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<th>On the mixed problems for the wave equation in an interior domain. II</th>
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1. Introduction. Let $\Gamma$ be a simple closed curve in $\mathbb{R}^2 = \{(x_1, x_2); x_j \in \mathbb{R}, j = 1, 2\}$ and $\Omega$ be its interior domain. Consider a mixed problem

\begin{equation}
\begin{cases}
\Box u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = 0 & \text{in } \Omega \times (0, \infty) \\
Bu = b_1(x) \frac{\partial u}{\partial x_1} + b_2(x) \frac{\partial u}{\partial x_2} + d(x) u(x) = 0 & \text{on } \Gamma \times (0, \infty) \\
u(x, 0) = u_0(x) \\
\frac{\partial u}{\partial t}(x, 0) = u_1(x),
\end{cases}
\end{equation}

(P)

where $b_j(x), j = 1, 2$ and $d(x)$ are $C^\infty$-functions defined in a neighborhood of $\Gamma$. We suppose that $b_j(x), j = 1, 2$, are real valued and satisfy

\begin{equation}
b_1(x)n_1(x) + b_2(x)n_2(x) = 1 \quad \text{on } \Gamma
\end{equation}

where $n(x) = (n_1(x), n_2(x))$ denotes the unit inner normal of $\Gamma$ at $x$.

Let $x(s), 0 \leq s \leq L$ be a representation of $\Gamma$ by the arc length $s$. Set

\begin{equation}
\tau(s) = [b_1(x)n_2(x) - b_2(x)n_1(x)]_{x=x(s)}.
\end{equation}

The result we want to show is the following

**Theorem.** Suppose that the curvature of $\Gamma$ never vanishes. In the case of $\tau(s) \neq 0$ in order that (P) is well posed in the sense of $C^\infty$ it must holds that

\begin{equation}
|\tau(s)| + \left| \frac{d\tau(s)}{ds} \right| \neq 0 \quad \text{for all } s.
\end{equation}

We should like to give some remarks on the theorem. If $\tau(s) \equiv 0$ the boundary condition is nothing but the Neumann condition or the boundary condition of the third kind. Then it is well known that (P) is well posed in the sense of $L^2$. And when $\tau(s) \neq 0$ for all $s$ the mixed problem (P) is also well posed in the sense of $C^\infty$, that is shown in [1]. In both cases the results are

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still valid without the assumption of the convexity of \( \Omega \).

In the preceding paper [5] we gave a necessary condition for the well posedness of \( (P) \). There we introduced an index \( I_B(P_0, \xi_0; T) \) of a broken ray according to the geometrical optics with respect to the coefficients of the boundary operator and it is proved that the condition

\[
I_B(P_0, \xi_0; T) < C_T, \quad \forall P_0 = (x_0, t_0) \in \Gamma \times (0, T), \ \xi_0 \in \Sigma_{x_0}
\]

is necessary for the well posedness. It is easy to verify that the supposition

\[
\sup_{\lambda_0, \xi_0} I_B(P_0, \xi_0; T) = \infty
\]

implies that \( \tau(s) \equiv 0 \) and \( \tau(s) \) has at least a zero of infinite order. Therefore the theorem of this paper is an improvement of the result of [5].

2. Asymptotic solutions with a caustic

From now on, we suppose that the curvature of \( \Gamma \) never vanishes. Then there exist functions \( \theta(x, \alpha) \) and \( \rho(x, \alpha) \) with the following properties:

(i) \( \theta \) and \( \rho \) are real valued \( C^\infty \) function defined in \( \{(x, \alpha); x \in R^2, \alpha \in [-\alpha_0, \alpha_0]\} \) where \( \alpha_0 \) is a positive constant.

(ii) \( \frac{\partial \rho}{\partial n} > c > 0 \) for \( x \in \Gamma \)

where \( \frac{\partial}{\partial n} = \sum_{j=1}^{2} n_j(x) \frac{\partial}{\partial x_j} \).

(iii) Let us set

\[
\Gamma_\alpha = \{x; \rho(x, \alpha) = \alpha\}
\]

\[
\omega_\alpha = \{x; \rho(x, \alpha) > 0\}.
\]

Then for all \( \alpha \) it holds that

\[
(\nabla \theta)^2 + \rho(\nabla \rho)^2 = 1 \quad \text{in } \omega_\alpha
\]

\[
\nabla \theta \cdot \nabla \rho = 0 \quad \text{in } \omega_\alpha
\]

and

\[
\rho(x, \alpha) \equiv \alpha \ (\text{mod } \alpha^\infty) \quad \text{on } \Gamma.
\]

For \( u(x, t) \in C^\infty(R^2 \times R) \) we set

\[
||u||_{(\omega), \alpha, \beta} = \sum_{\beta \leq \beta} \sup_{x \in \Gamma} |\partial_\alpha^\beta \partial_t^\beta u(x, t)|
\]

1) See, for example, Appendix C of Ludwig [7], §5 of Ikawa [4].

2) Hereafter, we will use \( c \) for various constants independent of \( \alpha \) and \( k \).
\[ \Phi(s) = \sum \sup |\partial_s^j \partial_s^k \phi(x, t)|, \]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^2 \) containing \( \Omega \) and

\[
\partial_t^s = \frac{\partial^s}{\partial t^s}, \quad \partial_\theta^s = \left( \sum_{j=1}^{\infty} \frac{\partial \theta}{\partial x_j} \frac{\partial}{\partial x_j} \right)^s \quad \text{and} \quad \partial_\rho^s = \left( \sum_{j=1}^{\infty} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial x_j} \right)^s.
\]

Let us denote

\[
|u|_{\alpha, s} = \sum_{|\beta| < s} \sup \left| D_\alpha^\beta u(x, t) \right|
\]

\[
|u|_{r, s} = \sum_{|\beta| < s} \sup \left| \partial_\alpha^\beta \partial_t^s u(x(t), t) \right|.
\]

Taking account of

\[
|D(\theta, \rho)| \quad \text{for all } \alpha
\]

it holds that for all \( u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}) \) and \( \alpha \)

\[
(2.3) \quad |u|_{\alpha, 2s} \leq C_\alpha ||u||_{(\alpha), s, s}
\]

where \( C_\alpha \) is independent of \( \alpha \).

Define

\[
\varphi^\pm(x, \alpha) = \theta(x, \alpha) \pm 2/3 \rho(x, \alpha)^{3/2}.
\]

Let \( v(x, t) \in C^\infty(\Omega \times \mathbb{R}) \) and set for \( \alpha > 0 \)

\[
m(x, t; \alpha, \kappa) = e^{i(k(\varphi^-(x, \alpha) - t))} v(x, t)
\]

We construct a function \( u(x, t; \alpha, \kappa) \) in the form

\[
(2.4) \quad u(x, t; \alpha, \kappa) = e^{i(k(\varphi^-(x, \alpha) - t))} \left\{ V(k^{2/3} \rho(x, \alpha)) g_0(x, t; \alpha, \kappa)
\right.
\]

\[
\left. + \frac{1}{ik^{2/3}} V'(k^{2/3} \rho(x, \alpha)) g_1(x, t; \alpha, \kappa) \right\}
\]

so that it may verify

\[
(2.5) \quad \left\{ \begin{array}{l} \square u = 0 \\
Bu|_{= 0} = m(x, t; \alpha, \kappa) \end{array} \right. \quad \text{in } \Omega \times \mathbb{R}
\]

\[
\left. \begin{array}{l} Bu|_{= 0} = m(x, t; \alpha, \kappa) \end{array} \right. \quad \text{on the support of } v
\]

asymptotically as \( k \to \infty \), where \( V(z) = Ai(-z) \) with the Airy function \( Ai(z) \).

Apply \( \square \) for \( u(x, t; \alpha, \kappa) \) of (2.4) and use \( V''(z) + zV(z) = 0, V'''(z) + zV'(z) + V(z) = 0 \). Then we have
\begin{equation}
\Box u = -e^{ik(\theta - t)} \left[ V(k^{3/2} \rho) \left\{ (ik)^2 ((\nabla \theta)^2 + \rho (\nabla \rho)^2 - 1)g_0 \right. \right.
\end{equation}

\begin{equation}
+ 2(ik)^2 \rho \nabla \rho \cdot \nabla \theta_1 + ik \left( \frac{1}{2} \frac{\partial g_0}{\partial t} + 2 \nabla \theta \cdot \nabla g_0 + \Delta \theta \cdot g_0 \right)
\end{equation}

\begin{equation}
+ 2 \rho \nabla \rho \cdot \nabla g_1 + (\nabla \rho)^2 g_1 + \rho \Delta \rho \cdot g_1 \right) - \Box g_1 \right] \right].
\end{equation}

Note that \( V(z) \) and \( V'(z) \) have the following asymptotic expansions for \( z \to +\infty \):

\begin{align*}
V(z) &= \frac{1}{2} \pi^{-1/2} z^{-1/4} \left\{ e^{(t-x/4)}(1 + \xi^{-1}P_1(\xi)) + e^{-i(t-x/4)}(1 + \xi^{-1}P_2(\xi)) \right\}, \\
V'(z) &= \frac{1}{2} \pi^{-1/2} z^{1/4} \left\{ e^{(t-x/4)}(1 + \xi^{-1}P_3(\xi)) - e^{-i(t-x/4)}(1 + \xi^{-1}P_4(\xi)) \right\},
\end{align*}

where \( \xi = \frac{2}{3} z^{3/2} \) and

\( P_j(\xi) \sim \sum_{j=0}^{\infty} p_{ji} \xi^{-1}, \quad p_{ji} \in C \).

Therefore the function \( u \) in the form \( (2.4) \) may be represented for large \( k^{3/2} \rho \) as follows

\begin{equation}
(2.7) \quad u(x, t; \alpha, k) = e^{ik(\theta - t)} \left( G^+ \right. + \frac{1}{ik} G^+ \left) \right. + e^{ik(\theta - t)} \left( G^- \right. + \frac{1}{ik} G^- \left) \right. \right.
\end{equation}

\begin{equation}
= u^+ + u^-
\end{equation}

where

\begin{align*}
G^+ &= \frac{1}{2\sqrt{\pi}} \rho^{-1/4} k^{-1/2} e^{-(t-x/4)}(g_0 \pm \sqrt{\rho} g_1) \\
G^+ &= \frac{3}{4} \pi^{-1/2} k^{-1/6} e^{-(t-x/4)}(P_0 g_0 + \sqrt{\rho} P_4 g_1) \\
G^- &= \frac{3}{4} \pi^{-1/2} k^{-1/6} e^{-(t-x/4)}(P_2 g_0 - \sqrt{\rho} P_4 g_1).
\end{align*}

From the form of \( G^\pm \) it holds that

\begin{equation}
(2.8) \quad |\partial^3_{\rho} \rho \hat{G}^\pm| \leq C_k k^{-1/3} \sum_{j=0}^{2} \rho^{-1/4} |g_j|_{(a)_{\rho, 1}} + \rho^{-1/4} |g_j|_{(a)_{\rho, 0}}
\end{equation}

\( 3 \) See Miller [8], page B 17.
when $k^{2/3} \rho > C$.

Applying the operator $B$ to $u$ of (2.7) we have

\[(2.9) \quad Bu = e^{i(k(p^+ - t))} \left\{ i k \Phi^+ \left( G^+ + \frac{1}{i k} G^+ \right) + B G^+ + \frac{1}{i k} B G^+ \right\} \]
\[+ e^{i(k(p^- - t))} \left\{ i k \Phi^- \left( G^- + \frac{1}{i k} G^- \right) + B G^- + \frac{1}{i k} B G^- \right\}, \]

where $\Phi^\pm = \sum_{j=1}^{2} b_j(x) \frac{\partial \varphi^\pm}{\partial x_j}$.

Suppose that $g_0$ and $g_1$ have the following asymptotic expansion with respect to $k^{-1}$ when $k \to \infty$

\[(2.10) \quad g_l(x, t; \alpha, k) \sim \sum_{j=0}^{\infty} g_{lj}(x, t; \alpha, k) k^{l/6 - 1 - j}, \quad l = 0, 1. \]

Denote by $\mathcal{L}_a$ a differential operator from $(C^\infty(\mathbb{R}^2 \times \mathbb{R}))^2$ into itself defined by for $\{a_1, a_2\}$

\[\mathcal{L}_a \{a_1, a_2\} = \left\{ 2 \frac{\partial a_1}{\partial t} + 2 \nabla \theta \cdot \nabla a_1 + \Delta \theta a_1 + 2 \rho \Delta \rho a_1 + (\nabla \rho)^2 a_2 \right\} \]
\[+ \rho \Delta \rho a_1, 2 \frac{\partial a_2}{\partial t} + 2 \nabla \theta \cdot \nabla a_2 + \Delta \theta a_2 + 2 \rho \Delta \rho a_2 \Delta a_1 + \rho \Delta \rho a_1 \right\}. \]

Substituting $g_0, g_1$ of (2.10) into (2.6) and (2.9) we claim that all the coefficients of $k^{-l}$ of (2.6) are equal to zero and those of $Bu-m$ are also equal to zero on the support of $\varphi$. Then it must hold that

\[(2.11)_0 \quad \mathcal{L}_a \{g_{00}, g_{10}\} = 0 \]
\[(2.12)_0 \quad i \Phi^- (g_{00} - \sqrt{\rho} g_{10}) = 2 \pi \alpha^{1/4} e^{i\alpha \varphi} \quad \text{on} \quad \Gamma_a \times \mathbb{R} \]

and for $j \geq 1$

\[(2.11)_j \quad \mathcal{L}_a \{g_{0j}, g_{1j}\} = \frac{1}{i} \left\{ \square g_{0j-1}, \square g_{1j-1} \right\} \]
\[(2.12)_j \quad i \Phi^- (g_{0j} - \sqrt{\rho} g_{1j}) = i \Phi^- \mathcal{G}_{i-1} + BG_{i-1} + \frac{1}{i k} B \mathcal{G}_{i-1} \quad \text{on} \quad \Gamma_a \times \mathbb{R} \]

where $G^\pm_1$ and $\mathcal{G}^\pm$ denote the $G^\pm$ and $\mathcal{G}^\pm$ corresponding to the pair of $k^{1/6} g_{0j}$ and $k^{1/6} g_{1j}$.

To obtain the existence and the estimates of $g_{0j}, g_{1j}$ satisfying (2.11) and (2.12), admit the following Lemma, whose proof will be given in the appendix.

**Lemma 2.1.** For $\{h_0, h_1\} \in (C^\infty(\mathbb{R}^2 \times \mathbb{R}))^2$ and $f \in C^\infty(\Gamma_a \times \mathbb{R})$ there exists $\{a_1, a_2\} \in (C^\infty(\mathbb{R}^2 \times \mathbb{R}))^2$ satisfying
and having the following properties:

(i) \[ \|a_j\|_{(a),e,b} \leq C_{a,b} \langle f \rangle_{(a),e+2b} + \sum_{i=0}^1 \sum_{j=0}^1 \|h_i\|_{(a),e+2(3-j)} \]

(ii) When \[ \bigcup_{i=0}^1 \text{supp } h_i \cap \omega_a \subseteq \{ L_i(x, t) \mid (x, t) \in \text{supp } f \} \], it holds that

\[ \bigcup_{i=0}^1 \text{supp } a_i \cap \omega_a \subseteq \{ L_i^-(x, t) \mid (x, t) \in \text{supp } f \} \]

(iii) When \{ h_0, h_1 \} \equiv 0, for \( (x, t) \in \Gamma_a \times R \)

\[ (a_0 + \sqrt{a_0})(x, t) = \gamma(x, t; \alpha) f(P_a(x, t)) \]

where \( \gamma(x, t; \alpha) \) is a \( C^\infty \) function on \( R^2 \times R \times [-\alpha_0, \alpha_0] \) such that

\[ \gamma(x, t; \alpha) \geq C > 0 \]

and \( P_a(x, t) \) denotes the point

\[ L_a^+(x, t) \cap (\Gamma_a \times R) - \{(x, t)\} , \]

where \( L_a^+(x, t) \) denotes a line passing \((x, t)\) defined by

\[ L_a^+(x, t) = \{(x + l \nabla \varphi^+(x, \alpha), t + l) \mid l \in R \} . \]

Let \( \Lambda_0 \) be an open set in \( \Gamma_a \times R \) such that \( \Lambda_0 \supset \text{supp } v \). Set

\[ \Lambda_1 = \{ L_a^-(x, t) \cap (\Gamma_a \times R) - \{(x, t)\} \mid (x, t) \in \Lambda_0 \} . \]

Suppose that

\[ (2.13) \quad \Lambda_0 \cap \Lambda_1 = \phi . \]

Let us set

\[ \beta = \inf_{(x, t) \in \Lambda_0} |\Phi^-| . \]

Using the above lemma we have \( g_{\beta 0} \) and \( g_{\beta 1} \) verifying

\[ \begin{align*}
\left\{ \begin{array}{ll}
-\mathcal{L}_a \{ g_{\beta 0}, g_{\beta 1} \} = 0 & \text{in } \omega_a \times R \\
g_{\beta 0} - \sqrt{a} g_{\beta 1} = \frac{2\pi \alpha^{1/4}e^{\xi(\epsilon_0)}}{i\Phi^-} & \text{on } \Gamma_a \times R
\end{array} \right.
\]

and the estimate

\[ \sum_{j=0}^1 \| g_{\beta j} \|_{(a),e,b} \leq C_{a,b} \langle 2\pi \alpha^{1/4}e^{\xi(\epsilon_0)} \rangle_{(a),e+2b+1} \]

Taking account of \( \langle \Phi^- \rangle_{(a),e} \leq C_a \) for all \( \alpha > 0 \), we have
Then it holds that

\[
\sum_{\ell=0}^{\infty} \| g_{\ell} \|_{(a), a, b} \leq C_{a, b} \alpha^{\frac{\ell}{4}} \sum_{\beta + \gamma = 2\beta + 1} \langle v \rangle_{(a), \beta} \langle (\Phi^{-1})_{(a), \beta} \rangle \\
\leq C_{a, b} \alpha^{\frac{\ell}{4}} \sum_{\beta + \gamma = 2\beta + 1} \langle v \rangle_{(a), \beta} \beta^{-(\beta + 1)} .
\]

Let us set

\[
E_{\alpha}(v, \beta; j) = \sum_{\ell=0}^{\infty} \langle v \rangle_{(a), \beta} \beta^{-(\beta + 1)} .
\]

Remark that the constant \( C_{a, b} \) depends on \( a \) and \( b \) but independent of \( \alpha \).

Next consider \( g_{01} \) and \( g_{11} \). Applying (2.8) to \( k^{1/3} g_{0} \) and using (2.14) we have

\[
| \partial_{x}^{\alpha} \partial_{t}^{\beta} G^{-1} | \leq C_{a} \{ \rho^{-3/4} \alpha^{\frac{\ell}{4}} E_{\alpha}(v, \beta; a + 3) + \rho^{-1/4} \alpha^{\frac{\ell}{4}} E_{\alpha}(v, \beta; a + 1) \}
\]

for \( \rho k^{2/3} > C \). Then, noting (2.2), it follows that

\[
\langle \Phi^{-1} \mathcal{G}^{-1} + B \mathcal{G}^{-1} + \frac{1}{ik} B \mathcal{G}^{-1} \rangle_{(a), a} \leq C_{a} \alpha^{-5/2} E_{\alpha}(v, \beta; a + 3) .
\]

Therefore

\[
\sum_{\ell=0}^{\infty} \| g_{\ell} \|_{(a), a, b} \leq C_{a, b} \alpha^{\frac{\ell}{4}} E_{\alpha}(v, \beta; a + 2b + 4 + 1) .
\]

With the aid of (2.15) and the above estimate Lemma 2.1 assures the existence \( g_{01} \) and \( g_{11} \) satisfying (2.11) in \( w_{a} \) and (2.12) such that

\[
\sum_{\ell=0}^{1} \| g_{\ell} \|_{(a), a, b} \leq C_{a, b} \{ C_{a, b+2} \alpha^{-5/2} E_{\alpha}(v, \beta; a + 2b + 5) + \sum_{q=0}^{b} \alpha^{1/4} E_{\alpha}(v, \beta; a + 2b - q + 2q + 5) \}
\]

\[
\leq C_{a, b} \alpha^{-5/2} E_{\alpha}(v, \beta; a + 2b + 5) .
\]

Now suppose that

\[
\sum_{\ell=0}^{1} \| g_{\ell} \|_{(a), a, b} \leq C_{a, b} \alpha^{-11/4} E_{\alpha}(v, \beta; a + 2b + 4j + 1) .
\]

Applying (2.8) to \( k^{1/3} g_{ij} \), \( l = 0, 1 \) we have


\[ \langle (\Phi - G_j + B\Phi^{-1} G_j^* + \frac{1}{ik} B G_j^*) \Phi^{-1} \rangle \rangle_{(a),a} \]

\[ \leq C_\epsilon \sum_{j \geq 0} (\alpha^{-\gamma/4} \sum_{j=0}^\infty \|g_j\|_{(a),p,1} + \alpha^{-1/4} \sum_{j=0}^\infty \|g_j\|_{(a),p,0}) \beta^{-j+1} \]

\[ \leq C_\epsilon \sum_{j \geq 0} C_j(a,1) \alpha^{-1/4} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; p+2j+1) \beta^{-j+1} \]

\[ \leq C_{j+1,a} \alpha^{-1/4} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; a+2j+1) . \]

And

\[ \|g_j\|_{(a),a,b} \leq C_{j,a,b} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; a+2b+4j+5) . \]

Then by using Lemma 2.1 we have \( g_{j+1,l} \), \( l=0,1 \) verifying (2.11)_j+1 in \( \omega_a \) and (2.12)_j+1 such that

\[ \sum_{j=0}^1 \|g_{j+1}\|_{(a),a,b} \leq C_{j,a,b} \{ C_{j+1,a,2j+1} \alpha^{-1/4} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; a+2b+4j+4) \]

\[ + \sum_{j=0}^b C_{j,a,b} \alpha^{-1/4} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; a+2b+2q+4j+5) \} \]

\[ \leq C_{j+1,a,b} \alpha^{-1/4} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; a+2b+4(j+1)+1) . \]

Thus by the method of induction we obtain

**Lemma 2.2.** For given \( \nu(x,t) \in C_{\infty}(\Gamma_x \times R) \) there exist \( g_{0,j}, g_{1,j}, j=0,1,2, \cdots \) verifying (2.11)_j in \( \omega_a \), (2.12)_j on \( \Gamma_x \times R \) and the estimate

\[ \sum_{j=0}^1 \|g_{j+1}\|_{(a),a,b} \leq C_{j,a,b} \alpha^{-1/4} \alpha^{-1/4} \alpha^{-1/4} E_a(\nu, \beta; a+2b+4j+1) , \]

where \( C_{j,a,b} \) depends on \( j \) and \( a, b \) but independent of \( \alpha \).

Let \( N \) be a positive integer. For \( g_{ij} \) of the above lemma we define \( g_{ij}^{(N)}, u^{(N)} \) by

\[ g^{(N)}(x, t; t; \alpha, k) = \sum_{j=0}^N g_{ij}(x, t; \alpha, k) k^{1/8 - 1/8 - i} , \quad l = 0,1 \]

\[ u^{(N)}(x, t; \alpha, k) = e^{ik(\theta-t)} \left\{ V(k^{2r} \beta) g^{(N)} \left( \frac{1}{i k^{1/8}} V'(k^{2r} \rho) g^{(N)} \right) \right\} . \]

Since

\[ \|e^{ik(\theta-t)} V(k^{2r} \rho)\|_{(a),a,b} \leq C_\epsilon j^a b^b \]

it holds that

\[ \|u^{(N)}\|_{(a),a,b} \leq C_N,a,b \sum_{j=0}^N k^b \sum_{j=0}^N k^{-j+1/8} E_a(\nu, \beta; 2l+4j+1) \]

\[ \leq C_N,a,b \sum_{j=0}^N k^b \sum_{j=0}^N k^{-j-1/8} E_a(\nu, \beta; 4j+1) . \]
Let us consider the estimates of \( \square u^{(N)} \). In \( \omega_a = \{ x; \rho \geq 0 \} \) it follows from (2.6) and the relations (2.11), \( j = 0, 1, \ldots, N \) that
\[
\square u^{(N)} = k^{-N-5/6} e^{i(k(\theta - t))} \left( V(k^{2/3} \rho) \square g_{0N} + \frac{1}{ik^{1/3}} V'(k^{2/3} \rho) \square g_{1N} \right).
\]
Using (2.16) and (2.17) we have in \( \omega_a \)
\[
|\partial_k^j \partial_\theta^b \square u^{(N)}| \leq C_{N, a, b} k^{-N-5/6} \sum_{r+s+\ell = j} \sum_{0 \leq k \leq L} k^{s+\ell} \sum_{\rho \leq \delta} \| g_{hN} \|_{(a), l, r}
\leq C_{N, a, b} k^{-N-5/6} \alpha^{-11/4} \sum_{r+s+\ell = j} \sum_{0 \leq k \leq L} k^{s+\ell} E_a(v, \beta; l + 2r + 4N + 1)
\leq C_{N, a, b} (k \alpha^{11/4})^{-N} \sum_{p=0}^{s+i+j'} \sum_{0 \leq k \leq L} k^p E_a(v, \beta; 2(a + b + b'-p) + 4N + 1).
\]
Next consider \( \square u^{(N)} \) in \( \{ x; \rho < 0 \} \). Note that
\[
D_{i,j} (e^{i(k(\theta - t))} V(k^{2/3} \rho) ((\nabla \theta)^2 + \rho(\nabla \rho)^2 - 1) g_{0j} k^{-i})
= k^{-i} \sum_{\gamma_1, \ldots, \gamma_4} \left( \gamma_{y_1} \ldots \gamma_{y_4} \right) D_{i} e^{i(k(\theta - t))} D_{i} V(k^{2/3} \rho) \cdot D_{i} ((\nabla \theta)^2 + \rho(\nabla \rho)^2 - 1) D_{i} g_{0j}.
\]
Since \( (\nabla \theta)^2 + \rho(\nabla \rho)^2 - 1 = 0 \) in \( \{ x; \rho \geq 0 \} \) we have for any \( M > 0 \) a constant \( C_M \gamma_3 \) such that
\[
|D_{i} ((\nabla \theta)^2 + \rho(\nabla \rho)^2 - 1)| \leq C_M, \gamma_3 (\rho)^{3M/2}
\]
for \( \rho \leq 0 \). On the other hand, since \( V(z) \) satisfies
\[
|(-z)^{3M/2} D_{i} (V(z))| \leq C_{\gamma_3, M} (\rho)^{3M/2}
\]
for all \( z < 0 \) it follows that for all \( k \geq 1 \) and \( \rho \leq 0 \)
\[
|(-\rho)^{3M/2} D_{i} V(k^{2/3} \rho)| \leq C_{\gamma_3, M} k^{-M}.
\]
By using (2.20)
\[
|e^{i(k(\theta - t))} V(k^{2/3} \rho) ((\nabla \theta)^2 + \rho(\nabla \rho)^2 - 1) g_{0j} k^{-i}|_{(a), \rho, b}
\leq C_{M, a, b} k^{s+\ell} k^{-M-j-5/6} \| g_{0j} \|_{(a), \rho, b}
\leq C_{M, a, b} k^{s+b-M-j-5/6} \alpha^{-11/4} E_a(v, \beta; 2a + b + 4j + 1).
\]
About \( e^{i(k(\theta - t))} V(k^{2/3} \rho) \nabla \theta \cdot \nabla \rho g_{ij} k^{-i} \) we can obtain the same estimate as (2.21) by taking account of the fact \( \nabla \theta \cdot \nabla \rho = 0 \) in \( \{ x; \rho \geq 0 \} \). Next consider terms of the type
\[
I_j = e^{i(k(\theta - t))} V(k^{2/3} \rho) J_j k^{-j+1-5/6}
\]
\[ J_j = 2 \frac{\partial g_{0j}}{\partial t} + 2 \nabla \theta \cdot \nabla g_{0j} + \Delta \theta g_{0j} + 2 \rho \nabla \rho \cdot \nabla g_{ij} + \nabla \rho \nabla g_{1j} + \frac{1}{t} \square g_{0j-1}. \]

Since \( \{g_{0j}, g_{1j}\} \) verify (2.11), in \( \omega_a \) we have for \( \rho < 0 \)
\[
| \partial_{x_i} \partial_{x_j} \partial_{\epsilon^j} J_j | \leq C_M (\rho^{3M/2} \{ ||g_{0j}||_{(a), a+b, b+3M/2+1} + ||g_{1j}||_{(a), a+b, b+3M/2+1} + ||g_{0j-1}||_{(a), a+b, b+3M/2+2} \}.
\]

Therefore
\[
||I_j||_{(a), a+b} \leq C_{j, a, b} k^{-M} k^{-j+1+5/6} \sum_{l, x \leq a + b} k^p \alpha^{-11j/4} \sum_{r, q \leq c_{a+b}} ||g_{0j}||_{(a), r, q+3M/2+1} + \alpha^{-11(j-1)/4} \sum_{r, q \leq c_{a+b}} ||g_{0j}||_{(a), r, q+3M/2+1} \}
\leq C_{j, a, b} k^{-M} k^{-j+1+5/6} \sum_{l, x \leq a + b} k^p \{ \alpha^{-11j/4} E_a(v, \beta; 2l+3M/2+4j+3) \}
\leq C_{j, a, b} k^{-M} k^{-j+1+5/6} \sum_{l, x \leq a + b} k^p \{ \alpha^{-11j/4} E_a(v, \beta; 2l+3M/2+4j+3) \},
\]

and setting \( M = N - (j-1) \) it follows that
\[
(2.22) \quad ||I_j||_{(a), a+b} \leq C_{j, a, b} k^{-M} k^{-j+1+5/6} \sum_{l, x \leq a + b} k^p \alpha^{-11j/4} E_a(v, \beta; 2l+4N+3) .
\]

Note that we have an estimate same as (2.22) for the other terms of \( \square u^{(N)}. \)
From (2.19), (2.21) and (2.22) we have an estimate
\[
(2.23) \quad ||\square u^{(N)}||_{(a), a+b} \leq C_{j, a, b} k^{\alpha^{11j/4}} \sum_{p, r \leq a+b} k^p \alpha^{-11N+1/4} E_a(v, \beta; 2l+4N+3) .
\]

We set about considering \( Bu^{(N)} |_{\Gamma_a \times R} \). Remark that from (ii) of Lemma 2.1
\[
\text{supp} \ Bu^{(N)} |_{\Gamma_a \times R} \subset \Lambda_0 \cup \Lambda_1 .
\]

On \( \Gamma_a \times R \)
\[
Bu^{(N)} - e^{ik(\theta - t)} u = e^{ik(\theta - t)} k^{-N} \left\{ \Phi \cdot G + B \cdot G \frac{1}{ik} \right\},
\]
from which it follows that
\[
(2.24) \quad \langle Bu^{(N)} - e^{ik(\theta - t)} u \rangle_{(a), a} \leq C_{j, a} k^{-N} \sum_{p, r \leq a} k^p \alpha^{-11(N+1)/4} E_a(v, \beta; 1+4N+3) .
\]

Since in \( \omega_a \)
\[
\square u^{(N)} = e^{ik(\theta - t)} \left\{ V(k^{1/3} \rho) \square g_{0N} + \frac{1}{ik^{1/3}} V'(k^{1/3} \rho) \square g_{1N} \right\} k^{-N-5/6} ,
\]
by applying the expansion of the type (2.7) to the right hand side of the above equality we may write near \( \Gamma_a \times R \)
\[ \Box u^{(N)} = e^{ik(\varphi - t)}H^{-k^{-N}} + e^{ik(\varphi + t)}H^{+k^{-N}}, \]

with \( H^\pm \) satisfying
\[ \left| \partial_t ^\pm \partial_\varphi ^\pm H^\pm \right| \leq C_{N, a, \beta} \alpha^{-11N/4}E_a(v, \beta; a + a' + 2b + 4N + 1). \]

On the other hand applying \( \Box \) to \( u^{(N)} \) of (2.7) we have in \( \omega_a \)
\[ \Box u^{(N)} = e^{ik(\varphi - t)} \left\{ ik \left( 2 \frac{\partial}{\partial t} + 2\nabla \varphi^- \cdot \nabla + \Delta \varphi^- \right) + \Box \right\} \left( G^{(N)}- + \frac{1}{ik} G^{(N)}- \right) \]
\[ + e^{ik(\varphi + t)} \left\{ ik \left( 2 \frac{\partial}{\partial t} + 2\nabla \varphi^+ \cdot \nabla + \Delta \varphi^+ \right) + \Box \right\} \left( G^{(N)}+ + \frac{1}{ik} G^{(N)}+ \right), \]
where \( G^{(N)}\pm, \ G^{(N)}\pm \) denote the terms corresponding to \( G^\pm, \ G^\pm \) of (2.7) when we substitute \( g_1^{(N)} \) and \( g_0^{(N)} \) into the places of \( g_0 \) and \( g_1 \) of (2.4). In the same meaning we will write the decomposition of (2.7) for \( u^{(N)} \) as \( u^{(N)} = u^{(N)+} + u^{(N)-} \). Since \( \nabla \varphi^+ \) and \( \nabla \varphi^- \) are linearly independent it follows that
\[ \left\{ ik \left( 2 \frac{\partial}{\partial t} + 2\nabla \varphi^\pm \cdot \nabla + \Delta \varphi^\pm \right) + \Box \right\} \left( G^{(N)}\pm + \frac{1}{ik} G^{(N)}\pm \right) = k^{-N}H^\pm, \]
from which we can derive an estimate in a neighborhood of \( \Lambda_0 \)
\[ \left| \partial_t ^\pm \partial_\varphi ^\pm \partial_\varphi ^\pm \left( G^{(N)}+ + \frac{1}{ik} G^{(N)}+ \right) \right| \leq C_{N, a, \beta} k^{-N+a+a'+a''+b} \alpha^{-11N/4}E_a(v, \beta; 4N+a+a'+2b+1), \]
by taking account of the location of the support of \( G^{(N)+} + \frac{1}{ik} G^{(N)+} \) and the equation \( G^{(N)+} + \frac{1}{ik} G^{(N)+} \) must satisfy. Then we have
\[ \left< Bu^{(N)+} \right|_{\Lambda_0} e^{ik(\varphi - t)} \left< v \right|_{(\alpha), \beta} \leq C_{N, a, \beta} \left( k\alpha^{11/4} \right)^{-N} \sum_{g \in G_a} k^p E_a(v, \beta; 4N+l+3). \]
Combining the above estimate with (2.24) it holds that
\[ (2.25) \quad \left< Bu^{(N)} \right|_{\Lambda_0} e^{ik(\varphi - t)} \left< v \right|_{(\alpha), \beta} \leq C_{N, a, \beta} \left( k\alpha^{11/4} \right)^{-N} \sum_{g \in G_a} k^p E_a(v, \beta; 4N+l+3). \]
Next consider \( Bu^{(N)} \) on \( \Lambda_1 \).
\[ Bu^{(N)+} \bigg|_{\Lambda_1} = e^{ik(\varphi + t)} \left\{ ik \Phi^+ \left( G^{(N)}+ + \frac{1}{ik} G^{(N)}+ \right) + BG^{(N)+} + \frac{1}{ik} BG^{(N)+} \right\} \]
where
\[ G^{(N)+} = \sum_{j=0}^{N} \pi^{-1/2} \alpha^{-1/4} e^{ij(k^j g_0 + \sqrt{\rho} g_1)} k^{-j-1}. \]
Let us set
Applying (iii) of Lemma 2.1 we have
\[ w_1(x, t) = \gamma_a(x) \Phi^\perp \left( \frac{v}{\Phi^-} \right)(P_a(x, t)). \]
Then it holds that
\[
\sup |w_1| \geq \frac{1}{2} \left( \inf_{(s, r) \in \Lambda_1} |\Phi^+| / \sup_{(s, r) \in \Lambda_0} |\Phi^-| \right) \sup |v|.
\]
\[
\langle w_1 \rangle_{a, \alpha} \leq C_a \left\{ \sup_{(s, r) \in \Lambda_1} |\Phi^+| E_a(v, \beta; a) + E_a(v, \beta; a-1) \right\}.
\]
Set
\[ w_2(x, t) = i\Phi^+ \sum_{j=1}^N (g_{ik} \sqrt{\rho}) k^{-j} + i\Phi^+ \bar{G}^{(N)+} + i \frac{1}{ik} B \bar{G}^{(N)+}. \]
Then
\[
\langle w_2 \rangle_{a, \alpha} \leq C_{N, a} \sum_{j=1}^N (k\alpha^{1/4})^{-j} E_a(v, \beta; 4j+a)
\]
By the same consideration as $u^{(N)+}$ in $\Lambda_0$ we have
\[
\langle Bu^{(N)-} \rangle_{a, \alpha} \leq C_{N, a} (k\alpha^{1/4})^{-N} \sum_{p + q < a} k^p E_a(v, \beta; 4N + l + 3).
\]
Summarizing the considerations in this section we have

**Proposition 2.3.** Let $\alpha > 0$ and $v(x, t) \in C^0(\Gamma_a \times R)$ such that $\Lambda_0 \cap \Lambda_1 = \phi$. For every positive integer $N$ there exists a function $u^{(N)}(x, t; \alpha, k) \in C^\infty(R^2 \times R)$ satisfying
\[
\sup u^{(N)} \cap (\omega_a \times R) \subset \{ L_a(x, t); (x, t) \in \sup \psi \},
\]
\[
\sup Bu^{(N)} \mid_{\Gamma_a \times R} \subset \Lambda_0 \cup \Lambda_1,
\]
and the estimates (2.18), (2.23) and (2.25). And
\[
\langle Bu^{(N)} \rangle_{\Lambda_1} \leq (\psi - \theta) \omega \rangle_{a, \alpha} \leq C_{N, a} (k\alpha^{1/4})^{-N} \sum_{p + q < a} k^p E_a(v, \beta; 4N + l + 3)
\]
where $w$ has the following properties
\[
\sup |w| \geq \frac{1}{2} \left( \inf_{(s, r) \in \Lambda_1} |\Phi^+| / \sup_{(s, r) \in \Lambda_0} |\Phi^-| \right) \sup |v|.
\]
\[
- C_a \sum_{j=1}^N (k\alpha^{1/4})^{-j} E_a(v, \beta; 4j)
\]
\[
\langle w \rangle_{a, \alpha} \leq C_a \{(\sup |\Phi^+| + \beta) E_a(v, \beta; a)
\]
\[
+C_{N, a} \sum_{j=1}^N (k\alpha^{1/4})^{-j} E_a(v, \beta; 4j+a) \right\},
\]
where all the constants are independent of $\alpha$.

3. Asymptotic solutions reflected $K$-time at $\Gamma_{\alpha}$

Let $v(x, t) \in C^\infty_0(\Gamma_\alpha \times \mathbb{R})$ and supp $v \subset \Lambda_0$. Define $\Lambda_1, \Lambda_2, \ldots, \Lambda_K$ successively by

$$\Lambda_{j+1} = \{L^- (x, t) \cap (\Gamma_\alpha \times \mathbb{R}) - \{(x, t)\}; (x, t) \in \Lambda_j\}.$$ 

Suppose that

$$\bar{\Lambda}_j \subset \Gamma_\alpha \times (t_j, t_{j+1}), \quad t_0 < t_1 < \cdots < t_{K+1}.$$ 

Set

$$\beta = \inf_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B\psi^{-}|,$$

$$\nu = \inf_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B\psi^{+}| \sup_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B\psi^{-}|.$$ 

We assume for some constant $C_K$

$$\sup_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B\psi^{+}|/|\beta| \leq C_K \nu.$$ 

Apply Proposition 2.3 for

$$m_0(x, t; \alpha, k) = e^{ik^2(x_{-\alpha} - t)}v(x, t)$$

and have $u_0^{(N)}(x, t; \alpha, k)$ with the properties

$$\|u_0^{(N)}\|_{(a, \beta, b)} \leq C_{N, a, b} \sum_{j=0}^{N+\alpha+3} k^{j+\beta+\beta_{-1}} E_a(v, \beta; 4j+1),$$

$$\|\big\|u_0^{(N)}\|_{(a, \beta, b)} \leq C_{N, a, b} (k\alpha^2)^{-N} \sum_{j=0}^{N+4N+3} k^j E_a(v, \beta; 2l+4N+3),$$

$$\langle Bu_0^{(N)} \rangle_{\Lambda_0} - m_0 \langle (a), \alpha \rangle + Bu_0^{(N)} |_{\Lambda_1} - m_{1} \langle (a), \alpha \rangle \leq C_{N, a} (k\alpha^2)^{-N} \sum_{j=0}^{N+4N+3} k^j E_a(v, \beta; 4N+1+3),$$

where

$$m_1 = e^{ik^2(x_{-\alpha} - t)}v_1,$$

$$\text{supp} v \subset \Lambda_1,$$

$$\|v_1\| \geq \frac{\nu}{2} \sup |v| - C \sum_{j=1}^{N} (k\alpha^2)^{-N} E_a(v, \beta; 4j),$$

$$\langle v_1 \rangle_{(a), \alpha} \leq C_{a} (\sup |\Phi^{+}| + |\beta|) E_a(v, \beta; a) + C_{N, a} \sum_{j=1}^{N} (k\alpha^2)^{-j} E_a(v, \beta; 4j+a).$$
Since $\rho = \alpha$ on $\Gamma$, we have
\[
\varphi^+ = \theta + \frac{2}{3} \rho^{3/2} = \theta - \frac{2}{3} \rho^{3/2} + \frac{4}{3} \alpha^{3/2}
\]
\[
= \varphi^- + \frac{4}{3} \alpha^{3/2} \quad \text{on } \Gamma,
\]
from which follows
\[
m_1 = e^{i(k(\varphi^- - t))} \vartheta_1, \quad \vartheta_1 = e^{i(4k\alpha^{3/2})} v_1.
\]
Then $\vartheta_1$ verifies the properties (3.6)~(3.8).

Now the application of Proposition 2.3 to $m_1$ gives the existence of a function $u_1^{(N)}(x, t; \alpha, k)$ with the properties
\[
\tag{3.3}_1 \| u_1^{(N)} \|_{(\alpha), \beta} \leq C_{N, \alpha, \beta} \sum_{j=0}^{N} k^{s+3/2-j/3} E_a(v_1, \beta; 4j+1)
\]
\[
\tag{3.4}_1 \| \Box u_1^{(N)} \|_{(\alpha), \beta} \leq C_{N, \alpha, \beta} (k\alpha)^{-N} \sum_{p \ll \delta \leq b} k^p E_a(v_1, \beta; 2l+4N+3)
\]
\[
\tag{3.5}_1 \langle Bu_1^{(N)} |_{\lambda} - m_1 \rangle_{(\alpha), \beta} + \langle Bu_1^{(N)} |_{\lambda} - m_2 \rangle_{(\alpha), \beta}
\leq C_{N, \alpha, \beta} \sum_{p \ll \delta \leq b} k^p E_a(v_1, \beta; 4N+1+3).
\]

From (3.8) and the definition of $E_a(v_1, \beta; a)$ it follows
\[
E_a(v_1, \beta; a) = \sum_{p \ll \delta \leq b} \langle v_1 \rangle_{(\alpha), \beta} \beta^{-l-1}
\]
\[
\leq C_{N, \alpha, \beta} \sum_{j=0}^{N} (k\alpha)^{-j} E_a(v_1, \beta; 4j+p) \beta^{-l-1}
\]
\[
\leq C_{N, \alpha, \beta} \sum_{j=0}^{N} (k\alpha)^{-j} E_a(v_1, \beta; 4j+p) \beta^{-l-1}
\]
By using $E_a(v, \beta; p) \beta^{-l} \leq E_a(v, \beta; p+l)$, we have
\[
\tag{3.9}_1 E_a(v_1, \beta; a) \leq C_{N, \alpha, \beta} \sum_{j=0}^{N} (k\alpha)^{-j} E_a(v, \beta; 4j+a).
\]

From the second part of Proposition 2.3 $m_2$ can be represented as
\[
m_2(x, t; \alpha, k) = e^{i(k(\varphi^- - t))} v_2(x, t; \alpha, k)
\[
= e^{i(k(\varphi^- - t))} e^{i(4k\alpha^{3/2})} v_2 = e^{i(k(\varphi^- - t))} \vartheta_2,
\]
and $\vartheta_2$ verifies from (2.7) and the above estimate (3.9).
\[(3.7)_2\] \[\sup |\vartheta_2| \geq \frac{1}{2} \nu \left( \frac{1}{2} \nu \sup |v| - C_N \sum_{j=1}^{\infty} (k \alpha^3)^{-i} E_a(v, \beta; 4j) \right) \]

\[-C_N \sum_{j=1}^{\infty} (k \alpha^3)^{-i} \{C_a \sup |\Phi^+| + \beta\} |E_a(v, \beta; 4j)\} + C_{N, a} \beta^{-1} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + 4h)\} \]

\[\geq \left( \frac{1}{2} \nu \right)^2 \sup |v| - C_N \sum_{j=1}^{\infty} (k \alpha^3)^{-i} E_a(v, \beta; 4j) \]

\[-C_{N, a} \beta^{-1} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j).\]

\[(3.8)_2\] \[\langle \vartheta_2(x), a \rangle \leq C_a \sup |\Phi^+| + \beta \} E_a(v, \beta; a) + C_{N, a} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \]

\[\leq C_a \sup |\Phi^+| + \beta \} \{C_a C_v E_a(v, \beta; a) + C_{N, a} \beta^{-1} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \} + \beta^{-1} C_{N, a} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \]

\[\leq C_a \sup |\Phi^+| + \beta \} \cdot v \cdot E_a(v, \beta; a) + C_{N, a} \nu \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \]

\[+ C_{N, a} \beta^{-1} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a).\]

Repeating this process we obtain \(u_j^{(N)}(x, t; \alpha, k), j=0, 1, 2, \ldots, K\) verifying

\[(3.3)_j\] \[\|u_j^{(N)}\|_{(\alpha), a, b} \leq C_{N, a, b} \sum_{k=0}^{\infty} (k \alpha^3)^{a+b-k-1/5} E_a(v, \beta; 4h+1)\]

\[(3.4)_j\] \[\|\square u_j^{(N)}\|_{(\alpha), a, b} \leq C_{N, a, b}(k \alpha^3)^{-N} \sum_{\beta+1 \leq h} k^b E_a(v, \beta; 2l+4N+3)\]

\[(3.5)_j\] \[\langle B u_j^{(N)} \mid \Lambda_j - m_j \rangle_{(\alpha), a} + \langle B u_j^{(N)} \mid \Lambda_j \rangle_{(\alpha), a} - m_j + 1 \rangle_{(\alpha), a} \leq C_{N, a}(k \alpha^3)^{-N} \sum_{\beta+1 \leq a} k^b E_a(v, \beta; 4N+l+3), \]

\[m_j = e^{ik(v-\tau-t)} \tilde{\vartheta}_j, \]

\[\supp \vartheta_j \subset \Lambda_j,\]

\[(3.7)_j\] \[\sup |\vartheta_j| \geq \left( \frac{1}{2} \nu \right)^2 \sup |v| \]

\[-C_j^{(p)} \sum_{j=1}^{\infty} \nu^{j-1} \sum_{k=1}^{\infty} (k \alpha^3)^{-b} E_a(v, \beta; 4h)\]
By using \( \nu \leq C \beta^{-1} \) it follows from (3.8) that

\[
\langle \mathcal{B}_j \rangle_{(\alpha), a} \leq C_j \left( \sum \right) \beta^{-j-1} \sum (k \alpha^3)^{-h} E_a(v, \beta; 4h + a)
\]

\[
+ C_j \beta^{-1} \sum (k \alpha^3)^{-h} E_a(v, \beta; 4h + a)
\]

Set

\[
U^{(N)}(x, t; \alpha, k) = \sum_{j=0}^{\infty} (-1)^j u^{(N)}(x, t; \alpha, k).
\]

Then we have from (3.3)\(-\sim(3.10)\)

**Proposition 3.1.** Let \( \nu(x, t) \in C^0(\Gamma_x \times \mathcal{R}) \) such that

\[
\text{supp } \nu \subset \Lambda_0.
\]

Suppose that (3.1) and (3.2). Then there exists a function \( U^{(N)}_K(x, t; \alpha, k) \) with the following properties:

\[
\supp U^{(N)}_K \cap (\Omega \times \mathcal{R}) \subset \Omega \times (t_0, \infty)
\]

\[
||U^{(N)}_K||_{(\alpha), a, b} \leq C_{N,K,a,b} \sum_{i=0}^{K} \beta^{-j-1/5} \sum_{k=0}^{i} \sum_{h=0}^{K} (k \alpha^3)^{-h} E_a(v, \beta; 4h + K - l + 4j + 2)
\]

\[
<\mathcal{B}U^{(N)}_K|_{\Gamma_x \times (t_0, t_k)} - m_0(\alpha), a \leq C_{N,K,a} \sum_{i=0}^{K} \beta^{-j-1/5} \sum_{k=0}^{i} \sum_{h=0}^{K} (k \alpha^3)^{-h} E_a(v, \beta; 4h + K - q + 2l + 4N + 3)
\]

\[
\sup_{\Gamma_x \times (t_0, t_k)} |U^{(N)}_K| \geq \left( \frac{1}{2} \right) \sup |\nu| \left( \begin{array}{c} \frac{1}{2} \nu^{j-1} \sum (k \alpha^3)^{-h} E_a(v, \beta; 4h) \\
- C_N \sum_{i=0}^{K} \beta^{-j-1/5} \sum_{k=0}^{i} \sum_{h=0}^{K} (k \alpha^3)^{-h} E_a(v, \beta; 4h)
\end{array} \right),
\]

where the constants \( C_{N,K,a,b} \) and \( C_{N,K,a} \) are independent of \( \alpha \).
4. Proof of the theorem

Lemma 4.1. Suppose that \( \tau(0) = \tau'(0) = 0 \) and
\[
\sup_{s_0 < s < t} \tau(s) > 0
\]
for any \( \varepsilon > 0 \). Then there exist a constant \( \delta > 1/2 \) and a sequence
\[
s_1 > s_2 > \cdots > s_n > s_{n+1} > \cdots > 0
\]
with the following properties:
\[
\begin{aligned}
s_n &\to 0 \quad \text{as } n \to \infty \\
\beta_n &= \tau(s_n) > 0
\end{aligned}
\]
and for any positive integer \( K \) there exists a constant \( C_K \) such that
\[
\sup_{s \in [s_n, s_{n+1}]} \sup_{0 < t < K} \frac{\tau(s + t\beta_n) - \beta_n}{\beta_n^{1+\delta}} \leq C_K.
\]

Proof. When \( s = 0 \) is a zero of finite order, namely for some \( q \geq 1 \)
\[
\tau(0) = \tau'(0) = \cdots = \tau^{(q)}(0) = 0, \quad \tau^{(q+1)}(0) > 0
\]
it holds that for some \( s_0 > 0 \)
\[
|\tau'(s)| \leq C \tau(s)^{q/(q+1)} \quad \text{for } 0 < s < s_0.
\]
Since for \( s > 0, t > 0, \)
\[
|\tau(s+t\tau(s)) - \tau(s)| \leq t\tau(s)|\tau'(s+t\tau(s))| \quad (0 < \eta < 1)
\]
\[
\leq t\tau(s)\{|\tau'(s)| + t\eta \tau(s) (\sup \tau'')\}
\]
\[
\leq C_K \tau(s)^{1+q/(q+1)} \quad (0 < t < K),
\]
\( \delta = q/(q+1) \) and the sequence \( s_n = 1/n \) are the desired one.

Next consider the case that \( s = 0 \) is a zero of infinite order.

Case 1. \( \tau(s) \) is monotonically increasing in \( 0 < s < \varepsilon_0 \) for some \( \varepsilon_0 > 0 \).

Suppose that for some \( 1 > \delta > 0 \) there is no sequence with property (4.1) verifying
\[
\tau'(s_n) < \tau(s_n)^\delta, \quad \forall n.
\]
This assumption implies that it holds that for some \( \varepsilon_1 > 0 \)
\[
\tau'(s) > \tau(s)^\delta \quad \text{for } 0 < s < \varepsilon_1,
\]
from which it follows
\[
\frac{d}{ds} \tau(s)^{1-\delta} = (1-\delta) \tau(s)\tau'(s) \geq (1-\delta) \quad \text{for } 0<s<\varepsilon_1.
\]

Then we have
\[
\tau(s)^{1-\delta} \geq (1-\delta)s \quad \text{for } 0<s<\varepsilon_1,
\]
namely \(\tau(s) \geq (1-\delta)^{s/(1-\delta)}\). This is contradict with the assumption that \(\tau(s)\) has a zero of infinite order at \(s=0\). Then we see that for any \(1>\delta>0\) there exists \(\{s_n\}\) verifying (4.1) and (4.3). By using (4.3) and
\[
\tau(s_n+t\beta_n) - \beta_n = t\beta_n \tau'(s_n + \eta t\beta_n), \quad 0<\eta<1
\]
\[
|\tau'(s_n + \eta t\beta_n) - \tau'(s_n)| \leq t\beta_n \sup |\tau''(s)|
\]
we have for all \(0 \leq t \leq K\)
\[
|\tau(s_n+t\beta_n) - \beta_n| \leq K\beta_n (\tau'(s_n) + CK\beta_n) \leq C_1 \beta_n^{1+\delta}.
\]
Thus (4.2) is proved.

Case 2. For some \(\varepsilon_0>0\)
\[
\tau(s) > 0 \quad \text{for } 0<s<\varepsilon_0
\]
and \(\tau(s)\) is not monotonically increasing in \(0<s<\varepsilon\) for any \(\varepsilon>0\). From the assumption for any \(\varepsilon>0\) there exists \(s\) such that \(0<s<\varepsilon\) and \(\tau'(s)=0\). Then we can choose \(s_n>0\) with the property (4.1) such that \(\tau'(s_n)=0\). Then
\[
|\tau(s_n+t\beta_n) - \beta_n| \leq |\tau'(s_n + \eta t\beta_n)| \cdot t\beta_n
\]
\[
\leq CK^2 \cdot \beta_n^2 \quad \forall n.
\]
Thus \(\{s_n\}_{n=0}^\infty\) is the desired one.

Case 3. \(\tau(s)\) does not verify the properties of the case 1 nor 2. Then there exists a sequence \(\theta_n>\theta_{n+1}>\cdots \rightarrow 0\) such that \(\tau(\theta_n)=0\) and \(\sup_{s \in [\theta_{n+1}, \theta_n]} \tau(s)>0\), since for any \(\varepsilon>0\) there exists \(0<s<\varepsilon\) such that \(\tau(s)>0\). If we choose \(s_n\) as
\[
\tau(s_n) = \max_{s \in [\theta_{n+1}, \theta_n]} \tau(s),
\]
it holds that \(\tau(s_n)>0\) and \(\tau'(s_n)=0\). Evidently \(s_n \rightarrow 0\). As case 2 we see that this \(\{s_n\}\) verifies (4.2). Q.E.D.

Since \(n(x)=(n_1(x), n_2(x))\) may be considered as a \(C^\infty\)-vector defined in a neighborhood of \(\Gamma\)
\[
\eta(x) = b_1(x)n_2(x) - b_2(x)n_1(x)
\]
is also a \(C^\infty\)-function defined in a neighborhood of \(\Gamma\). We show that \((P)\) is not well posed in the sense of \(C^\infty\) when \(\tau(s)\) of the introduction, i.e., \(\tau(s)=\)
\( \eta(x(s)) \) verifies the condition on \( \tau(s) \) of Lemma 4.1. Note that

\[
\begin{align*}
\nabla \varphi^\pm &= \pm \sqrt{\rho} \left( \nabla \rho_0 + \alpha \nabla \rho_1 + \cdots \right) + \nabla \theta_0 + \alpha \nabla \theta_1 + \cdots \\
\text{and } n(x) \cdot \nabla \rho_0 &= |\nabla \rho_0|, \quad n(x) \cdot \nabla \theta_0 = 0 \quad \text{on } \Gamma^0.
\end{align*}
\]

Then we have

\[
\begin{align*}
n(x) \cdot \nabla \varphi^-(x, \alpha) &= \alpha^{1/2} \frac{\partial \rho}{\partial n} + O(\alpha) \quad \text{on } \Gamma \\
\nabla \theta(x, 0) \cdot \nabla \varphi^-(x, \alpha) &= 1 + O(\alpha) \quad \text{on } \Gamma.
\end{align*}
\]

Therefore \( n(x) \cdot \nabla \varphi^-(x, \alpha)/\nabla \theta(x, \alpha) \cdot \nabla \varphi^-(x, \alpha) \) decreases monotonically to zero uniformly in \( x \in \Gamma \) when \( \alpha \to +0 \). Let \( \{s_n\} \) be the sequence with the property (4.1) for the above \( \tau(s) \).

For every \( n \) set \( y_n = x(s_n) \). Then \( \alpha_n > 0 \) is determined uniquely for large \( n \) by the relation

\[
\frac{n(y_n) \cdot \nabla \varphi^-(y_n, \alpha_n)}{\nabla \theta(y_n, 0) \cdot \nabla \varphi^-(y_n, \alpha_n)} = \beta_n + \beta_n^{1/2}.
\]

From the above relations we have

\[
\begin{align*}
c_1 \beta_n &\leq \alpha_n^{1/2} \leq c_2 \beta_n \quad \forall n,
\end{align*}
\]

where \( c_1, c_2 \) are positive constants.

Note that for \( \alpha = 0 \)

\[
\nabla \theta \cdot \nabla \rho = 0, \quad |\nabla \theta| = 1 \quad \text{on } \Gamma.
\]

On the other hand \( x(s) \in \Gamma \) and \( \left| \frac{dx}{ds} \right| = 1 \). Then it follows that

\[
\theta(x(s), 0) = s + \text{constant}.
\]

Without loss of generality we may pose the constant=0. Since we have from (2.1) and the property (ii) of \( \rho \)

\[
\begin{align*}
\text{rank} \begin{bmatrix}
\frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \\
\frac{\partial \rho}{\partial x_1} & \frac{\partial \rho}{\partial x_2}
\end{bmatrix}_{x=s(0)} &= 2,
\end{align*}
\]

there exists uniquely \( x_0(s) \) verifying \( x_0(s) \to x(s) \) as \( \alpha \to 0 \) and

\[
\begin{align*}
\theta(x_0(s), 0) &= s \\
\rho_0(x_0(s), \alpha) &= \alpha
\end{align*}
\]

4) See, for example, pages 70 and 71 of [4].
for small $s$ and $\alpha$. Moreover we have

$$|x_a(s) - x(s)| \leq C \{|\rho(x_a(s), \alpha) - \rho(x(s), \alpha)| + |\theta(x_a(s), 0) - \theta(x(s), 0)|\}$$

$$\leq C |\alpha - \rho(x(s), \alpha)|.$$

Using (2.2) and $x(s) \in \Gamma$, we obtain for any $P > 0$

$$|x_a(s) - x(s)| \leq C_P \alpha^p.$$

Then we have

$$|\langle B\varphi_\pm\rangle(x_a(s), \alpha) - \langle B\varphi_\pm\rangle(x(s), \alpha)| \leq C_P \alpha^p$$

for all $\alpha > 0$ and $s$. Note that

$$\langle B\varphi_\pm\rangle(x, \alpha) = n(x) \cdot \nabla \varphi_\pm(x, \alpha) - \eta(x) \nabla \theta_0(x) \cdot \nabla \varphi_\pm(x, \alpha).$$

Then we have

$$\langle B\varphi^-\rangle(y_n, \alpha_n) = (\beta_n + \beta_n^{1+\gamma/2} - \tau(s_n)) \nabla \theta_0(y_n) \cdot \nabla \varphi^-(y_n, \alpha_n)$$

$$= \beta_n^{1+\gamma/2} \nabla \theta_0(y_n) \cdot \nabla \varphi^-(y_n, \alpha_n)$$

$$= \beta_n^{1+\gamma/2} (1 + O(\beta_n)).$$

Taking account of (4.4) it holds that

$$n(x(t+s)) \cdot \nabla \varphi^\pm(x(s+t)) - n(x(s)) \cdot \nabla \varphi^\pm(x(s))$$

$$= \pm \sqrt{\alpha} (|\nabla \rho_0(x(s+t))| - |\nabla \rho_0(x(s))|) + O(\alpha).$$

Since $|\nabla \rho_0(x)|$ is $C^\infty$ we have

$$|n(x(s_n + t\beta_n)) \cdot \nabla \varphi^\pm(x(s_n + t\beta_n), \alpha_n) - n(x(s_n)) \cdot \nabla \varphi^\pm(x(s_n), \alpha_n)|$$

$$\leq C t \beta_n^2 \quad \forall n.$$

By the same consideration it holds that

$$|\nabla \theta_0(x(s_n + t\beta_n)) \cdot \nabla \varphi^\pm(x(s_n + t\beta_n), \alpha_n) - \nabla \theta_0(x(s_n)) \cdot \nabla \varphi^\pm(x(s_n), \alpha_n)|$$

$$\leq C t \alpha_n \leq C t \beta_n^2 \quad \forall n.$$

Therefore we have for $0 \leq t \leq K$

$$|\langle B\varphi^-\rangle(x(s_n + t\beta_n), \alpha_n) - \langle B\varphi^-\rangle(x(s_n), \alpha_n)|$$

$$\leq |\tau(s_n + t\beta_n) - \tau(s_n)| + C K \beta_n^2.$$

Combining (4.2) and (4.7) it follows that

$$|\langle B\varphi^-\rangle(x(s_n + t\beta_n), \alpha_n)| - \beta_n^{1+\delta} | \leq C_K \beta_n^{1+\delta}$$

for all $0 \leq t \leq K$ and $n$. By the same consideration we have
for all $0 \leq t \leq K$ and $n$. Then by using (4.6), (4.7) and (4.9) or (4.10) we have

**Lemma 4.2.** Suppose that $\tau(s)$ is equipped with the properties of Lemma 4.1. Then for any $K > 0$ there exists a constant $C_K$ such that

\[
|B\varphi^+(x(s_n + t\beta_n), \alpha_n) - 2\beta_n| \leq C_K \beta_n^{1 + \frac{1}{2}}
\]

and

\[
|B\varphi^-(x(s_n + t\beta_n), \alpha_n) - \beta_n^{1 + \frac{1}{2}}| \leq C_K \beta_n^{1 + \frac{1}{2}}
\]

for all $0 \leq t \leq K$ and $n$.

Suppose that the problem $(P)$ is well posed in the sense of $C^\infty$. Then for any $T$ there exist $q$ and $C_T$ such that for all $t \leq T$

\[
|u_{q, \Omega \times (-\infty, t)}| \leq C_T \{ \| u_{q, \Omega \times (-\infty, t)} \| + \| Bu_{q, \Gamma \times (-\infty, t)} \| \}
\]

for all $u(x, t) \in C^\infty(\overline{\Omega} \times (-\infty, T))$ verifying $u = 0$ for $t \leq 0$, where

\[
|v|_{q, \Omega \times (-\infty, t)} = \sum_{|\gamma| \leq q} \sup_{\Omega \times (-\infty, t)} |D^\gamma_t v|
\]

and

\[
|v|_{q, \Gamma \times (-\infty, t)} = \sum_{|\gamma| \leq q} \sup_{\Gamma \times (-\infty, t)} |D^\gamma_t (\nabla \theta_0(x) \cdot \nabla) v|.
\]

On the supposition on $\tau(s)$ of Lemma 4.1 we will show the existence of a sequence of functions which violates (4.13).

Let $h(s, t) \in C^\infty_0(\mathbb{R}^2)$ such that

\[
\sup |h| = 1, \quad \text{supp } h \subseteq [0, 1] \times [0, 1].
\]

For each $n$ define $v_n(x, t) \in C^\infty_0(\Gamma_{a_n} \times \mathbb{R})$ by

\[
v_n(x_{a_n}(s), t) = h\left(\frac{s - s_n}{\alpha_n}, \frac{t}{\alpha_n}\right).
\]

Put

\[
\Lambda_{n_0} = \{(x_{a_n}(s), t); \ |s - s_n| \leq \alpha_n, 0 \leq t \leq \alpha_n\},
\]

and define $\Lambda_{n_j}$, $j=1, 2, \ldots, K$ according to the description in the beginning of §3. Since $c_2 \sqrt{\alpha_n} \leq |P_{a_n}(x, t) - (x, t)| \leq c_1 \sqrt{\alpha_n}$ it holds that

\[
\Lambda_{n_j} \subset \Gamma_{a_n} \times (t_{n_j}, t_{n_j+1})
\]

and

\[
0 = t_{n_0} < t_{n_1} < \cdots < t_{n_K} < c_1 K \sqrt{\alpha_n}.
\]

From Lemma 4.2 we have

\[
\inf_{(x, t) \in \mathbb{U}_j \Lambda_{n_j}} |B\varphi^-| \geq C_K \beta_n^{1 + \frac{1}{2}} \geq C_K \alpha_n,
\]
and
\[
\sup_{(x,t) \in \bigcup_{j=0}^{K} \Lambda_{aj}} |B\varphi^+| \leq C_K \beta_n^{\delta/2},
\]
\[
\inf_{(x,t) \in \bigcup_{j=0}^{K} \Lambda_{aj}} |B\varphi^-| \geq C_K \beta_n^{\delta/2},
\]
where \(C_K\) and \(C_K\) are independent of \(n\).

Let us fix \(K\) as
\[
\frac{1}{2} K \delta \geq 20g + 1
\]
and \(N\) as
\[
6N > 2K + 6.
\]
For each \(n\) we apply Proposition 3.1 and obtain \(U^{(N)}_{nk}(x,t;\alpha,k)\). Note that it holds that
\[
\langle v_n, (\alpha_n)_{a} \rangle \leq C_a \alpha_{n}^{-2}
\]
where \(C_a\) is a constant independent of \(n\). Then
\[
E_{\alpha_n}(v_n, \alpha_n; a) \leq C_a \alpha_{n}^{-(a+1)}.
\]
Setting \(k = \beta_n^{-20}\) we have
\[
\|U^{(N)}_{nk}\|_{(\alpha_n)_{a,b}} \leq C_{N,K,a,b} \sum_{j=0}^{N} \beta_n^{-20(a+b-j)}
\]
\[
\cdot \sum_{j=0}^{K} \sum_{k=j}^{K} (\beta_n^{-\alpha_3})^{-2} \alpha_n^{-2h-K}\alpha_n^{-21}\alpha_n^{-21-4N-3}
\]
\[
\leq C_{N,K,a,b} \beta_n^{-20(a+b)}.
\]
\[
\|U^{(N)}_{nk}\|_{(\alpha_n)_{a,b}} \leq C_{N,a,b}(\beta_n^{-\alpha_3})^{-2N}
\]
\[
\cdot \sum_{j=0}^{K} \sum_{k=j}^{K} \beta_n^{-20a} \sum_{r=0}^{\alpha_3} \sum_{s=0}^{\alpha_3} (\beta_n^{-\alpha_3})^{-2} \alpha_n^{-2h-K+1} \alpha_n^{-2N-3}
\]
\[
\leq C_{N,a,b} \beta_n^{6N} \beta_n^{-2K+6} \leq C_{N,a,b}
\]
\[
\langle BU^{(N)}_{nk}, \Gamma_{\alpha_n} \times (\nu_{0},\nu_{K}) \rangle |m_{\nu}(\alpha_n)_{a} \rangle \leq C_{N,a,b}
\]
\[
\sup_{(x,t) \in (\nu_{0},\nu_{K})} |U^{(N)}_{nk}| \geq \left(\frac{1}{2}\right)^{K} \beta_n^{-K\delta/2}
\]
\[
-C_{N} \sum_{j=0}^{K-1} \beta_n^{-20a} \sum_{s=0}^{\alpha_3} (\beta_n^{-\alpha_3})^{-2} \alpha_n^{-2h-1}
\]
\[
-C_{N} \beta_n^{-20a} \sum_{s=0}^{\alpha_3} (\beta_n^{-\alpha_3})^{-2} \alpha_n^{-2h-1}
\]
\[
\geq \left(\frac{1}{2}\right)^{K} \beta_n^{-K\delta/2} - C_{N,K} \beta_n^{-2(K-1)/2}.
\]
Since
\[ \langle m_0 \langle a_\sigma \rangle_x, e \rangle \leq C_\varepsilon \beta_\sigma^{-2\varepsilon} \]
we obtain by using (4.16), (4.18) and (2.2)
\[ (4.20) \quad |BU_{JK}(\sigma)|_{t, t \times \epsilon} \leq C_\varepsilon \beta_\sigma^{-2\varepsilon}. \]
Taking account of (2.3) the substitution of (4.17), (4.19) and (4.20) into (4.13) gives
\[
\left( \frac{1}{2} \right)^K \beta_\sigma^{-KN/2} - C_{N,K}\beta_\sigma^{-(K-1)N/2} \leq C_\varepsilon \beta_\sigma^{-2\varepsilon},
\]
which shows a contradiction, because \( K \) verifies (4.14) and \( \beta_\sigma \to 0 \) as \( n \to \infty \). Thus the theorem is proved.

Appendix

By a change of variables
\[ \rho(x) = \sigma \]
the equation \( \mathcal{L}\{a_1, a_2\} = \{h_0, h_1\} \) turns to
\[ \begin{cases} 
2 \frac{\partial a_0}{\partial t} + 2(\nabla \theta)^2 \frac{\partial a_0}{\partial y} + \Delta \theta \cdot a_0 + 2\sigma(\nabla \rho)^2 \frac{\partial a_1}{\partial \sigma} + (\nabla \rho)^2 a_1 \\
+ \sigma \Delta \rho a_1 = h_0 & \text{in } \sigma \geq 0 \\
2 \frac{\partial a_1}{\partial t} + 2(\nabla \theta)^2 \frac{\partial a_1}{\partial y} + \Delta \theta \cdot a_1 + 2(\nabla \rho)^2 \frac{\partial a_0}{\partial \sigma} + \Delta \rho \cdot a_0 = h_1 & \text{in } \sigma \geq 0
\end{cases} \]  
(A.1)

First consider how \( a_{ij}(y, t) = \left( \frac{\partial a_i}{\partial \sigma_j} \right)(0, y, t) \) is determined. Let us set
\[ h_i(\sigma, y, t) \sim \sum_{j=0}^{\infty} h_{ij}(y, t) \sigma^j, \quad l = 0, 1 \]
\[ (\nabla \theta)^2(\sigma, y) \sim \sum_{j=0}^{\infty} A_j(y) \sigma^j, \quad (\Delta \theta)(\sigma, y) \sim \sum_{j=0}^{\infty} C_j(y) \sigma^j \]
\[ (\nabla \rho)^2(\sigma, y) \sim \sum_{j=0}^{\infty} B_j(y) \sigma^j, \quad (\Delta \rho)(\sigma, y) \sim \sum_{j=0}^{\infty} D_j(y) \sigma^j \]
and
\[ a_i(\sigma, y, t) \sim \sum_{j=0}^{\infty} a_{ij}(y, t) \sigma^j. \]
Note that the facts \( A_0(y) \geq \varepsilon > 0 \) and \( B_0(y) \geq \varepsilon > 0 \) follow from the the proper
of $\theta$ and $\rho$. Substitute the above expansions into (A.1) and set equal the coefficients of $\sigma^j$ of the both sides of the equations. Then we have

\[(A.2)_0 \quad 2\frac{\partial a_{00}}{\partial t} + 2A_0 \frac{\partial a_{00}}{\partial y} + C_0 a_{00} + B_0 a_{10} = h_{00}\]

\[(A.3)_0 \quad 2\frac{\partial a_{10}}{\partial t} + 2A_0 \frac{\partial a_{10}}{\partial y} + C_0 a_{10} + B_0 a_{01} + D_0 a_{00} = h_{10}\]

and for $j \geq 1$

\[(A.2)_j \quad 2\frac{\partial a_{0j}}{\partial t} + 2 \sum_{i=0}^{j} A_i \frac{\partial a_{0j-i}}{\partial y} + \sum_{i=0}^{j} C_i a_{0j-i} + 2 \sum_{i=0}^{j-1} (j-l) B_i a_{ij-l} + \sum_{i=0}^{j} D_i a_{ij-l} = h_{0j}\]

\[(A.3)_j \quad 2\frac{\partial a_{1j}}{\partial t} + 2 \sum_{i=0}^{j} A_i \frac{\partial a_{1j-i}}{\partial y} + \sum_{i=0}^{j} C_i a_{1j-i} + 2 \sum_{i=0}^{j} B_i (j+1-l) a_{0j+1-l} + \sum_{i=0}^{j} D_i a_{0j-l} = h_{1j}\]

Then if we set $a_0(y, t) = 0$, (A.2) determines $a_{10}$ and subsequently (A.3) determines $a_{01}$. In (A.2) all terms except $a_{10}$ are determined, therefore $a_{10}$ is determined, and next (A.3) determines $a_{01}$. Continuing this process we obtain successively $a_{ij}$, $j = 0, 1, \ldots$. By the manner of determining $a_{ij}$ it holds that

\[(A.4) \quad \sum_{|\gamma| \leq \sigma} \{\sup_{t \in \Delta} |D^\gamma a_{0j+1}(y, t)| + \sup_{t \in \Delta} |D^\gamma a_{1j}(y, t)|\} \leq C_a \sum_{|\gamma| \leq \sigma} \sum_{j \geq 0} \sup_{t \in \Delta} |D^\gamma h_j(y, t)|.\]

If we set $\hat{a}_i(\sigma, y, t) = \sum_{j=0}^{b} a_{ij}(y, t) \sigma^j$, the estimate (A.4) gives

**Lemma A.1.** For any $b$ positive integer there exists $\{a_0, a_1\}$ such that $a_0(0, y, t) = 0$ and

\[(A.5) \quad \sum_{|\gamma| \leq \sigma} \sum_{k \geq 0} \sup_{t \in \Delta} |D^\gamma f(t)D^k a_1| \leq C_a \sum_{|\gamma| \leq \sigma} \sum_{k \geq 0} \sup_{t \in \Delta} |D^\gamma f(t)D^k h_1|,\]

\[(A.6) \quad \sum_{|\gamma| \leq \sigma} \sup_{t \in \Delta} |D^\gamma f(t)\{-L_{\sigma}\{a_0, a_1\} - \{h_0, h_1\}\}| \leq |\sigma|^{b+1} C_a \sum_{|\gamma| \leq \sigma} \sum_{k \geq 0} \sup_{t \in \Delta} |D^\gamma f(t)D^k h_1(\sigma, y, t)|\]

Next consider that case

\[(A.7) \quad D^p h_1(0, y, t) = 0 \quad \text{for } p = 0, 1, 2, \ldots, b.\]

If we claim $a_0 = 0$ on $\{\sigma = 0\}$ the solution of (A.1) is given for $\sigma > 0$ by
\[ a_0(\sigma, y, t) = \frac{1}{2} \left\{ G^+(\sqrt{\sigma}, y, t) + G^+(-\sqrt{\sigma}, y, t) \right\} \]

\[ a_1(\sigma, y, t) = \frac{1}{2\sqrt{\sigma}} \left\{ G^+(\sqrt{\sigma}, y, t) - G^+(-\sqrt{\sigma}, y, t) \right\} , \]

where \( G^+(z, y, t) \) is the solution of

\[
\mathcal{L}^+ G^+ = \left( 2 \frac{\partial}{\partial t} + 2(\nabla \theta)^2(y, x^2) \frac{\partial}{\partial y} + 2(\nabla \rho)^2(y, x^2) \frac{\partial}{\partial z} \right) \]

\[ + (\Delta \theta)(y, x^2) + z(\Delta \tau)(y, x^2) \right) G^+(z, y, t) = H^+(z, y, t) \]

\[ G^+(0, y, t) = 0 \]

\[ H^+(z, y, t) = h_0(z^2, y, t) + z h_1(z^2, y, t). \]

The assumption (A.7) implies that for \( r \leq b, \ |\gamma| \leq a \)

\[ |D_{\gamma} D_{\gamma}^y, H^+(z, y, t)| \leq C_{a,b} K_{a,b} |z|^{2b+2-r} \]

\[ K_{a,b} = \sum_{|\gamma| \leq a} \sum_{|\mu| \leq a} \sup |D_{\mu} D_{\gamma}^y h_i(\sigma, y, t)|. \]

Therefore it holds that

\[ \sum_{|\gamma| \leq a} |D_{\gamma} D_{\gamma}^y, G^+(z, y, t)| \leq C_{a,b} K_{a,b} |z|^{2b+3-r}, \]

from which it follows immediately that

\[ \sum_{|\gamma| \leq a} \sum_{|\mu| \leq a} \sup |D_{\gamma} D_{\gamma}^y a_i(\sigma, y, t)| \leq C_{a,b} K_{a,b}, \sigma > 0. \]

Using \( (a_0 - \sqrt{\rho} a_i)(\alpha, y, t) = G^+(y, t, -\sqrt{\alpha}) \) we have

**Lemma A.2.** On the supposition (A.7) there exists a solution of (A.1) verifying \( a_0(0, y, t) = 0 \) and it holds that

\[ \sum_{|\gamma| \leq a} \sum_{|\mu| \leq a + 2b - r} \sup |D_{\mu} D_{\gamma}^y a_i(\sigma, y, t)| \leq C_{a,b} \sum_{|\gamma| \leq a} \sup |D_{\gamma} D_{\gamma}^y h_i(\sigma, y, t)| \]

and

\[ \sum_{|\gamma| \leq a} \sup |D_{\gamma}^y a_i(\alpha, y, t)| \leq C_{a,b} \sum_{|\gamma| \leq a} \sup |D_{\gamma}^y h_i(\sigma, y, t)|. \]

5) See, § 1 of Ludwig [6] and Lemma 5.2 of Ikawa [4].
When \( h_1 \equiv 0 \), the solution of (A.1) verifying 
\[
a_0 - \sqrt{\rho} a_1 \big|_{\sigma = 0} = f(y, t)
\]
is given by (A.8) where \( G^+ \) is the solution of
\[
\begin{align*}
\mathcal{L}^+ G^+ &= 0 \\
G^+(-\sqrt{\alpha}, y, t) &= f(y, t).
\end{align*}
\]
Evidently
\[
\sum_{|\gamma| < a} |D_\gamma^{\sigma} D_{\gamma, t} a_0| \leq \sum_{|\gamma| < a} |D_\gamma^{\sigma} D_{\gamma, t} G^+(x, y, t)|
\]
\[
\sum_{|\gamma| < a} |D_\gamma^{\sigma} D_{\gamma, t} a_1| \leq \sum_{|\gamma| < a} \sum_{|\eta| < \alpha} |D_\eta^{\sigma} D_{\gamma, t} G^+(x, y, t)|.
\]
And we see easily that
\[
\sum_{|\gamma| < a} |D_{\gamma, x, t} G^+(x, y, t)| \leq C_\sigma \sum_{|\gamma| < a} |D_{\gamma, t} f(y, t)|.
\]
Thus we have

**Lemma A.3.** When \( h_0, h_1 \equiv 0 \), the solution of (A.1) verifying \( a_0 - \sqrt{\rho} a_1 \big|_{\sigma = 0} = f \) has the estimate
\[
\sum_{j=0}^{\beta} \sum_{j=0}^{\beta} \sum_{|\gamma| < \alpha + 2(\beta - j)} \sup |D^{\sigma}_{\gamma, t} h_a(\sigma, y, t)| \leq C_{a, \delta} \sum_{j=0}^{\beta} \sum_{j=0}^{\beta} \sum_{|\gamma| < \alpha + 2(\beta - j)} \sup |D^{\sigma}_{\gamma, t} h_a(\sigma, y, t)|.
\]
To show (i) of Lemma 2.1 for fixed integer \( b \) first apply Lemma A.1 and we obtain \( \{a_0, a_1\} \) satisfying (A.6), and next apply Lemma A.2 to \( \mathcal{L}_a \{a_0, a_1\} - \{h_0, h_1\} \) then we have \( \{b_0, b_1\} \) verifying
\[
\mathcal{L}_a \{b_0, b_1\} = \{h_0, h_1\} - \mathcal{L}_a \{a_0, a_1\}.
\]
By using (A.5), (A.6) and (A.9) we have
\[
\sum_{j=0}^{\beta} \sum_{j=0}^{\beta} \sum_{|\gamma| < \alpha + 2(\beta - j)} \{|D^{\sigma}_{\gamma, t} D^{\sigma}_{\gamma, t} (a_0, a_1)| + |D^{\sigma}_{\gamma, t} D^{\sigma}_{\gamma, t} (\sigma, y, t)|\} \leq C_{a, \delta} \sum_{j=0}^{\beta} \sum_{j=0}^{\beta} \sum_{|\gamma| < \alpha + 2(\beta - j)} \sup |D^{\sigma}_{\gamma, t} h_a(\sigma, y, t)|.
\]
Moreover it follows form (A.5) and (A.10) that
\[
\sum_{|\gamma| < \alpha + 2a} \sup |D^{\sigma}_{\gamma, t} ((\bar{a}_0 + b_0) - \sqrt{\rho} (\bar{a}_1 + b_1))|_{\rho = a} \leq C_{a, \delta} \sum_{j=0}^{\beta} \sum_{j=0}^{\beta} \sum_{|\gamma| < \alpha + 2(\beta - j)} \sup |D^{\sigma}_{\gamma, t} h_a(\sigma, y, t)|.
\]
Then using Lemma A.3 we have \( \{c_0, c_1\} \) verifying
\[
\begin{align*}
\begin{cases}
-L_\omega^\tau \{e_0, e_1\} = 0 & \text{in } \rho \geq 0 \\
\partial_\rho e_0 - \sqrt{\rho} e_1 |_{\rho=a} = f - ((a_0 + b_0) - \sqrt{\rho} (a_1 + b_1)) |_{\rho=a}.
\end{align*}
\]

Then we see immediately that \(a_t = a_i + b_t + c_t, \ l = 0, 1\) are solutions of the problem (A.1) verifying the boundary condition and they satisfy the estimate of (i) of Lemma 2.1.

References


