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ON THE MIXED PROBLEMS FOR THE WAVE EQUATION IN AN INTERIOR DOMAIN. II

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1. Introduction. Let Γ be a simple closed curve in $\boldsymbol{R}^2=\{(x_1, x_2);\ x_j\in\boldsymbol{R},\ j=1, 2\}$ and Ω be its interior domain. Consider a mixed problem

$$
\text{(P)}\qquad \begin{cases} \Box u = \dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x_1^2}-\dfrac{\partial^2 u}{\partial x_2^2} = 0 & \text{in } \Omega\times(0, \infty) \\ Bu = b_1(x)\dfrac{\partial u}{\partial x_1}+b_2(x)\dfrac{\partial u}{\partial x_2}+d(x)u(x) = 0 & \text{on } \Gamma\times(0, \infty) \\ u(x, 0) = u_0(x) & \\ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x)\,, & \end{cases}
$$

where $b_j(x)$, $j=1, 2$ and $d(x)$ are C^∞-functions defined in a neighborhood of Γ. We suppose that $b_j(x)$, $j=1, 2$, are real valued and satisfy

$$
(1.1)\qquad b_1(x)n_1(x)+b_2(x)n_2(x) = 1 \qquad \text{on } \Gamma
$$

where $n(x)=(n_1(x), n_2(x))$ denotes the unit inner normal of Γ at x.

Let $x(s)$, $0\leqslant s\leqslant L$ be a representation of Γ by the arc length s. Set

$$
\tau(s) = [b_1(x)n_2(x)-b_2(x)n_1(x)]_{x=x(s)}\,.
$$

The result we want to show is the following

Theorem. *Suppose that the curvature of Γ never vanishes. In the case of $\tau(s)\not\equiv 0$ in order that (P) is well posed in the sense of C^∞ it must holds that*

$$
(1.2)\qquad |\tau(s)|+\left|\frac{d\tau(s)}{ds}\right|\neq 0 \qquad \textit{for all } s\,.
$$

We should like to give some remarks on the theorem. If $\tau(s)\equiv 0$ the boundary condition is nothing but the Neumann condition or the boundary condition of the third kind. Then it is well known that (P) is well posed in the sense of L^2. And when $\tau(s)\neq 0$ for all s the mixed problem (P) is also well posed in the sense of C^∞, that is shown in [1]. In both cases the results are

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still valid without the assumption of the convexity of Ω.

In the preceding paper [5] we gave a necessary condition for the well posedness of (P). There we introduced an index $I_B(p_0, \xi_0: T)$ of a broken ray according to the geometrical optics with respect to the coefficients of the boundary operator and it is proved that the condition

$$I_B(p_0, \xi_0: T) < C_T, \quad \forall p_0 = (x_0, t_0) \in \Gamma \times (0, T), \ \xi_0 \in \Sigma_{x_0}$$

is necessary for the well posedness. It is easy to verify that the supposition

$$\sup_{p_0, \xi_0} I_B(p_0, \xi_0: T) = \infty$$

implies that $\tau(s) \not\equiv 0$ and $\tau(s)$ has at least a zero of infinite order. Therefore the theorem of this paper is an improvement of the result of [5].

2. Asymptotic solutions with a caustic

From now on, we suppose that the curvature of Γ never vanishes. Then there exist functions $\theta(x, \alpha)$ and $\rho(x, \alpha)$ with the following properties:[1)]

(i) θ and ρ are real valued C^∞ function defined in $\{(x, \alpha); x \in \boldsymbol{R}^2, \alpha \in [-\alpha_0, \alpha_0]\}$ where α_0 is a positive constant.

(ii) $$\frac{\partial \rho}{\partial n} \geqslant c > 0^{2)} \qquad \text{for } x \in \Gamma$$

where $\dfrac{\partial}{\partial n} = \sum_{j=1}^{2} n_j(x) \dfrac{\partial}{\partial x_j}$.

(iii) Let us set

$$\Gamma_\alpha = \{x; \rho(x, \alpha) = \alpha\}$$
$$\omega_\alpha = \{x; \rho(x, \alpha) > 0\} .$$

Then for all α it holds that

$$\text{(2.1)} \qquad \begin{cases} (\nabla\theta)^2 + \rho(\nabla\rho)^2 = 1 & \text{in } \bar{\omega}_\alpha \\ \nabla\theta \cdot \nabla\rho = 0 & \text{in } \bar{\omega}_\alpha \end{cases}$$

and

$$\text{(2.2)} \qquad \rho(x, \alpha) \equiv \alpha \ (\bmod \alpha^\infty) \qquad \text{on } \Gamma .$$

For $u(x, t) \in C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R})$ we set

$$||u||_{(\alpha), a, b} = \sum_{\substack{p+r \leqslant a \\ q \leqslant b}} \sup_{\tilde{\Omega} \times \boldsymbol{R}} |\partial_t^r \partial_\theta^p \partial_\rho^q u(x, t)|$$

1) See, for example, Appendix C of Ludwig [7], § 5 of Ikawa [4].

2) Hereafter, we will use c for various constants independent of α and k.

$$\langle u\rangle_{(\alpha),a}=\sum_{p+q\leqslant a}\sup_{\Gamma_\alpha\times\boldsymbol{R}}|\partial_t^q\partial_\theta^p u(x,t)|,$$

where $\tilde{\Omega}$ is a bounded open set in $\boldsymbol{R}^2$ containg $\bar{\Omega}$ and

$$\partial_t^r=\frac{\partial^r}{\partial t^r},\quad \partial_\theta^p=\Big(\sum_{j=1}^2\frac{\partial\theta}{\partial x_j}\frac{\partial}{\partial x_j}\Big)^p\quad\text{and}\quad \partial_\rho^q=\Big(\sum_{j=1}^2\frac{\partial\rho}{\partial x_j}\frac{\partial}{\partial x_j}\Big)^q.$$

Let us denote

$$|u|_{\Omega,a}=\sum_{|\beta|\leqslant a}\sup_{\Omega\times\boldsymbol{R}}|D_{x,t}^\beta u(x,t)|$$
$$|u|_{\Gamma,a}=\sum_{p+q\leqslant a}\sup_{[0,1]\times\boldsymbol{R}}|\partial_s^p\partial_t^q u(x(s),t)|.$$

Taking account of

$$\left|\frac{D(\theta,\rho)}{D(x_1,x_2)}\right|\geqslant c>0\qquad\text{for all }\alpha$$

it holds that for all $u\in C^\infty(\boldsymbol{R}^2\times\boldsymbol{R}^1)$ and α

$$|u|_{\Omega,2a}\leqslant C_a\|u\|_{(\alpha),a,a}\tag{2.3}$$

where C_a is independent of α.

Define

$$\varphi^\pm(x,\alpha)=\theta(x,\alpha)\pm2/3\rho(x,\alpha)^{3/2}.$$

Let $v(x,t)\in C_0^\infty(\Gamma_\alpha\times\boldsymbol{R})$ and set for $\alpha>0$

$$m(x,t;\alpha,k)=e^{ik(\varphi^-(x,\alpha)-t)}v(x,t)$$

We construct a function $u(x,t;\alpha,k)$ in the form

$$u(x,t;\alpha,k)=e^{ik(\theta(x,\alpha)-t)}\Big\{V(k^{2/3}\rho(x,\alpha))g_0(x,t;\alpha,k)+\frac{1}{ik^{1/3}}V'(k^{2/3}\rho(x,\alpha))g_1(x,t;\alpha,k)\Big\}\tag{2.4}$$

so that it may verify

$$\begin{cases}\Box u=0 & \text{in }\Omega\times\boldsymbol{R}\\ Bu|_{\Gamma_\alpha}=m(x,t;\alpha,k) & \text{on the support of }v\end{cases}\tag{2.5}$$

asymptotically as $k\to\infty$, where $V(z)=Ai(-z)$ with the Airy function $Ai(z)$. Apply $\Box$ for $u(x,t;\alpha,k)$ of (2.4) and use $V''(z)+zV(z)=0$, $V'''(z)+zV'(z)+V(z)=0$. Then we have

$$(2.6)\qquad \Box u = -e^{ik(\theta-t)}\Big[V(k^{2/3}\rho)\Big\{(ik)^2((\nabla\theta)^2+\rho(\nabla\rho)^2-1)g_0 + 2(ik)^2\rho\nabla\rho\cdot\nabla\theta g_1 + ik\Big(2\frac{\partial g_0}{\partial t}+2\nabla\theta\cdot\nabla g_0+\Delta\theta\cdot g_0 + 2\rho\nabla\rho\cdot\nabla g_1+(\nabla\rho)^2 g_1+\rho\Delta\rho\cdot g_1\Big)-\Box g_0\Big\}$$
$$+\frac{1}{ik^{1/3}}V'(k^{2/3}\rho)\Big\{(ik)^2((\nabla\theta)^2+\rho(\nabla\rho)^2-1)g_1+2(ik)^2\nabla\theta\cdot\nabla\rho\cdot g_0 + ik\Big(2\frac{\partial g_1}{\partial t}+2\nabla\theta\cdot\nabla g_1+\Delta\theta g_1+2\nabla\rho\cdot\nabla g_0+\Delta\rho g_0\Big)-\Box g_1\Big\}\Big].$$

Note that $V(z)$ and $V'(z)$ have the following asymptotic expansions for $z\to+\infty$

$$V(z)=\frac{1}{2}\pi^{-1/2}z^{-1/4}\{e^{i(\xi-\pi/4)}(1+\xi^{-1}P_1(\xi))+e^{-i(\xi-\pi/4)}(1+\xi^{-1}P_2(\xi))\}$$
$$V'(z)=\frac{1}{2}i\pi^{-1/2}z^{1/4}\{e^{i(\xi-\pi/4)}(1+\xi^{-1}P_3(\xi))-e^{-i(\xi-\pi/4)}(1+\xi^{-1}P_4(\xi))\},$$

where $\xi=\frac{2}{3}z^{3/2}$ and

$$P_j(\xi)\sim\sum_{l=0}^{\infty}p_{jl}\xi^{-l},\qquad p_{jl}\in C.^{3)}$$

Therefore the function u in the form (2.4) may be represented for large $k^{2/3}\rho$ as follows

$$(2.7)\qquad u(x,t;\alpha,k)=e^{ik(\varphi^+-t)}\Big(G^++\frac{1}{ik}\tilde G^+\Big)+e^{ik(\varphi^--t)}\Big(G^-+\frac{1}{ik}\tilde G^-\Big) = u^++u^-$$

where

$$G^\pm=\frac{1}{2\sqrt{\pi}}\rho^{-1/4}k^{-1/6}e^{\mp\pi i/4}(g_0\pm\sqrt{\rho}\,g_1)$$
$$\tilde G^+=\frac{3}{4}\pi^{-1/2}k^{-1/6}\rho^{-7/4}e^{-\pi i/4}(P_1g_0+\sqrt{\rho}\,P_3g_1)$$
$$\tilde G^-=\frac{3}{4}\pi^{-1/2}k^{-1/6}\rho^{-7/4}e^{\pi i/4}(P_2g_0-\sqrt{\rho}\,P_4g_1).$$

From the form of $\tilde G^\pm$ it holds that

$$(2.8)\qquad |\partial_\theta^a\partial_\rho\tilde G^\pm|\leqslant C_ak^{-1/6}\sum_{l=0}^{1}\{\rho^{-7/4}\|g_l\|_{(\infty),a,1}+\rho^{-11/4}\|g_l\|_{(\infty),a,0}\}$$

3) See Miller [8], page B 17.

when $k^{2/3}\rho > C$.

Applying the operator B to u of (2.7) we have

$$Bu = e^{ik(\varphi^+ - t)}\left\{ik\Phi^+\left(G^+ + \frac{1}{ik}\tilde{G}^+\right) + BG^+ + \frac{1}{ik}B\tilde{G}^+\right\} + e^{ik(\varphi^- - t)}\left\{ik\Phi^-\left(G^- + \frac{1}{ik}\tilde{G}^-\right) + BG^- + \frac{1}{ik}B\tilde{G}^-\right\}, \tag{2.9}$$

where $\Phi^\pm = \sum_{j=1}^{2} b_j(x)\dfrac{\partial \varphi^\pm}{\partial x_j}$.

Suppose that g_0 and g_1 have the following asymptotic expansion with respect to k^{-1} when $k \to \infty$

$$g_l(x, t; \alpha, k) \sim \sum_{j=0}^{\infty} g_{lj}(x, t; \alpha, k) k^{1/6-1-j}, \qquad l = 0, 1. \tag{2.10}$$

Denote by $\mathcal{L}_\alpha$ a differential operator from $(C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R}))^2$ into itself defined by for $\{a_1, a_2\}$

$$\mathcal{L}_\alpha\{a_1, a_2\} = \left\{2\frac{\partial a_1}{\partial t} + 2\nabla\theta\cdot\nabla a_1 + \Delta\theta a_1 + 2\rho\nabla\rho\cdot\nabla a_2 + (\nabla\rho)^2 a_2 + \rho\Delta\rho a_2,\ 2\frac{\partial a_2}{\partial t} + 2\nabla\theta\cdot\nabla a_2 + \Delta\theta a_2 + 2\nabla\rho\cdot\nabla a_1 + \Delta\rho a_1\right\}.$$

Substituting g_0, g_1 of (2.10) into (2.6) and (2.9) we claim that all the coefficients of k^{-j} of (2.6) are equal to zero and those of Bu-m are also equal to zero on the support of v. Then it must hold that

$$\mathcal{L}_\alpha\{g_{00}, g_{10}\} = 0 \tag*{(2.11)$_0$}$$

$$i\Phi^-(g_{00} - \sqrt{\rho}\, g_{10}) = 2\pi\alpha^{1/4}e^{\pi i/4}v \qquad \text{on } \Gamma_\alpha \times \boldsymbol{R} \tag*{(2.12)$_0$}$$

and for $j \geq 1$

$$\mathcal{L}_\alpha\{g_{0j}, g_{1j}\} = \frac{1}{i}\{\square g_{0j-1}, \square g_{1j-1}\} \tag*{(2.11)$_j$}$$

$$i\Phi^-(g_{0j} - \sqrt{\rho}\, g_{1j}) = i\Phi^-\tilde{G}^-_{j-1} + BG^-_{j-1} + \frac{1}{ik}B\tilde{G}^-_{j-1} \qquad \text{on } \Gamma_\alpha \times \boldsymbol{R} \tag*{(2.12)$_j$}$$

where $G^\pm_j$ and $\tilde{G}^\pm_j$ denote the $G^\pm$ and $\tilde{G}^\pm$ corresponding to the pair of $k^{1/6}g_{0j}$ and $k^{1/6}g_{1j}$.

To obtain the existence and the estimates of g_{0j}, g_{1j} satisfying (2.11) and (2.12), admit the following Lemma, whose proof will be given in the appendix.

Lemma 2.1. *For* $\{h_0, h_1\} \in (C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R}))^2$ *and* $f \in C^\infty(\Gamma_\alpha \times \boldsymbol{R})$ *there exists* $\{a_1, a_2\} \in (C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R}))^2$ *satisfying*

$$\begin{cases} \mathcal{L}_\alpha\{a_1, a_2\} = \{h_0, h_1\} & \text{in } \bar{\omega}_\alpha \times \boldsymbol{R} \\ a_1 - \sqrt{\rho}\, a_2 = f & \text{on } \Gamma_\alpha \times \boldsymbol{R} \end{cases}$$

and having the following properties:

(i) $\quad ||a_j||_{(\alpha),a,b} \leqslant C_{a,b}\{\langle f\rangle_{(\alpha),a+2b+j} + \sum_{l=0}^{1}\sum_{q=0}^{b}||h_l||_{(\alpha),a+2(b-q),q}\}$

(ii) $\quad$ *When* $\bigcup_{l=0,1} \operatorname{supp} h_l \cap \omega_\alpha \subset \{L_\alpha^-(x, t);\ (x, t) \in \operatorname{supp} f\}$, *it holds that*

$$\bigcup_{l=0}^{1} \operatorname{supp} a_l \cap \bar{\omega}_\alpha \subset \{L_\alpha^-(x, t);\ (x, t) \in \operatorname{supp} f\}\ ,$$

(iii) $\quad$ *When* $\{h_0, h_1\} \equiv 0$, *for* $(x, t) \in \Gamma_\alpha \times \boldsymbol{R}$

$$(a_1 + \sqrt{\rho}\, a_2)(x, t) = \gamma(x, t;\ \alpha) f(P_\alpha(x, t))$$

where $\gamma(x, t;\ \alpha)$ *is a* C^∞ *function on* $\boldsymbol{R}^2 \times \boldsymbol{R} \times [-\alpha_0, \alpha_0]$ *such that*

$$\gamma(x, t;\ \alpha) \geqslant C > 0$$

and $P_\alpha(x, t)$ *denotes the point*

$$L_\alpha^+(x, t) \cap (\Gamma_\alpha \times \boldsymbol{R}) - \{(x, t)\}\ ,$$

where $L_\alpha^\pm(x, t)$ *denotes a line passing* (x, t) *defined by*

$$L^\pm(x, t) = \{(x + l\nabla\varphi^\pm(x, \alpha), t + l);\ l \in \boldsymbol{R}\}\ .$$

Let Λ_0 be an open set in $\Gamma_\alpha \times \boldsymbol{R}$ such that $\Lambda_0 \supset \operatorname{supp} v$. Set

$$\Lambda_1 = \{L_\alpha^-(x, t) \cap (\Gamma_\alpha \times \boldsymbol{R}) - \{(x, t)\}\,;\ (x, t) \in \Lambda_0\}\ .$$

Suppose that

(2.13) $$\Lambda_0 \cap \Lambda_1 = \phi\ .$$

Let us set

$$\beta = \inf_{(x,t)\in\Lambda_0} |\Phi^-|\,.$$

Using the above lemma we have g_{00} and g_{10} verifying

$$\begin{cases} \mathcal{L}_\alpha\{g_{00}, g_{10}\} = 0 & \text{in } \bar{\omega}_\alpha \times \boldsymbol{R} \\ g_{00} - \sqrt{\rho}\, g_{10} = \dfrac{2\pi\alpha^{1/4} e^{\pi i/4} v}{i\Phi^-} & \text{on } \Gamma_\alpha \times \boldsymbol{R} \end{cases}$$

and the estimate

$$\sum_{l=0}^{1} ||g_{l0}||_{(\alpha),a,b} \leqslant C_{a,b} \left\langle \frac{2\pi\alpha^{1/4} e^{\pi i/4} v}{i\Phi^-} \right\rangle_{(\alpha),a+2b+1}\,.$$

Taking account of $\langle\Phi^-\rangle_{(\alpha),a} \leqslant C_a$ for all $\alpha > 0$, we have

$$\langle(\Phi^-)^{-1}\rangle_{(\alpha),a}\leqslant C_a\beta^{-(a+1)}.$$

Then it holds that

$$\sum_{l=0}^{1}||g_{l0}||_{(\alpha),a,b}\leqslant C\alpha^{1/4}\sum_{p+l\leqslant a+2b+1}\langle v\rangle_{(\alpha),l}\langle(\Phi^-)^{-1}\rangle_{(\alpha),p} \tag{2.14}$$
$$\leqslant C_{a,b}\alpha^{1/4}\sum_{p+l\leqslant a+2b+1}\langle v\rangle_{(\alpha),l}\beta^{-(p+1)}.$$

Let us set

$$E_\alpha(v,\beta;j)=\sum_{p+l\leqslant 0}\langle v\rangle_{(\alpha),l}\beta^{-(p+1)}.$$

Remark that the constant $C_{a,b}$ depends on a and b but independent of α.

Next consider g_{01} and g_{11}. Applying (2.8) to $k^{1/6}g_{l0}$ and using (2.14) we have

$$|\partial_\theta^a\partial_\rho\tilde{G}_0^\pm|\leqslant C_a\{\rho^{-7/4}\alpha^{1/4}E_\alpha(v,\beta;a+3)+\rho^{-11/4}\alpha^{1/4}E_\alpha(v,\beta;a+1)\}$$

for $\rho k^{2/3}>C$. Then, noting (2.2), it follows that

$$\left\langle\Phi^-\tilde{G}_0^-+BG_0^-+\frac{1}{ik}B\tilde{G}_0^-\right\rangle_{(\alpha),a}\leqslant C_a\alpha^{-5/2}E_\alpha(v,\beta;a+3).$$

Therefore

$$\left\langle(\Phi^-\tilde{G}_0^-+BG_0^-+\frac{1}{ik}B\tilde{G}_0^-)(\Phi^-)^{-1}\right\rangle_{(\alpha),a} \tag{2.15}$$
$$\leqslant C'_a\sum_{l+p\leqslant a}\alpha^{-5/2}E_\alpha(v,\beta;l+3)\cdot\beta^{-(p+1)}$$
$$\leqslant C'_a\alpha^{-5/2}E_\alpha(v,\beta;a+4).$$

From (2.14) we have

$$||g_{l0}||_{(\alpha),a,b}\leqslant C_{a,b}\alpha^{1/4}E_\alpha(v,\beta;a+2b+4+1).$$

With the aid of (2.15) and the above estimate Lemma 2.1 assures the existence g_{01} and g_{11} satisfying $(2.11)_1$ in ϖ_α and $(2.12)_1$ such that

$$\sum_{l=0}^{1}||g_{l1}||_{(\alpha),a,b}\leqslant C_{a,b}\{C'_{a+2b+1}\alpha^{-5/2}E_\alpha(v,\beta;a+2b+5)$$
$$+\sum_{q=0}^{b}\alpha^{1/4}E_\alpha(v,\beta;a+2(b-q)+2q+5)\}$$
$$\leqslant C'_{a,b}\alpha^{-5/2}E_\alpha(v,\beta;a+2b+5).$$

Now suppose that

$$\sum_{l=0}^{1}||g_{lj}||_{(\alpha),a,b}\leqslant C_{j,a,b}\alpha^{-11j/4}E_\alpha(v,\beta;a+2b+4j+1).$$

Applying (2.8) to $k^{1/6}g_{lj}$, $l=0$, 1 we have

$$\left\langle (\Phi^- \tilde{G}_j^- + BG_j^- + \frac{1}{ik} B\tilde{G}_j^-)(\Phi^-)^{-1} \right\rangle_{(\alpha),a}$$
$$\leqslant C_a \sum_{p+l\leqslant a} (\alpha^{-7/4} \sum_{l=0}^{1} ||g_{lj}||_{(\alpha),p,1} + \alpha^{-11/4} \sum_{l=0}^{1} ||g_{lj}||_{(\alpha),p,0}) \cdot \beta^{-(l+1)}$$
$$\leqslant C_a \cdot \sum_{p+l\leqslant a} C_{j,p,1} \alpha^{-11/4} \alpha^{-11j/4} E_\alpha(v, \beta; p+2+4j+1) \beta^{-l-1}$$
$$\leqslant C_{j+1,a} \alpha^{-11(j+1)/4} E_\alpha(v, \beta; a+4j+1).$$

And

$$||\Box g_{lj}||_{(\alpha),a\,b} \leqslant C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2b+4j+5).$$

Then by using Lemma 2.1 we have g_{lj+1}, $l=0, 1$ verifying $(2.11)_{j+1}$ in ϖ_α and $(2.12)_{j+1}$ such that

$$\sum_{l=0}^{1} ||g_{lj+1}||_{(\alpha)\ a,b}$$
$$\leqslant C_{a,b} \{ C_{j+1,a+2b+1} \alpha^{-11(j+1)/4} E_\alpha(v, \beta; a+2b+1+4j+4)$$
$$+ \sum_{q=0}^{b} C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2(b-q)+2q+4j+5) \}$$
$$\leqslant C_{j+1,a,b} \alpha^{-11(j+1)/4} E_\alpha(v, \beta; a+2b+4(j+1)+1).$$

Thus by the method of induction we obtain

Lemma 2.2. *For given* $v(x, t) \in C_0^\infty(\Gamma_\alpha \times \boldsymbol{R})$ *there exist* g_{0j}, g_{1j}, $j=0, 1, 2, \cdots$ *verifying* $(2.11)_j$ *in* ϖ_α, $(2.12)_j$ *on* $\Gamma_\alpha \times \boldsymbol{R}$ *and the estimate*

$$\sum_{l=0}^{1} ||g_{lj}||_{(\alpha),a,b} \leqslant C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2b+4j+1), \tag{2.16}$$

where $C_{j,a,b}$ *depends on* j *and* a, b *but independent of* α.

Let N be a positive integer. For g_{lj} of the above lemma we define $g_l^{(N)}$, $u^{(N)}$ by

$$g_l^{(N)}(x, t; \alpha, k) = \sum_{j=0}^{N} g_{lj}(x, t; \alpha, k) k^{1/6-1-j}, \qquad l = 0, 1$$
$$u^{(N)}(x, t; \alpha, k) = e^{ik(\theta-t)} \left\{ V(k^{2/3}\rho) g_0^{(N)} + \frac{1}{ik^{1/3}} V'(k^{2/3}\rho) g_1^{(N)} \right\}.$$

Since

$$||e^{ik(\theta-t)} V(k^{2/3}\rho)||_{(\alpha),a,b} \leqslant C_{a\,b} k^{a+b} \tag{2.17}$$

it holds that

$$||u^{(N)}||_{(\alpha),a,b} \tag{2.18}$$
$$\leqslant C_{N,a,b} \sum_{p+l\leqslant a+b} k^p \sum_{j=0}^{N} k^{-j-1+1/6} E_\alpha(v, \beta; 2l+4j+1)$$
$$\leqslant C_{N,a,b} \sum_{j=0}^{N+a+b} k^{a+b-j-1/5} E_\alpha(v, \beta; 4j+1).$$

Let us consider the estimates of $\Box u^{(N)}$. In $\varpi_\alpha=\{x;\ \rho\geqslant 0\}$ it follows from (2.6) and the relations $(2.11)_j$, $j=0, 1, \cdots, N$ that

$$\Box u^{(N)}=k^{-N-5/6}e^{ik(\theta-t)}\left\{V(k^{2/3}\rho)\Box g_{0N}+\frac{1}{ik^{1/3}}V'(k^{2/3}\rho)\Box g_{1N}\right\}.$$

Using (2.16) and (2.17) we have in ϖ_α

$$\begin{aligned}(2.19)\qquad |\partial_t^{b'}\partial_\rho^b\partial_\theta^a\Box u^{(N)}| &\leqslant C_{N,a,b}k^{-N-5/6}\sum_{\substack{p+l\leqslant a\\ r+q\leqslant b+b'}}k^{p+q}\sum_{h=0}^{1}\|\Box g_{hN}\|_{(\alpha),l,r}\\ &\leqslant C_{N,a,b}k^{-N-5/6}\alpha^{-11N/4}\sum_{\substack{p+l\leqslant a\\ q+r\leqslant b+b'}}k^{p+q}E_\alpha(v,\ \beta;\ l+2r+4N+1)\\ &\leqslant C_{N,a,b}(k\alpha^{11/4})^{-N}\sum_{p=0}^{a+b+b'}k^pE_\alpha(v,\ \beta;\ 2(a+b+b'-p)+4N+1).\end{aligned}$$

Next consider $\Box u^{(N)}$ in $\{x;\ \rho<0\}$. Note that

$$\begin{aligned}&D_{x,t}^\gamma(e^{ik(\theta-t)}V(k^{2/3}\rho)((\nabla\theta)^2+\rho(\nabla\rho)^2-1)g_{0j}k^{-j})\\ &=k^{-j}\sum_{\gamma_1+\cdots\gamma_4=\gamma}\binom{\gamma}{\gamma_1\cdots\gamma_4}D^{\gamma_1}e^{ik(\theta-t)}D^{\gamma_2}V(k^{2/3}\rho)\cdot D^{\gamma_3}((\nabla\theta)^2+\rho(\nabla\rho)^2-1)D^{\gamma_4}g_{0j}.\end{aligned}$$

Since $(\nabla\theta)^2+\rho(\nabla\rho)^2-1=0$ in $\{x;\ \rho\geqslant 0\}$ we have for any $M.>0$ a constant $C_{M\ \gamma_3}$ such that

$$(2.20)\qquad |D^{\gamma_3}((\nabla\theta)^2+\rho(\nabla\rho)^2-1)|\leqslant C_{M,\gamma_3}(-\rho)^{3M/2})$$

for $\rho\leqslant 0$. On the other hand, since $V(z)$ satisfies

$$|(-z)^{3M/2}D^{\gamma_2}V(z)|\leqslant C_{\gamma_2,M}\qquad\text{for all } z<0$$

it follows that for all $k\geqslant 1$ and $\rho\leqslant 0$

$$|(-\rho)^{3M/2}D^{\gamma_2}V(k^{2/3}\rho)|\leqslant C_{\gamma_2,M}k^{-M}.$$

By using (2.20)

$$\begin{aligned}(2.21)\qquad &\|e^{ik(\theta-t)}V(k^{2/3}\rho)((\nabla\theta)^2+\rho(\nabla\rho)^2-1)g_{0j}k^{-j}\|_{(\alpha),a,b}\\ &\leqslant C_{M,a,b}k^{a+b}k^{-M}k^{-j-5/6}\|g_{0j}\|_{(\alpha),a,b}\\ &\leqslant C_{M,a,b}k^{a+b-M-j-5/6}\alpha^{-11j/4}E_\alpha(v,\ \beta;\ 2a+b+4j+1).\end{aligned}$$

About $e^{ik(\theta-t)}V(k^{2/3}\rho)\nabla\theta\cdot\nabla\rho g_{1j}k^{-j}$ we can obtain the same estimate as (2.21) by taking account of the fact $\nabla\theta\cdot\nabla\rho=0$ in $\{x;\ \rho\geqslant 0\}$. Next consider termes of the type

$$I_j=e^{ik(\theta-t)}V(k^{2/3}\rho)J_jk^{-j+1-5/6}$$

$$J_j = 2\frac{\partial g_{0j}}{\partial t} + 2\nabla\theta\cdot\nabla g_{0j} + \Delta\theta g_{0j} + 2\rho\nabla\rho\cdot\nabla g_{1j}$$
$$+(\nabla\rho)^2 g_{1j} + \rho\nabla\rho g_{1j} + \frac{1}{i}\Box g_{0j-1}\,.$$

Since $\{g_{0j}, g_{1j}\}$ verifyies $(2.11)_j$ in $\overline{\omega}_\alpha$ we have for $\rho<0$

$$|\partial_t^{b'}\partial_\rho^b\partial_\theta^a J_j| \leqslant C_M(-\rho)^{3M/2}\{||g_{0j}||_{(\alpha),a+b',b+3M/2+1}$$
$$+||g_{1j}||_{(\alpha),a+b',b+3M/2+1} + ||g_{0j-1}||_{(\alpha),a+b',b+3M/2+2}\}\,.$$

Therefore

$$||I_j||_{(\alpha),a,b} \leqslant C_{j,a,b}k^{-M}k^{-j+1+5/6}\sum_{l+p\leqslant a+b} k^p$$
$$\cdot\{\alpha^{-11j/4}\sum_{h=0}\sum_{r+q\leqslant l}||g_{hj}||_{(\alpha),r,q+3M/2+1} + \alpha^{-11(j-1)/4}\sum_{r+q\leqslant l}||g_{0j}||_{(\alpha)\,r,q+3M/2+1}\}$$
$$\leqslant C_{j,a,b}k^{-M}k^{-j+1-5/6}\sum_{l+p\leqslant a+b} k^p\{\alpha^{-11j/4}E_\alpha(v,\beta;\,2l+3M/2+4j+3)$$
$$+\alpha^{-11(j-1)/4}E_\alpha(v,\beta;\,2l+3M/2+4(j-1)+4)\}\,,$$

and setting $M=N-(j-1)$ it follows that

$$||I_j||_{(\alpha),a,b} \leqslant C_{j,a,b}k^{-N}\alpha^{-11(j-1)/4}\sum_{l+p\leqslant a+b} k^p E_\alpha(v,\beta;\,2l+4N+3)\,. \tag{2.22}$$

Note that we have an estimate same as (2.22) for the other terms of $\Box u^{(N)}$. From (2.19), (2.21) and (2.22) we have an estimate

$$||\Box u^{(N)}||_{(\alpha),a,b} \leqslant C_{N,a,b}(k\alpha^{11/4})^{-N}\sum_{p+l\leqslant a+b} k^p E_\alpha(v,\beta;\,2l+4N+3)\,. \tag{2.23}$$

We set about considering $Bu^{(N)}|_{\Gamma_\alpha\times\boldsymbol{R}}$. Remark that from (ii) of Lemma 2.1

$$\text{supp}\, Bu^{(N)}|_{\Gamma_\alpha\times\boldsymbol{R}} \subset \Lambda_0\cup\Lambda_1\,.$$

On $\Gamma_\alpha\times\boldsymbol{R}$

$$Bu^{(N)-} - e^{ik(\varphi^- -t)}v = e^{ik(\varphi^- -t)}k^{-N}\left\{\Phi^-\tilde{G}_N^- + BG_N^- + \frac{1}{ik}B\tilde{G}_N^-\right\},$$

from which it follows that

$$\langle Bu^{(N)-} - e^{ik(\varphi^- -t)}v\rangle_{(\alpha),a}$$
$$\leqslant C_{N,a}k^{-N}\sum_{p+l\leqslant a} k^p\alpha^{-11(N+1)/4}E_\alpha(v,\beta;\,l+4N+3)\,. \tag{2.24}$$

Since in ω_α

$$\Box u^{(N)} = e^{ik(\theta-t)}\left\{V(k^{2/3}\rho)\Box g_{0N} + \frac{1}{ik^{1/3}}V'(k^{2/3}\rho)\Box g_{1N}\right\}k^{-N-5/6},$$

by applying the expansion of the type (2.7) to the right hand side of the above equality we may write near $\Gamma_\alpha\times\boldsymbol{R}$

$$\square u^{(N)} = e^{ik(\varphi^- - t)}H^- k^{-N} + e^{ik(\varphi^+ - t)}H^+ k^{-N},$$

with $H^\pm$ satisfying

$$|\partial_t^{a'}\partial_\theta^a\partial_\rho^b H^\pm| \leqslant C_{N,a,b}\alpha^{-11N/4}E_\alpha(v, \beta; a+a'+2b+4N+1).$$

On the other hand applying $\square$ to $u^{(N)}$ of (2.7) we have in ω_α

$$\square u^{(N)} = e^{ik(\varphi^- - t)}\left\{ik\left(2\frac{\partial}{\partial t}+2\nabla\varphi^-\cdot\nabla+\Delta\varphi^-\right)+\square\right\}\left(G^{(N)-}+\frac{1}{ik}\tilde{G}^{(N)-}\right)$$
$$+e^{ik(\varphi^+ - t)}\left\{ik\left(2\frac{\partial}{\partial t}+2\nabla\varphi^+\cdot\nabla+\Delta\varphi^+\right)+\square\right\}\left(G^{(N)+}+\frac{1}{ik}\tilde{G}^{(N)+}\right),$$

where $G^{(N)\pm}$, $\tilde{G}^{(N)\pm}$ denote the terms corresponding to $G^\pm$, $\tilde{G}^\pm$ of (2.7) when we substitute $g_1^{(N)}$ and $g_1^{(N)}$ into the places of g_0 and g_1 of (2.4). In the same meaning we will write the decomposition of (2.7) for $u^{(N)}$ as $u^{(N)}=u^{(N)+}+u^{(N)-}$. Since $\nabla\varphi^+$ and $\nabla\varphi^-$ are linearly independent it follows that

$$\left\{ik\left(2\frac{\partial}{\partial t}+2\nabla\varphi^\pm\cdot\nabla+\Delta\varphi^\pm\right)+\square\right\}\left(G^{(N)\pm}+\frac{1}{ik}\tilde{G}^{(N)\pm}\right) = k^{-N}H^\pm,$$

from which we can derive an estimate in a neighborhood of Λ_0

$$\left|\partial_t^a\partial_\theta^{a'}\partial_\rho^b\left(G^{(N)+}+\frac{1}{ik}\tilde{G}^{(N)+}\right)\right|$$
$$\leqslant C_{N,a,b}k^{-N+a+a'+b}\alpha^{-11N/4}E_\alpha(v, \beta; 4N+a+a'+2b+1),$$

by taking account of the location of the support of $G^{(N)+}+\frac{1}{ik}\tilde{G}^{(N)+}$ and the equation $G^{(N)+}+\frac{1}{ik}\tilde{G}^{(N)+}$ must satisfy. Then we have

$$\langle Bu^{(N)+}|_{\Lambda_0}\rangle_{(\alpha),a} \leqslant C_{N,a}(k\alpha^{11/4})^{-N}\sum_{p+l\leqslant a} k^p E_\alpha(v, \beta; 4N+l+3).$$

Combining the above estimate with (2.24) it holds that

$$\text{(2.25)}\quad \langle Bu^{(N)}|_{\Lambda_0} - e^{ik(\varphi^- - t)}v\rangle_{(\alpha),a} \leqslant C_{N,a}(k\alpha^{11/4})^{-N}\sum_{p+l\leqslant a} k^p E_\alpha(v, \beta; 4N+l+3).$$

Next consider $Bu^{(N)}$ on Λ_1.

$$Bu^{(N)+}\Big|_{\Lambda_1} = e^{ik(\varphi^+ - t)}\left\{ik\Phi^+\left(G^{(N)+}+\frac{1}{ik}\tilde{G}^{(N)+}\right)+BG^{(N)+}+\frac{1}{ik}B\tilde{G}^{(N)+}\right\}$$

where

$$G^{(N)+} = \sum_{j=0}^{N}\pi^{-1/2}\alpha^{-1/4}e^{\pi i/4}(g_{0j}+\sqrt{\rho}\,g_{1j})k^{-j-1}.$$

Let us us set

$$w_1(x, t) = i\Phi^+(g_{00}+\sqrt{\rho}g_{10}) .$$

Applying (iii) of Lemma 2.1 we have

$$w_1(x, t) = \gamma_\alpha(x)\Phi^+\left(\frac{v}{\Phi^-}\right)(P_\alpha(x, t)) .$$

Then it holds that

$$\sup|w_1| \geqslant \frac{1}{2}(\inf_{(x,t)\in\Lambda_1} |\Phi^+|/\sup_{(x,t)\in\Lambda_0} |\Phi^-|)\sup|v| . \tag{2.26}$$

$$\langle w_1\rangle_{(\alpha),a} \leqslant C_a\{\sup_{(x,t)\in\Lambda_1} |\Phi^+|E_\alpha(v, \beta; a)+E_\alpha(v, \beta; a-1)\} . \tag{2.27}$$

Set

$$w_2(x, t) = i\Phi^+\sum_{j=1}^{N}(g_{0j}+\sqrt{\rho}g_{1j})k^{-j}+i\Phi^+\tilde{G}^{(N)+}+BG^{(N)+}+\frac{1}{ik}B\tilde{G}^{(N)+} .$$

Then

$$\langle w_2\rangle_{(\alpha),a} \leqslant C_{N,a}\sum_{j=1}^{N}(k\alpha^{11/4})^{-j}E_\alpha(v, \beta; 4j+a)$$

By the same consideration as $u^{(N)+}$ in Λ_0 we have

$$\langle Bu^{(N)-}|_{\Lambda_0}\rangle_{(\alpha),a} \leqslant C_{N,a}(k\alpha^{11/4})^{-N}\sum_{p+l\leqslant a} k^p E_\alpha(v, \beta; 4N+l+3) .$$

Summarizing the considerations in this section we have

Proposition 2.3. *Let $\alpha>0$ and $v(x, t)\in C_0^\infty(\Gamma_\alpha\times \boldsymbol{R})$ such that $\Lambda_0\cap\Lambda_1=\phi$. For every positive integer N there exists a function $u^{(N)}(x, t; \alpha, k)\in C^\infty(\boldsymbol{R}^2\times \boldsymbol{R})$ satisfying*

$$\operatorname{supp} u^{(N)}\cap(\overline{\omega}_\alpha\times \boldsymbol{R})\subset\{L_\alpha^-(x, t); (x, t)\in\operatorname{supp} v\} ,$$
$$\operatorname{supp} Bu^{(N)}|_{\Gamma_\alpha\times \boldsymbol{R}}\subset\Lambda_0\cup\Lambda_1 ,$$

and the estimates (2.18), (2.23) *and* (2.25). *And*

$$\langle Bu^{(N)}|_{\Lambda_1}-e^{ik(\varphi^+-t)}w\rangle_{(\alpha),a}$$
$$\leqslant C_{N,a}(k\alpha^{11/4})^{-N}\sum_{p+l\leqslant a} k^p E_\alpha(v, \beta; 4N+l+3)$$

where w has the following properties

$$\sup|w| \geqslant \frac{1}{2}(\inf_{(x,t)\in\Lambda_1} |\Phi^+|/\sup_{(x,t)\in\Lambda_0} |\Phi^-|)\cdot\sup|v|$$
$$-C\sum_{j=1}^{N}(k\alpha^{11/4})^{-j}E_\alpha(v, \beta;4j)$$

$$\langle w\rangle_{(\alpha)\ a} \leqslant C_a\{(\sup_{\Lambda_1}|\Phi^+|+\beta)E_\alpha(v, \beta; a)$$
$$+C_{N,a}\sum_{j=1}^{N}(k\alpha^{11/4})^{-j}E_\alpha(v,\beta; 4j+a)\} ,$$

where all the constants are independent of α.

3. Asymptotic solutions reflected K-time at Γ_α

Let $v(x, t)\in C_0^\infty(\Gamma_\alpha\times \boldsymbol{R})$ and supp $v\subset\Lambda_0$. Define $\Lambda_1, \Lambda_2, \cdots, \Lambda_K$ successively by

$$\Lambda_{j+1}=\{L^-(x, t)\cap(\Gamma_\alpha\times\boldsymbol{R})-\{(x, t)\}\,;\,(x, t)\in\Lambda_j\}\,.$$

Suppose that

$$\bar{\Lambda}_j\subset\Gamma_\alpha\times(t_j, t_{j+1}),\ t_0<t_1<\cdots<t_{K+1}\,. \tag{3.1}$$

Set

$$\beta=\inf_{(x,t)\in\bigcup_{j=0}^{K}\Lambda_j}|B\varphi^-|,$$

$$\nu=\inf_{(x,t)\in\bigcup_{j=0}^{K}\Lambda_j}|B\varphi^+|/\sup_{(x,t)\in\cup\Lambda_j}|B\varphi^-|.$$

We assume for some constant C_K

$$\sup_{(x,t)\in\cup\Lambda_j}|B\varphi^+|/\beta\leqslant C_K\nu\,. \tag{3.2}$$

Apply Proposition 2.3 for

$$m_0(x, t;\alpha, k)=e^{ik(\varphi^-(x,\alpha)-t)}v(x, t)$$

and have $u_0^{(N)}(x, t;\alpha, k)$ with the properties

$$\|u_0^{(N)}\|_{(\alpha),a,b}\leqslant C_{N,a,b}\sum_{j=0}^{N+a+b}k^{a+b-j-1/5}E_\alpha(v, \beta; 4j+1) \tag*{(3.3)$_0$}$$

$$\begin{aligned}&\|\Box u_0^{(N)}\|_{(\alpha),a,b}\\&\leqslant C_{N,a,b}(k\alpha^3)^{-N}\sum_{p+l\leqslant a+b}k^pE_\alpha(v, \beta; 2l+4N+3)\,,\end{aligned} \tag*{(3.4)$_0$}$$

$$\begin{aligned}&\langle Bu_0^{(N)}|_{\Lambda_0}-m_0\rangle_{(\alpha),a}+\langle Bu_0^{(N)}|_{\Lambda_1}-m_1\rangle_{(\alpha),a}\\&\leqslant C_{N,a}(k\alpha^4)^{-N}\sum_{p+l\leqslant a}k^pE_\alpha(v, \beta; 4N+l+3)\,,\end{aligned} \tag*{(3.5)$_0$}$$

where

$$m_1=e^{ik(\varphi^+-t)}v_1\,,$$

$$\mathrm{supp}_1\, v\subset\Lambda_1 \tag*{(3.6)$_1$}$$

$$\sup|v_1|\geqslant\frac{\nu}{2}\sup|v|-C\sum_{j=1}^{N}(k\alpha^3)^{-N}E_\alpha(v, \beta; 4j) \tag*{(3.7)$_1$}$$

$$\begin{aligned}\langle v_1\rangle_{(\alpha),a}\leqslant\ &C_a(\sup|\Phi^+|+\beta)E_\alpha(v, \beta; a)\\&+C_{N,a}\sum_{j=1}^{N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j+a)\,.\end{aligned} \tag{3.8}$$

Since $\rho=\alpha$ on Γ_α we have

$$\varphi^+ = \theta+\frac{2}{3}\rho^{3/2}=\theta-\frac{2}{3}\rho^{3/2}+\frac{4}{3}\alpha^{3/2}$$
$$= \varphi^- +\frac{4}{3}\alpha^{3/2} \qquad \text{on } \Gamma_\alpha\,,$$

from which follows

$$m_1 = e^{ik(\varphi^- -t)}\tilde{v}_1\,, \quad \tilde{v}_1 = e^{i4/3k\alpha^{3/2}}v_1\,.$$

Then $\tilde{v}_1$ verifies the properties $(3.6)_1\sim(3.8)_1$.

Now the application of Proposition 2.3 to m_1 gives the existence of a function $u_1^{(N)}(x, t; \alpha, k)$ with the properties

$$(3.3)_1 \qquad ||u_1^{(N)}||_{(\alpha),a,b}\leqslant C_{N,a,b}\sum_{j=0}^{N+a+b} k^{a+b-j-1/5}E_\alpha(v_1, \beta; 4j+1)$$

$$(3.4)_1 \qquad ||\square u_1^{(N)}||_{(\alpha),a,b}\leqslant C_{N,a,b}(k\alpha^3)^{-N}\sum_{p+l\leqslant a+b} k^p E_\alpha(v_1, \beta; 2l+4N+3)$$

$$(3.5)_1 \qquad \langle Bu_1^{(N)}|_{\Lambda_1}-m_1\rangle_{(\alpha),a}+\langle Bu_1^{(N)}|_{\Lambda_2}-m_2\rangle_{(\alpha),a}$$
$$\leqslant C_{N,a}(k\alpha^3)^{-N}\sum_{p+l\leqslant a}k^p E_\alpha(v_1, \beta; 4N+l+3)\,.$$

From $(3.8)_1$ and the definition of $E_\alpha(v_1, \beta; a)$ it follows

$$E_\alpha(v_1, \beta; a) = \sum_{p+l\leqslant a}\langle v_1\rangle_{(\alpha),p}\beta^{-l-1}$$
$$\leqslant \sum_{p+l\leqslant a}\{C_p(\sup|\Phi^+|+\beta)E_\alpha(v, \beta; p)$$
$$+C_{N,a}\sum_{j=1}^{N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j+p)\}\,\beta^{-l-1}$$
$$\leqslant C_a(\sup|\Phi^+|+\beta)\sum_{p+l\leqslant a}E_\alpha(v, \beta; p)\beta^{-l-1}$$
$$+C_{N,a}\sum_{j=1}^{N}(k\alpha^3)^{-j}\sum_{p+l\leqslant a}E_\alpha(v, \beta; 4j+p)\beta^{-l-1}\,.$$

By using $E_\alpha(v, \beta; p)\beta^{-l}\leqslant E_\alpha(v, \beta; p+l)$, we have

$$(3.9)_1 \qquad E_\alpha(v_1, \beta; a)\leqslant C_a(\sup|\Phi^+|+\beta)/\beta E_\alpha(v, \beta; a)$$
$$+C_{N,a}\beta^{-1}\sum_{j=1}^{N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j+a)\,.$$

From the second part of Proposition 2.3 m_2 can be represented as

$$m_2(x, t; \alpha, k) = e^{ik(\varphi^+ -t)}v_2(x, t; \alpha, k)$$
$$= e^{ik(\varphi^- -t)}e^{ik(4/3)\alpha^{3/2}}v_2 = e^{ik(\varphi^- -t)}\tilde{v}_2\,,$$

and $\tilde{v}_2$ verifies from (2.7) and the above estimate $(3.9)_1$

$$(3.7)_2 \qquad \sup|\tilde{v}_2| \geqslant \frac{1}{2}\nu\Big(\frac{1}{2}\nu\sup|v| - C_N\sum_{j=1}^{N}(k\alpha)^{3-j}E_\alpha(v, \beta; 4j)\Big)$$

$$-C\sum_{j=0}^{N}(k\alpha^3)^{-j}\{C_a(\sup|\Phi^+|+\beta)/\beta E_\alpha(v, \beta; 4j)$$

$$+C_{N,a}\beta^{-1}\sum_{h=1}^{N}(k\alpha^3)^{-h}E_\alpha(v, \beta; 4j+4h)\}$$

$$\geqslant\Big(\frac{1}{2}\nu\Big)^2\sup|v| - C\nu\sum_{j=1}^{N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j)$$

$$-C_{N,a}\beta^{-1}\sum_{j=2}^{2N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j)\,.$$

$$(3.8)_2 \qquad \langle\tilde{v}_2\rangle_{(\alpha),a} \leqslant C_a(\sup|\Phi^+|+\beta)E_\alpha(v_1, \beta; a)$$

$$+C_{N,a}\sum_{j=1}^{N}(k\alpha^3)^{-j}E_\alpha(v_1, \beta; 4j+a)$$

$$\leqslant C_a(\sup|\Phi^+|+\beta)\{C_aC\nu E_\alpha(v, \beta; a)$$

$$+C_{N,a}\beta^{-1}\sum_{j=2}^{N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j+a)\}$$

$$+C_{N,a}\sum_{j=1}^{N}(k\alpha^3)^{-j}\{C_a\cdot C\nu E_\alpha(v, \beta; 4j+a)$$

$$+\beta^{-1}C_{N,a}\sum_{h=2}^{N}(k\alpha^3)^{-h}E_\alpha(v, \beta; 4h+4j+a)\}$$

$$\leqslant C_a'(\sup|\Phi^+|+\beta)\cdot\nu\cdot E_\alpha(v, \beta; a)$$

$$+C_{N,a}'\nu\sum_{j=1}^{N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j+a)$$

$$+C_{N,a}'\beta^{-1}\sum_{j=2}^{2N}(k\alpha^3)^{-j}E_\alpha(v, \beta; 4j+a)\,.$$

Repeating this process we obtain $u_j^{(N)}(x, t; \alpha, k)$, $j=0, 1, 2, \cdots, K$ verifying

$$(3.3)_j \qquad \|u_j^{(N)}\|_{(\alpha),a,b} \leqslant C_{N,a,b}\sum_{h=0}^{N+a+b}k^{a+b-h-1/5}E_\alpha(v_j, \beta; 4h+1)$$

$$(3.4)_j \qquad \|\Box u_j^{(N)}\|_{(\alpha),a,b} \leqslant C_{N,a,b}(k\alpha^3)^{-N}\sum_{p+l\leqslant a+b}k^pE_\alpha(v_j, \beta; 2l+4N+3)$$

$$(3.5)_j \qquad \langle Bu_j^{(N)}|_{\Lambda_j} - m_j\rangle_{(\alpha),a} + \langle Bu_j^{(N)}|_{\Lambda_{j+1}} - m_{j+1}\rangle_{(\alpha),a}$$

$$\leqslant C_{N,a}(k\alpha^3)^{-N}\sum_{p+l\leqslant a}k^pE_\alpha(v_j, \beta; 4N+l+3)\,,$$

$$m_j = e^{ik(\varphi^- - t)}\tilde{v}_j$$

$$\operatorname{supp}\tilde{v}_j \subset \Lambda_j$$

$$(3.7)_j \qquad \sup|\tilde{v}_j| \geqslant \Big(\frac{1}{2}\nu\Big)^j\sup|v|$$

$$-C_N^{(j)}\sum_{l=1}^{j-1}\nu^{j-l}\sum_{h=l}^{lN}(k\alpha^3)^{-h}E_\alpha(v, \beta; 4h)$$

$$-C_N^{(j)}\beta^{-1}\sum_{h=j}^{jN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h)\,,$$

$$(3.8)_j \qquad \langle\tilde{v}_j\rangle_{(\alpha),a}\leqslant C_a^{(j)}(\sup|\Phi^+|+\beta)\cdot\nu^{j-1}E_\alpha(v,\beta;a)$$
$$+C_{N,a}^{(j)}\sum_{l=1}^{j-1}\nu^{j-l}\sum_{h=l}^{lN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+a)$$
$$+C_{N,a}^{(j)}\beta^{-1}\sum_{h=j}^{jN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+a)\,.$$

By using $\nu\leqslant C\beta^{-1}$ it follows from $(3.8)_j$ that

$$(3.10)_j \qquad \langle\tilde{v}_j\rangle_{(\alpha),a}\leqslant C_{N,a}^{(j)}\sum_{l=0}^{j}\beta^{-(j-l)}\sum_{h=l}^{lN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+a)$$
$$\leqslant C_{N,a}^{(j)}\sum_{l=0}^{j}\sum_{h=l}^{lN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+j-l+a)\,.$$

Set

$$U_K^{(N)}(x,t;\alpha,k)=\sum_{j=0}^{N}(-1)^j u_j^{(N)}(x,t;\alpha,k)\,.$$

Then we have from $(3.3)_j\sim(3.10)_j$

Proposition 3.1. *Let $v(x,t)\in C_0^\infty(\Gamma_\alpha\times\boldsymbol{R})$ such that*

$$\operatorname{supp} v\subset\Lambda_0\,.$$

Suppose that (3.1) *and* (3.2). *Then there exists a function $U_K^{(N)}(x,t;\alpha,k)$ with the following properties*:

$$(3.11) \qquad \operatorname{supp} U_K^{(N)}\cap(\bar{\Omega}\times\boldsymbol{R})\subset\bar{\Omega}\times(t_0,\infty)$$

$$(3.12) \qquad \|U_K^{(N)}\|_{(\alpha),a,b}\leqslant C_{N,K,a,b}\sum_{j=0}^{N+a+b}k^{a+b-j-1/5}$$
$$\cdot\sum_{l=0}^{K}\sum_{h=l}^{lN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+K-l+4j+2)$$

$$(3.13) \qquad \|\square U_K^{(N)}\|_{(\alpha),a,b}$$
$$\leqslant C_{N,K,a,b}(k\alpha^3)^{-N}\sum_{p+l\leqslant a+b}k^p\sum_{q=0}^{K}\sum_{h=q}^{qN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+K-q+2l+4N+3)$$

$$(3.14) \qquad \langle BU_K^{(N)}|_{\Gamma_\alpha\times(t_0,t_K)}-m_0\rangle_{(\alpha),a}$$
$$\leqslant C_{N,K,a}(k\alpha^3)^{-N}\sum_{p+l\leqslant a+b}k^p\sum_{q=0}^{K}\sum_{h=q}^{qN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h+K-q+2l+4N+3)$$

$$(3.15) \qquad \sup_{\Gamma_\alpha\times(t_0,t_k)}|U_K^{(N)}|\geqslant\left(\frac{1}{2}\nu\right)^K\sup|v|$$
$$-C_N\sum_{l=1}^{K-1}\nu^{j-l}\sum_{h=l}^{lN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h)$$
$$-C_N\beta^{-1}\sum_{h=K}^{KN}(k\alpha^3)^{-h}E_\alpha(v,\beta;4h)\,,$$

where the constants $C_{N,K,a,b}$ and $C_{N,K,a}$ are independent of α.

4. Proof of the theorem

Lemma 4.1. *Suppose that* $\tau(0)=\tau'(0)=0$ *and*

$$\sup_{0<s<\varepsilon} \tau(s)>0$$

for any $\varepsilon>0$. *Then there exist a constant* $\delta\geqslant 1/2$ *and a sequence*

$$s_1>s_2>\cdots>s_n>s_{n+1}>\cdots>0$$

with the following properties:

$$\left\{\begin{array}{ll} s_n \to 0 & \text{as } n\to\infty \\ \beta_n=\tau(s_n)>0 & \end{array}\right. \tag{4.1}$$

and for any positive integer K *there exists a constant* C_K *such that*

$$\sup_n \sup_{0\leqslant t\leqslant K} \frac{|\tau(s_n+t\beta_n)-\beta_n|}{\beta_n^{1+\delta}}\leqslant C_K\,. \tag{4.2}$$

Proof. When $s=0$ is a zero of finite order, namely for some $q\geqslant 1$

$$\tau(0)=\tau'(0)=\cdots=\tau^{(q)}(0)=0\,,\qquad \tau^{(q+1)}(0)>0$$

it holds that for some $s_0>0$

$$|\tau'(s)|\leqslant C\tau(s)^{q/(q+1)}\qquad \text{for } 0<s<s_0\,.$$

Since for $s>0$, $t>0$,

$$\begin{aligned} |\tau(s+t\tau(s))-\tau(s)| &\leqslant t\tau(s)|\tau'(s+\eta t\tau(s))| \qquad (0<\eta<1)\\ &\leqslant t\tau(s)\{|\tau'(s)|+t\eta\tau(s)(\sup\tau'')\}\\ &\leqslant C_K\tau(s)^{1+q/(q+1)} \qquad (0<t\leqslant K)\,, \end{aligned}$$

$\delta=q/(q+1)$ and the sequence $s_n=1/n$ are the desired one.

Next consider the case that $s=0$ is a zero of infinite order.

Case 1. $\tau(s)$ is monotonically increasing in $0<s<\varepsilon_0$ for some $\varepsilon_0>0$. Suppose that for some $1>\delta>0$ there is no sequence with property (4.1) verifying

$$\tau'(s_n)<\tau(s_n)^{\delta}\,,\qquad \forall n\,. \tag{4.3}$$

This assumption implies that it holds that for some $\varepsilon_1>0$

$$\tau'(s)\geqslant\tau(s)^{\delta}\qquad \text{for } 0<s<\varepsilon_1\,,$$

from which it follows

$$\frac{d}{ds}\tau(s)^{1-\delta}=(1-\delta)\tau(s)^{-\delta}\tau'(s)\geqslant(1-\delta) \qquad \text{for } 0<s<\varepsilon_1\,.$$

Then we have

$$\tau(s)^{1-\delta}\geqslant(1-\delta)s \qquad \text{for } 0<s<\varepsilon_1\,,$$

namely $\tau(s)\geqslant(1-\delta)s^{1/(1-\delta)}$. This is contradict with the assumption that $\tau(s)$ has a zero of infinite order at $s=0$. Then we see that for any $1>\delta>0$ there exists $\{s_n\}$ verifying (4.1) and (4.3). By using (4.3) and

$$\begin{aligned}&\tau(s_n+t\beta_n)-\beta_n=t\beta_n\tau'(s_n+\eta t\beta_n)\,, \qquad 0<\eta<1\\ &|\tau'(s_n+\eta t\beta_n)-\tau'(s_n)|\leqslant t\beta_n\sup|\tau''(s)|\end{aligned}$$

we have for all $0\leqslant t\leqslant K$

$$|\tau(s_n+t\beta_n)-\beta_n|\leqslant K\beta_n(\tau'(s_n)+CK\beta_n)\leqslant C_K\beta_n^{1+\delta}\,.$$

Thus (4.2) is proved.

Case 2. For some $\varepsilon_0>0$

$$\tau(s)>0 \qquad \text{for } 0<s<\varepsilon_0$$

and $\tau(s)$ is not monotonically increasing in $0<s<\varepsilon$ for any $\varepsilon>0$. From the assumption for any $\varepsilon>0$ there exists s such that $0<s<\varepsilon$ and $\tau'(s)=0$. Then we can choose $s_n>0$ with the propertiy (4.1) such that $\tau'(s_n)=0$. Then

$$\begin{aligned}|\tau(s_n+t\beta_n)-\beta_n|&\leqslant|\tau'(s_n+\eta t\beta_n)|\cdot t\beta_n\\ &\leqslant CK^2\cdot\beta_n^2 \qquad \forall n\,.\end{aligned}$$

Thus $\{s_n\}_{n=0}^\infty$ is the desired one.

Case 3. $\tau(s)$ does not verify the properties of the case 1 nor 2. Then there exists a sequence $\theta_n>\theta_{n+1}>\cdots\to 0$ such that $\tau(\theta_n)=0$ and $\sup_{s\in[\theta_{n+1},\theta_n]}\tau(s)>0$, since for any $\varepsilon>0$ there exists $0<s<\varepsilon$ such that $\tau(s)>0$. If we choose s_n as

$$\tau(s_n)=\max_{s\in[\theta_{n+1},\theta_n]}\tau(s)\,,$$

it holds that $\tau(s_n)>0$ and $\tau'(s_n)=0$. Evidently $s_n\to 0$. As case 2 we see that this $\{s_n\}$ verifies (4.2). Q.E.D.

Since $n(x)=(n_1(x), n_2(x))$ may be considered as a C^∞-vector defined in a neighborhood of Γ

$$\eta(x)=b_1(x)n_2(x)-b_2(x)n_1(x)$$

is also a C^∞-function defined in a neighborhood of Γ. We show that (P) is not well posed in the sense of C^∞ when $\tau(s)$ of the introduction, *i.e.*, $\tau(s)=$

$\eta(x(s))$ verifies the condition on $\tau(s)$ of Lemma 4.1. Note that

$$\text{(4.4)} \qquad \begin{cases} \nabla\varphi^{\pm} = \pm\sqrt{\rho}(\nabla\rho_0+\alpha\nabla\rho_1+\cdots)+\nabla\theta_0+\alpha\nabla\theta_1+\cdots \\ \text{and } n(x)\cdot\nabla\rho_0 = |\nabla\rho_0|,\ n(x)\cdot\nabla\theta_0 = 0 \qquad \text{on } \Gamma^{4)}. \end{cases}$$

Then we have

$$n(x)\cdot\nabla\varphi^{-}(x,\alpha) = \alpha^{1/2}\frac{\partial\rho}{\partial n}+O(\alpha) \qquad \text{on } \Gamma$$

$$\nabla\theta(x,0)\cdot\nabla\varphi^{-}(x,\alpha) = 1+O(\alpha) \qquad \text{on } \Gamma.$$

Therefore $n(x)\cdot\nabla\varphi^{-}(x,\alpha)/\nabla\theta(x,\alpha)\cdot\nabla\varphi^{-}(x,\alpha)$ decreases monotonically to zero uniformly in $x\in\Gamma$ when $\alpha\to+0$. Let $\{s_n\}$ be the sequence with the property (4.1) for the above $\tau(s)$

For every n set $y_n=x(s_n)$. Then $\alpha_n>0$ is determined uniquely for large n by the relation

$$\text{(4.5)} \qquad \frac{n(y_n)\cdot\nabla\varphi^{-}(y_n,\alpha_n)}{\nabla\theta(y_n,0)\cdot\nabla\varphi^{-}(y_n,\alpha_n)} = \beta_n+\beta_n^{1+\delta/2}.$$

From the above relations we have

$$\text{(4.6)} \qquad c_1\beta_n \leqslant \alpha_1^{1/2} \leqslant c_2\beta_n, \qquad \forall n,$$

where c_1, c_2 are positive constants.

Note that for $\alpha=0$

$$\nabla\theta\cdot\nabla\rho = 0, \quad |\nabla\theta| = 1 \qquad \text{on } \Gamma.$$

On the other hand $x(s)\in\Gamma$ and $\left|\frac{dx}{ds}\right|=1$. Then it follows that

$$\theta(x(s),0) = s+\text{constant}.$$

Without loss of generality we may pose the constant$=0$. Since we have from (2.1) and the property (ii) of ρ

$$\operatorname{rank}\begin{pmatrix} \frac{\partial\theta}{\partial x_1} & \frac{\partial\theta}{\partial x_2} \\ \frac{\partial\rho}{\partial x_1} & \frac{\partial\rho}{\partial x_2} \end{pmatrix}_{\substack{\alpha=0\\ x=x(0)}} = 2,$$

there exists uniquely $x_\alpha(s)$ verifying $x_\alpha(s)\to x(s)$ as $\alpha\to 0$ and

$$\begin{cases} \theta(x_\alpha(s),0) = s \\ \rho_\alpha(x_\alpha(s),\alpha) = \alpha \end{cases}$$

4) See, for example, pages 70 and 71 of [4].

for small s and α. Moreover we have

$$\begin{aligned}|x_\alpha(s)-x(s)| &\leqslant C\{|\rho(x_\alpha(s),\alpha)-\rho(x(s),\alpha)|+|\theta(x_\alpha(s),0)-\theta(x(s),0)|\}\\ &\leqslant C|\alpha-\rho(x(s),\alpha)|.\end{aligned}$$

Using (2.2) and $x(s)\in\Gamma$, we obtain for any $P>0$

$$|x_\alpha(s)-x(s)|\leqslant C_P\alpha^P.$$

Then we have

$$|(B\varphi^\pm)(x_\alpha(s),\alpha)-(B\varphi^\pm)(x(s),\alpha)|\leqslant C_P\alpha^P \tag{4.7}$$

for all $\alpha>0$ and s. Note that

$$(B\varphi^\pm)(x,\alpha)=n(x)\cdot\nabla\varphi^\pm(x,\alpha)-\eta(x)\nabla\theta_0(x)\cdot\nabla\varphi^\pm(x,\alpha).$$

Then we have

$$\begin{aligned}(B\varphi^-)(y_n,\alpha_n) &= (\beta_n+\beta_n^{1+\delta/2}-\tau(s_n))\nabla\theta_0(y_n)\cdot\nabla\varphi^-(y_n,\alpha_n)\\ &= \beta_n^{1+\delta/2}\nabla\theta_0(y_n)\cdot\nabla\varphi^-(y_n,\alpha_n)\\ &= \beta_n^{1+\delta/2}(1+O(\beta_n)).\end{aligned} \tag{4.8}$$

Taking account of (4.4) it holds that

$$\begin{aligned}&n(x(t+s))\cdot\nabla\varphi^\pm(x(s+t))-n(x(s))\cdot\nabla\varphi^\pm(x(s))\\ &=\pm\sqrt{\alpha}(|\nabla\rho_0(x(s+t))|-|\nabla\rho_0(x(s)|)+O(\alpha).\end{aligned}$$

Since $|\nabla\rho_0(x)|$ is C^∞ we have

$$\begin{aligned}&|n(x(s_n+t\beta_n))\cdot\nabla\varphi^\pm(x(s_n+t\beta_n),\alpha_n)-n(x(s_n))\cdot\nabla\varphi^\pm(x(s_n),\alpha_n)|\\ &\leqslant Ct\beta_n^2 \qquad \forall n.\end{aligned}$$

By the same consideration it holds that

$$\begin{aligned}&|\nabla\theta_0(x(s_n+t\beta_n))\cdot\nabla\varphi^\pm(x(s_n+t\beta_n),\alpha_n)-\nabla\theta_0(x(s_n))\cdot\nabla\varphi^\pm(x(s_n),\alpha_n)|\\ &\leqslant Ct\alpha_n\leqslant Ct\beta_n^2, \qquad \forall n.\end{aligned}$$

Therefore we have for $0\leqslant t\leqslant K$

$$\begin{aligned}&|(B\varphi^-)(x(s_n+t\beta_n),\alpha_n)-(B\varphi^-)(x(s_n),\alpha_n)|\\ &\leqslant|\tau(s_n+t\beta_n)-\tau(s_n)|+CK\beta_n^2.\end{aligned}$$

Combinig (4.2) and (4.7) it follows that

$$|(B\varphi^-)(x(s_n+t\beta_n),\alpha_n)-\beta_n^{1+\delta/2}|\leqslant C_K\beta_n^{1+\delta} \tag{4.9}$$

for all $0\leqslant t\leqslant K$ and n. By the same consideration we have

(4.10) $$|(B\varphi^+)(x(s_n+t\beta_n),\ \alpha_n)-2\beta_n|\leqslant G_K\beta_n^{1+\delta/2}$$

for all $0\leqslant t\leqslant K$ and n. Then by using (4.6), (4.7) and (4.9) or (4.10) we have

Lemma 4.2. *Suppose that $\tau(s)$ is equipped with the properties of Lemma 4.1. Then for any $K>0$ there exists a constant C_K such that*

(4.11) $$|(B\varphi^-)(x_{\alpha_n}(s_n+t\beta_n),\ \alpha_n)-\beta_n^{1+\delta/2}|\leqslant C_K\beta_n^{1+\delta}$$

(4.12) $$|(B\varphi^+)(x_{\alpha_n}(s_n+t\beta_n),\ \alpha_n)-2\beta_n|\leqslant C_K\beta_n^{1+\delta/2}$$

for all $0\leqslant t\leqslant K$ and n.

Suppose that the problem (P) is well posed in the sense of C^∞. Then for any T there exist q and C_T such that for all $t\leqslant T$

(4.13) $$|u|_{0,\Omega\times(-\infty,t)}\leqslant C_T\{|\Box u|_{q,\Omega\times(-\infty,t)}+|Bu|_{q,\Gamma\times(-\infty,t)}\}$$

for all $u(x, t)\in C^\infty(\bar{\Omega}\times(-\infty,\ T))$ verifying $u=0$ for $t\leqslant 0$, where

$$|v|_{q,\Omega\times(-\infty,t)}=\sum_{|\gamma|\leqslant q}\sup_{\Omega\times(\infty,t)}|D_{x,t}^\gamma v|$$

$$|v|_{q,\Gamma\times(-\infty\ t)}=\sum_{p+r\leqslant q}\sup_{\Gamma\times(-\infty,t)}|D_t^p(\nabla\theta_0(x)\cdot\nabla)^r v|.$$

On the supposition on $\tau(s)$ of Lemma 4.1 we will show the existence of a sequence of functions which violates (4.13).

Let $h(s, t)\in C_0^\infty(\boldsymbol{R}^2)$ such that

$$\sup|h|=1\,,\quad \operatorname{supp} h\subset[0,\ 1]\times[0,\ 1]\,.$$

For each n define $v_n(x, t)\in C_0^\infty(\Gamma_{\alpha_n}\times\boldsymbol{R})$ by

$$v_n(x_{\alpha_n}(s),\ t)=h\Big(\frac{s-s_n}{\alpha_n},\ \frac{t}{\alpha_n}\Big).$$

Put

$$\Lambda_{n0}=\{(x_{\alpha_n}(s),\ t);\ |s-s_n|\leqslant\alpha_n,\ 0\leqslant t\leqslant\alpha_n\}\,,$$

and define Λ_{nj}, $j=1, 2, \cdots, K$ according to the description in the beginning of §3. Since $c_2\sqrt{\alpha_n}\leqslant|P_{\alpha_n}(x, t)-(x, t)|\leqslant c_1\sqrt{\alpha_n}$ it holds that

$$\Lambda_{nj}\subset\Gamma_{\alpha_n}\times(t_{nj},\ t_{nj+1})$$
$$0=t_{n0}<t_{n1}<\cdots<t_{nK}<c_1K\sqrt{\alpha_n}\,.$$

From Lemma 4.2 we have

$$\inf_{(x,t)\in\bigcup_{j=0}^K\Lambda_{nj}}|B\varphi^-|\geqslant C_K\beta_n^{1+\delta/2}\geqslant C_K\alpha_n\,,$$

$$\inf_{(x,t)\in\bigcup_{j=0}^{K}\Lambda_{nj}} |B\varphi^+| / \sup_{(x,t)\in\bigcup_{j=0}^{K}\Lambda_{nj}} |B\varphi^-| \geqslant C_K\beta_n^{\delta/2}$$

and

$$\sup_{(x,t)\in\bigcup_{j=0}^{K}\Lambda_{nj}} |B\varphi^+| / \inf_{(x,t)\in\bigcup_{j=0}^{K}\Lambda_{nj}} |B\varphi^-| \leqslant C'_K\beta_n^{\delta/2},$$

where C_K and C'_K are independent of n.

Let us fix K as

$$\frac{1}{2}K\delta \geqslant 20q+1 \tag{4.14}$$

and N as

$$6N > 2K+6\,. \tag{4.15}$$

For each n we apply Proposition 3.1 and obtain $U_{nK}^{(N)}(x, t; \alpha, k)$. Note that it holds that

$$\langle v_n\rangle_{(\alpha_n),a} \leqslant C_a\alpha_n^{-a}$$

where C_a is a constant independent of n. Then

$$E_{\alpha_n}(v_n, \alpha_n; a) \leqslant C_a\alpha_n^{-(a+1)}\,.$$

Setting $k=\beta_n^{-20}$ we have

$$\begin{aligned} \|U_{nK}^{(N)}\|_{(\alpha_n),a,b} &\leqslant C_{N,K,a,b}\sum_{j=0}^{N+a+b}\beta_n^{-20(a+b-j)} \\ &\quad\cdot\sum_{l=0}^{K}\sum_{h=l}^{lN}(\beta_n^{-20}\alpha_n^3)^{-h}\alpha_n^{-4h-K+l-4j-2-1} \\ &\leqslant C_{N,K,a,b}\beta_n^{-20(a+b)}\,. \end{aligned} \tag{4.16}$$

$$\begin{aligned} \|\Box U_{nK}^{(N)}\|_{(\alpha_n),a,b} &\leqslant C_{N,a,b}(\beta_n^{-20}\alpha_n^3)^{-N} \\ &\quad\cdot\sum_{p+l\leqslant a+b}\beta_n^{-20p}\sum_{r=0}^{K}\sum_{h=r}^{rN}(\beta_n^{-20}\alpha_n^3)^{-h}\alpha_n^{-4h-K+r-2l-4N-3} \\ &\leqslant C_{N,a,b}\beta_n^{6N}\beta_n^{-2K-6} \leqslant C_{N,a,b} \end{aligned} \tag{4.17}$$

$$\langle BU_{nN}^{(N)}|_{\Gamma_{\alpha_n}\times(t_{n0},t_{nK})} - m_0\rangle_{(\alpha_n),a} \leqslant C_{N,a,b} \tag{4.18}$$

$$\begin{aligned} \sup_{\Omega\times(t_{n0},t_{nK})}|U_{nK}^{(N)}| &\geqslant \left(\frac{1}{2}\right)^K\beta_n^{-K\delta/2} \\ &\quad -C_N\sum_{l=0}^{K-1}\beta_n^{-(K-j)\delta}\sum_{h=l}^{lN}(\beta_n^{-20}\alpha_n^3)^{-h}\alpha_n^{-4h-1} \\ &\quad -C_N\beta_n^{-1}\sum_{h=K}^{KN}(\beta_n^{-20}\alpha_n^3)^{-h}\alpha_n^{-4h-1} \\ &\geqslant \left(\frac{1}{2}\right)^K\beta_n^{-K\delta/2} - C_{N,K}\beta_n^{-(K-1)\delta/2}\,. \end{aligned} \tag{4.19}$$

Since

$$\langle m_0\rangle_{(\alpha_n),a} \leqslant C_a \beta_n^{-20a}$$

we obtain by using (4.16), (4.18) and (2.2)

$$|BU_{nK}^{(N)}|_{q,\Gamma\times(-\infty,t_{nK})} \leqslant C_q \beta_n^{-20q}. \tag{4.20}$$

Taking acount of (2.3) the substitution of (4.17), (4.19) and (4.20) into (4.13) gives

$$\left(\frac{1}{2}\right)^K \beta_n^{-K\delta/2} - C_{N,K}\beta_n^{-(K-1)\delta/2} \leqslant C_q \beta_n^{-20q},$$

which shows a contradiction, because K verifies (4.14) and $\beta_n \to 0$ as $n\to\infty$. Thus the theorem is proved.

Appendix

By a change of variavhles

$$\begin{cases} \theta(x) = y \\ \rho(x) = \sigma \end{cases}$$

the equation $\mathcal{L}_\alpha\{a_1, a_2\} = \{h_0, h_1\}$ turns to

$$\begin{cases} 2\dfrac{\partial a_0}{\partial t} + 2(\nabla\theta)^2 \dfrac{\partial a_0}{\partial y} + \Delta\theta\cdot a_0 + 2\sigma(\nabla\rho)^2 \dfrac{\partial a_1}{\partial \sigma} + (\nabla\rho)^2 a_1 \\ \qquad + \sigma\Delta\rho a_1 = h_0 \qquad \text{in } \sigma\geqslant 0 \\ 2\dfrac{\partial a_1}{\partial t} + 2(\nabla\theta)^2 \dfrac{\partial a_1}{\partial y} + \Delta\theta\cdot a_1 + 2(\nabla\rho)^2 \dfrac{\partial a_0}{\partial \sigma} + \Delta\rho\cdot a_0 = h_1 \qquad \text{in } \sigma\geqslant 0 \end{cases} \tag{A.1}$$

First consider how $a_{lj}(y, t) = \left(\dfrac{\partial a_l}{\partial \sigma_j}\right)(0, y, t)$ is determined. Let us set

$$h_l(\sigma, y, t) \sim \sum_{j=0}^{\infty} h_{lj}(y, t)\sigma^j, \qquad l = 0, 1$$

$$(\nabla\theta)^2(\sigma, y) \sim \sum_{j=0}^{\infty} A_j(y)\sigma^j, \qquad (\Delta\theta)(\sigma, y) \sim \sum_{j=0}^{\infty} C_j(y)\sigma^j$$

$$(\nabla\rho)^2(\sigma, y) \sim \sum_{j=0}^{\infty} B_j(y)\sigma^j, \qquad (\Delta\rho)(\sigma, y) \sim \sum_{j=0}^{\infty} D_j(y)\sigma^j$$

and

$$a_l(\sigma, y, t) \sim \sum_{j=0}^{\infty} a_{lj}(y, t)\sigma^j.$$

Note that the facts $A_0(y) \geqslant c > 0$ and $B_0(y) \geqslant c > 0$ follow from the the proper of θ and ρ. Substitute the above expansions into (A.1) and set equal the coefficients of σ^j of the both sides of the equations. Then we have

$$(\text{A.2})_0 \qquad 2\frac{\partial a_{00}}{\partial t}+2A_0\frac{\partial a_{00}}{\partial y}+C_0a_{00}+B_0a_{10}=h_{00}$$

$$(\text{A.3})_0 \qquad 2\frac{\partial a_{10}}{\partial t}+2A_0\frac{\partial a_{10}}{\partial y}+C_0a_{10}+B_0a_{01}+D_0a_{00}=h_{10}$$

and for $j\geqslant 1$

$$(\text{A.2})_j \qquad 2\frac{\partial a_{0j}}{\partial t}+2\sum_{l=0}^{j}A_l\frac{\partial a_{0j-l}}{\partial y}+\sum_{l=0}^{j}C_la_{0j-l}+2\sum_{l=0}^{j-1}(j-l)B_la_{1j-l}$$
$$+\sum_{l=1}^{j}B_la_{1j-l}+(2j+1)B_0a_{1j}+\sum_{l=0}^{j-1}D_la_{1j-1-l}=h_{0j}$$

$$(\text{A.3})_j \qquad 2\frac{\partial a_{1j}}{\partial t}+2\sum_{l=0}^{j}A_l\frac{\partial a_{1j-l}}{\partial y}+\sum_{l=0}^{j}C_la_{1j-l}+2\sum_{l=0}^{j}B_l(j+1-l)a_{0j+1-l}$$
$$+\sum_{l=0}^{j}D_la_{0j-l}=h_{1j}\,.$$

Then if we set $a_{00}(y,t)=0$, $(\text{A.2})_0$ determines a_{10} and subsequently $(\text{A.3})_0$ determines a_{01}. In $(\text{A.2})_1$ besides a_{11} all terms are determined, therefore a_{11} is determined, and next $(\text{A.3})_1$ determines a_{02}. Continuing this process we obtain successively a_{lj}, $j=0, 1, \cdots$. By the manner of determing a_{lj} it holds that

$$(\text{A.4}) \qquad \sum_{|\gamma|\leqslant a}\{\sup|D^{\gamma}_{y,t}a_{0j+1}(y,t)|+\sup|D^{\gamma}_{y,t}a_{1j}(y,t)|\}$$
$$\leqslant C_a\sum_{k=0}^{j}\sum_{l=0}^{1}\sum_{|\gamma|\leqslant a+2(j-k)}\sup|D^{\gamma}_{y,t}h_{lk}(y,t)|\,.$$

If we set $\tilde{a}_l(\sigma,y,t)=\sum_{j=0}^{b}a_{lj}(y,t)\sigma^j$, the estimate (A.4) gives

Lemma A.1. *For any b positive integer there exists $\{a_0, a_1\}$ such that $a_0(0,y,t)=0$ and*

$$(\text{A.5}) \qquad \sum_{k=0}^{b}\sum_{|\gamma|\leqslant a+2(b-k)}\sup|D^{\gamma}_{y,t}D^k\tilde{a}_l|\leqslant C_{a,b}\sum_{l=0}^{1}\sum_{k=0}^{b}\sum_{|\gamma|\leqslant a+2(b-k)}\sup|D^{\gamma}_{y,t}D^k_{\sigma}h_l|\,,$$

$$(\text{A.6}) \qquad \sum_{|\gamma|\leqslant a}\sup|D^{\gamma}_{y,t}(\mathcal{L}_{\alpha}\{a_0, a_1\}-\{h_0, h_1\})|$$
$$\leqslant|\sigma|^{b+1}C_{a,b}\sum_{l=0}^{1}\sum_{k=0}^{b}\sum_{|\gamma|\leqslant a+2(b-k)}\sup|D^{\gamma}_{y,t}D^k_{\sigma}h_l(\sigma,y,t)|$$

Next consider that case

$$(\text{A.7}) \qquad D^p_{\sigma}h_l(0,y,t)=0 \qquad \text{for } p=0, 1, 2, \cdots, b\,.$$

If we claim $a_0=0$ on $\{\sigma=0\}$ the solution of (A.1) is given for $\sigma>0$ by

$$a_0(\sigma, y, t) = \frac{1}{2}\{G^+(\sqrt{\sigma}, y, t)+G^+(-\sqrt{\sigma}, y, t)\}$$

$$a_1(\sigma, y, t) = \frac{1}{2\sqrt{\sigma}}\{G^+(\sqrt{\sigma}, y, t)-G^+(-\sqrt{\sigma}, y, t)\},$$

where $G^+(z, y, t)$ is the solution of

$$\mathcal{L}^+G^+ = \Big(2\frac{\partial}{\partial t}+2(\nabla\theta)^2(y, z^2)\frac{\partial}{\partial y}+2(\nabla\rho)^2(y, z^2)\frac{\partial}{\partial z}$$

$$+(\Delta\theta)(y, z^2)+z(\Delta\tau)(y, z^2)\Big)G^+(z, y, t) = H^+(z, y, t)$$

$$G^+(0, y, t) = 0$$

$$H^+(z, y, t) = h_0(z^2, y, t)+zh_1(z^2, y, t).$$ [5)]

The assumption (A.7) implies that for $r\leqslant b$, $|\gamma|\leqslant a$

$$|D_z^r D_{y,t}^\gamma H^+(z, y, t)|\leqslant C_{a,b}K_{a,b}|z|^{2b+2-r}$$

$$K_{a,b} = \sum_{l=0}^{1}\sum_{|\gamma|\leqslant a}\sup|D_{y,t}^\gamma D_\sigma^b h_l(\sigma, y, t)|.$$

Therefore it holds that

$$\sum_{|\gamma|\leqslant a}|D_z^r D_{y,t}^\gamma G^+(z, y, t)|\leqslant C_{a,b}K_{a,b}|z|^{2b+3-r},$$

from which it follows immediately that

$$\sum_{r=0}^{b+1}\sum_{|\gamma|\leqslant a+2(b+1-r)}\sup|D_\sigma^r D_{y,t}^\gamma a_l(\sigma, y, t)|\leqslant C_{a,b}K_{a,b},\ \sigma>0.$$

Using $(a_0-\sqrt{\rho}\,a_1)(\alpha, y, t)=G^+(y, t, -\sqrt{\alpha})$ we have

Lemma A·2. *On the supposition* (A.7) *there exists a solution of* (A.1) *veriying $a_0(0, y, t)=0$ and it holds that*

$$\text{(A.9)}\qquad \sum_{r=0}^{b}\sum_{|\gamma|\leqslant a+2(d-r)}\sup|D_\sigma^r D_{y,t}^\gamma a_l(\sigma, y, t)|$$

$$\leqslant C_{a,b}\sum_{l=0}^{1}\sum_{|\gamma|\leqslant a}\sup|D_{y,t}^\gamma D_\sigma^b h_l(\sigma, y, t)|$$

and

$$\text{(A.10)}\qquad \sum_{|\gamma|\leqslant a+2b+2}\sup|D_{y,t}^\gamma(a_0-\sqrt{\rho}\,a_1)(\alpha, y, t)|$$

$$\leqslant C_{a,b}\sum_{l=0}^{1}\sum_{|\gamma|\leqslant a+2b+1}\sup|D_{y,t}^\gamma h_l(\sigma, y, t)|.$$

5) See, § 1 of Ludwig [6] and Lemma 5.2 of Ikawa [4].

When $h_l \equiv 0$, the solution of (A.1) verifying

$$a_0 - \sqrt{\rho}\, a_1 |_{\sigma=\alpha} = f(y, t)$$

is given by (A.8) where G^+ is the solution of

$$\begin{cases} \mathcal{L}^+ G^+ = 0 \\ G^+(-\sqrt{\alpha}, y, t) = f(y, t). \end{cases}$$

Evidently

$$\sum_{|\gamma|\leq a} |D_\sigma^j D_{y,t}^\gamma a_0| \leq \sum_{|p|\leq 2j} \sum_{|\gamma|\leq a} \sup |D_{y,t}^\gamma D_z^p G^+(z, y, t)|$$

$$\sum_{|\gamma|\leq a} |D_\sigma^j D_{y,t}^\gamma a_1| \leq \sum_{|p|\leq 2j+1} \sum_{|\gamma|\leq a} \sup |D_{y,t}^\gamma D_z^p G^+(z, y, t)|.$$

And we see easily that

$$\sum_{|\gamma|\leq a} \sup |D_{z,y,t}^\gamma G^+(z, y, t)| \leq C_a \sum_{|\gamma|\leq a} \sup |D_{y,t}^\gamma f(y, t)|.$$

Thus we have

Lemma A.3. *When h_0, $h_1 \equiv 0$, the solution of* (A.1) *verifying $a_0 - \sqrt{\rho}\, a_1 |_{\sigma=a}$ $= f$ has the estimate*

$$\text{(A.11)} \qquad \sum_{l=0}^{1} \sum_{j=0}^{b} \sum_{|\gamma|\leq a+2(b-j)} \sup |D_\sigma^r D_{y,t}^\gamma a_l(\sigma, y, t)|$$
$$\leq C_{a,b} \sum_{j=0} \sum_{|\gamma|\leq 2a+b+1} \sup |D_{y,t}^\gamma f(y, t)|.$$

To show (i) of Lemma 2.1 for fixed integer b first apply Lemma A.1 and we obtain $\{\tilde{a}_0, \tilde{a}_1\}$ satisfying (A.6), and next apply Lemma A.2 to $\mathcal{L}_\alpha\{\tilde{a}_0, \tilde{a}_1\} - \{h_0, h_1\}$ then we have $\{b_0, b_1\}$ verifying

$$\mathcal{L}_\alpha\{b_0, b_1\} = \{h_0, h_1\} - \mathcal{L}_\alpha\{\tilde{a}_0, \tilde{a}_1\}.$$

By using (A.5), (A.6) and (A.9) we have

$$\sum_{j=0}^{b} \sum_{|p|\leq a+2(b-j)} \{|D_{y,t}^\gamma D_\sigma^j \tilde{a}_l(\sigma, y, t)| + |D_{y,t}^\gamma D_\sigma^j(\sigma, y, t)|\}$$
$$\leq C_{a,b} \sum_{j=0}^{b} \sum_{l=0}^{1} \sum_{|\gamma|\leq a+2(b-j)} \sup |D_\sigma^j D_{y,t}^\gamma h_l(\sigma, y, t)|.$$

Moreover it follows form (A.5) and (A.10) that

$$\sum_{|\gamma|\leq a+2b} \sup |D_{y,t}^\gamma((\tilde{a}_0 + b_0) - \sqrt{\rho}(\tilde{a}_1 + b_1))|_{\rho=\alpha}|$$
$$\leq C_{a,b} \sum_{l=0}^{1} \sum_{j=0}^{b} \sum_{|\gamma|\leq a+2(b-j)} \sup |D_\sigma^j D_{y,t}^\gamma h_l(\sigma, y, t)|.$$

Then using Lemma A.3 we have $\{c_0, c_1\}$ verifying

$$\begin{cases} \mathcal{L}_\alpha\{c_0, c_1\} = 0 & \text{in } \rho \geqslant 0 \\ c_0 - \sqrt{\rho}\, c_1 |_{\rho=\alpha} = f - ((\tilde{a}_0 + b_0) - \sqrt{\rho}(\tilde{a}_1 + b_1))|_{\rho=\alpha}\,. \end{cases}$$

Then we see immediately that $a_l = \tilde{a}_l + b_l + c_l$, $l=0, 1$ are solutions of the problem (A.1) verifying the boundary condition and they satisfy the estimate of (i) of Lemma 2.1.

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