\
&\leqq 8II^1_a+128E_{e^a}[(\boldsymbol{X}(l)(M^{m-l}-I)\boldsymbol{\lambda})^4]\,.
\end{aligned}
$$

Then by the Markov property and (2.35) we have

$$
\begin{aligned}
II^1_a &= E_{e^a}[E_{X(l)}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda}-\boldsymbol{X}(0)M^{m-l}\boldsymbol{\lambda})^4]] \\
&\leqq 3E_{e^a}[(\sum_{b=1}^d X_b(l)\,E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda}-\boldsymbol{e}^bM^{m-l}\boldsymbol{\lambda})^2])^2] \\
&\quad +E_{e^a}[\sum_{b=1}^d X_b(l)\,E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda}-\boldsymbol{e}^bM^{m-l}\boldsymbol{\lambda})^4]] \\
&\leqq 3d\sum_{b=1}^d E_{e^a}[X_b(l)^2]\,E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^2]^2 \\
&\quad +16\sum_{b=1}^d E_{e^a}[X_b(l)]\,E_{e^b}[(\boldsymbol{X}(m-l)\boldsymbol{\lambda})^4]\,,
\end{aligned}
$$

and the proof is completed.

We can show the following lemma by the same method and the proof is omitted.

Lemma 2.6. *Let* (A.1)–(A.3) *be satisfied and* $m\geqq l+1$. *Then we have*

$$
\begin{aligned}
&E_x[(\sum_{k=l}^{m-1}(\boldsymbol{X}(k)\boldsymbol{\lambda}-\boldsymbol{x}M^k\boldsymbol{\lambda}))^2] \\
&\quad\leqq \sum_{a=1}^d\sum_{b=1}^d x_a\,E_{e^a}[X_b(l)]\,E_{e^b}[(\sum_{k=l}^{m-1}\boldsymbol{X}(k)\boldsymbol{\lambda})^2] \\
&\qquad +\sum_{a=1}^d x_a\,E_{e^a}[(\boldsymbol{X}(l)\sum_{k=0}^{m-l-1}M^k\boldsymbol{\lambda})^2]\,,
\end{aligned}
\tag{2.36}
$$

$$
\begin{aligned}
&E_x[(\sum_{k=l}^{m-1}(\boldsymbol{X}(k)\boldsymbol{\lambda}-\boldsymbol{x}M^k\boldsymbol{\lambda}))^4] \\
&\quad\leqq 3E_x[(\sum_{k=l}^{m-1}(\boldsymbol{X}(k)\boldsymbol{\lambda}-\boldsymbol{x}M^k\boldsymbol{\lambda}))^2]^2 \\
&\qquad +24d\sum_{a=1}^d\sum_{b=1}^d x_a\,E_{e^a}[X_b(l)^2]\,E_{e^b}[(\sum_{k=0}^{m-l-1}\boldsymbol{X}(k)\boldsymbol{\lambda})^2]^2 \\
&\qquad +128\sum_{a=1}^d\sum_{b=1}^d x_a\,E_{e^a}[X_b(l)]\,E_{e^b}[(\sum_{k=0}^{m-l-1}\boldsymbol{X}(k)\boldsymbol{\lambda})^4] \\
&\qquad +128\sum_{a=1}^d x_a\,E_{e^a}[(\boldsymbol{X}(k)\sum_{k=0}^{m-l-1}M^k\boldsymbol{\lambda})^4]\,.
\end{aligned}
\tag{2.37}
$$

3. Proof of the tightness

We shall show the tightness part in Theorem A at first. Let $\{\boldsymbol{x}^n\}_{n\geqq 1}$ be a sequence of nonnegative integer valued vectors satisfying

$$
\lim_{n\to\infty} n^{-\alpha}\,\boldsymbol{x}^n_\alpha = x_\alpha\,\boldsymbol{v}_\alpha,\ 1\leqq\alpha\leqq N\,,
\tag{3.1}
$$

for some nonnegative numbers $x_1, x_2, \cdots, x_N$. Let β nad $\boldsymbol{\lambda}^\beta$ be fixed arbitrarily. Set

$$\text{(3.2)}\qquad \begin{cases} \phi_n(t) = E_{\boldsymbol{x}^n}[n^{-\beta}\,\boldsymbol{X}^n_\beta(t)\,\boldsymbol{\lambda}^\beta] \\ \qquad\quad = n^{-\beta}\,\boldsymbol{x}^n M^{[nt]}(I+(nt-[nt])\,(M-I))\,\boldsymbol{\lambda}^\beta\,, \\ \psi_n(t) = E_{\boldsymbol{x}^n}[n^{-\beta}\,\boldsymbol{Y}^n_\beta(t)\,\boldsymbol{\lambda}^\beta] \\ \qquad\quad = n^{-\beta}\,\boldsymbol{x}^n(\sum_{k=0}^{[nt]-1} M^k+(nt-[nt])\,M^{[nt]})\,\bar{\boldsymbol{\lambda}}^\beta\,. \end{cases}$$

By Lemma 2.1 we have

$$\lim_{n\to\infty} \phi_n(t) = \sum_{\alpha=1}^{\beta} x_\alpha\,\boldsymbol{v}_\alpha\,P^\alpha_\beta\,\boldsymbol{\lambda}^\beta\,t^{\beta-\alpha}\,,$$

$$\lim_{n\to\infty} \psi_n(t) = \sum_{\alpha=1}^{\beta} x_\alpha\,\boldsymbol{v}_\alpha\,Q^\alpha_\beta\,\boldsymbol{\lambda}^\beta\,t^{\beta-\alpha}\,,$$

uniformly on each finite interval. Set

$$U_n(t) = n^{-\beta}\,\boldsymbol{X}^n_\beta(t)\,\boldsymbol{\lambda}^\beta-\phi_n(t),\ V_n(t) = n^{-\beta}\,\boldsymbol{Y}^n_\beta(t)\,\boldsymbol{\lambda}^\beta-\psi_n(t)\,.$$

Then it suffices to show the following lemma.

Lemma 3.1. *Let* (A.1)–(A.3) *be satisfied. Then for each fixed* $T>0$ *there exists* $C(T)>0$ *satisfying*

$$\text{(3.3)}\qquad E_{\boldsymbol{x}^n}[(U_n(t)-U_n(s))^4+(V_n(t)-V_n(s))^4] \leqq C(T)\,(t-s)^2\,, \quad n\geqq 1,\ 0\leqq s\leqq t\leqq T\,.$$

Proof. Set $l=[ns]$ and $m=[nt]$. Then by (2.1) and (2.4) we have

$$\text{(3.4)}\qquad n^\beta(U_n(t)-U_n(s)) = \begin{cases} n(t-s)\,(\boldsymbol{X}(l+1)-\boldsymbol{x}^n M^{l+1}-\boldsymbol{X}(l)+\boldsymbol{x}^n M^l)\,\boldsymbol{\lambda}^\beta & \text{if}\quad m=l\,, \\ (\boldsymbol{X}(m)-\boldsymbol{x}^n M^m-\boldsymbol{X}(l)+\boldsymbol{x}^n M^l)\,\boldsymbol{\lambda}^\beta & \\ \quad +(nt-m)\,(\boldsymbol{X}(m+1)-\boldsymbol{x}^n M^{m+1}-\boldsymbol{X}(m)+\boldsymbol{x}^n M^m)\,\boldsymbol{\lambda}^\beta & \\ \quad +(ns-l)\,(\boldsymbol{X}(l+1)-\boldsymbol{x}^n M^{l+1}-\boldsymbol{X}(l)+\boldsymbol{x}^n M^l)\,\boldsymbol{\lambda}^\beta & \text{if}\quad m\geqq l+1\,, \end{cases}$$

$$\text{(3.5)}\qquad n^\beta(V_n(t)-V_n(s)) = \begin{cases} n(t-s)\,(\boldsymbol{X}(l)-\boldsymbol{x}^n M^l)\,\bar{\boldsymbol{\lambda}}^\beta & \text{if}\quad m=l\,, \\ \sum_{k=l}^{m-1}(\boldsymbol{X}(k)-\boldsymbol{x}^n M^k)\,\bar{\boldsymbol{\lambda}}^\beta+(nt-m)\,(\boldsymbol{X}(m)-\boldsymbol{x}^n M^m)\,\bar{\boldsymbol{\lambda}}^\beta & \\ \quad -(ns-l)\,(\boldsymbol{X}(l)-\boldsymbol{x}^n M^l)\,\bar{\boldsymbol{\lambda}}^\beta & \text{if}\quad m\geqq l+1\,. \end{cases}$$

Hence it suffices to show that

$$\text{(3.6)}\qquad \begin{aligned} &E_{\boldsymbol{x}^n}[(\boldsymbol{X}_\beta(m)\,\boldsymbol{\lambda}^\beta-\boldsymbol{x}^n M^m\,\boldsymbol{\lambda}^\beta-\boldsymbol{X}_\beta(l)\,\boldsymbol{\lambda}^\beta+\boldsymbol{x}^n M^l\,\boldsymbol{\lambda}^\beta)^4] \\ &\quad\leqq C(T)\,(m-l)^2\,n^{4\beta-2}\,|\boldsymbol{\lambda}^\beta|^4\,, \end{aligned}$$

$$(3.7)\qquad E_{x^n}[(\sum_{k=l}^{m-1}(X_\beta(k)\bar{\lambda}^\beta-x^n M^k\bar{\lambda}^\beta))^4] \leqq C(T)(m-l)^2 n^{4\beta-2}|\lambda^\beta|^4,$$

hold for any $n\geqq 1$ and $l+1\leqq m\leqq nT$.

We shall show (3.6) only, since (3.7) is shown by the same method. Let $x=x^n$, $\lambda=\lambda^\beta$ in Lemma 2.5. Since $x_\alpha^n=O(n^\alpha)$, we have

$$E_{x^n}[(X(m)\lambda^\beta-x^n M^m\lambda^\beta-X(l)\lambda^\beta+x^n M^l\lambda^\beta)^2] \leqq C(T)n^\beta(m-l)^{2\beta-1}|\lambda^\beta|^2,$$

by (2.6), (2.9), (2.28) for $l\leqq(m-1)T$ and (2.32). Then (3.6) follows from (2.6), (2.9), (2.28) for $l\leqq(m-1)T$ and (2.33).

We can show the tightness part in Theorem B by the same method and the proof is omitted.

4. An auxiliary limit theorem

In this section we shall show a theorem which will be used to prove the convergence of finite dimensional distributions.

Theorem 1. *Let* (.A.1)-(A.3) *be satisfied,* α *and* $t>0$ *be fixed. Then we have*

$$(4.1)\qquad \lim_{n\to\infty} n^{\beta-\alpha+1}(E_{e^a}[\exp(i\sum_{\gamma=\alpha}^{N} n^{\alpha-\gamma-1}(X_\gamma^n(t)\lambda^\gamma+Y_\gamma^n(t)\mu^\gamma))]-1)_{a\in C_\beta} = \begin{cases} \psi_\alpha(t:\lambda,\mu)\,u^\alpha+iQ_\alpha^\alpha\mu^\alpha & \text{if } \beta=\alpha, \\ i\sum_{\gamma=\beta}^{N}(P_\gamma^\beta\lambda^\gamma+Q_\gamma^\beta\mu^\gamma)\,t^{\gamma-\beta} & \text{if } \beta\geqq\alpha+1, \end{cases}$$

where $\psi_\alpha(t)=\psi_\alpha(t:\lambda,\mu)$ *is the solution of*

$$(4.2)\qquad \begin{cases} \psi_\alpha(0)=iv_\alpha\lambda^\alpha, \\ \dfrac{d}{dt}\psi_\alpha(t) \\ \quad =\dfrac{1}{2}\sum_{a\in C_\alpha} v_\alpha E_{e^a}[((X_\alpha(1)-X_\alpha(0)M_\alpha^\alpha)(\psi_\alpha(t)u^\alpha+iQ_\alpha^\alpha\mu^\alpha))^2] \\ \quad\quad +i\sum_{\beta=\alpha+1}^{N}(\beta-\alpha)(v_\alpha P_\beta^\alpha\lambda^\beta+v_\alpha Q_\beta^\alpha\mu^\beta)\,t^{\beta-\alpha-1}. \end{cases}$$

Especially if C_α *is a final class then*

$$(4.3)\qquad \psi_\alpha(t:\lambda,\mu) = -\frac{1}{2}\sum_{a\in C_\alpha} v_a\bar{\mu}^a(\bar{\mu}^a+2(M_\alpha^\alpha Q_\alpha^\alpha\mu^\alpha)^a)+i\sum_{\beta=\alpha}^{N} v_\alpha(P_\beta^\alpha\lambda^\beta+Q_\beta^\alpha\mu^\beta)\,t^{\beta-\alpha}.$$

Proof. Let $\alpha\leqq\beta$, $b\in C_\beta$ and $t>0$ be fixed and set

$$U_n(t)=\sum_{\gamma=1}^{N} n^{\alpha-\gamma-1}(\boldsymbol{X}_\gamma^n(t)\,\boldsymbol{\lambda}^\gamma+\boldsymbol{Y}_\gamma^n(t)\,\boldsymbol{\mu}^\gamma)\,.$$

Then

$$U_n\Big(\frac{[nt]}{n}\Big)=\sum_{\gamma=1}^{N} n^{\alpha-\gamma-1}(\boldsymbol{X}_\gamma([nt])\,\boldsymbol{\lambda}^\gamma+\sum_{j=1}^{[nt]-1}\boldsymbol{X}_\gamma(k)\,\boldsymbol{\mu}_\gamma)\,.$$

By Lemma 2.2 and Lemma 2.3 we obtain

$$(4.4)\qquad \begin{aligned}&|E_{e^b}[\exp(iU_n\Big(\frac{[nt]}{n}\Big))]-1-E_{e^b}[iU_n\Big(\frac{[nt]}{n}\Big)]|\\ &\leqq E_{e^b}[U_n\Big(\frac{[nt]}{n}\Big)^2]\\ &\leqq 2N\sum_{\gamma=\beta}^{N} n^{2(\alpha-\gamma-1)}\,E_{e^b}[(\boldsymbol{X}_\gamma([nt])\,\boldsymbol{\lambda}^\gamma)^2+(\sum_{k=0}^{[nt]-1}\boldsymbol{X}_\gamma(k)\,\bar{\boldsymbol{\mu}}^\gamma)^2]\\ &=O(n^{-2(\beta-\alpha)-1})\,.\end{aligned}$$

Hence (4.1) in the csae $\beta\geqq\alpha+1$ follows from Lemma 2.1 and (4.4). Set

$$(4.5)\qquad G^a(n:\boldsymbol{\lambda},\boldsymbol{\mu})=E_{e^a}[\exp(i\boldsymbol{X}(n)\,\boldsymbol{\lambda}+i\sum_{k=0}^{n}\boldsymbol{X}(k)\,\boldsymbol{\mu})]\,,$$

$$(4.6)\qquad I(\boldsymbol{\lambda})=(\delta_b^a\exp(i\lambda^a))_{1\leqq a,b\leqq d}\,.$$

Then $\boldsymbol{G}(n:\boldsymbol{\lambda},\boldsymbol{\mu})=(G^a(n:\boldsymbol{\lambda},\boldsymbol{\mu}))_{1\leqq a\leqq d}$ satisfies

$$(4.7)\qquad \begin{cases}\boldsymbol{G}(0:\boldsymbol{\lambda},\boldsymbol{\mu})=I(\boldsymbol{\lambda}+\boldsymbol{\mu})\,\mathbf{1}\,,\\ \boldsymbol{G}(n:\boldsymbol{\lambda},\boldsymbol{\mu})=I(\boldsymbol{\mu})\,\boldsymbol{F}(\boldsymbol{G}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}))\quad n\geqq1\,,\end{cases}$$

where $\boldsymbol{F}(\boldsymbol{s})$ is the vector of generating functions of $\boldsymbol{X}(1)$. To treat this excursion formula we expand the generating functions as follows,

$$(4.8)\qquad F^a(\boldsymbol{s})-1=\sum_{b=1}^{d} m_b^a(s^b-1)+\frac{1}{2}\sum_{b,c=1}^{d} D_b\,D_c\,F^a(\boldsymbol{s}+\theta_a(\mathbf{1}-\boldsymbol{s}))\,(s^b-1)\,(s^c-1)\,,\qquad 0<\theta_a<1\,.$$

Set

$$(4.9)\qquad \boldsymbol{B}(\boldsymbol{s}:\boldsymbol{\lambda},\boldsymbol{\lambda})=(\sum_{b,c=1}^{d} D_b\,D_c\,F^a(\boldsymbol{s}+\theta_a(\mathbf{1}-\boldsymbol{s}))\,\lambda^b\,\lambda^c)_{1\leqq a\leqq d}\,.$$

Then (4.8) becomes

$$(4.10)\qquad \boldsymbol{F}(\boldsymbol{s})-\mathbf{1}=M(\boldsymbol{s}-\mathbf{1})+\frac{1}{2}\boldsymbol{B}(\boldsymbol{s}:\boldsymbol{s}-\mathbf{1},\boldsymbol{s}-\mathbf{1})\,.$$

Combining this with (4.7) we obtain

$$
\begin{aligned}
&\boldsymbol{G}(n:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1}\\
&= M(\boldsymbol{G}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1})+(I(\boldsymbol{\mu})-I)\,\boldsymbol{F}(\boldsymbol{G}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}))\\
&\quad+\frac{1}{2}\boldsymbol{B}(\boldsymbol{G}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}):\boldsymbol{G}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1},\boldsymbol{G}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1})\,,
\end{aligned}
\tag{4.11}
$$

and hence

$$
\begin{aligned}
&\boldsymbol{G}(n:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1}\\
&= M^n(I(\boldsymbol{\lambda}+\boldsymbol{\mu})-I)\,\mathbf{1}+\sum_{k=0}^{n-1}M^{n-k-1}(I(\boldsymbol{\mu})-I)\,\boldsymbol{F}(\boldsymbol{G}(k:\boldsymbol{\lambda},\boldsymbol{\mu}))\\
&\quad+\frac{1}{2}\sum_{k=0}^{n-1}M^{n-k-1}\,\boldsymbol{B}(\boldsymbol{G}(k:\boldsymbol{\lambda},\boldsymbol{\mu}):\boldsymbol{G}(k:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1},\boldsymbol{G}(k:\boldsymbol{\lambda},\boldsymbol{\mu})-\mathbf{1})\,.
\end{aligned}
\tag{4.12}
$$

Set

$$
\boldsymbol{\lambda}_n=(n^{\alpha-\beta-1}\boldsymbol{\lambda}^\beta)_{1\leq\beta\leq N}\,,
\tag{4.13}
$$

$$
\boldsymbol{B}^\alpha_{\beta,\gamma}(\boldsymbol{s}:\boldsymbol{\lambda}^\beta,\boldsymbol{\lambda}^\gamma)=\Big(\sum_{b\in C_\beta}\sum_{c\in C_\gamma}D_b\,D_c\,F^a(\boldsymbol{s}+\theta_a(\mathbf{1}-\boldsymbol{s}))\,\lambda^b\,\lambda^c\Big)_{a\in C_\alpha}\,.
\tag{4.14}
$$

Then we have

$$
\begin{aligned}
&\Big(E_{e^a}\Big[\exp\Big(i\sum_{\beta=1}^{N}n^{\alpha-\beta-1}\Big(\boldsymbol{X}^n_\beta\Big(\frac{[nt]}{n}\Big)\boldsymbol{\lambda}^\beta+\boldsymbol{Y}^n_\beta\Big(\frac{[nt]+1}{n}\Big)\boldsymbol{\mu}^\beta\Big)\Big)\Big]-1\Big)_{a\in C_\alpha}\\
&=\boldsymbol{G}^\alpha([nt]:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n)-\mathbf{1}^\alpha\\
&=\sum_{\beta=\alpha}^{N}(M^{[nt]})^\alpha_\beta\,(I^\beta_\beta(\boldsymbol{\lambda}_n+\bar{\boldsymbol{\mu}}_n)-I)\,\mathbf{1}^\beta\\
&\quad+\sum_{k=0}^{[nt]-1}\sum_{\beta=\alpha}^{N}(M^{[nt]-k-1})^\alpha_\beta\,(I^\beta_\beta(\bar{\boldsymbol{\mu}}_n)-I)\,\boldsymbol{F}^\beta(\boldsymbol{G}(k:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n))\\
&\quad+\frac{1}{2}\sum_{k=0}^{[nt]-1}\sum_{\beta=\alpha}^{N}\sum_{\gamma,\delta=\beta}^{N}(M^{[nt]-k-1})^\alpha_\beta\,\boldsymbol{B}^\beta_{\gamma,\delta}(\boldsymbol{G}(k:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n):\boldsymbol{G}^\gamma(k:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n)\\
&\qquad\qquad-\mathbf{1}^\gamma,\boldsymbol{G}^\delta(k:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n)-\mathbf{1}^\delta)\,.
\end{aligned}
\tag{4.15}
$$

We shall estimate the last three terms. We remark at first

$$
\begin{cases}
I^\beta_\beta(\boldsymbol{\lambda}_n+\bar{\boldsymbol{\mu}}_n)-I=in^{\alpha-\beta-1}(\delta^a_b(\lambda^a+\bar{\mu}^a))_{a,b\in C_\beta}+O(n^{2(\alpha-\beta-1)})\,,\\
I^\beta_\beta(\bar{\boldsymbol{\mu}}_n)-I=in^{\alpha-\beta-1}(\delta^a_b\,\bar{\mu}^b)_{a,b\in C_\beta}-\dfrac{1}{2}n^{2(\alpha-\beta-1)}(\delta^a_b(\bar{\mu}^b)^2)_{a,b\in C_\beta}\\
\qquad\qquad\qquad+O(n^{3(\alpha-\beta-1)})\,.
\end{cases}
\tag{4.16}
$$

Then the first term is

$$
in^{-1}\sum_{\beta=\alpha}^{N}P^\alpha_\beta\,\boldsymbol{\lambda}^\beta\,t^{\beta-\alpha}+o(n^{-1})
$$

by Lemma 2.1. By (4.4) and Lemma 2.1 we have

$$G^{\beta}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)=\mathbf{1}^{\beta}+O(n^{\alpha-\beta-1}) \quad k\leqq nt\,, \tag{4.17}$$

and hence

$$\boldsymbol{F}^{\beta}(\boldsymbol{G}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n))=\mathbf{1}^{\beta}+O(n^{\alpha-\beta-1}) \quad k\leqq nt\,. \tag{4.18}$$

Then by (4.16) we obtain

$$\begin{aligned}&\sum_{k=0}^{[nt]-1}\sum_{\beta=\alpha+1}^{N}(M^k)^{\alpha}_{\beta}(I^{\beta}_{\beta}(\overline{\boldsymbol{\mu}}_n)-I)\,\boldsymbol{F}^{\beta}(\boldsymbol{G}([nt]-k-1:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n))\\ &\quad=in^{-1}\sum_{\beta=\alpha+1}^{N}Q^{\alpha}_{\beta}\,\boldsymbol{\mu}^{\beta}\,t^{\beta-\alpha}+o(n^{-1})\,.\end{aligned}$$

By (4.10) and (4.17) we have

$$\boldsymbol{F}^{\alpha}(\boldsymbol{G}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n))=\mathbf{1}^{\alpha}+M^{\alpha}_{\alpha}(\mathbf{1}^{\alpha}-\boldsymbol{G}^{\alpha}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n))+o(n^{-2})\,.$$

Hence by (4.16) and Lemma 2.1 we obtain

$$\begin{aligned}&\sum_{k=0}^{[nt]-1}(M^k)^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\overline{\boldsymbol{\mu}}_n)-I)\,\boldsymbol{F}^{\alpha}(\boldsymbol{G}([nt]-k-1:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_o))\\ &\quad=in^{-1}\sum_{k=0}^{[nt]-1}(M^k)^{\alpha}_{\alpha}(\delta^a_b\overline{\boldsymbol{\mu}}^b)_{a,b\in C_\alpha}\,\mathbf{1}^{\alpha}\\ &\qquad-\frac{1}{2}n^{-2}\sum_{k=0}^{[nt]-1}(M^k)^{\alpha}_{\alpha}(\delta^a_b(\overline{\mu}^b)^2)_{a,b\in C_\alpha}\,\mathbf{1}^{\alpha}\\ &\qquad+\sum_{k=0}^{[nt]-1}(M^k)^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\boldsymbol{\mu}_n)-I)\,M^{\alpha}_{\alpha}(\boldsymbol{G}^{\alpha}([nt]-k-1:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)-\mathbf{1}^{\alpha})\\ &\qquad\qquad+o(n^{-1})\\ &\quad=in^{-1}\,Q^{\alpha}_{\alpha}\,\boldsymbol{\mu}^{\alpha}-\frac{1}{2}\,n^{-1}\,t\sum_{a\in C_\alpha}v_a(\overline{\mu}^a)^2\,\boldsymbol{u}^{\alpha}\\ &\qquad+\sum_{k=0}^{[nt]-1}P^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\overline{\boldsymbol{\mu}}_n)-I)\,M^{\alpha}_{\alpha}(\boldsymbol{G}^{\alpha}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)-\mathbf{1}^{\alpha})+o(n^{-1})\,.\end{aligned}$$

Then the second term in (4.15) is

$$\begin{aligned}&-\frac{1}{2}\,n^{-1}\,t\sum_{a\to C_\alpha}v_a(\overline{\mu}^a)^2\,\boldsymbol{u}^{\alpha}+in^{-1}\sum_{\beta=\alpha}^{N}Q^{\alpha}_{\beta}\,\boldsymbol{\mu}^{\beta}\,t^{\beta-\alpha}\\ &\quad+\sum_{k=0}^{[nt]-1}P^{\alpha}_{\alpha}(I^{\alpha}_{\alpha}(\overline{\boldsymbol{\mu}}_n)-I)\,M^{\alpha}_{\alpha}(\boldsymbol{G}^{\alpha}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)-\mathbf{1}^{\alpha})+o(n^{-1})\,.\end{aligned}$$

By (4.17) and Lemma 2.1 the last term in (4.15) is

$$\frac{1}{2}\sum_{k=0}^{[nt]-1}P^{\alpha}_{\alpha}\,\boldsymbol{B}^{\alpha}_{\alpha,\alpha}(\mathbf{1}:\boldsymbol{G}^{\alpha}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)-\mathbf{1}^{\alpha},\,\boldsymbol{G}^{\alpha}(k:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)-\mathbf{1}^{\alpha})+o(n^{-1})\,.$$

Hence it follows that

$$n(\boldsymbol{G}^{\alpha}([nt]:\boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n)-\mathbf{1}^{\alpha}) \tag{4.19}$$

$$= i \sum_{\beta=\alpha}^{N} (P_{\beta}^{\alpha} \boldsymbol{\lambda}^{\beta} + Q_{\beta}^{\alpha} \boldsymbol{\mu}^{\beta})\, t^{\beta-\alpha} - \frac{1}{2}\, t \sum_{a \in C_{\alpha}} v_a (\overline{\mu}^a)^2 \boldsymbol{u}^{\alpha}$$
$$+ \sum_{k=0}^{[nt]-1} P_{\alpha}^{\alpha} (I_{\alpha}^{\alpha}(\overline{\boldsymbol{\mu}}_n) - I)\, M_{\alpha}^{\alpha} \cdot n(\boldsymbol{G}^{\alpha}(k: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n) - \mathbf{1}^{\alpha})$$
$$+ \frac{1}{2n} \sum_{k=0}^{[nt]-1} P_{\alpha}^{\alpha} \boldsymbol{B}_{\alpha,\alpha}^{\alpha}(\mathbf{1}: n(\boldsymbol{G}^{\alpha}(k: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n) - \mathbf{1}^{\alpha}), n(\boldsymbol{G}^{\alpha}(k: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n) - \mathbf{1}^{\alpha}))$$
$$+ o(1)\,.$$

Set

$$\begin{aligned}\psi_n(t) &= n\boldsymbol{v}_{\alpha}(\boldsymbol{G}^{\alpha}([nt]: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n) - \mathbf{1}^{\alpha}) \\ &\quad + n(nt - [nt])\, \boldsymbol{v}_{\alpha}(\boldsymbol{G}^{\alpha}([nt]+1: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n) - \boldsymbol{G}^{\alpha}([nt]: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n))\,. \end{aligned} \tag{4.20}$$

Then by (2.1), (2.2) and (4.19) we obtain

$$n(\boldsymbol{G}^{\alpha}([nt]: \boldsymbol{\lambda}_n, \overline{\boldsymbol{\mu}}_n) - \mathbf{1}^{\alpha}) = \psi_n\Big(\frac{[nt]}{n}\Big) \boldsymbol{u}^{\alpha} + iQ_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha} + o(1)\,. \tag{4.21}$$

But by (4.4), (4.11) and (4.13) we have

$$\Big|\psi_n\Big(\frac{[nt]+1}{n}\Big) - \psi_n\Big(\frac{[nt]}{n}\Big)\Big| = O(n^{-1})\,. \tag{4.22}$$

Hence $\{\psi_n(t)\}_{n\geq 1}$ is equicontinuous on each finite interval. Let $\psi(t)$ be any limit of $\{\psi_n(t)\}_{n\geq 1}$. Then by (4.19) and (4.21) ψ must satisfy

$$\begin{aligned}\psi(t) &= i \sum_{\beta=\alpha}^{N} \boldsymbol{v}_{\alpha}(P_{\beta}^{\alpha} \boldsymbol{\lambda}^{\beta} + Q_{\beta}^{\alpha} \boldsymbol{\mu}^{\beta})\, t^{\beta-\alpha} - \frac{t}{2} \sum_{a \in C_{\alpha}} v_a (\overline{\mu}^a)^2 \\ &\quad + i\int_0^t \boldsymbol{v}_{\alpha}(\delta_b^a\, \overline{\mu}^b)_{a,b\in C_{\alpha}} M_{\alpha}^{\alpha}(\psi(s)\, \boldsymbol{u}^{\alpha} + iQ_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha})\, ds \\ &\quad + \frac{1}{2}\int_0^t \boldsymbol{v}_{\alpha} \boldsymbol{B}_{\alpha,\alpha}^{\alpha}(\mathbf{1}: \psi(s)\, \boldsymbol{u}^{\alpha} + iQ_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha}, \psi(s)\, \boldsymbol{u}^{\alpha} + iQ_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha})\, ds\,. \end{aligned} \tag{4.23}$$

Then it suffices to show the equivalence of (4.2) and (4.23). Since $\psi(0) = i\boldsymbol{v}_{\alpha}(P_{\alpha}^{\alpha} \boldsymbol{\lambda}^{\alpha} + Q_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha}) = i\boldsymbol{v}_{\alpha} \boldsymbol{\lambda}^{\alpha}$ we have only to show that ψ satisfies the differential equation in (4.2). Remark that

$$\boldsymbol{v}_{\alpha}(\delta_b^a\, \overline{\mu}^b)_{a,b\in C_{\alpha}} = (v_a\, \overline{\mu}^a)_{a\in C_{\alpha}}\,. \tag{4.24}$$

By (2.2) and $P_{\alpha}^{\alpha} Q_{\alpha}^{\alpha} = O$ it follows that

$$M_{\alpha}^{\alpha} Q_{\alpha}^{\alpha} = Q_{\alpha}^{\alpha} - I + P_{\alpha}^{\alpha}\,, \tag{4.25}$$

$$M_{\alpha}^{\alpha} Q_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha} = Q_{\alpha}^{\alpha} \boldsymbol{\mu}^{\alpha} - \overline{\boldsymbol{\mu}}^{\alpha}\,. \tag{4.26}$$

For any α we have

$$
\begin{aligned}
&v_\alpha\, \boldsymbol{B}^\alpha_{\alpha,\alpha}(1:\boldsymbol{\lambda}^\alpha, \boldsymbol{\lambda}^\alpha) \\
&\quad = \sum_{a\in C_\alpha} \sum_{b,c\in C_\alpha} v_a\, E_{e^a}[X_b(1)\,(X_c(1)-\delta^b_c)]\,\lambda^b\lambda^c \\
&\quad = \sum_{a\in C_\alpha} v_a\, E_{e^a}[(\boldsymbol{X}_\alpha(1)\,\boldsymbol{\lambda}^\alpha)^2] - \sum_{a\in C_\alpha} v_a(\lambda^a)^2 \\
&\quad = \sum_{a\in C_\alpha} v_a\, E_{e^a}[((\boldsymbol{X}_\alpha(1)-\boldsymbol{X}_\alpha(0)\,M^\alpha_\alpha)\,\boldsymbol{\lambda}^\alpha)^2] \\
&\qquad + \sum_{a\in C_\alpha} v_a(\boldsymbol{e}^a\, M^\alpha_\alpha \boldsymbol{\lambda}^\alpha)^2 - \sum_{a\in C_\alpha} v_a(\lambda^a)^2\,.
\end{aligned} \tag{4.27}
$$

Hence ψ satisfies

$$
\begin{aligned}
&\frac{d\psi}{dt}(t) \\
&\quad = \sum_{a\in C_\alpha} v_a\, E_{e^a}[((\boldsymbol{X}_\alpha(1)-\boldsymbol{X}_\alpha(0)\,M^\alpha_\alpha)\,(\psi(t)\,\boldsymbol{u}^\alpha+iQ^\alpha_\alpha\,\boldsymbol{\mu}^\alpha))^2] \\
&\qquad + \sum_{\beta=\alpha+1}^{N} i(\beta-\alpha)\,(P^\alpha_\beta \boldsymbol{\lambda}^\beta + Q^\alpha_\beta\,\boldsymbol{\mu}^\beta)\,t^{\beta-\alpha-1} - \frac{1}{2}\sum_{a\in C_\alpha} v_a(\overline{\mu}^a)^2 \\
&\qquad + i(v_a\,\overline{\mu}^a)_{a\in C_\alpha}\, M^\alpha_\alpha(\psi(t)\,\boldsymbol{u}^\alpha + iQ^\alpha_\alpha\,\boldsymbol{\mu}^\alpha) \\
&\qquad + \frac{1}{2}\sum_{a\in C_\alpha} v_a(\boldsymbol{e}^a\, M^\alpha_\alpha(\psi(t)\,\boldsymbol{u}^\alpha+iQ^\alpha_\alpha\,\boldsymbol{\mu}^\alpha))^2 \\
&\qquad - \frac{1}{2}\sum_{a\in C_\alpha} v_a(\boldsymbol{e}^a(\psi(t)\,\boldsymbol{u}^\alpha+iQ^\alpha_\alpha\,\boldsymbol{\mu}^\alpha))^2 \\
&\quad = \sum_{k=1}^{6} I_k\,.
\end{aligned} \tag{4.28}
$$

By (4.26) and $M^\alpha_\alpha \boldsymbol{u}^\alpha = \boldsymbol{u}^\alpha$ we have

$$
I_4 = i\sum_{a\in C_\alpha} v_a\,\overline{\mu}^a u^a \psi(t) - \sum_{a\in C_\alpha} v_a\,\overline{\mu}^a(\boldsymbol{e}^a\,Q^\alpha_\alpha\,\boldsymbol{\mu}^\alpha) + \sum_{a\in C_\alpha} v_a(\overline{\mu}^a)^2\,,
$$

$$
I_5+I_6 = -i\sum_{a\in C_\alpha} v_a\,u^a\overline{\mu}^a\psi(t) + \frac{1}{2}\sum_{a\to C_\alpha} v_a(2\boldsymbol{e}^a\,Q^\alpha_\alpha\,\boldsymbol{\mu}^\alpha - (\overline{\mu}^a)^2)\,.
$$

Then it is easy to see that $\sum_{k=3}^{6} I_k=0$ and we have shown the equivalence of (4.2) and (4.23).

5. Proof of Theorem A

Before proceeding to the proof we state Theorem A more precisely.

Theorem 2. *Let* (A.1)–(A.3) *be satisfied,* $\boldsymbol{x}=(x_\alpha)_{1\leq\alpha\leq N}$ *be a nonnegative vector and* $\boldsymbol{y}$ *be a d-dimensional vector. Assume that* $\{\boldsymbol{x}^n\}_{n\geq 1}$ *is a sequence of nonnegative integer valued vectors satisfying* $\lim_{n\to\infty} n^{-\alpha}\,\boldsymbol{x}^n_\alpha = x_\alpha\,\boldsymbol{v}_\alpha$, $1\leq\alpha\leq N$. *Then the sequence of processes* $\{((n^{-\alpha}\,\boldsymbol{X}^n_\alpha(t),\, \boldsymbol{y}_\alpha + n^{-\alpha}\,\boldsymbol{Y}^n_\alpha(t))_{1\leq\alpha\leq N}, P_{\boldsymbol{x}^n})\}_{n\geq 1}$ *converges to some diffusion process* $((\boldsymbol{X}_\alpha(t), \boldsymbol{Y}_\alpha(t))_{1\leq\alpha\leq N}, P_{(\boldsymbol{x},\boldsymbol{y})})$ *and*

$$(5.1)\quad E_{(x,y)}[\exp(i\sum_{\alpha=1}^{N}(\boldsymbol{X}_\alpha(t)\boldsymbol{\lambda}^\alpha+\boldsymbol{Y}_\alpha(t)\boldsymbol{\mu}^\alpha))]$$
$$=\exp(i\sum_{\alpha=1}^{N}\boldsymbol{y}_\alpha\boldsymbol{\mu}^\alpha+x_1\psi_1(t:\boldsymbol{\lambda},\boldsymbol{\mu})+i\sum_{\alpha=2}^{N}\sum_{\beta=\alpha}^{N}x_\alpha\boldsymbol{v}_\alpha(P_\beta^\alpha\boldsymbol{\lambda}^\beta+Q_\beta^\alpha\boldsymbol{\mu}^\beta)\,t^{\beta-\alpha}).$$

Proof. First remark that (5.1) is clear if $t=0$ and if $t>0$ then (5.1) follows from Theorem 1 and the branching property. If we can show the convergence of any finite dimensional distributions then it is easy to see that the limit process is a diffusion process by (5.1) (cf. section 7). Hence it suffices to show that for any $p\geqq 2$ and $0<t_1<\cdots<t_p$.

$$(5.2)\qquad E_{x^n}[\exp(i\sum_{q=1}^{p}\sum_{\alpha=1}^{N}n^{-\alpha}(\boldsymbol{X}_\alpha([nt_q])\boldsymbol{\lambda}^\alpha(q)+\sum_{k=0}^{[nt_q]}\boldsymbol{X}_\alpha(k)\bar{\boldsymbol{\mu}}^\alpha(q)))]$$

converges to a continuous function of $(\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\lambda}(p),\boldsymbol{\mu}(1),\cdots,\boldsymbol{\mu}(p))$.

Set

$$(5.3)\qquad \begin{aligned} &G_{n,q}^a(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))\\ &\quad=E_{e^a}[\exp(i\sum_{q=1}^{p}\sum_{\alpha=1}^{N}n^{-\alpha}(\boldsymbol{X}_\alpha([nt_q])\boldsymbol{\lambda}^\alpha(q)+\sum_{k=0}^{[nt_q]}\boldsymbol{X}_\alpha(k)\bar{\boldsymbol{\mu}}^\alpha(q)))].\end{aligned}$$

Then, by the branching property, (5.2) becomes

$$(5.4)\qquad \prod_{\alpha=1}^{N}\prod_{a\in C_\alpha}G_{n,p}^a(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))^{x_a^n}.$$

Hence it suffices to show that there exists a vector of continuous functions $(\psi_p^\alpha(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p)))_{1\leqq\alpha\leqq N}$ such that

$$(5.5)\qquad \begin{aligned} &\lim_{n\to\infty}n^\alpha(G_{n,p}^a(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))-\mathbf{1})_{a\in C_\alpha}\\ &\quad=\psi_p^\alpha(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p)),\ 1\leqq\alpha\leqq N.\end{aligned}$$

But by Markov property and the branching property we have

$$(5.6)\quad \begin{aligned} &G_{n,p}^a(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))\\ &\quad=G_{n,p-1}^a(t_1,\cdots,t_{p-1}:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\lambda}(p-2),\boldsymbol{\lambda}_n(p-1),\boldsymbol{\mu}(1),\cdots,\boldsymbol{\mu}(p-1)),\end{aligned}$$

in which

$$(5.7)\qquad \begin{aligned} &\lambda_n^a(p-1)\\ &\quad=\lambda^a(p-1)\\ &\qquad+n^\alpha\log E_{e^a}[\exp(i\sum_{\beta=1}^{N}n^{-\beta}(\boldsymbol{X}_\beta([nt_p]-[nt_{p-1}])\boldsymbol{\lambda}^\beta(p)\\ &\qquad\qquad+\sum_{k=0}^{[qt_p]-[at_{p-1}]}\boldsymbol{X}_\beta(k)\bar{\boldsymbol{\mu}}^\beta(p)))],\ a\in C_\alpha.\end{aligned}$$

Combining this with Theorem 1, (5.5) is easily seen by the induction argument with respect to p.

6. Proof of Theorem B

We state Theorem B more precisely.

Theorem 3. *Let* (A.1)–(A.3) *be satisfied,* C_1 *be a final class,* $\boldsymbol{x}=(x_\alpha)_{2\leqq\alpha\leqq N}$ *be a nonnegative vector and* $\boldsymbol{y}=(\boldsymbol{y}_\alpha)_{2\leqq\alpha\leqq N}$ *be a real vector. Assume that* $\{\boldsymbol{x}^n\}_{n\geqq 1}$ *is a sequence of nonnegative integer valued vectors satisfying* $x_1^n=1$, $x_a^n=0$ *for* $a\in C_1$, $a\neq 1$ *and* $\lim_{n\to\infty} n^{1-\alpha}\boldsymbol{x}_\alpha^n=x_\alpha v_\alpha$ *for* $\alpha\geqq 2$. *Then the sequence of processes* $\{((n^{1-\alpha}\boldsymbol{X}_\alpha^n(t), \boldsymbol{y}_\alpha+n^{1-\alpha}\boldsymbol{Y}_\alpha^n(t))_{2\leqq\alpha\leqq N}, P_{x^n})\}_{n\geqq 1}$ *converges to some diffusion processes* $((\hat{\boldsymbol{X}}_\alpha(t), \hat{\boldsymbol{Y}}_\alpha(t))_{2\leqq\alpha\leqq N}$ $\hat{P}_{(x,y)})$ *and*

$$
\begin{aligned}
&\hat{E}_{(x,y)}[\exp(i\sum_{\alpha=2}^{N}(\hat{\boldsymbol{X}}_\alpha(t)\boldsymbol{\lambda}^\alpha+\hat{\boldsymbol{Y}}_\alpha(t)\boldsymbol{\mu}^\alpha))]\\
&\qquad=\exp(i\sum_{\alpha=2}^{N}\boldsymbol{y}_\alpha\boldsymbol{\mu}^\alpha+\boldsymbol{v}_1M_2^1\boldsymbol{u}^2\int_0^t\psi_2(s:\boldsymbol{\lambda},\boldsymbol{\mu})\,ds+it\,\boldsymbol{v}_1M_2^1Q_2^2\boldsymbol{\mu}^2)\\
&\qquad\quad\cdot\exp(x_2\psi_2(t:\boldsymbol{\lambda},\boldsymbol{\mu})+i\sum_{\alpha=3}^{N}\sum_{\beta=\alpha}^{N}x_\alpha\boldsymbol{v}_\alpha(P_\beta^\alpha\boldsymbol{\lambda}^\beta+Q_\beta^\alpha\boldsymbol{\mu}^\beta)\,t^{\beta-\alpha}).
\end{aligned}\tag{6.1}
$$

Proof. As the proof of Theorem 2 it suffices to show (6.1) and the convergence of finite dimensional distributions. To show the second part we proceed as follows. Let $p\geqq 1$ and $0<t_1<\cdots<t_p$ be fixed arbitrarily. Then it suffices to show that

$$
E_{x^n}[\exp(i\sum_{q=1}^{p}\sum_{\alpha=2}^{N}n^{1-\alpha}(\boldsymbol{X}_\alpha([nt_q])\boldsymbol{\lambda}^\alpha(q)+\sum_{k=0}^{[nt_q]}\boldsymbol{X}_\alpha(k)\bar{\boldsymbol{\mu}}^\alpha(q)))]\tag{6.2}
$$

converges to a continuous function of $(\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))$. Set

$$
\begin{aligned}
&H_{n,p}^\alpha(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))\\
&\qquad=E_{e^a}[\exp(i\sum_{q=1}^{p}\sum_{\alpha=2}^{N}n^{1-\alpha}(\boldsymbol{X}_\alpha([nt_q])\boldsymbol{\lambda}^\alpha(q)+\sum_{k=0}^{[nt_q]}\boldsymbol{X}_\alpha(k)\bar{\boldsymbol{\mu}}^\alpha(q)))].
\end{aligned}\tag{6.3}
$$

Then by the branching property (6.2) becomes

$$
H_{n}^{1}{}_{,p}(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))\prod_{\alpha=2}^{N}\prod_{a\in C_\alpha}H_{n,p}^{a}(t_1,\cdots,t_p:\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\mu}(p))^{x_b^n}.\tag{6.4}
$$

In section 5 we have already shown that the second term converges to a continuous function. Transform the first term as in (5.6). Then applying the following lemma the convergence of the first term is easily seen by the induction argument with respect to p. Also (6.1) follows from this lemma and Theorem 1 applied to (6.4).

Lemma 6.1. *Let* (A.1)–(A.3) *be satisfied,* C_1 *be a final class and* $t>0$ *be* *fixed. Then for any* $a\in C_1$ *we have*

$$\lim_{n\to\infty} E_{e^a}[\exp(i\sum_{\alpha=2}^{N} n^{1-\alpha}(\boldsymbol{X}_\alpha^n(t)\boldsymbol{\lambda}^\alpha+\boldsymbol{Y}_\alpha^n(t)\boldsymbol{\mu}^\alpha))] \tag{6.5}$$
$$= \exp(it\,\boldsymbol{v}_1 M_2^1 Q_2^2 \boldsymbol{\mu}^2+\boldsymbol{v}_1 M_2^1 \boldsymbol{u}^2\int_0^t \phi_2(s:\boldsymbol{\lambda},\boldsymbol{\mu})\,ds)\,.$$

Proof. We define $\boldsymbol{G}(n:\boldsymbol{\lambda},\boldsymbol{\mu})$ by (4.9). Let $\boldsymbol{\lambda}^1=\boldsymbol{\mu}^1=\boldsymbol{0}$ and set

$$\boldsymbol{\lambda}_n = (n^{1-\alpha}\boldsymbol{\lambda}^\alpha)_{1\leq\alpha\leq N}\,. \tag{6.6}$$

Then it is necessary to estimate $\boldsymbol{G}^1([nt]:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n)$. To this end we expand the generating functions as follows.

Set $\boldsymbol{s}^{[2,N]}=(\boldsymbol{s}^\alpha)_{2\leq\alpha\leq N}$. Since C_1 is a final class we have

$$\boldsymbol{F}_1(\boldsymbol{s}) = M_1^1(\boldsymbol{s}^{[2,N]})\,\boldsymbol{s}^1 \tag{6.7}$$

for some $M_1^1(\boldsymbol{s}^{[2,N]})=(m_b^a(\boldsymbol{s}^{[2,N]}))_{a,b\in C_1}$. Since $m_b^a=m_b^a(\boldsymbol{1})$ we obtain

$$m_b^a(\boldsymbol{s}^{[2,N]}) = m_b^a+l_b^a(\boldsymbol{s}^{[2,N]})\,, \tag{6.8}$$

where

$$l_b^a(\boldsymbol{s}^{[2,N]}) = \sum_{\alpha=2}^{N}\sum_{c\in C_\alpha} D_c\, m_b^a(\boldsymbol{s}^{[2,N]}+\theta_{a,b}(\boldsymbol{1}-\boldsymbol{s}^{[2,N]}))\,(s^c-1)\,, \quad 0<\theta_{a,b}<1\,. \tag{6.9}$$

Set $L_1^1(\boldsymbol{s}^{[2,N]})=(l_b^a(\boldsymbol{s}^{[2,N]}))_{a,b\in C_1}$. Then we have

$$M_1^1(\boldsymbol{s}^{[2,N]}) = M_1^1+L_1^1(\boldsymbol{s}^{[2,N]}) \tag{6.10}$$

and hence

$$\begin{cases} \boldsymbol{G}^1(0:\boldsymbol{\lambda},\boldsymbol{\mu}) = \boldsymbol{1}^1\,, \\ \boldsymbol{G}^1(n:\boldsymbol{\lambda},\boldsymbol{\mu}) = M_1^1(\boldsymbol{G}^{[2,N]}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}))\,\boldsymbol{G}^1(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}),\ n\geqq 1\,. \end{cases} \tag{6.11}$$

Combining this with (6.10) we obtain

$$\boldsymbol{G}^1(n:\boldsymbol{\lambda},\boldsymbol{\mu}) = M_1^1\,\boldsymbol{G}^1(n-1:\boldsymbol{\lambda},\boldsymbol{\mu})+L_1^1(\boldsymbol{G}^{[2,N]}(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}))\,\boldsymbol{G}^1(n-1:\boldsymbol{\lambda},\boldsymbol{\mu}) \tag{6.12}$$

and it follows that

$$\boldsymbol{G}^1(n:\boldsymbol{\lambda},\boldsymbol{\mu}) = \boldsymbol{1}^1+\sum_{k=0}^{n-1}(M_1^1)^{n-k-1}\,L_1^1(\boldsymbol{G}^{[2,N]}(k:\boldsymbol{\lambda},\boldsymbol{\mu}))\,\boldsymbol{G}^1(k:\boldsymbol{\lambda},\boldsymbol{\mu})\,. \tag{6.13}$$

Since $(M_1^1)^n-P_1^1=O(\rho^n)$ for some $0<\rho<1$, by (6.6), (6.13) and Theorem 1 we have

$$\boldsymbol{G}^1([nt]:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n) = (\boldsymbol{v}_1\,\boldsymbol{G}^1([nt]:\boldsymbol{\lambda}_n,\bar{\boldsymbol{\mu}}_n))\,\boldsymbol{1}^1+O(n^{-1})\,. \tag{6.14}$$

Hence it is sufficient to estimate $\boldsymbol{v}_1\, \boldsymbol{G}^1([nt]: \boldsymbol{\lambda}_n, \bar{\boldsymbol{\mu}}_n)$. Set

$$
\begin{aligned}
(6.15)\qquad & \psi_n(t) \\
&= \boldsymbol{v}_1\, \boldsymbol{G}^1([nt]: \boldsymbol{\lambda}_n, \bar{\boldsymbol{\mu}}_n)+(nt-[nt])\, \boldsymbol{v}_1(\boldsymbol{G}^1([nt]+1: \boldsymbol{\lambda}_n, \bar{\boldsymbol{\mu}}_n) \\
&\qquad\qquad -\boldsymbol{G}^1([nt]: \boldsymbol{\lambda}_n, \bar{\boldsymbol{\mu}}_n))\,.
\end{aligned}
$$

Then, by (6.12) and Theorem 1, it follows that

$$
(6.16)\qquad |\psi_n\Big(\frac{[nt]+1}{n}\Big)-\psi_n\Big(\frac{[nt]}{n}\Big)| = O(n^{-1})\,,
$$

and $\{\psi_n(t)\}_{n\geq 1}$ is equicontinuous on each finite interval. Let $\psi(t)$ be any limit. Then by Theorem 1 and (6.14) we have

$$
\begin{aligned}
(6.17)\qquad & \lim_{n\to\infty} nL_1^1(\boldsymbol{G}^{[2,N]}([nt]: \boldsymbol{\lambda}_n, \bar{\boldsymbol{\mu}}_n))\, \boldsymbol{G}^1([nt]: \boldsymbol{\lambda}_n, \bar{\boldsymbol{\mu}}_n) \\
&= (\sum_{c\in C_2}\sum_{b\in C_1} D_c\, m_b^a(\boldsymbol{1})\, (\psi_2(t: \boldsymbol{\lambda}, \boldsymbol{\mu})\, u^c+i(Q_2^2\, \boldsymbol{\mu}^2)^c))_{a\in C_1}\, \psi(t)\,.
\end{aligned}
$$

Remark that $\sum_{b\in C_1} D_c\, m_b^a(\boldsymbol{1})=D_c\, F^a(\boldsymbol{1})=m_c^a$. Then by (6.13) ψ satisfies

$$
(6.18)\qquad \psi(t) = \int_0^t \boldsymbol{v}_1\, M_2^1(\psi_2(s: \boldsymbol{\lambda}, \boldsymbol{\mu})\, \boldsymbol{u}^2+iQ_2^2\, \boldsymbol{\mu}^2)\, \psi(s)\, ds+1\,,
$$

i.e., $\psi(t)$ is given by (6.5).

7. Some remarks

Set

$$
(7.1)\qquad X_\alpha(t) = \boldsymbol{X}_\alpha(t)\, \boldsymbol{u}^\alpha,\ \hat{X}_\alpha(t) = \hat{\boldsymbol{X}}_\alpha(t)\, \boldsymbol{u}^\alpha\,.
$$

Then by the preceding three theorems it is easy to see that

$$
(7.2)\quad P_{(x,y)}(\boldsymbol{X}_\alpha(t) = X_\alpha(t)\, \boldsymbol{v}_\alpha,\ t\geqq 0) = \hat{P}_{(x,y)}(\hat{\boldsymbol{X}}_\alpha(t) = \hat{X}_\alpha(t)\, \boldsymbol{v}_\alpha,\ t\geqq 0) = 1\,.
$$

Set $X(t)=(X_\alpha(t))_{1\leqq\alpha\leqq N}$, $\hat{X}(t)=(\hat{X}_\alpha(t))_{2\leqq\alpha\leqq N}$ and $B^\alpha=(B_{a,b}^\alpha)_{a,b\in C_\alpha\cup\{0\}}$ be the symmetric and non-negative definite matrix defined by

$$
(7.3)\quad \sum_{a,b\in C_\alpha\cup\{0\}} B_{a,b}^\alpha\, \lambda^a\, \lambda^b = \sum_{a\in C_\alpha} v_a\, E_{e^a}[((\boldsymbol{X}_\alpha(1)-\boldsymbol{X}_\alpha(0)\, M_\alpha^\alpha)\, (\lambda^0\, \boldsymbol{u}^\alpha+Q_\alpha^\alpha\, \boldsymbol{\lambda}^\alpha))^2]\,.
$$

Set $\boldsymbol{\lambda}^\alpha=\lambda^\alpha\, \boldsymbol{u}^\alpha$ for $1\leqq\alpha\leqq N$. Then by (7.2) we have

$$
\sum_{\alpha=1}^N X_\alpha(t)\, \lambda^\alpha = \sum_{\alpha=1}^N \boldsymbol{X}_\alpha(t)\, \boldsymbol{\lambda}^\alpha\,.
$$

Then (5.1) becomes

$$
(7.4)\qquad E_{(x,y)}[\exp(i\sum_{\alpha=1}^N (X_\alpha(t)\, \lambda^\alpha+\boldsymbol{Y}_\alpha(t)\, \boldsymbol{\mu}^\alpha)]
$$

$$= \exp\left(i \sum_{\alpha=1}^{N} \boldsymbol{y}_\alpha \boldsymbol{\mu}^\alpha + x_1 \psi_1(t: \boldsymbol{\lambda}, \boldsymbol{\mu})\right.$$
$$\left. + i \sum_{\alpha=2}^{N} \sum_{\beta=\alpha}^{N} x_\alpha (\lambda^\beta \boldsymbol{v}_\alpha P_\beta^\alpha \boldsymbol{u}^\beta + \boldsymbol{v}_\alpha Q_\beta^\alpha \boldsymbol{\mu}^\beta) t^{\beta-\alpha}\right).$$

By (4.2) we obtain

$$\frac{d}{dt} \log E_{(x,y)}[\exp(i \sum_{\alpha=1}^{N} (X_\alpha(t) \lambda^\alpha + \boldsymbol{Y}_\alpha(t) \boldsymbol{\mu}^\alpha))] |_{t=0}$$
$$= -\frac{1}{2} x_1 (B_{0,0}^1 (\lambda^1)^2 + 2\lambda^1 \sum_{a \in C_1} B_{0,a}^1 \mu^a + \sum_{a,b \in C_1} B_{a,b}^1 \mu^a \mu^b)$$
$$+ i \sum_{\alpha=1}^{N} x_\alpha (\lambda^{\alpha+1} \boldsymbol{v}_\alpha P_{\alpha+1}^\alpha \boldsymbol{u}^{\alpha+1} + \boldsymbol{v}_\alpha Q_{\alpha+1}^\alpha \boldsymbol{\mu}^{\alpha+1}).$$

Then $((\boldsymbol{X}(t), \boldsymbol{Y}(t)), P_{(x,y)})$ is a diffusion process on the state space $[0, \infty)^N \times \boldsymbol{R}^d$ and the generator is given by

$$Af(\boldsymbol{x}, \boldsymbol{y})$$
$$= \frac{1}{2} x_1 (B_{0,0}^1 D_{x_1}^2 + 2 \sum_{a \to C_1} B_{0,a}^1 D_{x_1} D_{y_a} + \sum_{a,b \in C_1} B_{a,b}^1 D_{y_a} D_{y_b}) f(\boldsymbol{x}, \boldsymbol{y}) \tag{7.5}$$
$$+ \sum_{\alpha=1}^{N-1} x_\alpha (\boldsymbol{v}_\alpha P_{\alpha+1}^\alpha \boldsymbol{u}^{\alpha+1} D_{x_{\alpha+1}} + \sum_{a \in C_{\alpha+1}} (\boldsymbol{v}_\alpha Q_{\alpha+1}^\alpha)_a D_{y_a}) f(\boldsymbol{x}, \boldsymbol{y})$$

where D_x denotes the partial differentiation with respect to x.

By the same method it follows that $((\hat{\boldsymbol{X}}(t), \hat{\boldsymbol{Y}}(t)), \hat{P}_{(x,y)})$ is a diffusion process on the state space $[0, \infty)^{N-1} \times \boldsymbol{R}^{d-d_1}$ (d_1 is the number of elements in C_1) and the generator is given by

$$\hat{A}f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{v}_1 M_2^1 \boldsymbol{u}^2 D_{x_2} f(\boldsymbol{x}, \boldsymbol{y})$$
$$+ \frac{1}{2} x_2 (B_{0,0}^2 D_{x_2}^2 + 2 \sum_{a \in C_2} B_{0,a}^2 D_{x_2} D_{y_a} + \sum_{a,b \to C_2} B_{a,b}^2 D_{y_a} D_{y_b}) f(\boldsymbol{x}, \boldsymbol{y}) \tag{7.6}$$
$$+ \sum_{\alpha=2}^{N-1} x_\alpha (\boldsymbol{v}_\alpha P_{\alpha+1}^\alpha \boldsymbol{u}^{\alpha+1} D_{x_{\alpha+1}} + \sum_{a \in C_{\alpha+1}} (\boldsymbol{v}_\alpha Q_{\alpha+1}^\alpha)_a D_{y_a}) f(\boldsymbol{x}, \boldsymbol{y}).$$

8. An example

Let $p>0$, $0<q\leqq 1$ be fixed and set $m=[p]+1$. In this section we study the 4-type branching process $(\boldsymbol{X}(n))$ whose generating functions are given by

$$\begin{cases} F^1(\boldsymbol{s}) = \dfrac{1}{m} s^1(m-p+p(s^2)^m), \\ F^2(\boldsymbol{s}) = \dfrac{1}{2} (s^2 s^3 (s^4)^2 + 1), \\ F^3(\boldsymbol{s}) = q s^2 + (1-q) s^3, \\ F^4(\boldsymbol{s}) = s^4. \end{cases} \tag{8.1}$$

Then the mean matrix is

$$(8.2)\qquad M=\frac{1}{2}\begin{pmatrix} 2 & 2p & 0 & 0\\ 0 & 1 & 1 & 2\\ 0 & 2q & 2-2q & 0\\ 0 & 0 & 0 & 2\end{pmatrix}.$$

Hence $C_1=\{1\}$, $C_2=\{2, 3\}$, $C_3=\{4\}$. Remark that C_1 and C_3 are final classes. By elementary calculations we have

$$(8.3)\qquad \boldsymbol{v}_2=\frac{1}{2q+1}(2q, 1),\ \boldsymbol{u}^2=\begin{pmatrix}1\\1\end{pmatrix},$$

$$(8.4)\qquad \begin{cases} P_2^2=\dfrac{1}{2q+1}\begin{pmatrix}2q & 1\\ 2q & 1\end{pmatrix},\ Q_2^2=\dfrac{2}{(2q+1)^2}\begin{pmatrix}1 & -1\\ -2q & 2q\end{pmatrix},\\ P_2^1=\dfrac{p}{2q+1}(2q, 1),\ Q_2^1=\dfrac{2p}{(2q+1)^2}(1, -1),\\ P_3^2=\dfrac{2q}{2q+1}\begin{pmatrix}1\\1\end{pmatrix},\ Q_3^2=\begin{pmatrix}0\\0\end{pmatrix},\ P_3^1=\dfrac{pq}{2q+1},\ Q_3^1=0.\end{cases}$$

Since

$$(8.5)\qquad \begin{cases} E_{e^2}[(\boldsymbol{X}_2(1)\boldsymbol{\lambda}^2-\boldsymbol{X}_2(0)M_2^2\boldsymbol{\lambda}^2)^2]=\dfrac{1}{4}(\lambda^2+\lambda^3)^2,\\ E_{e^3}[(\boldsymbol{X}_2(1)\boldsymbol{\lambda}^2-\boldsymbol{X}_2(0)M_2^2\boldsymbol{\lambda}^2)^2]=q(1-q)(\lambda^2-\lambda^3)^2,\end{cases}$$

the bilinear form (7.3) is

$$(8.6)\qquad \begin{aligned}&\sum_{a=2}^{3} v_a E_{e^a}[((\boldsymbol{X}_2(1)-\boldsymbol{X}_2(0)M_2^2)(\lambda^0\boldsymbol{u}^2+Q_2^2\boldsymbol{\lambda}^2))^2]\\ &\quad=B_0(q)(\lambda^0)^2+2B_1(q)\lambda^0(\lambda^2-\lambda^3)+B_2(q)(\lambda^2-\lambda^3)^2,\end{aligned}$$

where

$$(8.7)\qquad B_0(q)=\frac{2q}{2q+1},\ B_1(q)=\frac{2q(1-2q)}{(2q+1)^3},\ B_2(q)=\frac{2q(3+2q+4q^2-8q^3)}{(2q+1)^5}.$$

By (4.2) $\psi_2=\psi_2(t:\boldsymbol{\lambda},\boldsymbol{\mu})$ is the solution of

$$(8.8)\qquad \begin{cases}\psi_2(0:\boldsymbol{\lambda},\boldsymbol{\mu})=\dfrac{i}{2q+1}(2q\lambda^2+\lambda^3),\\ \dfrac{d}{dt}\psi_2=\dfrac{1}{2}(B_0(q)\psi_2^2+2iB_1(q)(\mu^2-\mu^3)\psi_2-B_2(q)(\mu^2-\mu^3)^2)+\dfrac{2iq}{2q+1}\lambda^4.\end{cases}$$

Let $x_0\geqq 0$, $x_4\geqq 0$, y_2, y_3 be fixed arbitrarily and set $x_2=\dfrac{2q}{2q+1}x_0$, $x_3=\dfrac{1}{2q+1}x_0$, $\boldsymbol{x}_2$

$=(x_2, x_3)$, $\boldsymbol{y}_2=(y_2, y_3)$. Then by Theorem 2, the sequence of processes

$$\{((n^{-1}\boldsymbol{X}_2^n(t), n^{-2}X_4^n(t), \boldsymbol{y}_2+n^{-2}\boldsymbol{Y}_2^n(t)), P_{(0,[nx_2],[nx_3],[n^2x_4])})\}_{n\geq 1}$$

converges to a diffusion process $((\boldsymbol{X}_2(t), X_4(t), \boldsymbol{Y}_2(t)), P_{(x_0,x_4,\boldsymbol{y}_2)})$ and

$$\begin{aligned} E_{(x_5,x_4,\boldsymbol{y}_2)}&[\exp(i(\boldsymbol{X}_2(t)\boldsymbol{\lambda}^2+X_4(t)\lambda^4+\boldsymbol{Y}_2(t)\boldsymbol{\mu}^2))] \\ &= \exp(i\boldsymbol{y}_2\boldsymbol{\mu}^2+x_0\psi_2(t:\boldsymbol{\lambda},\boldsymbol{\mu})+ix_4\lambda^4)\,. \end{aligned} \tag{8.9}$$

By Theorem 3 the sequence of processes

$$\{((n^{-1}\boldsymbol{X}_2^n(t), n^{-2}X_4^n(t), \boldsymbol{y}_2+n^{-1}\boldsymbol{Y}_2^n(t)), P_{(1,[nx_2],[nx_3],[n^2x_4])})\}_{n\geq 1}$$

converges to a diffusion process $((\hat{\boldsymbol{X}}_2(t), \hat{X}_4(t), \hat{\boldsymbol{Y}}_2(t)), \hat{P}_{(x_0,x4,\boldsymbol{y}_2)})$ and

$$\begin{aligned} \hat{E}_{(x_5,x_4,\boldsymbol{y}_2)}&[\exp(i(\hat{\boldsymbol{X}}_2(t)\boldsymbol{\lambda}^2+\hat{X}_4(t)\lambda^4+\hat{\boldsymbol{Y}}_2(t)\boldsymbol{\mu}^2))] \\ &= \exp\Big(i\boldsymbol{y}_2\boldsymbol{\mu}^2+p\int_0^t\psi_2(s:\boldsymbol{\lambda},\boldsymbol{\mu})\,ds+x_0\psi_2(t:\boldsymbol{\lambda},\boldsymbol{\mu}) \\ &\qquad +\frac{2ipt}{(2q+1)^2}(\mu^2-\mu^3)+ix_4\lambda^4\Big)\,. \end{aligned} \tag{8.10}$$

We shall clarify these limit processes applying the remarks in section 7. Set $X(t)=\boldsymbol{X}_2(t)\boldsymbol{u}^2=X_2(t)+X_3(t)$. Then $\boldsymbol{X}_2(t)=\dfrac{X(t)}{2q+1}(2q, 1)$. By (8.8) and (8.9) it follows that $Y_2(t)+Y_3(t)=y_2+y_3$. Hence $\boldsymbol{Y}_2(t)$ is determined by $Y(t)=Y_2(t)$ and y_3. For the convenience we ste $Z(t)=X_4(t)$, $x=x_0$, $z=x_4$. Let $P_{(x,y,z)}$ be the probability measure induced from $P_{(x,z,y,0)}$ by the diffusion process $(X(t), Y(t), Z(t))$. Then by (8.9) it follows that

$$\begin{aligned} E_{(x,y,z)}&[\exp(i(X(t)\lambda+Y(t)\mu+Z(t)\nu))] \\ &= \exp(iy\mu+x\phi(t:\lambda,\mu,\nu)+iz\nu)\,, \end{aligned} \tag{8.11}$$

where $\phi=\phi(t:\lambda,\mu,\nu)$ is the solution of

$$\begin{cases} \phi(0:\lambda,\mu,\nu)=i\lambda\,, \\ \dfrac{d\phi}{dt}(t:\lambda,\mu,\nu)=\dfrac{1}{2}(B_0(q)\phi^2+2iB_1(q)\mu\phi-B_2(q)\mu^2)+\dfrac{2iq}{2q+1}\nu\,. \end{cases} \tag{8.12}$$

By the same method we can define the process $(\hat{X}(t), \hat{Y}(t), \hat{Z}(t))$ and let $\hat{P}_{(x,y,z)}$ be the measure induced from $\hat{P}_{(x,z,y,0)}$ by this process. Then by (8.10) we have

$$\begin{aligned} \hat{E}_{(x,x,z)}&[\exp(i(\hat{X}(t)\lambda+\hat{Y}(t)\mu+\hat{Z}(t)\nu)] \\ &= \exp\Big(iy\mu+p\int_0^t\phi(s:\lambda,\mu,\nu)\,ds+x\phi(t:\lambda,\mu,\nu)+\frac{2ipt}{(2q+1)^2}\mu+iz\nu\Big)\,. \end{aligned} \tag{8.13}$$

Hence the generator A of the first process is

$$Af(x,y,z)=\frac{1}{2}x(B_0(q)D_x^2+2B_1(q)D_xD_y+B_2(q)D_y^2)f(x,y,z)+\frac{2q}{2q+1}xD_zf(x,y,z)\,, \tag{8.14}$$

and the generator $\hat{A}$ of the second process is

$$\hat{A}f(x,y,z)=Af(x,y,z)+pD_xf(x,y,z)+\frac{2p}{(2q+1)^2}D_zf(x,y,z)\,. \tag{8.15}$$

We shall end this section by giving the forms of characteristic functions for $Y(t)$ and $\hat{Y}(t)$ in some special cases. (The forms of Laplace transforms for $X(t)$, $\hat{X}(t)$, $Z(t)$ and $\hat{Z}(t)$ are given in [5]). If $q=1$ then $\phi(t\colon 0,\mu,0)=\frac{-t\mu^2}{9(27+it\mu)}$. Hence we have

$$E_{(x,0,0)}[\exp(iY(t)\mu)]=\exp\left(\frac{-xt\mu^2}{9(27+it\mu)}\right), \tag{8.16}$$

$$\hat{E}_{(x,0,0)}[\exp(i\hat{Y}(t)\mu)]=\left(1+\frac{it}{27}\mu\right)^{-3p}\exp\left(\frac{i}{9}pt\mu-\frac{xt\mu^2}{9(27+it\mu)}\right). \tag{8.17}$$

If $q=\frac{1}{2}$ then $\phi(t\colon 0,\mu,0)=-\frac{\mu}{2}\tanh\left(\frac{\mu}{8}t\right)$. Hence we ha have

$$E_{(x,0,0)}[\exp(iY(t)\mu)]=\exp\left(-\frac{x}{2}\mu\tanh\left(\frac{\mu}{8}t\right)\right), \tag{8.18}$$

$$\hat{E}_{(x,0,0)}[\exp(i\hat{Y}(t)\mu)]=\left(\cosh\left(\frac{\mu}{8}t\right)\right)^{-4p}\exp\left(-\frac{x}{2}\mu\tanh\left(\frac{\mu}{8}t\right)+\frac{i}{2}pt\mu\right). \tag{8.19}$$

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