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Lowig, H. F. J.
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On Some Representations of Lattices of Law Relations

By H. F. J. Lowig

In this paper, I am going to consider some representations of the set of all law relations on a freely generated algebra. It is understood that all conventions concerning terminology and notation introduced in (II), (III) and (IV) hold in this paper. (See the bibliography at the end of the paper.) In particular, it is understood that the following conventions hold:

S is a set, and σ is an S-system of sets; hence if $s \in S$, the value σs of σ at s is a set. By an operator system of species σ on a set A is meant a representation F of S such that, for $s \in S$, Fs is a σs-operator on A. (See (II), Definitions 1.1 and 1.2.) An algebra of species σ (or shortly: an algebra) on A is determined if σ, A, and an operator system of species σ on A are given. $\mathfrak{A}$ and $\mathfrak{B}$ are algebras. The operator system corresponding to $\mathfrak{A}$ is denoted by $\langle\mathfrak{A}\rangle$. Hence if $\mathfrak{A}$ is an algebra on A, $s \in S$, and $a \in A^{\sigma s}$, $\langle\mathfrak{A}\rangle s$ is a σs-operator on A, and $(\langle\mathfrak{A}\rangle s)a$ is an element of A. If there exists a subset Q of A which generates $\mathfrak{A}$ and has the property that

$$(\langle\mathfrak{A}\rangle s)a \notin Q \quad \text{for } s \in S, \qquad a \in A^{\sigma s},$$

and

$$(\langle\mathfrak{A}\rangle s_1)a_1 \neq (\langle\mathfrak{A}\rangle s_2)a_2$$

$$\text{if } s_\nu \in S \text{ and } a \in A^{\sigma s_\nu} \text{ for } \nu = 1, 2, \text{ and } s_1 \neq s_2 \text{ or } a_1 \neq a_2,$$

then $\mathfrak{A}$ is called *freely generated*. In this case, Q is unique; it is called the *free basis* of $\mathfrak{A}$. $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are freely generated algebras on the sets C, C_1 and C_2; their free bases are D, D_1 and D_2. $\mathfrak{M}$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are sets of algebras.

The title of this paper is justified by (III), Satz 2.10.

§1. The operators $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ and $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$.

DEFINITION 1.1. *By* $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is meant* $\{(\Phi\mathfrak{C}_2)(\mathfrak{C}_1/r)\,;\ r \in L\mathfrak{C}_1\}$. (See (III), Definitions 2.1 and 3.1.)

By (III), Satz 3.4, $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is a representation of $L\mathfrak{C}_1$ into $L\mathfrak{C}_2$.

Theorem 1.1. *$\Psi\{\mathfrak{C}, \mathfrak{C}\}$ is the identical representation of $L\mathfrak{C}$.*

Proof. See (III), Satz 3.8.

If ψ is a representation of $L\mathfrak{C}_1$ into the set of all relations on C_2, let us say that ψ is *monotonic* if

$$\psi r' \subset \psi r'' \quad \text{for} \quad r' \in L\mathfrak{C}_1, \quad r'' \in L\mathfrak{C}_1, \quad r' \subset r''.$$

Theorem 1.2. *$\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is monotonic.*

Proof. See (III), Satz 3.9.

Theorem 1.3.

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Phi\mathfrak{C}_2)\mathfrak{M}) \supset (\Phi\mathfrak{C}_1)\mathfrak{M}.$$

Proof. Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 with $\mathfrak{c}_1'((\Phi\mathfrak{C}_1)\mathfrak{A})\mathfrak{c}_1''$. Let h_2 be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ and h be a homomorphism of $\mathfrak{C}_2$ into $\mathfrak{A}$. Then $h \cdot h_2$ is a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{A}$. Hence, by (III), Definition 3.1 and and Satz 3.6,

$$h(h_2\mathfrak{c}_1') = h(h_2\mathfrak{c}_1''),$$
$$(h_2\mathfrak{c}_1')((\Phi\mathfrak{C}_2)\mathfrak{A})(h_2\mathfrak{c}_1''),$$

and

$$\mathfrak{c}_1'((\Phi\mathfrak{C}_1)(\mathfrak{C}_2/(\Phi\mathfrak{C}_2)\mathfrak{A}))\mathfrak{c}_1''.$$

This proves that

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Phi\mathfrak{C}_2)\mathfrak{A}) \supset (\Phi\mathfrak{C}_1)\mathfrak{A}.$$

The theorem now follows from (III), Satz 3.10.

Theorem 1.4. *Let $r \in L\mathfrak{C}_1$. Then*

$$(\Psi(\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset r.$$

Proof. In Theorem 1.3, put $M = [\mathfrak{C}_1/r]$.

DEFINITION 1.2. *By $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is meant the representation of $L\mathfrak{C}_1$ into the set of all congruence relations on $\mathfrak{C}_2$ defined by the condition that, for $r \in L\mathfrak{C}_1$, $(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r$ is the intersection of all congruence relations r^* on $\mathfrak{C}_2$ such that*

$$(\Phi\mathfrak{C}_1)(\mathfrak{C}_2/r^*) \supset r. \tag{1.1}$$

Theorem 1.5.

$$(\Phi\mathfrak{C}_1)(\mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset r \quad \textit{for} \quad r \in L\mathfrak{C}_1.$$

Proof. See (III), Satz 3.11.

Theorem 1.6. *Let r be a law relation on $\mathfrak{C}_1$, let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 with $\mathfrak{c}_1' r \mathfrak{c}_1''$, and let h be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Then*

$$(h\mathfrak{c}_1')((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)(h\mathfrak{c}_1'') .$$

Theorem 1.7. *Let r be a law relation on $\mathfrak{C}_1$ and r^* be a congruence relation on $\mathfrak{C}_2$. Then*

$$r^* \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r$$

if and only if (1.1) *holds.*

Theorem 1.8. *Let $r_\nu \in L\mathfrak{C}_\nu$ for $\nu = 1, 2$. Then*

$$r_2 \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1$$

if and only if

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2 \supset r_1 .$$

Theorem 1.9.

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \subset r \qquad \text{for} \quad r \in L\mathfrak{C}_2 .$$

Theorem 1.10. *Let r be a law relation on $\mathfrak{C}_1$. Then $(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r$ is a law relation on $\mathfrak{C}_2$.* (In other words: $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is a representation of $L\mathfrak{C}_1$ into $L\mathfrak{C}_2$.)

Proof. Let $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ be elements of C_2 such that

$$\mathfrak{c}_2'((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$$

and h be a homomorphism of $\mathfrak{C}_2$ into $\mathfrak{C}_2$. Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 with $\mathfrak{c}_1' r \mathfrak{c}_1''$ and r^* be a congruence relation on $\mathfrak{C}_2$ satisfying (1.1). Then

$$\mathfrak{c}_1'((\Phi\mathfrak{C}_1)(\mathfrak{C}_2/r^*))\mathfrak{c}_1'' .$$

Let h_2 be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Then

$$(h(h_2\mathfrak{c}_1'))r^*(h(h_2\mathfrak{c}_1'')) .$$

Hence

$$(Hr^*)(h(h_2\mathfrak{c}_1')) = (Hr^*)(h(h_2\mathfrak{c}_1''))$$

(see (III), p. 132, line 16) and

$$(h_2\mathfrak{c}_1')(R((Hr^*)\cdot h))(h_2\mathfrak{c}_1'') .$$

(See (III), p. 133, line 14.) Hence

$$\mathfrak{c}_1'((\Phi\mathfrak{C}_1)(\mathfrak{C}_2/R((Hr^*)\cdot h)))\mathfrak{c}_1'' .$$

Hence

$$(\Phi\mathfrak{C}_1)(\mathfrak{C}_2/R((Hr^*)\cdot h))\supset r.$$

By Theorem 1.7,

$$R((Hr^*)\cdot h)\supset(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r.$$

Hence

$$\mathfrak{c}_2'(R((Hr^*)\cdot h))\mathfrak{c}_2'',$$
$$(Hr^*)(h\mathfrak{c}_2')=(Hr^*)(h\mathfrak{c}_2'');$$

and

$$(h\mathfrak{c}_2')r^*(h\mathfrak{c}_2'').$$

By Definition 1.2,

$$(h\mathfrak{c}_2')((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)(h\mathfrak{c}_2'').$$

This proves that $(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r$ is a law relation on $\mathfrak{C}_2$.

Reformulation of Theorem 1.5.

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\supset r \qquad \textit{for} \quad r\in L\mathfrak{C}_1.$$

Theorem 1.11. *$\psi\{\mathfrak{C}, \mathfrak{C}\}$ is the identical representation of $L\mathfrak{C}$.*

Proof. See (III), Satz 3.13.

Theorem 1.12. *$\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is monotonic.*

Proof. Let $r'\in L\mathfrak{C}_1$, $r''\in L\mathfrak{C}_1$ and $r''\supset r'$. Then, by Theorem 1.5,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r'')\supset r'',$$

hence

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r'')\supset r',$$

hence

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r''\supset(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r'$$

by Theorem 1.8.

Theorem 1.13.

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r\subset(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r \qquad \textit{for} \quad r\in L\mathfrak{C}_1.$$

Theorem 1.13 is obvious from Theorems 1.4 and 1.8.

Theorem 1.14. *Let $|C_1|\leq 1$. Let $r\in L\mathfrak{C}_1$. Then $(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r$ is the all relation on C_2, and $(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r$ is the equality relation on C_2.*

Proof. The first part of the theorem is obvious. If r^* is any congruence relation on $\mathfrak{C}_2$ then (1.1) holds. Hence the intersection of all congruence relations on $\mathfrak{C}_2$ satisfying (1.1) is the equality relation on C_2. This proves the second part of the theorem.

Theorem 1.15.

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Phi\mathfrak{C}_2)\mathfrak{M}) \subset (\Phi\mathfrak{C}_1)\mathfrak{M}.$$

Proof. By Theorem 1.3,

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Phi\mathfrak{C}_1)\mathfrak{M}) \supset (\Phi\mathfrak{C}_2)\mathfrak{M}.$$

By Theorem 1.8,

$$(\Phi\mathfrak{C}_1)\mathfrak{M} \supset (\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Phi\mathfrak{C}_2)\mathfrak{M}).$$

Definition 1.3. *By $\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is meant the representation of $L\mathfrak{C}_1$ into the set of all relations on C_2 defined by the following condition: if $r \in L\mathfrak{C}_1$, $\mathfrak{c}_2' \in C_2$ and $\mathfrak{c}_2'' \in C_2$ then $\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$ holds if and only if there exist elements $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ of C_1 with $\mathfrak{c}_1' r \mathfrak{c}_1''$ and $\{\mathfrak{c}_1', \mathfrak{c}_1''\}(\{\mathfrak{C}_1, \mathfrak{C}_2\}$-conf$)\{\mathfrak{c}_2', \mathfrak{c}_2''\}$.* (See (III), Definition 1.1.)

Theorem 1.16. *Let $r \in L\mathfrak{C}_1$, $\mathfrak{c}_2' \in C_2$ and $\mathfrak{c}_2'' \in C_2$. Then $\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$ holds if and only if the following two conditions are satisfied:*

(1.2) $|\mathfrak{S}[(P\mathfrak{C}_2)\mathfrak{c}_2', (P\mathfrak{C}_2)\mathfrak{c}_2'']| \leqq |D_1|$. (See (II), Definition 3.2.)

(1.3) *There exist elements $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ of C_1 with $\mathfrak{c}_1' r \mathfrak{c}_1''$ and a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h\mathfrak{c}_1' = \mathfrak{c}_2'$ and $h\mathfrak{c}_1'' = \mathfrak{c}_2''$.*

Proof. Let (1.2) and (1.3) hold. Let $\mathfrak{c}'$ and $\mathfrak{c}''$ be elements of C_1 with $\{\mathfrak{c}', \mathfrak{c}''\}(\{\mathfrak{C}_1, \mathfrak{C}_2\}$-conf$)\{\mathfrak{c}_2', \mathfrak{c}_2''\}$ and h_1 be a homomorphism of $\mathfrak{C}_2$ into $\mathfrak{C}_1$ with $h_1\mathfrak{c}_2' = \mathfrak{c}'$ and $h_1\mathfrak{c}_2'' = \mathfrak{c}''$. (See (III), Satz 1.2 and Satz 1.5.) Then, by (III), Definition 2.1,

$$(h_1(h\mathfrak{c}_1'))\, r (h_1(h\mathfrak{c}_1'')).$$

Hence $\mathfrak{c}' r \mathfrak{c}''$. By Definition 1.3, $\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$.

The converse implication follows from (III), Satz 1.1 and Satz 1.2.

Theorem 1.17. *Let $r \in L\mathfrak{C}_1$. Then*

$$(\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r \subset (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r.$$

Proof. If $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ are elements of C_2 with $\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$ then $\mathfrak{c}_2'((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$ holds by Theorems 1.16 and 1.6.

By Theorems 1.17 and 1.13,

$$(\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r \qquad \text{for} \quad r \in L\mathfrak{C}_1.$$

Theorem 1.18. *Let r be a law relation on $\mathfrak{C}_1$. Let $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ be elements of C_2 with*

$$|\mathfrak{S}[(P\mathfrak{C}_2)\mathfrak{c}_2', (P\mathfrak{C}_2)\mathfrak{c}_2'']| \leqq |D_1|$$

and

$$\mathfrak{c}_2'((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''.$$

Then $\mathfrak{c}_2'((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$.

Proof. Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 with $\{\mathfrak{c}_1', \mathfrak{c}_1''\}(\{\mathfrak{C}_1, \mathfrak{C}_2\}\text{-conf})\{\mathfrak{c}_2', \mathfrak{c}_2''\}$. Then there exists a homomorphism h of $\mathfrak{C}_2$ into $\mathfrak{C}_1$ with $h\mathfrak{c}_2' = \mathfrak{c}_1'$ and $h\mathfrak{c}_2'' = \mathfrak{c}_1''$. By (III), Satz 3.6, $\mathfrak{c}_1' r \mathfrak{c}_1''$. Hence $\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r)\mathfrak{c}_2''$.

Theorem 1.19. *Let r be a law relation on $\mathfrak{C}_2$. Let $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ be elements of C_2 with*

$$|\mathfrak{S}[(P\mathfrak{C}_2)\mathfrak{c}_2', (P\mathfrak{C}_2)\mathfrak{c}_2'']| \leqq |D_1|.$$

and

$$\mathfrak{c}_2'((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r))\mathfrak{c}_2''.$$

Then $\mathfrak{c}_2' r \mathfrak{c}_2''$.

Proof. By Theorem 1.18,

$$\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r))\mathfrak{c}_2''.$$

By Theorems 1.17 and 1.9, $\mathfrak{c}_2' r \mathfrak{c}_2''$.

Theorem 1.20. *Let*

$$|D_2| \leqq |D_1| \quad \textit{or} \quad 2m_\sigma \leqq |D_1|.$$

(See (II), Definition 3.4.) *Then*

$$\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}.$$

Proof. It is obvious that

$$|\mathfrak{S}[(P\mathfrak{C}_2)\mathfrak{c}_2', (P\mathfrak{C}_2)\mathfrak{c}_2'']| \leqq |D_1| \quad \text{for} \quad \mathfrak{c}_2' \in C_2,\ \mathfrak{c}_2'' \in C_2.$$

After this is said the present theorem follows from Theorem 1.18.

Theorem 1.21. *Let r_1 be a law relation on $\mathfrak{C}_1$. Then*

$$(\Theta\mathfrak{C}_2)((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1) = (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1.$$

(See (III), Definition 2.2.)

Proof. The assertion follows from the preceding theorem if $|D_2| \leqq |D_1|$. Let $|D_2| > |D_1|$. Let r_2 be a law relation on $\mathfrak{C}_2$ such that

$$(\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1 \subset r_2.$$

Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 with $\mathfrak{c}_1' r_1 \mathfrak{c}_1''$ and $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ be elements of C_2 with $\{\mathfrak{c}_1', \mathfrak{c}_1''\}(\{\mathfrak{C}_1, \mathfrak{C}_2\}\text{-conf})\{\mathfrak{c}_2', \mathfrak{c}_2''\}$. Then

$$\mathfrak{c}_2'((\psi_0\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1)\mathfrak{c}_2'' ,$$

$$\mathfrak{c}_2' r_2 \mathfrak{c}_2'' ,$$

$$\mathfrak{c}_1'((\psi_0\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2)c_1'' ,$$

and

$$\mathfrak{c}_1'((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2)\mathfrak{c}_1'' .$$

Hence

$$r_1 \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2 .$$

By Theorem 1.8,

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1 \subset r_2 .$$

This proves the assertion.

Our concept of a relation of C corresponds to Birkhoff's concept of a set of equations between functions of the given species. (See (I), p. 440, line 2 from bottom.) If B is a relation on C_1, and $|D_2| = m$, then our $(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)$ corresponds to Birkhoff's set of all equations in m primitive symbols following from B, and our $\mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)$ corresponds to Birkhoff's $F(B, m)$.

Theorem 1.22. *Let B be a relation on C_1. Then*

$$(\Phi\mathfrak{C}_1)(\mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)) \supset B .$$

Theorem 1.22 is obvious from Theorem 1.5.

Compare Theorem 1.22 with the following statement occurring in (I), p. 441, line 2: "$F(B, m)$ satisfies all the laws of B."

Theorem 1.23. *Let B be a relation on C_1. Let Q be a basis of $\mathfrak{A}$. Let $|Q| = |D_2|$. Let $(\Phi\mathfrak{C}_1)\mathfrak{A} \supset B$. Then $\mathfrak{A}$ is a homomorphic image of $\mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)$.*

Proof. If $|\mathfrak{A}| = 0$ then $|C_2| = 0$ by (II), Theorem 1.3, and the assertion is obvious. Let $|\mathfrak{A}| \neq 0$. By Theorem 1.15,

$$(\Phi\mathfrak{C}_2)\mathfrak{A} \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Phi\mathfrak{C}_1)\mathfrak{A}) \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B) .$$

The assertion now follows from (III), Satz 3.22.

Compare Theorem 1.23 with the following statement occurring in (I), p. 441, lines 3 and 4: "Every algebra of species Σ generated by m elements and of which B is a set of laws is a homomorphic image of $F(B, m)$."

§ 2. The relation $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$.

DEFINITION 2. 1. *We define a cardinal l_σ in the following way:*

(2. 1) *If there exists an element s_0 of S with $|\sigma s_0|=0$, and $|\sigma s|\leqq 1$ for all elements s of S, then $l_\sigma=1$.*

(2. 2) *If $|\sigma s|=1$ for all elements s of S then $l_\sigma=2$.*

(2. 3) *If there exists an element s of S with $|\sigma s|\geqq 2$ then $l_\sigma=m_\sigma$.*

It follows from this definition and from (II), Definition 3. 4, that

$$l_\sigma = m_\sigma$$

except the case considered under (2. 2). Also,

$$l_\sigma = 2m_\sigma$$

except the case considered under (2. 1).

Theorem 2. 1. *Let $|D|\geqq l_\sigma$. Then $|C|\geqq 2$.*

Proof. The assertion is obvious if $|D|\geqq 2$. If $|D|=l_\sigma=1$ then the hypotheses of (2. 1) are satisfied, S is not void, and $|C|\geqq 2$ by (II), Theorem 2. 8.

DEFINITION 2. 2. $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ *or* $\mathfrak{C}_2 \sqsupset \mathfrak{C}_1$ *means that* $|D_1|\leqq|D_2|$, *or* $l_\sigma\leqq|D_2|$, *or* $|S|=0$ *and* $|D_1|\leqq 1$.

From this definition it is obvious that $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ or $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$ for any two freely generated algebras $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

Theorem 2. 2. *Let $\mathfrak{M}$ be a set of freely generated algebras. Then $\sqsubset$ is a quasi-ordering of $\mathfrak{M}$.*

Theorem 2. 3. *Let $l_\sigma\leqq|D_1|$ and $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. Then $l_\sigma\leqq|D_2|$.*

The proof of Theorems 2. 2 and 2. 3 is left to the reader.

DEFINITION 2. 3. $\mathfrak{C}_1 \sqsubset\!\sqsupset \mathfrak{C}_2$ *means that* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ *as well as* $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$.

Hence $\mathfrak{C}_1 \sqsubset\!\sqsupset \mathfrak{C}_2$ if and only if $|D_1|=|D_2|$, or $l_\sigma\leqq|D_\nu|$ for $\nu=1,2$, or $|S|=0$ and $|D_\nu|\leqq 1$ for $\nu=1,2$.

Theorem 2. 4. *Let $\mathfrak{M}$ be a set of freely generated algebras. Then $\sqsubset\!\sqsupset$ is an equivalence relation on $\mathfrak{M}$.*

Theorem 2. 4 is obvious from Theorem 2. 2.

Lemma 2. 1. *Let $|D_2|\leqq|D_1|$. Let $\mathfrak{c}_0\in C_1$. Let r be the relation on C_1 defined by the condition that, for $\mathfrak{c}_1'\in C_1$, $\mathfrak{c}_1''\in C_1$, $\mathfrak{c}_1' r \mathfrak{c}_1''$ holds if and only if*

$$|\mathfrak{S}[(P\mathfrak{C}_1)\mathfrak{c}_1', (P\mathfrak{C}_1)\mathfrak{c}_1'']|\leqq|D_2| , \tag{2. 4}$$

and there exist homomorphisms h' and h'' of $\mathfrak{C}_1$ into $\mathfrak{C}_1$ such that $h'\mathfrak{c}_0=\mathfrak{c}_1'$ and $h''\mathfrak{c}_0=\mathfrak{c}_1''$. Let h_1' and h_1'' be homomorphisms of $\mathfrak{C}_1$ into $\mathfrak{C}_1$. Then

$$(2.5)\qquad (h_1'\mathfrak{c}_0)((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)r)))(h_1''\mathfrak{c}_0)\,.$$

Proof. Let h_2 be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ and h_1 be a homomorphism of $\mathfrak{C}_2$ into $\mathfrak{C}_1$. For abbreviation, let us put

$$h_2(h_1'\mathfrak{c}_0)=\mathfrak{c}_2'$$

and

$$h_2(h_1''\mathfrak{c}_0)=\mathfrak{c}_2''\,.$$

Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 such that

$$\{\mathfrak{c}_1', \mathfrak{c}_1''\}(\{\mathfrak{C}_1, \mathfrak{C}_2\}\text{-conf})\{\mathfrak{c}_2', \mathfrak{c}_2''\}\,.$$

Then (2.4) holds. Let h_1^* be a homomorphism of $\mathfrak{C}_2$ into $\mathfrak{C}_1$ such that $h_1^*\mathfrak{c}_2'=\mathfrak{c}_1'$ and $h_1^*\mathfrak{c}_2''=\mathfrak{c}_1''$. Let h_2^* be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h_2^*\mathfrak{c}_1'=\mathfrak{c}_2'$ and $h_2^*\mathfrak{c}_1''=\mathfrak{c}_2''$. Then

$$h_1^*(h_2(h_1'\mathfrak{c}_0))=\mathfrak{c}_1'$$

and

$$h_1^*(h_2(h_1''\mathfrak{c}_0))=\mathfrak{c}_1''\,.$$

Hence $\mathfrak{c}_1' r\mathfrak{c}_1''$. Hence

$$\mathfrak{c}_1'((\Theta\mathfrak{C}_1)r)\mathfrak{c}_1''\,.$$

Because $(\Theta\mathfrak{C}_1)r$ is a law relation on $\mathfrak{C}_1$,

$$(h_1(h_2^*\mathfrak{c}_1'))((\Theta\mathfrak{C}_1)r)(h_1(h_2^*\mathfrak{c}_1''))\,.$$

Hence

$$(h_1\mathfrak{c}_2')((\Theta\mathfrak{C}_1)r)(h_1\mathfrak{c}_2'')\,.$$

Hence

$$(h_1(h_2(h_1'\mathfrak{c}_0)))((\Theta\mathfrak{C}_1)r)(h_1(h_2(h_1''\mathfrak{c}_0)))\,.$$

By (III), Satz 3.6,

$$(h_2(h_1'\mathfrak{c}_0))((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)r))(h_2(h_1''\mathfrak{c}_0))\,.$$

Applying Satz 3.6 of (III) again, we find that (2.5) holds.

DEFINITION 2.4. *By $\rho\mathfrak{C}$ is meant the element of $(\mathfrak{a}\mu_\sigma)^C$ defined by the condition that, for $\mathfrak{c}\in C$, $(\rho\mathfrak{C})\mathfrak{c}$ is that element α of $\mathfrak{a}\mu_\sigma$ for which $\mathfrak{c}\in((E\mathfrak{C})D)'\alpha$. If $\mathfrak{c}\in C$, $(\rho\mathfrak{C})\mathfrak{c}$ is called the rank of $\mathfrak{c}$ with respect to $\mathfrak{C}$.* (See (IV), Definitions 1 and 2 and Theorems 2 and 3.)

Lemma 2.2. *Let $\mathfrak{c}$ be an element of C_1 and h be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Then*

$$(2.6) \qquad (\rho\mathfrak{C}_2)(h\mathfrak{c}) \geqq (\rho\mathfrak{C}_1)\mathfrak{c}\,.$$

Proof. (2.6) obviously holds if $\mathfrak{c}\in D_1$. Let s be an element of S and c be an element of $C_1^{\sigma s}$ such that

$$(\rho\mathfrak{C}_2)(h(ck)) \geqq (\rho\mathfrak{C}_1)(ck) \qquad \text{for} \quad k\in\sigma s\,.$$

If $|\sigma s|=0$ then

$$(\rho\mathfrak{C}_2)(h((\langle\mathfrak{C}_1\rangle s)c)) = (\rho\mathfrak{C}_2)((\langle\mathfrak{C}_2\rangle s)(h\cdot c)) = 1 = (\rho\mathfrak{C}_1)((\langle\mathfrak{C}_1\rangle s)c)\,.$$

If $|\sigma s|\neq 0$ then

$$\begin{aligned}(\rho\mathfrak{C}_2)(h((\langle\mathfrak{C}_1\rangle s)c)) &= (\rho\mathfrak{C}_2)((\langle\mathfrak{C}_2\rangle s)(h\cdot c)) = \mathrm{Suc}((\rho\mathfrak{C}_2)\cdot(h\cdot c))\\ &\geqq \mathrm{Suc}((\rho\mathfrak{C}_1)c) = (\rho\mathfrak{C}_1)((\langle\mathfrak{C}_1\rangle s)c)\,.\end{aligned}$$

In both cases,

$$(\rho\mathfrak{C}_2)(h((\langle\mathfrak{C}_1\rangle s)c)) \geqq (\rho\mathfrak{C}_1)((\langle\mathfrak{C}_1\rangle s)c)\,.$$

This proves that (2.6) holds generally.

Lemma 2.3. *Let m be a cardinal. Let r be the relation on C defined by the condition that, for $\mathfrak{c}'\in C$, $\mathfrak{c}''\in C$, $\mathfrak{c}'r\mathfrak{c}''$ holds if and only if $\mathfrak{c}'=\mathfrak{c}''$, or $(\rho\mathfrak{C})\mathfrak{c}'\geqq m$ and $(\rho\mathfrak{C})\mathfrak{c}''\geqq m$. Then r is a law relation on $\mathfrak{C}$.*

Proof. It is obvious that r is an equivalence relation on C. Let s be an element of S, and let c' and c'' be elements of $C^{\sigma s}$ such that

$$(c'k)\,r\,(c''k) \qquad \text{for} \quad k\in\sigma s\,.$$

Let

$$(\langle\mathfrak{C}\rangle s)c' \neq (\langle\mathfrak{C}\rangle s)c''\,.$$

Then $c'\neq c''$. Let k_0 be an element of σs such that

$$c'k_0 \neq c''k_0\,.$$

Then $(\rho\mathfrak{C})(c'k_0)\geqq m$ and $(\rho\mathfrak{C})(c''k_0)\geqq m$. Also,

$$(\rho\mathfrak{C})((\langle\mathfrak{C}\rangle s)c') = \mathrm{Suc}((\rho\mathfrak{C})\cdot c') > (\rho\mathfrak{C})(c'k_0)\,.$$

Hence

$$(\rho\mathfrak{C})((\langle\mathfrak{C}\rangle s)c') \geqq m\,.$$

For a similar reason,

$$(\rho\mathfrak{C})((\langle\mathfrak{C}\rangle s)c'') \geqq m\,.$$

Hence

$$((\langle\mathfrak{C}\rangle s)c')\,r\,((\langle\mathfrak{C}\rangle s)c'')\,.$$

Hence r is homomorphic with respect to $\mathfrak{C}$.

Let $\mathfrak{c}'$ and $\mathfrak{c}''$ be elements of C with $\mathfrak{c}'r\mathfrak{c}''$. Let h be a homomorphism of $\mathfrak{C}$ into $\mathfrak{C}$. Let

$$h\mathfrak{c}' \neq h\mathfrak{c}'' .$$

Then $c' \neq c''$. Hence $(\rho\mathfrak{C})\mathfrak{c}' \geqq m$ and $(\rho\mathfrak{C})\mathfrak{c}'' \geqq m$. By Lemma 2.2,

$$(\rho\mathfrak{C})(h\mathfrak{c}') \geqq m$$

and

$$(\rho\mathfrak{C})(h\mathfrak{c}'') \geqq m .$$

Hence

$$(h\mathfrak{c}')\, r\, (h\mathfrak{c}'') .$$

Hence r is a law relation on C.

Lemma 2.4. *Let r be a relation on C. Let $\mu = \omega$ if $\mu_\sigma = 2$, and $\mu = \mu_\sigma$ if $\mu_\sigma \neq 2$. Let $\mathfrak{x}$ be the $\mathfrak{b}_\mu$-system of relations on C which has the following two properties*:

I. $\mathfrak{x}0 = r$.

II. *If $\gamma \in \mathfrak{b}\{1, \mu\}$, $\mathfrak{c}' \in C$ and $\mathfrak{c}'' \in C$ then $\mathfrak{c}'(\mathfrak{x}\gamma)\mathfrak{c}''$ holds if and only if at least one of the following conditions* (2.7) *to* (2.12) *is satisfied*:

(2.7) $$\mathfrak{c}'(\mathfrak{x}\alpha)\,\mathfrak{c}'' \quad \textit{for some} \quad \alpha \in \mathfrak{a}\gamma .$$

(2.8) $$\mathfrak{c}' = \mathfrak{c}'' .$$

(2.9) $$\mathfrak{c}''(\mathfrak{x}\alpha)\mathfrak{c}' \quad \textit{for some} \quad \alpha \in \mathfrak{a}\gamma .$$

(2.10) *There exist an element $\mathfrak{c}$ of C and elements α_1 and α_2 of $\mathfrak{a}\gamma$ with $\mathfrak{c}'(\mathfrak{x}\alpha_1)\mathfrak{c}$ and $\mathfrak{c}(\mathfrak{x}\alpha_2)\mathfrak{c}''$.*

(2.11) *There exist an element s of S, elements $\mathfrak{c}'$ and $\mathfrak{c}''$ of $C^{\sigma s}$ and an element β of $(\mathfrak{a}\gamma)^{\sigma s}$ with $(c'k)(\mathfrak{x}(\beta k))(c''k)$ for $k \in \sigma s$, $(\langle\mathfrak{C}\rangle s)c' = \mathfrak{c}'$ and $(\langle\mathfrak{C}\rangle s)c'' = \mathfrak{c}''$,*

(2.12) *There exist elements $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ of C, an element α of $\mathfrak{a}\gamma$ and a homomorphism h of $\mathfrak{C}$ into $\mathfrak{C}$ with $\mathfrak{c}_1'(\mathfrak{x}\alpha)\mathfrak{c}_1''$, $h\mathfrak{c}_1' = c'$ and $h\mathfrak{c}_1'' = c''$.*

Then

$$\mathfrak{x}\mu = (\Theta\mathfrak{C})r .$$

The proof is left to the reader.

Compare Lemma 2.4 with Definition 5 on p. 440 of (I).

Lemma 2.5. *Let $\mathfrak{c}_0$ be an element of C and m be a cardinal which is* $< (\pi\mathfrak{C})\mathfrak{c}_0$. (See (II), Definition 3.3) *Let r be the relation on C defined by the condition that, for $\mathfrak{c}' \in C$, $\mathfrak{c}'' \in C$, $\mathfrak{c}' r \mathfrak{c}''$ holds if and only if $|\mathfrak{S}[(P\mathfrak{C})\mathfrak{c}', (P\mathfrak{C})\mathfrak{c}'']| \leqq m$, and there exist homomorphisms h' and h'' of $\mathfrak{C}$ into $\mathfrak{C}$ with $h'\mathfrak{c}_0 = \mathfrak{c}'$ and $h''\mathfrak{c}_0 = \mathfrak{c}''$. Let $\mathfrak{c}_1$ be an element of C with $\mathfrak{c}_1((\Theta\mathfrak{C})r)\mathfrak{c}_0$. Then $\mathfrak{c}_1 = \mathfrak{c}_0$.*

Proof. If $\mathfrak{c}'$ and $\mathfrak{c}''$ are elements of C with $\mathfrak{c}' r \mathfrak{c}''$ then, by Lemma 2.2,

$$(\rho\mathfrak{C})\,\mathfrak{c}' \geqq (\rho\mathfrak{C})\,\mathfrak{c}_0$$

and

$$(\rho\mathfrak{C})\,\mathfrak{c}'' \geqq (\rho\mathfrak{C})\,\mathfrak{c}_0 .$$

Let r_1 be the relation on C defined by the condition that, for $\mathfrak{c}' \in C$, $\mathfrak{c}'' \in C$, $\mathfrak{c}' r_1 \mathfrak{c}''$ holds if and only if $\mathfrak{c}' = \mathfrak{c}''$, or $(\rho\mathfrak{C})\mathfrak{c}' \geqq (\rho\mathfrak{C})\mathfrak{c}_0$ and $(\rho\mathfrak{C})\mathfrak{c}'' \geqq (\rho\mathfrak{C})\mathfrak{c}_0$. Then

$$r_1 \supset r ,$$

and r_1 is a law relation on $\mathfrak{C}$ by Lemma 2. 3. Hence

$$r_1 \supset (\Theta\mathfrak{C})r .$$

Hence

(2. 13) if $\mathfrak{c}'$ and $\mathfrak{c}''$ are elements of C with $\mathfrak{c}'((\Theta\mathfrak{C})r)\mathfrak{c}''$ then $\mathfrak{c}' = \mathfrak{c}''$, or $(\rho\mathfrak{C})\mathfrak{c}' \geqq (\rho\mathfrak{C})\mathfrak{c}_0$ and $(\rho\mathfrak{C})\mathfrak{c}'' \geqq (\rho\mathfrak{C})\mathfrak{c}_0$.

Let us now define an ordinal μ and a $\mathfrak{b}\mu$-system $\mathfrak{r}$ of relations on C in the same way as in Lemma 2. 4. Then

$$\mathfrak{r}\mu = (\Theta\mathfrak{C})r .$$

I assert that the following proposition holds:

(2. 14) Let $\gamma \in \mathfrak{b}\{1, \mu\}$, h_0 be a homomorphism of $\mathfrak{C}$ into $\mathfrak{C}$, and $\mathfrak{c}'$ and $\mathfrak{c}''$ be elements of C such that $\mathfrak{c}' \neq \mathfrak{c}''$, $\mathfrak{c}'(\mathfrak{r}\gamma)\mathfrak{c}''$ and $h_0\mathfrak{c}'' = \mathfrak{c}_0$. Then there exist an element α of $\mathfrak{a}\gamma$, a homomorphism h_1 of $\mathfrak{C}$ into $\mathfrak{C}$ and elements $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ of C such that $\mathfrak{c}_1' \neq \mathfrak{c}_1''$, $\mathfrak{c}_1'(\mathfrak{r}\alpha)\mathfrak{c}_1''$ and $h_1\mathfrak{c}_1'' = \mathfrak{c}_0$.

To prove (2. 14), we first observe that

$$(\rho\mathfrak{C})\,\mathfrak{c}_0 \geqq (\rho\mathfrak{C})\,\mathfrak{c}''$$

by Lemma 2. 2. By (2. 13),

$$(\rho\mathfrak{C})\,\mathfrak{c}'' \geqq (\rho\mathfrak{C})\,\mathfrak{c}_0 .$$

Hence

$$(\rho\mathfrak{C})\,\mathfrak{c}'' = (\rho\mathfrak{C})\,\mathfrak{c}_0 . \tag{2. 15}$$

By the definition of r, and because $\mathfrak{c}' \neq \mathfrak{c}''$, at least one of the propositions (2. 7) and (2. 9) to (2. 12) holds. If (2. 7), (2. 9) or (2. 10) holds, (2. 14) is obvious. If (2. 11) holds then, by (2. 13),

$$(\rho\mathfrak{C})((\langle\mathfrak{C}\rangle s)c'') = \mathrm{Suc}\,((\rho\mathfrak{C})\cdot c'') > (\rho\mathfrak{C})\mathfrak{c}_0 ,$$

and

$$(\rho\mathfrak{C})\,\mathfrak{c}'' > (\rho\mathfrak{C})\,\mathfrak{c}_0 ,$$

contrary to (2. 15). Hence (2. 11) does not hold. If (2. 12) holds then $\mathfrak{c}_1' \neq \mathfrak{c}_1''$, and

$$(h_0 \cdot h)\mathfrak{c}_1'' = h_0(h\mathfrak{c}_1'') = h_0\mathfrak{c}'' = \mathfrak{c}_0 .$$

This completes the proof of (2. 14).

It is now obvious that the assertion of (2. 14) can be improved by adding that α can be made $= 0$. I. e., under the hypotheses of (2. 14), there exist a homomorphism h_1 of $\mathfrak{C}$ into $\mathfrak{C}$ and elements $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ of C such that $\mathfrak{c}_1' \neq \mathfrak{c}_1''$, $\mathfrak{c}_1' r \mathfrak{c}_1''$ and $h_1 \mathfrak{c}'' = \mathfrak{c}_0$.

Let us now assume that

$$\mathfrak{c}_1 \neq \mathfrak{c}_0 .$$

Then the hypotheses of (2. 14) are satisfied if we put $\gamma = \mu$, $\mathfrak{c}' = \mathfrak{c}_1$ and $\mathfrak{c}'' = \mathfrak{c}_0$ and let h_0 be the identical representation of C. Let h_1 be a homomorphism of $\mathfrak{C}$ into $\mathfrak{C}$ and $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C such that $\mathfrak{c}_1' \neq \mathfrak{c}_1''$, $\mathfrak{c}_1' r \mathfrak{c}_1''$ and $h_1 \mathfrak{c}_1'' = \mathfrak{c}_0$. Let h'' be a homomorphism of $\mathfrak{C}$ into $\mathfrak{C}$ with $h''\mathfrak{c}_0 = \mathfrak{c}_1''$. ($h''$ exists by the definition of r.) Then

$$(h_1 \cdot h'')\mathfrak{c}_0 = \mathfrak{c}_0 .$$

By (II), Theorem 3. 10,

$$(h_1 \cdot h'')\mathfrak{d} = \mathfrak{d} \qquad \text{for} \quad \mathfrak{d} \in (P\mathfrak{C})\mathfrak{c}_0 .$$

Hence

$$h''\mathfrak{d} \in D \qquad \text{for} \quad \mathfrak{d} \in (P\mathfrak{C})\mathfrak{c}_0 .$$

By (II), Theorems 3. 11 and 3. 1,

$$(P\mathfrak{C})(h''\mathfrak{c}_0) = [h''\mathfrak{d}\, ;\ \mathfrak{d} \in (P\mathfrak{C})\mathfrak{c}_0] .$$

Hence

$$(\pi\mathfrak{C})\mathfrak{c}_1'' = (\pi\mathfrak{C})\mathfrak{c}_0 .$$

Hence

$$(\pi\mathfrak{C})\mathfrak{c}_1'' > m . \tag{2. 16}$$

But by the definition of r,

$$|\mathfrak{S}[(P\mathfrak{C})\mathfrak{c}_1', (P\mathfrak{C})\mathfrak{c}_1'']| \leqq m .$$

Hence

$$(\pi\mathfrak{C})\mathfrak{c}_1'' \leqq m ,$$

contrary to (2. 16). Hence our assumption that $\mathfrak{c}_1 \neq \mathfrak{c}_0$ is wrong. Hence $\mathfrak{c}_1 = \mathfrak{c}_0$ as asserted.

Theorem 2. 5. $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ *if and only if*

$$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} . \tag{2. 17}$$

Proof. Let $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. We wish to prove that (2.17) holds. This is obvious if $|S|=0$ and $|D_1| \leqq 1$. Let $|S| \neq 0$ or $|D_1|>1$. If $|D_1| \leqq |D_2|$ then (2.17) holds by Theorem 1.20. Let $|D_2|<|D_1|$. By Definition 2.2,

$$l_\sigma \leqq |D_2|.$$

If $2m_\sigma \leqq |D_2|$ then (2.17) holds by Theorem 1.20. Let

$$|D_2|<2m_\sigma .$$

Then $l_\sigma<2m_\sigma$, and we have the case considered under (2.1). Hence

$$l_\sigma = m_\sigma = 1 ,$$

and

$$|D_2| = 1 .$$

Let s_0 be an element of S such that $|\sigma s_0|=0$. Let r be a law relation on $\mathfrak{C}_2$, and let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of C_1 with

$$\mathfrak{c}_1'((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r)\mathfrak{c}_1'' . \tag{2.18}$$

We wish to prove that

$$\mathfrak{c}_1'((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r)\mathfrak{c}_1'' \tag{2.19}$$

holds. By (II), Theorem 3.15, $(\pi\mathfrak{C}_1)\mathfrak{c}_1' \leqq 1$ and $(\pi\mathfrak{C}_1)\mathfrak{c}_1'' \leqq 1$. Hence

$$|\mathfrak{S}[(P\mathfrak{C}_1)\mathfrak{c}_1', (P\mathfrak{C}_1)\mathfrak{c}_1'']| \leqq 2 .$$

If $|\mathfrak{S}[(P\mathfrak{C}_1)\mathfrak{c}_1', (P\mathfrak{C}_1)\mathfrak{c}_1'']|<2$, (2.19) holds by Theorems 1.18 and 1.17. Let

$$|\mathfrak{S}[(P\mathfrak{C}_1)\mathfrak{c}_1', (P\mathfrak{C}_1)\mathfrak{c}_1'']| = 2 .$$

Then $(\pi\mathfrak{C}_1)\mathfrak{c}_1' = (\pi\mathfrak{C}_1)\mathfrak{c}_1'' = 1$. Let $\mathfrak{d}'$ be the only element of $(P\mathfrak{C}_1)\mathfrak{c}_1'$ and $\mathfrak{d}''$ be the only element of $(P\mathfrak{C}_1)\mathfrak{c}_1''$. Then $\mathfrak{d}' \neq \mathfrak{d}''$. Let h be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Let c_0 be the void system, h_0 be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_1$ such that

$$h_0\mathfrak{d}' = h_0\mathfrak{d}'' = (\langle\mathfrak{C}_1\rangle s_0)c_0 ,$$

and h' and h'' be homomorphisms of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that

$$h'\mathfrak{d}' = h\mathfrak{d}' , \quad h'\mathfrak{d}'' = (\langle\mathfrak{C}_2\rangle s_0)c_0 ,$$
$$h''\mathfrak{d}' = (\langle\mathfrak{C}_2\rangle s_0)c_0 , \quad h''\mathfrak{d}'' = h\mathfrak{d}'' .$$

By (II), Theorem 3.8,

$$h'\mathfrak{c}_1' = h\mathfrak{c}_1' \tag{2.20}$$

and

$$h''\mathfrak{c}_1'' = h\mathfrak{c}_1''. \tag{2.21}$$

Also

$$h(h_0\mathfrak{d}') = h(h_0\mathfrak{d}'') = h((\langle\mathfrak{C}_1\rangle s_0)\, c_0) = (\langle\mathfrak{C}_2\rangle s_0)\, c_0\,.$$

Hence

$$h(h_0\mathfrak{d}') = h''\mathfrak{d}'$$

and

$$h(h_0\mathfrak{d}'') = h'\mathfrak{d}''\,.$$

By (II), Theorem 3. 8,

(2. 22) $$h(h_0\mathfrak{c}_1') = h''\mathfrak{c}_1'$$

and

(2. 23) $$h(h_0\mathfrak{c}_1'') = h'\mathfrak{c}_1''\,.$$

By (2. 18),

(2. 24) $$(h'\mathfrak{c}_1')\, r\, (h'\mathfrak{c}_1'')$$

and

(2. 25) $$(h''\mathfrak{c}_1')\, r\, (h''\mathfrak{c}_1'')\,.$$

By (2. 20), (2. 23) and (2. 24),

$$(h\mathfrak{c}_1')\, r\, (h(h_0\mathfrak{c}_1''))\,.$$

By (2. 21), (2. 22) and (2. 25),

$$(h(h_0\mathfrak{c}_1'))\, r\, (h\mathfrak{c}_1'')\,.$$

Hence

(2. 26) $$\mathfrak{c}_1'((\Psi\{\mathfrak{C}_2,\, \mathfrak{C}_1\})r)(h_0\mathfrak{c}_1'')$$

and

(2. 27) $$(h_0\mathfrak{c}_1')((\Psi\{\mathfrak{C}_2,\, \mathfrak{C}_1\})r)\mathfrak{c}_1''\,.$$

Also,

(2. 28) $$(h_0\mathfrak{c}_1')((\Psi\{\mathfrak{C}_2,\, \mathfrak{C}_1\})r)(h_0\mathfrak{c}_1'')\,.$$

By (II), Theorems 3. 11 and 3. 2,

$$(P\mathfrak{C}_1)(h_0\mathfrak{c}_1') = (P\mathfrak{C}_1)(h_0\mathfrak{d}') = (P\mathfrak{C}_1)((\langle\mathfrak{C}_1\rangle s_0)c_0) = \mathfrak{S}((P\mathfrak{C}_1)\cdot c_0)\,.$$

Hence $(P\mathfrak{C}_1)(h_0\mathfrak{c}_1')$ is void. For a similar reason, $(P\mathfrak{C}_1)(h_0\mathfrak{c}_1'')$ is void. Hence (2. 26), (2. 27) and (2. 28) imply

(2. 29) $$\mathfrak{c}_1'((\psi\{\mathfrak{C}_2,\, \mathfrak{C}_1\})r)(h_0\mathfrak{c}_1'')\,,$$

(2. 30) $$(h_0\mathfrak{c}_1')((\psi\{\mathfrak{C}_2,\, \mathfrak{C}_1\})r)\mathfrak{c}_1''\,,$$

and

(2. 31) $$(h_0\mathfrak{c}_1')((\psi\{\mathfrak{C}_2,\, \mathfrak{C}_1\})r)(h_0c_1'')\,.$$

(2.29), (2.30) and (2.31) imply (2.19). Hence

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r \subset (\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r .$$

By Theorem 1.13,

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r = (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r .$$

Hence (2.17) holds.

Let us now start from the hypothesis that $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ does not hold. Hence

$$|S| \neq 0 \quad \text{or} \quad |D_1| > 1$$
$$|D_2| < |D_1| ,$$

and

$$|D_2| < l_\sigma .$$

Also,

$$|C_1| \geqq 2 ;$$

for if $|D_1| = 1$ then $|S| \neq 0$. We wish to prove that (2.17) does not hold. This follows from Theorem 1.14 if C_2 is void. Therefore let us assume that C_2 is not void.

Let us, firstly, treat the case that

$$|\sigma s| = 1 \qquad \text{for} \quad s \in S .$$

Then $m_\sigma = 1$, $l_\sigma = 2$, and

$$|D_2| < 2 .$$

If $|D_2| = 0$ then $|C_2| = 0$, contrary to hypothesis. Hence

$$|D_2| = 1$$

and

$$|D_1| > 1 .$$

By (II), Theorems 3.12 and 3.15,

$$(\pi\mathfrak{C}_1)\mathfrak{c}_1 = 1 \qquad \text{for} \quad \mathfrak{c}_1 \in C_1 ,$$

and

$$(\pi\mathfrak{C}_2)\mathfrak{c}_2 = 1 \qquad \text{for} \quad \mathfrak{c}_2 \in C_2 .$$

Let r_2 be the all relation on C_2, and let

$$r_1 = R(P\mathfrak{C}_1) .$$

Then, for $\mathfrak{c}_1' \in C_1$, $\mathfrak{c}_1'' \in C_1$, the following propositions (2.32) to (2.36) are equivalent:

(2.32) $$\mathfrak{c}_1' r_1 \mathfrak{c}_1'' .$$

(2.33) $$(P\mathfrak{C}_1)\mathfrak{c}_1' = (P\mathfrak{C}_1)\mathfrak{c}_1''.$$

(2.34) $$|\mathfrak{S}[(P\mathfrak{C}_1)\mathfrak{c}_1', (P\mathfrak{C}_1)\mathfrak{c}_1'']| \leqq 1.$$

(2.35) There exist elements $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ of C_2 with

$$\{\mathfrak{c}_2', \mathfrak{c}_2''\}(\{\mathfrak{C}_2, \mathfrak{C}_1\}\text{-conf})\{\mathfrak{c}_1', \mathfrak{c}_1''\}.$$

(2.36) $$\mathfrak{c}_1'((\psi_0\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2)\mathfrak{c}_1''.$$

Hence

$$(\psi_0\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2 = r_1.$$

By (III), Satz 2.8, r_1 is a law relation on $\mathfrak{C}_1$. Hence

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2 = r_1.$$

Hence $(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2$ is not the all relation on C_1. On the other hand, it is obvious that $(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2$ is the all relation on C_1. Hence (2.17) does not hold.

Let us, secondly, treat the case that

$$|\sigma s| \neq 1 \quad \text{for some element } s \text{ of } S.$$

Then

$$l_\sigma = m_\sigma.$$

By (II), Theorem 3.19,

$$|D_2| < \sup(\pi\mathfrak{C}_1).$$

Let $\mathfrak{c}_0$ be an element of C_1 with

$$(\pi\mathfrak{C}_1)\mathfrak{c}_0 > |D_2|.$$

Let a be an element of $C_1^{P_1}$ such that

$$a\mathfrak{d} \neq \mathfrak{d} \quad \text{for some element } \mathfrak{d} \text{ of } (P\mathfrak{C}_1)\mathfrak{c}_0.$$

Put

$$(\varphi\{\mathfrak{C}_1, \mathfrak{C}_1\})a = h.$$

Then

$$h\mathfrak{d} \neq \mathfrak{d} \quad \text{for some element } \mathfrak{d} \text{ of } (P\mathfrak{C}_1)\mathfrak{c}_0.$$

Hence

$$h\mathfrak{c}_0 \neq \mathfrak{c}_0.$$

Define a relation r on C_1 by requiring that, for $\mathfrak{c}_1' \in C_1$, $\mathfrak{c}_1'' \in C_1$, $\mathfrak{c}_1' r \mathfrak{c}_1''$ holds if and only if

$$|\mathfrak{S}[(P\mathfrak{C}_1)\mathfrak{c}_1', (P\mathfrak{C}_1)\mathfrak{c}_1'']| \leqq |D_2|,$$

and there exist homomorphisms h' and h'' of $\mathfrak{C}_1$ into $\mathfrak{C}_1$ with $h'\mathfrak{c}_0 = \mathfrak{c}_1'$ and $h''\mathfrak{c}_0 = \mathfrak{c}_1''$. Then, by Lemma 2.1,

$$(hc_0)((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)r)))\,c_0 .$$

By Lemma 2. 5, $(hc_0)((\Theta\mathfrak{C}_1)r)c_0$ does not hold. By Theorem 1. 9,

$$(hc_0)((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)r)))\,c_0$$

does not hold. Hence (2. 17) does not hold.

This completes the proof of Theorem 2. 5.

§ 3. Further theorems on the operators $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ and $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$.

Theorem 3. 1. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Then*

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Phi\mathfrak{C}_2)M) = (\Phi\mathfrak{C}_1)M = (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Phi\mathfrak{C}_2)M) .$$

Proof. See Theorems 1. 3, 1. 15 and 2. 5.

Theorem 3. 2. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Let* $\mathfrak{M}_1$ *as well as* $\mathfrak{M}_2$ *be an algebra or a set of algebras. Let*

$$(\Phi\mathfrak{C}_2)\mathfrak{M}_1 = (\Phi\mathfrak{C}_2)\mathfrak{M}_2 .$$

Then

$$(\Phi\mathfrak{C}_1)\mathfrak{M}_1 = (\Phi\mathfrak{C}_1)\mathfrak{M}_2 .$$

Proof. See Theorem 3. 1.

Theorem 3. 2 represents an improvement of Satz 3. 3 of (III).

Theorem 3. 3. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Then each of* $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$, $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$, $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *and* $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is the identical representation of* $L\mathfrak{C}_1$.

Proof. Let $r \in L\mathfrak{C}_1$. Then, putting $\mathfrak{M} = [\mathfrak{C}_1/r]$ in Theorem 3. 1, we find that

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) = r = (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) .$$

Hence, by Theorems 1. 2 and 1. 13,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset r .$$

By Theorem 1. 5,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) = r .$$

By Theorem 2. 5,

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) = r .$$

Theorem 3. 4.

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$$

and

$$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_2, \mathfrak{C}_1\}\cdot\psi\{\mathfrak{C}_1, \mathfrak{C}_2\} .$$

Proof. For the case that $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$, Theorem 3.4 follows from Theorem 3.3. For the case that $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$, it follows from Theorem 2.5.

Theorem 3.5. *Let* $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *be the identical representation of* $L\mathfrak{C}_1$. *Then* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$.

Proof. Let $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$. Then, by Theorem 3.3, $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \cdot \psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ is the identical representation of $L\mathfrak{C}_2$. Hence

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \cdot \psi\{\mathfrak{C}_2, \mathfrak{C}_1\}) = \psi\{\mathfrak{C}_2, \mathfrak{C}_1\} .$$

By Theorem 2.5, $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$.

Theorem 3.6. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Let* $r_\nu \in L\mathfrak{C}_\nu$ *for* $\nu = 1, 2$. *Then*

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2 = r_1 \tag{3.1}$$

if and only if

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1 \subset r_2 \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1 . \tag{3.2}$$

Proof. (3.1) implies (3.2) by Theorems 1.4 and 1.9. If (3.2) holds then

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1) \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r_2 \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1)$$

by Theorem 1.2, and (3.1) holds by Theorem 3.3.

Theorem 3.7. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Let* r_1 *and* r_2 *be law relations on* C_1. *Then the following propositions* (3.3) *to* (3.5) *are equivalent*:

$$r_1 \subset r_2 . \tag{3.3}$$

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1 \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_2 . \tag{3.4}$$

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_1 \subset (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r_2 . \tag{3.5}$$

Theorem 3.7 is obvious from Theorems 1.2, 1.12 and 3.3.

Theorem 3.8. *Let*

$$\chi_1 \in [\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}, \ \psi\{\mathfrak{C}_2, \mathfrak{C}_1\}]$$

and

$$\chi_2 \in [\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}, \ \psi\{\mathfrak{C}_1, \mathfrak{C}_2\}] .$$

Then the following propositions (3.6) *to* (3.8) *are equivalent*:

$$\chi_2 \text{ is simple.} \tag{3.6}$$

$$\chi_1 \text{ takes all values of } L\mathfrak{C}_1. \tag{3.7}$$

$$\chi_1 \cdot \chi_2 \text{ is the identical representation of } L\mathfrak{C}_1 . \tag{3.8}$$

Proof. By Theorem 3.3, $\chi_1 \cdot \chi_2$ is the identical representation of $L\mathfrak{C}_1$, or $\chi_2 \cdot \chi_1$ is the identical representation of $L\mathfrak{C}_2$. In the latter case, it is obvious that (3.6) as well as (3.7) implies (3.8). Conversely, (3.8) implies both (3.6) and (3.7).

Theorem 3.9. *The following propositions* (3.9) *to* (3.22) *are equivalent*:

(3.9) $$\mathfrak{C}_1 \sqsubset \mathfrak{C}_2 .$$

(3.10) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} .$$

(3.11) $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is simple.*

(3.12) $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is simple.*

(3.13) $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *takes all values of* $L\mathfrak{C}_1$.

(3.14) $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *takes all values of* $L\mathfrak{C}_1$.

(3.15) $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is the identical representation of* $L\mathfrak{C}_1$.

(3.16) $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is the identical representation of* $L\mathfrak{C}_1$.

(3.17) $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is the identical representation of* $L\mathfrak{C}_1$.

(3.18) $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is the identical representation of* $L\mathfrak{C}_1$.

(3.19) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} .$$

(3.20) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} .$$

(3.21) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} .$$

(3.22) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} .$$

Proof. (3.11) to (3.18) are equivalent by Theorem 3.8. (3.9) is equivalent to (3.10) by Theorem 2.5 and equivalent to (3.15) by Theorems 3.3 and 3.5. (3.19) to (3.22) are equivalent by Theorem 3.4. (3.9) implies (3.19) by Theorem 2.5. (3.19) implies (3.15) by Theorems 1.4 and 1.9.

Theorem 3.10. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Then* $|L\mathfrak{C}_1| \leqq |L\mathfrak{C}_2|$.

Proof. By Theorem 3.9, $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is simple. Hence $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is a one-to-one representation of $L\mathfrak{C}_1$ onto a sub-set of $L\mathfrak{C}_2$.

Theorem 3.11. *Let* $\mathfrak{C}_1 \sqsupset\sqsubset \mathfrak{C}_2$. *Then* $|L\mathfrak{C}_1| = |L\mathfrak{C}_2|$.

It follows that, in particular, $|L\mathfrak{C}_1| = |L\mathfrak{C}_2|$ if $|D_1| \geqq l_\sigma$ and $|D_2| \geqq l_\sigma$.

DEFINITION 3.1. *By* w_σ *is meant* $|L\mathfrak{C}|$ *if* $|D| = l_\sigma$. (See (IV), Theorem 11.)

Theorem 3.12. $w_\sigma \geqq 2$.

Proof. Let $|D| = l_\sigma$. Then, by Theorem 2.1, $|C| \geqq 2$. Hence the equality relation on C is different from the all relation on C. Hence $|L\mathfrak{C}| \geqq 2$.

Theorem 3.13. $|L\mathfrak{C}| \leqq w_\sigma$.

Theorem 3.14. *Let* $|D| \geqq l_\sigma$. *Then* $|L\mathfrak{C}| = w_\sigma$.

Theorem 3.15. *The relation* $\subset$ *on* $L\mathfrak{C}$ *determines a* w_σ*-lattice-algebra on* $L\mathfrak{C}$. (See (III), p. 133, line 5 from bottom.)

Proof. See (III), Satz 2.9.

DEFINITION 3.2. *By* $\mathfrak{L}\mathfrak{C}$ *is meant the* w_σ*-lattice-algebra on* $L\mathfrak{C}$ *determined by the relation* $\subset$ *on* $L\mathfrak{C}$.

Theorem 3.16. *Let* $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. *Then* $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *is a homomorphism of* $\mathfrak{L}\mathfrak{C}_2$ *onto* $\mathfrak{L}\mathfrak{C}_1$.

Proof. By Theorem 3.9, $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ takes all values of $L\mathfrak{C}_1$. Let $b \in (L\mathfrak{C}_2)^{\{\{1, \Omega w_\sigma\}}$. Then, by (III), Satz 3.10,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\langle\mathfrak{L}\mathfrak{C}_2\rangle 1)b) = (\langle\mathfrak{L}\mathfrak{C}_1\rangle 1)(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot b) .$$

Let r be a law relation on $\mathfrak{C}_1$ such that

$$r \supset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})(b\alpha) \qquad \text{for} \quad \alpha \in \mathfrak{b}\{1, \Omega_{w_\sigma}\} .$$

Then, by Theorems 2.5 and 1.8,

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r \supset b\alpha \qquad \text{for} \quad \alpha \in \mathfrak{b}\{1, \Omega_{w_\sigma}\} .$$

Hence

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r \supset (\langle L\mathfrak{C}_2\rangle 2)b ,$$

and

$$r \supset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\langle L\mathfrak{C}_2\rangle 2)b) .$$

Hence

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\langle L\mathfrak{C}_2\rangle 2)b) = (\langle L\mathfrak{C}_1\rangle 2)(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot b) .$$

Theorem 3.17. *The following propositions* (3.23) *to* (3.32) *are equivalent*:

(3.23) $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$.

(3.24) $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *and* $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$.

(3.25) $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is simple and takes all values of* $L\mathfrak{C}_2$.

(3.26) $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *is simple and takes all values of* $L\mathfrak{C}_2$.

(3.27) $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *is simple and takes all values of* $L\mathfrak{C}_1$.

(3.28) $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *is simple and takes all values of* $L\mathfrak{C}_1$.

(3.29) $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *and* $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *are inverse to each other.*

(3.30) $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *and* $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *are inverse to each other.*

(3.31) $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *and* $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *are inverse to each other.*

(3.32) $\psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ *and* $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *are inverse to each other.*

Theorem 3.17 is obvious from Theorem 3.9.

Theorem 3.18. *Let* $\mathfrak{C}_1 \sqsupset \mathfrak{C}_2$. *Then* $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ *is an isomorphism of* $L\mathfrak{C}_2$ *onto* $L\mathfrak{C}_1$.

Proof. By Theorem 3.17, $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ is simple. See Theorem 3.16.

Department of Mathematics,
University of Alberta.

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