On the mixed problems for the wave equation in an interior domain. II

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1. Introduction. Let $\Gamma$ be a simple closed curve in $\mathbb{R}^2 = \{(x_1, x_2); x_j \in \mathbb{R}, j=1, 2\}$ and $\Omega$ be its interior domain. Consider a mixed problem

\[
\begin{aligned}
\Box u &= \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{in } \Omega \times (0, \infty) \\
Bu &= b_1(x) \frac{\partial u}{\partial x_1} + b_2(x) \frac{\partial u}{\partial x_2} + d(x)u(x) = 0 \quad \text{on } \Gamma \times (0, \infty) \\
u(x, 0) &= u_0(x) \\
\frac{\partial u}{\partial t}(x, 0) &= u_1(x),
\end{aligned}
\]

where $b_j(x), j=1, 2$ and $d(x)$ are $C^\infty$-functions defined in a neighborhood of $\Gamma$. We suppose that $b_j(x), j=1, 2,$ are real valued and satisfy

\[(1.1) \quad b_1(x)n_1(x) + b_2(x)n_2(x) = 1 \quad \text{on } \Gamma \]

where $n(x) = (n_1(x), n_2(x))$ denotes the unit inner normal of $\Gamma$ at $x$.

Let $x(s), 0 \leq s \leq L$ be a representation of $\Gamma$ by the arc length $s$. Set

\[\tau(s) = [b_1(x)n_2(x) - b_2(x)n_1(x)]_{s=x(s)}.\]

The result we want to show is the following

**Theorem.** Suppose that the curvature of $\Gamma$ never vanishes. In the case of $\tau(s) \neq 0$ in order that $(P)$ is well posed in the sense of $C^\infty$ it must holds that

\[(1.2) \quad |\tau(s)| + \left| \frac{d\tau(s)}{ds} \right| \neq 0 \quad \text{for all } s.\]

We should like to give some remarks on the theorem. If $\tau(s) \equiv 0$ the boundary condition is nothing but the Neumann condition or the boundary condition of the third kind. Then it is well known that $(P)$ is well posed in the sense of $L^2$. And when $\tau(s) \neq 0$ for all $s$ the mixed problem $(P)$ is also well posed in the sense of $C^\infty$, that is shown in [1]. In both cases the results are

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still valid without the assumption of the convexity of $\Omega$.

In the preceding paper [5] we gave a necessary condition for the well posedness of $(P)$. There we introduced an index $I_B(p_0, \xi_0; T)$ of a broken ray according to the geometrical optics with respect to the coefficients of the boundary operator and it is proved that the condition

$$I_B(p_0, \xi_0; T) < C_T, \quad \forall p_0 = (x_0, t_0) \in \Gamma \times (0, T), \quad \xi_0 \in \Sigma_{x_0}$$

is necessary for the well posedness. It is easy to verify that the supposition

$$\sup_{p_0, \xi_0} I_B(p_0, \xi_0; T) = \infty$$

implies that $\tau(s) \equiv 0$ and $\tau(s)$ has at least a zero of infinite order. Therefore the theorem of this paper is an improvement of the result of [5].

2. Asymptotic solutions with a caustic

From now on, we suppose that the curvature of $\Gamma$ never vanishes. Then there exist functions $\theta(x, \alpha)$ and $\rho(x, \alpha)$ with the following properties:

(i) $\theta$ and $\rho$ are real valued $C^\infty$ function defined in \{(x, \alpha); x \in \mathbb{R}^2, \alpha \in [-\alpha_0, \alpha_0]\} where $\alpha_0$ is a positive constant.

(ii) $\frac{\partial \rho}{\partial n} > c > 0$ for $x \in \Gamma$

where $\frac{\partial}{\partial n} = \sum_{j=1}^{2} n_j(x) \frac{\partial}{\partial x_j}$.

(iii) Let us set

$$\Gamma_\alpha = \{x; \rho(x, \alpha) = \alpha\}$$

$$\omega_\alpha = \{x; \rho(x, \alpha) > 0\}.$$  

Then for all $\alpha$ it holds that

$$(2.1) \begin{cases} (\nabla \theta)^2 + \rho(\nabla \rho)^2 = 1 & \text{in } \omega_\alpha \\ \nabla \theta \cdot \nabla \rho = 0 & \text{in } \omega_\alpha \end{cases}$$

and

$$(2.2) \quad \rho(x, \alpha) \equiv \alpha \pmod{\alpha^n} \quad \text{on } \Gamma.$$  

For $u(x, t) \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$ we set

$$||u||_{(\omega), \alpha, \beta} = \sum_{\alpha < \beta} \sup_{x \in \mathbb{R}} |\partial_\alpha^\beta \partial_x^\delta u(x, t)|$$

1) See, for example, Appendix C of Ludwig [7], §5 of Ikawa [4].

2) Hereafter, we will use $c$ for various constants independent of $\alpha$ and $k$. 
\[ \langle u \rangle_{(\alpha),a} = \sum_{\frac{-\pi}{a} \leq \varphi \leq \frac{\pi}{a}} \sup_{t \leq s \leq T_a \times R} \left| \partial_t^\alpha \partial_s^2 u(x, t) \right|, \]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^2 \) containing \( \Omega \) and

\[
\partial_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}, \quad \partial_s^2 = \left( \sum_{j=1}^2 \frac{\partial^\alpha}{\partial x_j} \frac{\partial}{\partial x_j} \right)^2 \quad \text{and} \quad \partial_t^\alpha = \left( \sum_{j=1}^2 \frac{\partial^\alpha}{\partial x_j} \frac{\partial}{\partial x_j} \right)^\alpha.
\]

Let us denote

\[
|u|_{\alpha,a} = \sum_{|\beta| \leq \alpha} \sup_{\Omega \times R} \left| D_{(\beta)}^\alpha u(x, t) \right|
\]

\[
|u|_{\Gamma_a} = \sum_{\frac{-\pi}{a} \leq \varphi \leq \frac{\pi}{a}} \sup_{[0,1]} \left| \partial_t \partial_s^2 u(x(t), t) \right|.
\]

Taking account of

\[
\frac{|D(\theta, \rho)|}{|D(x_1, x_2)|} \geq c > 0 \quad \text{for all } \alpha
\]

it holds that for all \( u \in C^\infty(\Omega \times (\mathbb{R}^2 \times \mathbb{R})) \) and \( \alpha \)

\[
(2.3) \quad |u|_{\alpha, 2\alpha} \leq C_\alpha \|u\|_{(\alpha),a,a}
\]

where \( C_\alpha \) is independent of \( \alpha \).

Define

\[
\varphi^\pm(x, \alpha) = \varphi(x, \alpha) \pm 2/3 \rho(x, \alpha)^{3/2}.
\]

Let \( v(x, t) \in C^\infty(\Gamma_a \times \mathbb{R}) \) and set for \( \alpha > 0 \)

\[
m(x, t; \alpha, k) = e^{ik(\varphi^-(x, \alpha)-t)} v(x, t)
\]

We construct a function \( u(x, t; \alpha, k) \) in the form

\[
(2.4) \quad u(x, t; \alpha, k) = e^{i\theta(\theta(x, \alpha)-t)} \left\{ V(k^{2/3} \rho(x, \alpha)) g_0(x, t; \alpha, k) + \frac{1}{tk^{1/3}} V''(k^{2/3} \rho(x, \alpha)) g_1(x, t; \alpha, k) \right\}
\]

so that it may verify

\[
(2.5) \quad \left\{ \begin{array}{l}
\Box u = 0 \\
Bu |_{\Gamma_a} = m(x, t; \alpha, k)
\end{array} \right. \quad \text{in } \Omega \times \mathbb{R}
\]

\[
Bu |_{\Gamma_a} = m(x, t; \alpha, k) \quad \text{on the support of } v
\]

asymptotically as \( k \to \infty \), where \( V(z) = Ai(-z) \) with the Airy function \( Ai(z) \).

Apply \( \Box \) for \( u(x, t; \alpha, k) \) of (2.4) and use \( V''(z) + zV(z) = 0, \ V'''(z) + zV'(z) + V(z) = 0 \). Then we have
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(2.6) \( \square u = -e^{ik(\theta-t)} \left[ \frac{1}{(k^{3/2} \rho)} \left\{ (ik)^2(\nabla \theta)^2 + \rho(\nabla \rho)^2 - 1 \right\} g_0 + 2(ik)^2 \rho \nabla \rho \cdot \nabla \theta \int g_1 + ik \left( 2 \frac{\partial g_0}{\partial t} + 2 \nabla \theta \cdot \nabla g_0 + \Delta \theta \cdot g_0 \right) + 2\rho \nabla \rho \cdot \nabla g_1 + (\nabla \rho)^2 g_1 + \rho \Delta \rho \cdot g_1 \right] \)

Note that \( V(z) \) and \( V'(z) \) have the following asymptotic expansions for \( z \to +\infty \):

\[
V(z) = \frac{1}{2} \pi^{-1/2} z^{-1/4} \left\{ e^{i(t - \pi/4)}(1 + \xi^{-1} P_4(\xi)) + e^{-i(t - \pi/4)}(1 + \xi^{-1} P_4(\xi)) \right\}
\]

\[
V'(z) = \frac{1}{2} \pi^{-1/2} z^{1/4} \left\{ e^{i(t - \pi/4)}(1 + \xi^{-1} P_3(\xi)) - e^{-i(t - \pi/4)}(1 + \xi^{-1} P_3(\xi)) \right\},
\]

where \( \xi = \frac{2}{3} z^{3/2} \) and \( \rho = \frac{1}{2} \pi^{-1/2} z^{-1/4} \left\{ e^{i(t - \pi/4)}(1 + \xi^{-1} P_4(\xi)) + e^{-i(t - \pi/4)}(1 + \xi^{-1} P_4(\xi)) \right\} \),

\[
P_x(\xi) \sim \sum_{p_{ji}\in \mathcal{C}} p_{ji}, \quad p_{ji} \in \mathcal{C}.
\]

Therefore the function \( u \) in the form (2.4) may be represented for large \( k^{3/2} \rho \) as follows:

(2.7) \[
U(x, t; \alpha, k) = e^{ik(\theta-t)} \left( G^+ + \frac{1}{ik} G^+ \right) + e^{ik(\theta-t)} \left( G^- + \frac{1}{ik} G^- \right)
\]

where

\[
G^+ = \frac{1}{2\sqrt{\pi}} \rho^{-1/4} k^{-1/6} e^{-\pi i/4} (g_0 + \sqrt{\rho} g_1)
\]

\[
G^+ = \frac{3}{4} \pi^{-1/2} k^{-1/6} \rho^{-7/4} e^{-\pi i/4} (P_4 g_0 + \sqrt{\rho} P_3 g_1)
\]

\[
G^- = \frac{3}{4} \pi^{-1/2} k^{-1/6} \rho^{-7/4} e^{\pi i/4} (P_2 g_0 - \sqrt{\rho} P_4 g_1)
\]

From the form of \( G^\pm \) it holds that

(2.8) \[
|\partial_x^\alpha \partial_x^\beta \bar{G}^\pm| \leq C_4 k^{-1/8} \sum_{j=0}^3 \left\{ \rho^{-7/4} ||g||_{(4,1)} + \rho^{-11/4} ||g||_{(4,0)} \right\}
\]

3) See Miller [8], page B 17.
when $k^{2/3} \rho > C$.

Applying the operator $B$ to $u$ of (2.7) we have

\[(2.9) \quad Bu = e^{ik(\varphi + t)} \left\{ ik\Phi^+ \left( G^+ + \frac{1}{ik} G^+ \right) + B G^+ + \frac{1}{ik} B G^+ \right\} + e^{ik(\varphi - t)} \left\{ ik\Phi^- \left( G^- + \frac{1}{ik} G^- \right) + B G^- + \frac{1}{ik} B G^- \right\},\]

where $\Phi^\pm = \sum_{j=1}^{L} b_j(x) \frac{\partial \varphi^\pm}{\partial x_j}$.

Suppose that $g_0$ and $g_1$ have the following asymptotic expansion with respect to $k^{-1}$ when $k \to \infty$

\[(2.10) \quad g_l(x, t; \alpha, k) \sim \sum_{j=0}^{\infty} g_{lj}(x, t; \alpha, k)k^{1/6-j}, \quad l = 0, 1.
\]

Denote by $\mathcal{L}_a$ a differential operator from $(C^\omega(\mathbb{R}^2 \times \Gamma))^2$ into itself defined by for $\{a_1, a_2\}$

\[
\mathcal{L}_a \{a_1, a_2\} = \left\{ \frac{\partial a_1}{\partial t} + 2\nabla \theta \cdot \nabla a_1 + \Delta \theta a_1 + 2\rho \nabla \rho \cdot \nabla a_1 + (\nabla \rho)^2 a_2 \\
+ \rho \Delta \rho a_2, \frac{\partial a_2}{\partial t} + 2\nabla \theta \cdot \nabla a_2 + \Delta \theta a_2 + 2\nabla \rho \cdot \nabla a_1 + \Delta \rho a_1 \right\}.
\]

Substituting $g_0, g_1$ of (2.10) into (2.6) and (2.9) we claim that all the coefficients of $k^{-i}$ of (2.6) are equal to zero and those of $Bu - m$ are also equal to zero on the support of $v$. Then it must hold that

\[(2.11)_0 \quad \mathcal{L}_a \{g_{00}, g_{10}\} = 0 \]

\[(2.12)_0 \quad i\Phi^- (g_{10} - \sqrt{\rho} g_{10}) = 2\pi \alpha \sqrt{\rho} e^{i\varphi_0} \quad \text{on } \Gamma_a \times \mathbb{R}
\]

and for $j \geq 1$

\[(2.11)_j \quad \mathcal{L}_a \{g_{0j}, g_{1j}\} = \frac{1}{i} \left\{ \square g_{0j-1}, \square g_{1j-1} \right\}
\]

\[(2.12)_j \quad i\Phi^- (g_{0j} - \sqrt{\rho} g_{1j}) = i\Phi^- G^-_{j-1} + B G^-_{j-1} + \frac{1}{ik} B G^-_{j-1} \quad \text{on } \Gamma_a \times \mathbb{R}
\]

where $G^\pm$ and $G^\pm$ denote the $G^\pm$ and $G^\pm$ corresponding to the pair of $k^{1/6} g_j$ and $k^{1/6} g_{1j}$.

To obtain the existence and the estimates of $g_{0j}, g_{1j}$ satisfying (2.11) and (2.12), admit the following Lemma, whose proof will be given in the appendix.

**Lemma 2.1.** For $\{h_0, h_1\} \in (C^\omega(\mathbb{R}^2 \times \Gamma))^2$ and $f \in C^\omega(\Gamma_a \times \mathbb{R})$ there exists $\{a_1, a_2\} \in (C^\omega(\mathbb{R}^2 \times \Gamma))^2$ satisfying
\begin{align*}
&\begin{cases}
-L_a \{a_1,a_2\} = \{h_0, h_3\} & \text{in } \omega_a \times R \\
a_1 - \sqrt{\rho} a_2 = f & \text{on } \Gamma_a \times R
\end{cases}
\end{align*}

and having the following properties:

(i) \( \|a_j\|_{(a),\alpha,\beta} \leq C_{a,\beta} \langle (f)_{(a),\alpha+2b+j} + \sum_{j=0}^{\delta} \sum_{\ell=0}^{\delta} \|h_k\|_{(a),\alpha+2(3-\beta)\ell} \rangle \)

(ii) When \( \bigcup_{i=0}^{1} \text{supp } h_i \cap \omega_a \subset \{L^*(x, t); (x, t) \in \text{supp } f\} \), it holds that
\[ \bigcup_{i=0}^{1} \text{supp } a_i \cap \omega_a \subset \{L^*(x, t); (x, t) \in \text{supp } f\} , \]

(iii) When \( \{h_0, h_1\} \equiv 0 \), for \( (x, t) \in \Gamma_a \times R \)
\[ a_1 + \sqrt{\rho} a_2 (x, t) = \gamma(x, t; \alpha)f(P_a(x, t)) \]
where \( \gamma(x, t; \alpha) \) is a \( C^\infty \) function on \( R^2 \times R \times [-\alpha_0, \alpha_0] \) such that
\[ \gamma(x, t; \alpha) \geq C > 0 \]
and \( P_a(x, t) \) denotes the point
\[ L^+(x, t) \cap (\Gamma_a \times R) - \{(x, t)\} , \]
where \( L^+(x, t) \) denotes a line passing \( (x, t) \) defined by
\[ L^+(x, t) = \{(x + l \nabla \varphi^+(x, \alpha), t + l); l \in R\} . \]

Let \( \Lambda_0 \) be an open set in \( \Gamma_a \times R \) such that \( \Lambda_0 \supset \text{supp } v \). Set
\[ \Lambda_1 = \{L^-(x, t) \cap (\Gamma_a \times R) - \{(x, t)\}; (x, t) \in \Lambda_0\} . \]

Suppose that
\[ (2.13) \quad \Lambda_0 \cap \Lambda_1 = \phi . \]

Let us set
\[ \beta = \inf_{(x, t) \in \Lambda_0} |\Phi^-| . \]

Using the above lemma we have \( g_{00} \) and \( g_{10} \) verifying
\[ \begin{cases}
-L_a \{g_{00}, g_{10}\} = 0 & \text{in } \omega_a \times R \\
g_{00} - \sqrt{\rho} g_{10} = \frac{2\pi \alpha^{1/4} e^{\xi(x, t)}}{i\Phi^-} & \text{on } \Gamma_a \times R
\end{cases}
\]
and the estimate
\[ \sum_{j=0}^{1} \|g_{10}\|_{(a),\alpha,\beta} \leq C_{a,\beta} \langle 2\pi \alpha^{1/4} e^{\xi(x, t)} \rangle_{(a),\alpha+2b+1} . \]
Taking account of \( \langle \Phi^- \rangle_{(a),\alpha} \leq C_{a} \) for all \( \alpha > 0 \), we have
Then it holds that
\[ \sum_{j=0}^{l} \| g_{i,j} \|_{(a),a,b} \leq C \alpha^{1/4} \sum_{v=1}^{x \leq 2b+1} \langle v \rangle_{(a)}, \langle (\Phi^{-1}) \rangle_{(a),b} \leq C_{a,b} \alpha^{1/4} \sum_{v=1}^{x \leq 2b+1} \langle v \rangle_{(a),b} \beta^{-1/4}. \]

Let us set
\[ E_{a}(v, \beta; j) = \sum_{\beta \leq j} \langle v \rangle_{(a),b} \beta^{-1/4}. \]

Remark that the constant \( C_{a,b} \) depends on \( a \) and \( b \) but independent of \( \alpha \).

Next consider \( g_{01} \) and \( g_{11} \). Applying (2.8) to \( k^{1/6} g_{01} \) and using (2.14) we have
\[ \rho \left( \begin{array}{c} C \alpha^{1/4} E_{a}(v, \beta; a+3) + \beta^{-1/4} E_{a}(v, \beta; a+1) \end{array} \right) \]
for \( \rho k^{2/3} > C \). Then, noting (2.2), it follows that
\[ \langle (\Phi^{-1}) G_0 + B G_0 + \frac{1}{ik} B G_0 \rangle_{(a),a} \leq C_{a} \alpha^{-5/2} E_{a}(v, \beta; a+3). \]

Therefore
\[ \langle (\Phi^{-1}) G_0 + B G_0 + \frac{1}{ik} B G_0 \rangle_{(a),a} \leq C_{a} \sum_{v \leq x \leq a+1} \alpha^{-5/2} E_{a}(v, \beta; l+3) \cdot \beta^{-1/4} \]
\[ \leq C_{a} \alpha^{-5/2} E_{a}(v, \beta; a+4). \]

From (2.14) we have
\[ \| g_{i,j} \|_{(a),a,b} \leq C_{a,b} \alpha^{1/4} E_{a}(v, \beta; a+2b+4+1). \]

With the aid of (2.15) and the above estimate Lemma 2.1 assures the existence \( g_{01} \) and \( g_{11} \) satisfying (2.11) in \( \omega_{a} \) and (2.12) such that
\[ \sum_{j=0}^{l} \| g_{i,j} \|_{(a),a,b} \leq C_{a,b} \{ C_{a+2b+1} \alpha^{-5/2} E_{a}(v, \beta; a+2b+5) + \sum_{q=0}^{b} \alpha^{1/4} E_{a}(v, \beta; a+2(b-q)+2q+5) \}. \]

Now suppose that
\[ \sum_{j=0}^{l} \| g_{i,j} \|_{(a),a,b} \leq C_{a,b} \alpha^{-11/4} E_{a}(v, \beta; a+2b+4j+1). \]

Applying (2.8) to \( k^{1/6} g_{i,j} \), \( l=0 \), 1 we have
\[
\left( \Phi^* G_j + BG_j + \frac{1}{ik} BG_j \right) (\Phi^-)^{-1} \right)_{(\alpha), a} \\
\leq C_{j+1,a} \sum_{j=0}^{\infty} \left( \alpha^{-7/4} \sum_{j=0}^{\infty} ||g_{ij}||_{(\alpha), p, 1} + \alpha^{-11/4} \sum_{j=0}^{\infty} ||g_{ij}||_{(\alpha), p, 1} \right) \beta^{-l-1} \\
\leq C_{j+1,a} \sum_{j=0}^{\infty} C_{j+1,a} \alpha^{-11/4} \alpha^{-11/4} E_\alpha(v, \beta; p+2+4j+1) \beta^{-l-1} \\
\leq C_{j+1,a} \alpha^{-11(j+1)/4} E_\alpha(v, \beta; a+4j+1). 
\]

And

\[
||g_{ij}||_{(\alpha), a} \leq C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2b+4j+5). 
\]

Then by using Lemma 2.1 we have \(g_{ij}, l=0, 1\) verifying (2.11) in \(\omega_a\) and (2.12) in such that

\[
\sum_{j=0}^{\infty} ||g_{ij}||_{(\alpha), a} \leq C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2b+4j+4) \\
+ \sum_{j=0}^{\infty} C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2b+4j+5) \\
\leq C_{j+1,a} \alpha^{-11(j+1)/4} E_\alpha(v, \beta; a+2b+4j+1). 
\]

Thus by the method of induction we obtain

**Lemma 2.2.** For given \(v(x, t) \in C_0^\infty(\Gamma_a \times \mathbb{R})\) there exist \(g_{ij}, j=0, 1, 2, \ldots\) verifying (2.11) in \(\omega_a\), (2.12) on \(\Gamma_a \times \mathbb{R}\) and the estimate

(2.16)

\[
\sum_{j=0}^{\infty} ||g_{ij}||_{(\alpha), a} \leq C_{j,a,b} \alpha^{-11j/4} E_\alpha(v, \beta; a+2b+4j+1), 
\]

where \(C_{j,a,b}\) depends on \(j\) and \(a, b\) but independent of \(\alpha\).

Let \(N\) be a positive integer. For \(g_{ij}\) of the above lemma we define \(g_{ij}^{(N)}, u^{(N)}\) by

\[
g^{(N)}(x, t; \alpha, k) = \sum_{l=0}^{N} g_{ij}(x, t; \alpha, k) k^{1/5} - 1 - i, \quad l = 0, 1 \\
u^{(N)}(x, t; \alpha, k) = e^{i(k^2/4)t} \left[ V(k^2/4)g^{(N)} + \frac{1}{ik^{1/3}} V'(k^2/4)g^{(N)} \right].
\]

Since

(2.17)

\[
||e^{i(k^2/4)t} V(k^2/4) ||_{(\alpha), a} \leq C_{a} \ k^{a+b},
\]

it holds that

(2.18)

\[
||u^{(N)}||_{(\alpha), a} \leq C_{a} \ k^{a+b} \\
\leq C_{N,a,b} \sum_{j=0}^{N} k^b \sum_{j=0}^{N} k^{-j-1/5} E_\alpha(v, \beta; 2l+4j+1) \\
\leq C_{N,a,b} \sum_{j=0}^{N} k^{a+b-j-1/5} E_\alpha(v, \beta; 4j+1).
\]
Let us consider the estimates of $\Box u^{(N)}$. In $\omega_a = \{x; \rho \geq 0\}$ it follows from (2.6) and the relations (2.11)$_{j}$, $j = 0, 1, \ldots, N$ that

$$\Box u^{(N)} = k^{-N-5/6}e^{ik(\theta-i)}\left\{V(k^{2/3}\rho)\Box g_{0N} + \frac{1}{i k^{1/3}} V'(k^{2/3}\rho)\Box g_{1N}\right\}.$$  

Using (2.16) and (2.17) we have in $\omega_a$

$$\Box u^{(N)} = k^{-N-5/6}e^{ik(\theta-i)}\sum_{r+s+t+\lambda'=0} k^{s+t}e^{ik(\theta-i)}\sum_{r+s+t+\lambda'=0} \Box g_{4N} \mid (v, \beta; 1+2r+4N+1)$$

Next consider $\Box u^{(N)}$ in $\{x; \rho < 0\}$. Note that

$$\begin{align*}
\Box u^{(N)} &= \sum_{r+s+t+\lambda'} k^{s+t}e^{ik(\theta-i)}\sum_{r+s+t+\lambda'} \Box g_{4N} \mid (v, \beta; 2(a+b+b'-\rho)+4N+1). 
\end{align*}$$

Since $(\nabla \theta)^2 + \rho(\nabla \rho)^2 = 1$ in $\{x; \rho \geq 0\}$ we have for any $M > 0$ a constant $C_M \gamma_0$ such that

$$\Box u^{(N)} \leq C_M \gamma_0 (\rho)^{M/2}$$

for $\rho \leq 0$. On the other hand, since $V(z)$ satisfies

$$\Box u^{(N)} \leq C_M \gamma_0 (\rho)^{M/2}$$

it follows that for all $k \geq 1$ and $\rho \leq 0$

$$\Box u^{(N)} \leq C_M \gamma_0 k^{-M}.$$  

By using (2.20)

$$\Box u^{(N)} \leq C_M \gamma_0 k^{-M}.$$  

About $\Box u^{(N)} \leq C_M \gamma_0 k^{-M}$ we can obtain the same estimate as (2.21) by taking account of the fact $\nabla \theta \cdot \nabla \rho = 0$ in $\{x; \rho \geq 0\}$. Next consider terms of the type

$$I_j = e^{ik(\theta-i)} V(k^{2/3}\rho)J_j k^{-j+1-5/6}.$$
\begin{align*}
J_j &= 2 \frac{\partial g_{0j}}{\partial t} + 2 \nabla \theta \cdot \nabla g_{0j} + \Delta \theta g_{0j} + 2 \rho \nabla \rho \cdot \nabla g_{1j} \\
&\quad + (\nabla \rho)^2 g_{1j} + \rho \nabla \rho g_{1j} + \frac{1}{i} \Box g_{0j-1}.
\end{align*}

Since \{g_{0j}, g_{1j}\} verifies (2.11), in \omega, we have for \( \rho < 0 \)
\begin{align*}
|\partial_t \partial_\nu^2 \partial^m_j J_j| &\leq C_M(\rho)^{3M/2}\left\{\|g_{0j}\|_{(a), \alpha + \beta', \beta + 3M/2 + 1} \\
&\quad + \|g_{1j}\|_{(a), \alpha + \beta', \beta + 3M/2 + 1} + \|g_{0j-1}\|_{(a), \alpha + \beta', \beta + 3M/2 + 2}\right\}.
\end{align*}

Therefore
\begin{align*}
\|I_j\|_{(a), \alpha, \beta} &\leq C_{j, \alpha, \beta}k^{-M}k^{-j + 1 + \frac{5}{6}} \sum_{l = j}^{\infty} k^p \\
&\quad \cdot \left\{\alpha^{\frac{11}{4}} \sum_{l = j}^{\infty} \|g_{0j}\|_{(a), \alpha + \beta', \beta + 3M/2 + 1} + \alpha^{\frac{11}{4} - \frac{1}{4}} \sum_{l = j}^{\infty} \|g_{1j}\|_{(a), \alpha + \beta', \beta + 3M/2 + 1}\right\} \\
&\leq C_{j, \alpha, \beta}k^{-M}k^{-j + 1 + \frac{5}{6}} \sum_{l = j}^{\infty} k^p \left\{\alpha^{\frac{11}{4}} E_a(v, \beta; 2l + 3M/2 + 4j + 3) \\
&\quad + \alpha^{\frac{11}{4} - \frac{1}{4}} E_a(v, \beta; 2l + 3M/2 + 4(j - 1) + 4)\right\},
\end{align*}

and setting \( M = N - (j - 1) \) it follows that
\begin{equation}
\|I_j\|_{(a), \alpha, \beta} \leq C_{j, \alpha, \beta}k^{-N} \alpha^{-\frac{11}{4}} \sum_{l = j}^{\infty} k^p E_a(v, \beta; 2l + 4N + 3). \tag{2.22}
\end{equation}

Note that we have an estimate same as (2.22) for the other terms of \( \Box u^{(N)} \).

From (2.19), (2.21) and (2.22) we have an estimate
\begin{equation}
\|\Box u^{(N)}\|_{(a), \alpha, \beta} \leq C_{N, \alpha, \beta}(k\alpha^{1/4})^{-N} \sum_{l = j}^{\infty} k^p E_a(v, \beta; 2l + 4N + 3). \tag{2.23}
\end{equation}

We set about considering \( B u^{(N)} \mid_{\Gamma_a \times R} \). Remark that from (ii) of Lemma 2.1

\( \text{supp } B u^{(N)} \mid_{\Gamma_a \times R} \subseteq \Lambda_0 \cup \Lambda_1 \).

On \( \Gamma_a \times R \)
\begin{equation}
B u^{(N)} - e^{ik(\varphi - t)v} = e^{ik(\varphi - t)}k^{-N} \left\{\Phi^{-}\hat{G}_N + B\hat{G}_N + \frac{1}{ik} B\hat{G}_N\right\},
\end{equation}

from which it follows that
\begin{equation}
\left\langle B u^{(N)} - e^{ik(\varphi - t)v}\right\rangle_{(a), \alpha, \beta} \leq C_{N, \alpha, \beta}k^{-N} \sum_{l = j}^{\infty} k^p \alpha^{-\frac{11}{4} + \frac{1}{4} - \frac{1}{4}} E_a(v, \beta; l + 4N + 3). \tag{2.24}
\end{equation}

Since in \( \omega, \)
\begin{equation}
\Box u^{(N)} = e^{ik(\theta - t)} \left\{\left(V I^{3/3} \rho\right)\Box g_{0N} + \frac{1}{ik^{1/3}} V'(k^{3/3} \rho)\Box g_{1N}\right\}k^{-N - 5/6},
\end{equation}

by applying the expansion of the type (2.7) to the right hand side of the above equality we may write near \( \Gamma_a \times R \)
\[ \Box u^{(N)} = e^{ik(\varphi^{-t})}H^{-k^{-N}} + e^{ik(\varphi^{+t})}H^{+k^{-N}}, \]

with \( H^{\pm} \) satisfying

\[ |\partial_t^\delta \partial_x^\delta H^{\pm}| \leq C_{N,\alpha,\beta} \alpha^{-11N}E_\alpha(v, \beta; a \pm a' + 2b + 4N + 1). \]

On the other hand applying \( \Box \) to \( u^{(N)} \) of (2.7) we have in \( \omega_\alpha \)

\[ \Box u^{(N)} = e^{ik(\varphi^{-t})}\left\{ ik\left( 2\frac{\partial}{\partial t} + 2\nabla \varphi^{-} \cdot \nabla + \Delta \varphi^{-}\right) + \Box\right\}(G^{(N)^{-}} + \frac{1}{ik}G^{(N)^{-}}) \]

\[ + e^{ik(\varphi^{+t})}\left\{ ik\left( 2\frac{\partial}{\partial t} + 2\nabla \varphi^{+} \cdot \nabla + \Delta \varphi^{+}\right) + \Box\right\}(G^{(N)^{+}} + \frac{1}{ik}G^{(N)^{+}}), \]

where \( G^{(N)^{\pm}}, \tilde{G}^{(N)^{\pm}} \) denote the terms corresponding to \( G^{\pm}, \tilde{G}^{\pm} \) of (2.7) when we substitute \( g_1^{(N)} \) and \( \tilde{g}_1^{(N)} \) into the places of \( g_0 \) and \( g_1 \) of (2.4). In the same meaning we will write the decomposition of (2.7) for \( u^{(N)} \) as \( u^{(N)} = u^{(N)^{+}} + u^{(N)^{-}} \). Since \( \nabla \varphi^{+} \) and \( \nabla \varphi^{-} \) are linearly independent it follows that

\[ \left\{ ik\left( 2\frac{\partial}{\partial t} + 2\nabla \varphi^{\pm} \cdot \nabla + \Delta \varphi^{\pm}\right) + \Box\right\}(G^{(N)^{\pm}} + \frac{1}{ik}G^{(N)^{\pm}}) = k^{-N}H^{\pm}, \]

from which we can derive an estimate in a neighborhood of \( \Lambda_0 \)

\[ \left| \partial_t^\delta \partial_x^\delta \left( G^{(N)^{+}} + \frac{1}{ik}G^{(N)^{+}} \right) \right| \leq C_{N,\alpha,\beta} k^{-N+a+a'+b+4N+1}, \]

by taking account of the location of the support of \( G^{(N)^{+}} + \frac{1}{ik}G^{(N)^{+}} \) and the equation \( G^{(N)^{+}} + \frac{1}{ik}G^{(N)^{+}} \) must satisfy. Then we have

\[ \langle Bu^{(N)^{+}} \rangle_{\Lambda_0} \leq C_{N,\alpha,\beta}(k\alpha^{-1/4})^{-N} \sum_{\rho \in \Omega} k^{p}E_\alpha(v, \beta; 4N + I + 3). \]

Combining the above estimate with (2.24) it holds that

(2.25) \[ \langle Bu^{(N)} \rangle_{\Lambda_0} \leq e^{ik(\varphi^{+t})}\left\{ ik\Phi^+ \left( G^{(N)^{+}} + \frac{1}{ik}G^{(N)^{+}} \right) + BG^{(N)^{+}} + \frac{1}{ik}BG^{(N)^{+}} \right\} \]

where

\[ G^{(N)^{+}} = \sum_{j=0}^{N} \pi^{-1/2} \alpha^{-1/4} e^{i\pi(j)}(g_{0j} + \sqrt{\rho} g_{1j})k^{-j-1}. \]

Let us set
Applying (iii) of Lemma 2.1 we have

\[ w_1(x, t) = \gamma_a(x) \Phi^+ \left( \frac{v}{\Phi^-} \right) (P_a(x, t)). \]

Then it holds that

\begin{align}
(2.26) & \quad \sup |w_1| \geq \frac{1}{2} \left( \inf_{(\alpha, \beta) \in \Lambda_1} |\Phi^+|/ \sup_{(\alpha, \beta) \in \Lambda_0} |\Phi^-| \right) \sup |v| . \\
(2.27) & \quad \langle w_1 \rangle_{(\alpha), \beta} \leq C_e \left\{ \sup_{(\alpha, \beta) \in \Lambda_1} |\Phi^+| E_a(v, \beta; a) + E_a(v, \beta; a) \right\} .
\end{align}

Set

\[ w_2(x, t) = i \Phi^+ \sum_{j=1}^{N} (g_{i,j} + \sqrt{\rho_{i,j}})k^{-j} + i \Phi^+ G^{(N)+} + BG^{(N)+} + \frac{1}{ik} BG^{(N)+}. \]

Then

\[ \langle w_2 \rangle_{(\alpha), \beta} \leq C_{N, a} \sum_{j=1}^{N} (k\alpha^{1/4})^{-j} E_a(v, \beta; 4j+a) \]

By the same consideration as \( u^{(N)+} \) in \( \Lambda_0 \) we have

\[ \langle Bu^{(N)+} \rangle_{(\alpha), \beta} \leq C_{N, a} (k\alpha^{1/4})^{-N} \sum_{j \in \mathbb{Z}} k^j E_a(v, \beta; 4N+l+3). \]

Summarizing the considerations in this section we have

**Proposition 2.3.** Let \( \alpha > 0 \) and \( v(x, t) \in C_0(\Gamma_a \times \mathbb{R}) \) such that \( \Lambda_0 \cap \Lambda_1 = \phi \). For every positive integer \( N \) there exists a function \( u^{(N)}(x, t; \alpha, k) \in C_0(\mathbb{R}^2 \times \mathbb{R}) \) satisfying

\[ \sup \{ u^{(N)} \} \cap (\omega_a \times \mathbb{R}) \subset \{ L^\infty(x, t); (x, t) \in \sup \{ v \} \}, \]

\[ \sup \{ B u^{(N)} \} |_{\Gamma_a \times \mathbb{R}} \subset \Lambda_0 \cup \Lambda_1 , \]

and the estimates (2.18), (2.23) and (2.25). And

\[ \langle Bu^{(N)} \rangle_{(\alpha), \beta} \leq C_{N, a} (k\alpha^{1/4})^{-N} \sum_{j \in \mathbb{Z}} k^j E_a(v, \beta; 4N+l+3) \]

where \( w \) has the following properties

\[ \sup |w| \geq \frac{1}{2} \left( \inf_{(\alpha, \beta) \in \Lambda_1} |\Phi^+| / \sup_{(\alpha, \beta) \in \Lambda_0} |\Phi^-| \right) \cdot \sup |v| \]

\[ - C \sum_{j=1}^{N} (k\alpha^{1/4})^{-j} E_a(v, \beta; 4j) \]

\[ \langle w \rangle_{(\alpha), \beta} \leq C_a \left\{ \left( \sup_{(\alpha, \beta) \in \Lambda_1} |\Phi^+| + \beta \right) E_a(v, \beta; a) \right\} \]

\[ + C_{N, a} \sum_{j=1}^{N} (k\alpha^{1/4})^{-j} E_a(v, \beta; 4j+a) \].
where all the constants are independent of \( \alpha \).

3. Asymptotic solutions reflected \( K \)-time at \( \Gamma \)

Let \( v(x, t) \in C_0^\omega(\Gamma_a \times \mathbb{R}) \) and \( \text{supp } v \subseteq \Lambda_0 \). Define \( \Lambda_1, \Lambda_2, \ldots, \Lambda_K \) successively by

\[
\Lambda_{j+1} = \{ L^{-}(x, t) \cap (\Gamma_a \times \mathbb{R}) \setminus \{(x, t)\}; (x, t) \in \Lambda_j \}.
\]

Suppose that

\[
(3.1) \quad \Lambda_j \subset \Gamma_a \times (t_j, t_{j+1}), \quad t_0 < t_1 < \cdots < t_K.
\]

Set

\[
\beta = \inf_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B \varphi^-|,
\]

\[

\nu = \inf_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B \varphi^+| / \sup_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B \varphi^-|.
\]

We assume for some constant \( C_K \)

\[
(3.2) \quad \sup_{(x, t) \in \bigcup_{j=0}^K \Lambda_j} |B \varphi^+| / \beta \leq C_K \nu.
\]

Apply Proposition 2.3 for

\[
m_0(x, t; \alpha, k) = e^{i(k\varphi^-(x, \alpha) - t)} v(x, t)
\]

and have \( u_0^{(N)}(x, t; \alpha, k) \) with the properties

\[
(3.3) \quad \|u_0^{(N)}\|_{(\alpha), a, b} \leq C_{N, a, b} \sum_{j=0}^{N+3} k^{N+1} E_a(v, \beta; 4j+1)
\]

\[
(3.4) \quad \|\Box u_0^{(N)}\|_{(\alpha), a, b}
\leq C_{N, a, b} (k\alpha^3)^{-N} \sum_{j=0}^{N+3} k^j E_a(v, \beta; 2l+4N+3),
\]

\[
(3.5) \quad \langle Bu_0^{(N)} \rangle_{(\alpha), a} = m_0(x, t; \alpha, a) + \langle Bu_0^{(N)} \rangle_{(\alpha), a}
\leq C_{N, a} (k\alpha^4)^{-N} \sum_{j=0}^{N+3} k^j E_a(v, \beta; 4N+l+3),
\]

where

\[
m_1 = e^{i(k\varphi^-(\alpha) - t)} v_1,
\]

\[
\text{supp } v \subset \Lambda_1,
\]

\[
(3.7) \quad \sup |v_1| \geq \frac{\nu}{2} \sup|v| - C \sum_{j=1}^{N} (k\alpha^3)^{-N} E_a(v, \beta; 4j)
\]

\[
(3.8) \quad \langle v_1 \rangle_{(\alpha), a} \leq C_a \left( \sup |\Phi^+| + \beta \right) E_a(v, \beta; a)
\]

\[
+ C_{N, a} \sum_{j=1}^{N} (k\alpha^3)^{-j} E_a(v, \beta; 4j+a).
\]
Since \( \rho = \alpha \) on \( \Gamma_a \) we have

\[
\varphi^+ = \theta + \frac{2}{3} \rho^{3/2} = \theta - \frac{2}{3} \rho^{3/2} + \frac{4}{3} \alpha^{3/2}
\]

\[
= \varphi^- + \frac{4}{3} \alpha^{3/2} \quad \text{on } \Gamma_a,
\]

from which follows

\[
m_1 = e^{ik(\varphi^- - t)} \vartheta_1, \quad \vartheta_1 = e^{i\theta(4/3\alpha^{3/2})} v_1.
\]

Then \( \vartheta_1 \) verifies the properties (3.6) \( \sim \) (3.8).

Now the application of Proposition 2.3 to \( m_1 \) gives the existence of a function \( u^{(N)}(x, t; \alpha, k) \) with the properties

\[
\begin{align*}
(3.3) & \quad \|u^{(N)}\|_{(\omega), \beta} \leq C_{N, \alpha, \beta} \sum_{j=0}^N k^{2j+b} \beta^{j-1/2} E_{\alpha}(v_1, \beta; 4j+1) \\
(3.4) & \quad \|\|^N u^{(N)}\|_{(\omega), \beta} \leq C_{N, \alpha, \beta} (k\alpha^3)^{-N} \sum_{j=0}^N k^p E_{\alpha}(v_1, \beta; 2l+4N+3) \\
(3.5) & \quad <Bu^{(N)}|_{\lambda_1} - m_1>_{(\omega), \alpha} + <Bu^{(N)}|_{\lambda_2} - m_2>_{(\omega), \alpha} \\
& \leq C_{N, \alpha} (k\alpha^3)^{-N} \sum_{p+i<\alpha} k^p E_{\alpha}(v_1, \beta; 4N+l+3).
\end{align*}
\]

From (3.8) and the definition of \( E_{\alpha}(v_1, \beta; a) \) it follows

\[
\begin{align*}
E_{\alpha}(v_1, \beta; a) & = \sum_{p+i<\alpha} \langle v_1 \rangle_{(\omega), \beta} \beta^{-i-1} \\
& \leq \sum_{p+i<\alpha} \{C_p (\sup |\Phi^+| + \beta) E_{\alpha}(v, \beta; p) \\
& \quad + C_{N, \alpha} \sum_{j=1}^N (k\alpha^3)^{-j} E_{\alpha}(v, \beta; 4j+p) \} \beta^{-i-1} \\
& \leq C_{\alpha} (\sup |\Phi^+| + \beta) \sum_{p+i<\alpha} E_{\alpha}(v, \beta; p) \beta^{-i-1} \\
& \quad + C_{N, \alpha} \sum_{j=1}^N (k\alpha^3)^{-j} \sum_{p+i<\alpha} E_{\alpha}(v, \beta; 4j+p) \beta^{-i-1}.
\end{align*}
\]

By using \( E_{\alpha}(v, \beta; p) \beta^{-i} \leq E_{\alpha}(v, \beta; p+i) \), we have

\[
(3.9) \quad E_{\alpha}(v_1, \beta; a) \leq C_{\alpha} (\sup |\Phi^+| + \beta) |\beta| E_{\alpha}(v, \beta; a) \\
\quad + C_{N, \alpha} \beta^{-1} \sum_{j=1}^N (k\alpha^3)^{-j} E_{\alpha}(v, \beta; 4j+a).
\]

From the second part of Proposition 2.3 \( m_2 \) can be represented as

\[
m_2(x, t; \alpha, k) = e^{ik(\varphi^- - t)} v_2(x, t; \alpha, k) \\
= e^{ik(\varphi^- - t)} e^{i\theta(4/3\alpha^{3/2})} v_2 = e^{i\theta(\varphi^- - t)} \vartheta_2,
\]

and \( \vartheta_2 \) verifies from (2.7) and the above estimate (3.9)
\[ (3.7)_2 \sup |\vartheta_2| \geq \frac{1}{2} \nu \left( \frac{1}{2} \nu \sup |v| - C_N \sum_{j=1}^{\infty} (k \alpha^3)^{-i} E_a(v, \beta; 4j) \right) \]
\[ - C_N \sum_{j=1}^{\infty} (k \alpha^3)^{-i} \{ C_a(\sup |\Phi^+| + \beta) E_a(v, \beta; 4j) \} \]
\[ + C_{N,a} \beta^{-1} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + 4h) \} \]
\[ \geq \left( \frac{1}{2} \nu \right)^2 \sup |v| - C \nu \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j) \]
\[ - C_{N,a} \beta^{-1} \sum_{j=2}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j). \]

\[ (3.8)_2 \langle \vartheta_2 \rangle_{(a),a} \leq C_a(\sup |\Phi^+| + \beta) E_a(v, \beta; a) \]
\[ + C_{N,a} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \]
\[ \leq C_a(\sup |\Phi^+| + \beta) \{ C_a C \nu E_a(v, \beta; a) \}
\[ + C_{N,a} \beta^{-1} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \}
\[ + \beta^{-1} C_{N,a} \sum_{j=1}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \}
\[ \leq C_a(\sup |\Phi^+| + \beta) \cdot \nu \cdot E_a(v, \beta; a) \]
\[ + C_{N,a} \nu \sum_{j=2}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a) \]
\[ + C_{N,a} \beta^{-1} \sum_{j=2}^{\infty} (k \alpha^3)^{-j} E_a(v, \beta; 4j + a). \]

Repeating this process we obtain \( u_j^{(N)}(x, t; \alpha, k), j=0, 1, 2, \ldots, K \) verifying

\[ (3.3)_j \| u_j^{(N)}(x, t; \alpha, k), j=0, 1, 2, \ldots, K \]
\[ (3.4)_j \| u_j^{(N)}(x, t; \alpha, k) \leq C_{N,a,b} \sum_{k=0}^{k^{x+b}} k^{x+b} \cdot E_a(v, \beta; 4h+1) \]
\[ (3.5)_j \langle Bu_j^{(N)} \rangle_{(a),a} \leq C_{N,a} (k \alpha^3)^{-N} \sum_{\alpha \in \Lambda_j} k^{\alpha} E_a(v, \beta; 4N + l + 3) \]
\[ m_j = e^{i(k \alpha - t)} \vartheta_j \sup \vartheta_j \subset \Lambda_j \]
\[ (3.7)_j \sup |\vartheta_j| \geq \left( \frac{1}{2} \nu \right)^j \sup |v| \]
\[ - C_{j,1} \sum_{i=1}^{j-1} v_{i-1} \sum_{\alpha \in \Lambda_j} (k \alpha^3)^{-j} E_a(v, \beta; 4h) \]
\[-C[^{(i)}]_j \beta^{-1} \sum_{k=2}^{iN} (k\alpha\beta^{-1}E_{\alpha}(v, \beta; 4h), \]

\[(3.8) \quad \langle \hat{v}_j \rangle_{(a), a} \leq C[^{(i)}]_j \sum_{j=1}^{i-1} \sum_{k=2}^{iN} (k\alpha\beta^{-1}E_{\alpha}(v, \beta; 4h+a)\]

\[+ C[^{(i)}]_j \beta^{-1} \sum_{k=2}^{iN} (k\alpha\beta^{-1}E_{\alpha}(v, \beta; 4h+a)).\]

By using \(v \leq C \beta^{-1}\) it follows from (3.8) that

\[(3.10) \quad \langle \hat{v}_j \rangle_{(a), a} \leq C[^{(i)}]_j \sum_{j=0}^{i-1} \sum_{k=2}^{iN} (k\alpha\beta^{-1}E_{\alpha}(v, \beta; 4h+a)\]

\[= C[^{(i)}]_j \sum_{j=0}^{i-1} \sum_{k=2}^{iN} (k\alpha\beta^{-1}E_{\alpha}(v, \beta; 4h+j+1+a)).\]

Set

\[U^{(N)}(x, t; \alpha, k) = \sum_{j=0}^{i-1} (-1)^j u^{(N)}_j(x, t; \alpha, k).\]

Then we have from (3.3) \((3.10)\)

**Proposition 3.1.** Let \(v(x, t) \in C_0(\Gamma_\alpha \times R)\) such that

\[\text{supp } v \subset \Lambda_0.\]

Suppose that (3.1) and (3.2). Then there exists a function \(U^{(N)}_k(x, t; \alpha, k)\) with the following properties:

\[(3.11) \quad \text{supp } U^{(N)}_k \cap (\Omega \times R) \subset \Omega \times (t_0, \infty)\]

\[(3.12) \quad \| U^{(N)}_k \|_{(a), a, b} \leq C_{N,K,a,b} \sum_{j=0}^{iN} (k\alpha\beta^{-1}E_{\alpha}(v, \beta; 4h+K-1+4j+2))\]

\[(3.13) \quad \| \Box U^{(N)}_k \|_{(a), a, b} \leq C_{N,K,a,b} (k\alpha\beta^{-1})^{N-1} \sum_{p_1, q_1, p_2, q_2} \sum_{s_1, t_1} (k\alpha\beta^{-1})^{N-1} E_{\alpha}(v, \beta; 4h+K-q+4l+4N+3)\]

\[(3.14) \quad \langle BU^{(N)}_k \rangle_{(a), a} \leq C_{N,K,a} (k\alpha\beta^{-1})^{N-1} \sum_{p_1, q_1, p_2, q_2} \sum_{s_1, t_1} (k\alpha\beta^{-1})^{N-1} E_{\alpha}(v, \beta; 4h+K-q+4l+4N+3)\]

\[(3.15) \quad \sup_{\Omega_\alpha \times (t_0, t_k)} \| U^{(N)}_k \| \geq \left( \frac{1}{2} \right)^k \| v \| \]

\[-C_N \sum_{j=1}^{iN} (k\alpha\beta^{-1})^{N-1} E_{\alpha}(v, \beta; 4h), \]

where the constants \(C_{N,K,a,b}\) and \(C_{N,K,a}\) are independent of \(\alpha\).
4. Proof of the theorem

Lemma 4.1. Suppose that \( \tau(0) = \tau'(0) = 0 \) and
\[
\sup_{0 \leq t < \xi} \tau(s) > 0
\]
for any \( \varepsilon > 0 \). Then there exist a constant \( \delta \geq 1/2 \) and a sequence
\[
s_1 > s_2 > \cdots > s_n > s_{n+1} > \cdots > 0
\]
with the following properties:
\[
(4.1) \quad \left\{ \begin{array}{l}
 s_n \to 0 \quad \text{as } n \to \infty \\
 \beta_n = \tau(s_n) > 0
\end{array} \right.
\]
and for any positive integer \( K \) there exists a constant \( C_K \) such that
\[
(4.2) \quad \sup_{0 \leq t < \xi} \sup_{0 \leq \beta < \delta} \frac{\mid \tau(s_n + t\beta_n) - \beta_n \mid}{\beta_n^{\delta+\delta}} \leq C_K.
\]

Proof. When \( \xi = 0 \) is a zero of finite order, namely for some \( q \geq 1 \)
\[
\tau(0) = \tau'(0) = \cdots = \tau^{(q)}(0) = 0, \quad \tau^{(q+1)}(0) > 0
\]
it holds that for some \( s_0 > 0 \)
\[
|\tau'(s)| \leq C \tau(s)^{\delta/(q+1)} \quad \text{for } 0 < s < s_0.
\]
Since for \( s > 0, t > 0, \)
\[
|\tau(s+t\tau(s)) - \tau(s)| \leq t\tau(s)|\tau'(s+\eta t\tau(s))| \quad (0 < \eta < 1)
\]
\[
\leq t\tau(s)\{ |\tau'(s)| + t\eta \tau(s) (\sup \tau'') \}
\]
\[
\leq C_K \tau(s)^{1+\delta/(q+1)} \quad (0 < t \leq K),
\]
\( \delta = q/(q+1) \) and the sequence \( s_n = 1/n \) are the desired one.

Next consider the case that \( s = 0 \) is a zero of infinite order.

Case 1. \( \tau(s) \) is monotonically increasing in \( 0 < s < \varepsilon_0 \) for some \( \varepsilon_0 > 0 \).
Suppose that for some \( 1 > \delta > 0 \) there is no sequence with property (4.1) verifying
\[
(4.3) \quad \tau'(s_n) < \tau(s_n)^{\delta}, \quad \forall n.
\]
This assumption implies that it holds that for some \( \varepsilon_1 > 0 \)
\[
\tau'(s) \geq \tau(s)^{\delta} \quad \text{for } 0 < s < \varepsilon_1,
\]
from which it follows
\[ \frac{d}{ds} \tau(s)^{1-\delta} = (1-\delta)\tau(s)^{-\delta} \tau'(s) \geq (1-\delta) \quad \text{for } 0<s<\varepsilon_1. \]

Then we have

\[ \tau(s)^{1-\delta} \geq (1-\delta)s \quad \text{for } 0<s<\varepsilon_1, \]

namely \( \tau(s) \geq (1-\delta)^{s/(1-\delta)} \). This is contradict with the assumption that \( \tau(s) \) has a zero of infinite order at \( s=0 \). Then we see that for any \( 1>\delta>0 \) there exists \( \{s_n\} \) verifying (4.1) and (4.3). By using (4.3) and

\[ \tau(s_n+t\beta_n) - \beta_n = t\beta_n \tau'(s_n+\eta t\beta_n), \quad 0<\eta<1 \]

we have for all \( 0\leq t\leq K \)

\[ |\tau(s_n+t\beta_n) - \beta_n| \leq t\beta_n \sup |\tau''(s)| \]

Thus (4.2) is proved.

**Case 2.** For some \( \varepsilon_0>0 \)

\[ \tau(s)>0 \quad \text{for } 0<s<\varepsilon_0 \]

and \( \tau(s) \) is not monotonically increasing in \( 0<s<\varepsilon \) for any \( \varepsilon>0 \). From the assumption for any \( \varepsilon>0 \) there exists \( s \) such that \( 0<s<\varepsilon \) and \( \tau'(s)=0 \). Then we can choose \( s_n > 0 \) with the property (4.1) such that \( \tau'(s_n)=0 \). Then

\[ |\tau(s_n+t\beta_n) - \beta_n| \leq |\tau'(s_n+\eta t\beta_n)| \cdot t\beta_n \]

\[ \leq CK^2 \cdot \beta_n^2 \quad \forall n. \]

Thus \( \{s_n\}_{n=0}^\infty \) is the desired one.

**Case 3.** \( \tau(s) \) does not verify the properties of the case 1 nor 2. Then there exists a sequence \( \theta_n>\theta_{n+1}>\cdots \rightarrow 0 \) such that \( \tau(\theta_n)=0 \) and \( \sup_{s=\theta_n+1, \theta_n} \tau(s)>0 \), since for any \( \varepsilon>0 \) there exists \( 0<s<\varepsilon \) such that \( \tau(s)>0 \). If we choose \( s_n \) as

\[ \tau(s_n) = \max_{s=\theta_n+1, \theta_n} \tau(s), \]

it holds that \( \tau(s_n)>0 \) and \( \tau'(s_n)=0 \). Evidently \( s_n \rightarrow 0 \). As case 2 we see that this \( \{s_n\} \) verifies (4.2). Q.E.D.

Since \( n(x)=(n_1(x), n_2(x)) \) may be considered as a \( C^m \)-vector defined in a neighborhood of \( \Gamma \)

\[ n(x) = b_1(x)n_2(x) - b_2(x)n_1(x) \]

is also a \( C^m \)-function defined in a neighborhood of \( \Gamma \). We show that \( (P) \) is not well posed in the sense of \( C^m \) when \( \tau(s) \) of the introduction, i.e., \( \tau(s)=\)
\( \eta(x(s)) \) verifies the condition on \( \tau(s) \) of Lemma 4.1. Note that

\[
\begin{align*}
\varphi^\pm &= \pm \sqrt{\rho} (\nabla \rho + \alpha \nabla \rho_0 + \cdots) + \nabla \theta_0 + \alpha \nabla \theta_1 + \cdots \\
\text{and } n(x) \cdot \nabla \rho_0 &= \left| \nabla \rho_0 \right|, \ n(x) \cdot \nabla \theta_0 = 0 \quad \text{on } \Gamma^0.
\end{align*}
\]

Then we have

\[
n(x) \cdot \nabla \varphi^-(x, \alpha) = \alpha^{1/2} \frac{\partial \rho}{\partial \nu} + O(\alpha) \quad \text{on } \Gamma
\]

\[
\nabla \theta(x, 0) \cdot \nabla \varphi^-(x, \alpha) = 1 + O(\alpha) \quad \text{on } \Gamma.
\]

Therefore \( n(x) \cdot \nabla \varphi^-(x, \alpha) / \nabla \theta(x, \alpha) \cdot \nabla \varphi^-(x, \alpha) \) decreases monotonically to zero uniformly in \( x \in \Gamma \) when \( \alpha \to +0 \). Let \( \{s_n\} \) be the sequence with the property (4.1) for the above \( \tau(s) \).

For every \( n \) set \( y_n = x(s_n) \). Then \( \alpha_n > 0 \) is determined uniquely for large \( n \) by the relation

\[
\frac{n(y_n) \cdot \nabla \varphi^-(y_n, \alpha_n)}{\nabla \theta(y_n, 0) \cdot \nabla \varphi^-(y_n, \alpha_n)} = \beta_n + \beta_n^{1/2}.
\]

From the above relations we have

\[
c_1 \beta_n \leq \alpha_n^{1/2} \leq c_2 \beta_n, \quad \forall n,
\]

where \( c_1, c_2 \) are positive constants.

Note that for \( \alpha = 0 \)

\[
\nabla \theta \cdot \nabla \rho = 0, \quad |\nabla \theta| = 1 \quad \text{on } \Gamma.
\]

On the other hand \( x(s) \in \Gamma \) and \( \left| \frac{dx}{ds} \right| = 1 \). Then it follows that

\[
\theta(x(s), 0) = s + \text{constant}.
\]

Without loss of generality we may pose the constant = 0. Since we have from (2.1) and the property (ii) of \( \rho \)

\[
\text{rank} \begin{pmatrix} \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \\ \frac{\partial \rho}{\partial x_1} & \frac{\partial \rho}{\partial x_2} \end{pmatrix}_{\nabla \theta = 0, \ n(x) = 0} = 2,
\]

there exists uniquely \( x_\alpha(s) \) verifying \( x_\alpha(s) \to x(s) \) as \( \alpha \to 0 \) and

\[
\begin{align*}
\theta(x_\alpha(s), 0) &= s \\
\rho_\alpha(x_\alpha(s), \alpha) &= \alpha
\end{align*}
\]

4) See, for example, pages 70 and 71 of [4].
for small $s$ and $\alpha$. Moreover we have
\[ |x_a(s) - x(s)| \leq C \left\{ |\rho(x_a(s), \alpha) - \rho(x(s), \alpha)| + |\theta(x_a(s), 0) - \theta(x(s), 0)| \right\} \leq C |\alpha - \rho(x(s), \alpha)|. \]

Using (2.2) and $x(s) \in \Gamma$, we obtain for any $P > 0$
\[ |x_a(s) - x(s)| \leq C_{P} \alpha^P. \]

Then we have
\[ (4.7) \quad \left| (B\varphi^-)(x_a(s), \alpha) - (B\varphi^-)(x(s), \alpha) \right| \leq C_{P} \alpha^P \]
for all $\alpha > 0$ and $s$. Note that
\[ (B\varphi^-)(x, \alpha) = n(x) \cdot \nabla \varphi^-(x, \alpha) - \eta(x) \nabla \vartheta_0(x) \cdot \nabla \varphi^-(x, \alpha). \]

Then we have
\[ (4.8) \quad (B\varphi^-)(y_n, \alpha_n) = (\beta_n + \beta_n^{1+\eta/2} - \tau(s_n)) \nabla \theta_0(y_n) \cdot \nabla \varphi^-(y_n, \alpha_n) \]
\[ = \beta_n^{1+\eta/2} \nabla \theta_0(y_n) \cdot \nabla \varphi^-(y_n, \alpha_n) \]
\[ = \beta_n^{1+\eta/2}(1 + O(\beta_n)). \]

Taking account of (4.4) it holds that
\[ n(x(t+s)) \cdot \nabla \varphi^-(x(s+t)) - n(x(s)) \cdot \nabla \varphi^-(x(s)) \]
\[ = \pm \sqrt{\alpha} \left( |\nabla \rho_0(x(s+t))| - |\nabla \rho_0(x(s))| \right) + O(\alpha). \]

Since $|\nabla \rho_0(x)|$ is $C^\infty$ we have
\[ |n(x(s_n + t\beta_n)) \cdot \nabla \varphi^+(x(s_n + t\beta_n), \alpha_n) - n(x(s_n)) \cdot \nabla \varphi^+(x(s_n), \alpha_n)| \leq C t \beta_n^2 \forall n. \]

By the same consideration it holds that
\[ |\nabla \theta_0(x(s_n + t\beta_n)) \cdot \nabla \varphi^+(x(s_n + t\beta_n), \alpha_n) - \nabla \theta_0(x(s_n)) \cdot \nabla \varphi^+(x(s_n), \alpha_n)| \leq C t \alpha_n \leq C t \beta_n^2 \forall n. \]

Therefore we have for $0 \leq t \leq K$
\[ \left| (B\varphi^-)(x(s_n + t\beta_n), \alpha_n) - (B\varphi^-)(x(s_n), \alpha_n) \right| \leq |\tau(s_n + t\beta_n) - \tau(s_n)| + C K \beta_n^2. \]

Combining (4.2) and (4.7) it follows that
\[ (4.9) \quad \left| (B\varphi^-)(x(s_n + t\beta_n), \alpha_n) - \beta_n^{1+\eta/2} \right| \leq C_K \beta_n^{1+\delta} \]
for all $0 \leq t \leq K$ and $n$. By the same consideration we have
for all $0 \leq t \leq K$ and $n$. Then by using (4.6), (4.7) and (4.9) or (4.10) we have

**Lemma 4.2.** Suppose that $\tau(s)$ is equipped with the properties of Lemma 4.1. Then for any $K > 0$ there exists a constant $C_K$ such that

\[
(B\varphi^-)(x(s_n + t\beta_n), \alpha_n) - 2\beta_n \leq C_K \beta_n^{1+b/2}
\]

and

\[
(B\varphi^+)(x(s_n + t\beta_n), \alpha_n) - 2\beta_n \leq C_K \beta_n^{1+b/2}
\]

for all $0 \leq t \leq K$ and $n$.

Suppose that the problem $(P)$ is well posed in the sense of $C^\infty$. Then for any $T$ there exist $q$ and $C_\tau$ such that for all $t \leq T$

\[
|u|_{q, \Omega \times (-\infty, 0)} \leq C_T \left\{ |\Box u|_{q, \Omega \times (-\infty, 0)} + |B u|_{q, \Gamma \times (-\infty, 0)} \right\}
\]

for all $u(x, t) \in C^\infty(\Omega \times (0, T))$ verifying $u=0$ for $t \leq 0$, where

\[
|v|_{q, \Omega \times (-\infty, 0)} = \sum |D_{y_1}^q v|, \quad |v|_{q, \Gamma \times (-\infty, 0)} = \sum |D_{y_0}^q (\nabla v)\cdot \nu|.
\]

On the supposition on $\tau(s)$ of Lemma 4.1 we will show the existence of a sequence of functions which violates (4.13).

Let $h(s, t) \in C^\infty_0(\mathbb{R}^2)$ such that

$$\sup |h| = 1, \quad \text{supp } h \subset [0, 1] \times [0, 1].$$

For each $n$ define $v_n(x, t) \in C^\infty_0(\Gamma \times \mathbb{R})$ by

$$v_n(x, t) = h\left(\frac{s-s_n}{\alpha_n}, \frac{t}{\alpha_n}\right).$$

Put

$$\Lambda_n = \{(x, s, t); |s-s_n| \leq \alpha_n, 0 \leq t \leq \alpha_n\},$$

and define $\Lambda_{nj}$, $j=1, 2, \ldots, K$ according to the description in the beginning of §3. Since $c_2/\alpha_n \leq |P_{a_n}(x, t)-(x, t)| \leq c_1/\alpha_n$ it holds that

$$\Lambda_{nj} \cap \Gamma_{a_n} \times (t_{nj}, t_{nj+1}) \neq \emptyset, \quad 0 = t_{00} < t_{11} < \cdots < t_{nK} < c_1 K \alpha_n.$$

From Lemma 4.2 we have

$$\inf_{(x, t) \in \bigcup_{j=1}^K \Lambda_{nj}} |B \varphi^-| \geq C_K \beta_n^{1+b/2} \geq C_K \alpha_n,$$
\[
\inf_{(x,t) \in \bigcup_{j=0}^{K} \Delta_{K_j}} |B\phi^+| \leq \sup_{(x,t) \in \bigcup_{j=0}^{K} \Delta_{K_j}} |B\phi^-| \geq C_K \beta_n^{3/2}
\]

and
\[
\sup_{(x,t) \in \bigcup_{j=0}^{K} \Delta_{K_j}} |B\phi^+| \leq \inf_{(x,t) \in \bigcup_{j=0}^{K} \Delta_{K_j}} |B\phi^-| \leq C_K \beta_n^{3/2},
\]

where \( C_K \) and \( C_K' \) are independent of \( n \).

Let us fix \( K \) as
\[
\frac{1}{2} K \delta \geq 20 + 1
\]

and \( N \) as
\[
6N > 2K + 6.
\]

For each \( n \) we apply Proposition 3.1 and obtain \( U^{(N)}(x, t; \alpha, k) \). Note that it holds that
\[
\langle v_n \rangle(\alpha_n), a \leq C_n \alpha_n^{-a}
\]

where \( C_n \) is a constant independent of \( n \). Then
\[
E_{\alpha_n}(v_n, \alpha_n; a) \leq C_n \alpha_n^{-(a+1)}.
\]

Setting \( k = \beta_n^{20} \) we have
\[
||U^{(N)}(x, t; \alpha, k)\|_{\alpha_n, a, b} \leq C_{N, K, \alpha_n, \alpha_n} \sum_{j=0}^{N} \beta_n^{20(a+b-j)}
\]

\[
\cdot \left( \sum_{j=0}^{K} \sum_{k-j}^{N} (\beta_n^{20} \alpha_n)^{-h} \alpha_n^{-4hK + r - 21 - 4N - 3} \right)
\]

\[
\leq C_{N, K, \alpha_n, \alpha_n} \beta_n^{20(a+b)}.
\]

For each \( n \) we apply Proposition 3.1 and obtain \( U^{(N)}(x, t; \alpha, k) \). Note that it holds that
\[
\langle v_n \rangle(\alpha_n), a \leq C_n \alpha_n^{-a}
\]

where \( C_n \) is a constant independent of \( n \). Then
\[
E_{\alpha_n}(v_n, \alpha_n; a) \leq C_n \alpha_n^{-(a+1)}.
\]

Setting \( k = \beta_n^{20} \) we have
\[
||U^{(N)}||_{\alpha_n, a, b} \leq C_{N, K, \alpha_n, \alpha_n} \sum_{j=0}^{N} \beta_n^{20(a+b-j)}
\]

\[
\cdot \left( \sum_{j=0}^{K} \sum_{k-j}^{N} (\beta_n^{20} \alpha_n)^{-h} \alpha_n^{-4hK + r - 21 - 4N - 3} \right)
\]

\[
\leq C_{N, K, \alpha_n, \alpha_n} \beta_n^{20(a+b)}.
\]

For each \( n \) we apply Proposition 3.1 and obtain \( U^{(N)}(x, t; \alpha, k) \). Note that it holds that
\[
\langle v_n \rangle(\alpha_n), a \leq C_n \alpha_n^{-a}
\]

where \( C_n \) is a constant independent of \( n \). Then
\[
E_{\alpha_n}(v_n, \alpha_n; a) \leq C_n \alpha_n^{-(a+1)}.
\]

Setting \( k = \beta_n^{20} \) we have
\[
||U^{(N)}||_{\alpha_n, a, b} \leq C_{N, K, \alpha_n, \alpha_n} \sum_{j=0}^{N} \beta_n^{20(a+b-j)}
\]

\[
\cdot \left( \sum_{j=0}^{K} \sum_{k-j}^{N} (\beta_n^{20} \alpha_n)^{-h} \alpha_n^{-4hK + r - 21 - 4N - 3} \right)
\]

\[
\leq C_{N, K, \alpha_n, \alpha_n} \beta_n^{20(a+b)}.
\]
Since
\[ m_{0}(\sigma_{n}) \leq C_{\sigma} \beta_{n}^{-2\sigma} \]
we obtain by using (4.16), (4.18) and (2.2)
\[ |BU^{(N)}_{e,\tau}| \leq C_{\sigma} \beta_{n}^{-2\sigma}. \]
Taking account of (2.3) the substitution of (4.17), (4.19) and (4.20) into (4.13) gives
\[ \left( \frac{1}{2} \right)^{K} \beta_{n}^{-KN/2} - C_{N,K} \beta_{n}^{-(K-1)N/2} \leq C_{\sigma} \beta_{n}^{-2\sigma}, \]
which shows a contradiction, because \( K \) verifies (4.14) and \( \beta_{n} \to 0 \) as \( n \to \infty \). Thus the theorem is proved.

Appendix

By a change of variables
\[ p(x) = \sigma \]
the equation \( \mathcal{L}_{\sigma} \{ a_{1}, a_{2} \} = \{ h_{0}, h_{1} \} \) turns to
\[ (A.1) \]
First consider how \( a_{ij}(y, t) = \left( \frac{\partial a_{ij}}{\partial \sigma} \right)(0, y, t) \) is determined. Let us set
\[ h_{i}(\sigma, y, t) \sim \sum_{j=0}^{\infty} h_{ij}(y, t) \sigma^{j}, \quad l = 0, 1 \]
\[ (\nabla \sigma)^{2}(\sigma, y) \sim \sum_{j=0}^{\infty} A_{j}(y) \sigma^{j}, \quad (\Delta \sigma)(\sigma, y) \sim \sum_{j=0}^{\infty} C_{j}(y) \sigma^{j} \]
\[ (\nabla \rho)^{2}(\sigma, y) \sim \sum_{j=0}^{\infty} B_{j}(y) \sigma^{j}, \quad (\Delta \rho)(\sigma, y) \sim \sum_{j=0}^{\infty} D_{j}(y) \sigma^{j} \]
and
\[ a_{ij}(\sigma, y, t) \sim \sum_{j=0}^{\infty} a_{ij}(y, t) \sigma^{j}. \]
Note that the facts \( A_{0}(y) \geq \epsilon > 0 \) and \( B_{0}(y) \geq \epsilon > 0 \) follow from the proper...
of $\theta$ and $\rho$. Substitute the above expansions into (A.1) and set equal the coefficients of $\sigma^j$ of the both sides of the equations. Then we have

\[(A.2)\]  
\[2 \frac{\partial a_{00}}{\partial t} + 2A_0 \frac{\partial a_{00}}{\partial y} + C_\theta a_{00} + B_\rho a_{10} = h_{00}\]

\[(A.3)\]  
\[2 \frac{\partial a_{10}}{\partial t} + 2A_0 \frac{\partial a_{10}}{\partial y} + C_\theta a_{10} + B_\rho a_{01} + D_\theta a_{00} = h_{10}\]

and for $j \geq 1$

\[(A.2)_j\]  
\[2 \frac{\partial a_{0j}}{\partial t} + 2 \sum_{l=0}^j A_l \frac{\partial a_{0j-l}}{\partial y} + \sum_{l=0}^j C_l a_{0j-l} + 2 \sum_{l=0}^{j-1} (j-l)B_l a_{1j-l} + \sum_{l=0}^{j-1} B_l a_{0j-l} + (2j+1)B_l a_j + \sum_{l=0}^{j-1} D_l a_{1j-l-1} = h_{0j}\]

\[(A.3)_j\]  
\[2 \frac{\partial a_{1j}}{\partial t} + 2 \sum_{l=0}^j A_l \frac{\partial a_{1j-l}}{\partial y} + \sum_{l=0}^j C_l a_{1j-l} + 2 \sum_{l=0}^{j-1} B_l (j+1-l) a_{0j+1-l} + \sum_{l=0}^{j-1} D_l a_{0j-l} = h_{1j}.\]

Then if we set $a_{00}(y, t) = 0$, (A.2) determines $a_{10}$ and subsequently (A.3) determines $a_{01}$. In (A.2)_j besides $a_{11}$ all terms are determined, therefore $a_{11}$ is determined, and next (A.3)_j determines $a_{10}$. Continuing this process we obtain successively $a_{ij}, j=0, 1, \ldots$. By the manner of determining $a_{ij}$ it holds that

\[(A.4)\]  
\[\sum_{|\gamma| \leq \sigma} \sup_{(y, t)} |D_i^\gamma a_{0j+1}(y, t)| + \sup_{(y, t)} |D_i^\gamma a_j(y, t)| \leq C_a \sum_{|\gamma| \leq \sigma} \sum_{|\gamma| \leq \sigma} \sup_{(y, t)} |D_i^\gamma h_{ik}(y, t)|.\]

If we set $a_i(\sigma, y, t) = \sum_{j=0}^b a_{ij}(y, t) \sigma^j$, the estimate (A.4) gives

**Lemma A.1.** For any $b$ positive integer there exists $\{a_0, a_1\}$ such that $a_{00}(0, y, t) = 0$ and

\[(A.5)\]  
\[\sum_{|\gamma| \leq \sigma} \sum_{k \geq 0} |D_i^\gamma a| \leq C_{a, b} \sum_{|\gamma| \leq \sigma} \sum_{k \geq 0} |D_i^\gamma h_1|,\]

\[(A.6)\]  
\[\sum_{|\gamma| \leq \sigma} |D_i^\gamma h_1(x, a_0, a_1) - \{h_{00}, h_0\}| \leq |\sigma|^{b+1} C_{a, b} \sum_{|\gamma| \leq \sigma} \sum_{k \geq 0} |D_i^\gamma h_1(\sigma, y, t)|.\]

Next consider that case

\[(A.7)\]  
\[D_i^ph_1(0, y, t) = 0 \quad \text{for } p = 0, 1, 2, \ldots, b.\]

If we claim $a_{00} = 0$ on $\{\sigma = 0\}$ the solution of (A.1) is given for $\sigma > 0$ by
\begin{align*}
a_0(\sigma, y, t) &= \frac{1}{2} \left\{ G^+(\sqrt{\sigma}, y, t) + \text{sgn}(\sigma) G^+(\sqrt{-\sigma}, y, t) \right\} \\
a_1(\sigma, y, t) &= \frac{1}{2\sqrt{\sigma}} \left\{ G^+(\sqrt{\sigma}, y, t) - \text{sgn}(\sigma) G^+(\sqrt{-\sigma}, y, t) \right\},
\end{align*}

where \( G^+(z, y, t) \) is the solution of

\[
\mathcal{L}^+ G^+ = \left( 2 \frac{\partial}{\partial t} + 2(\nabla \theta)^2(y, z^2) \frac{\partial}{\partial y} + 2(\nabla \rho)^2(y, z^2) \frac{\partial}{\partial z} + (\Delta \theta)(y, z^2) + \alpha(\Delta \tau)(y, z^2) \right) G^+(z, y, t) = H^+(z, y, t)
\]

\( G^+(0, y, t) = 0 \)

\( H^+(z, y, t) = h_0(z^2, y, t) + zh_1(z^2, y, t). \)

The assumption (A.7) implies that for \( r \leq b, |\gamma| \leq a \)

\[
|D_{x}D_{y}H^+(z, y, t)| \leq C_{a,b} |x|^{2b+1-r}
\]

\[
K_{a,b} = \sum_{l=0}^{\infty} \sum_{|\gamma|<a} \sup_{\sigma>0} |D_{x}D_{y}h_{l}(\sigma, y, t)|.
\]

Therefore it holds that

\[
\sum_{|\gamma|<a} |D_{x}D_{y}G^+(z, y, t)| \leq C_{a,b} K_{a,b} |x|^{2b+3-r},
\]

from which it follows immediately that

\[
\sum_{l=0}^{\infty} \sum_{|\gamma|<a} \sup_{\sigma>0} |D_{x}D_{y}a_{l}(\sigma, y, t)| \leq C_{a,b} K_{a,b}, \quad \sigma > 0.
\]

Using \((a_{0} - \sqrt{\rho}) a_{1}(\alpha, y, t) = G^+(y, t, -\sqrt{\alpha})\) we have

**Lemma A.2.** On the supposition (A.7) there exists a solution of (A.1) verifying \( a_{0}(0, y, t) = 0 \) and it holds that

\[
\sum_{l=0}^{\infty} \sum_{|\gamma|<a} \sup_{\sigma>0} |D_{x}D_{y}a_{l}(\sigma, y, t)| \leq C_{a,b} K_{a,b} \sum_{l=0}^{\infty} \sum_{|\gamma|<a} \sup_{\sigma>0} |D_{x}D_{y}h_{l}(\sigma, y, t)|
\]

and

\[
\sum_{|\gamma|<a} \sum_{l=0}^{\infty} \sup_{\sigma>0} |D_{x}D_{y}(a_{0} - \sqrt{\rho}) a_{1}(\alpha, y, t)| \leq C_{a,b} \sum_{l=0}^{\infty} \sum_{|\gamma|<a} \sum_{|\gamma|<b+1} \sup_{\sigma>0} |D_{x}D_{y}h_{l}(\sigma, y, t)|.
\]

5) See, § 1 of Ludwig [6] and Lemma 5.2 of Ikawa [4].
When $h_i \equiv 0$, the solution of (A.1) verifying

$$a_0 - \sqrt{\rho} \; a_1 |_{\sigma=a} = f(y, t)$$

is given by (A.8) where $G^+$ is the solution of

$$\begin{cases}
\mathcal{L}^+ G^+ = 0 \\
G^+(-\sqrt{\alpha}, y, t) = f(y, t).
\end{cases}$$

Evidently

$$\sum_{|\gamma| < a} |D^\gamma_\sigma a_0| \leq \sum_{|\gamma| < a} \sum_{j=1} \sup |D^\gamma_\sigma D^j_\tau G^+(\sigma, y, t)|$$

and

$$\sum_{|\gamma| < a} |D^\gamma_\sigma a_1| \leq \sum_{|\gamma| < a+1} \sum_{j=1} \sup |D^\gamma_\sigma D^j_\tau G^+(\sigma, y, t)|.$$

And we see easily that

$$\sum_{|\gamma| < a} \sup |D^\gamma_\sigma G^+(\sigma, y, t)| \leq C_\sigma \sum_{|\gamma| < a} \sup |D^\gamma_\sigma f(y, t)|.$$

Thus we have

**Lemma A.3.** When $h_0, h_1 \equiv 0$, the solution of (A.1) verifying $a_0 - \sqrt{\rho} \; a_1 |_{\sigma=a}$

$$= f$$

has the estimate

(A.11)

$$\sum_{j=0}^\gamma \sum_{j=0}^{\gamma_j} \sum_{|\gamma| < a + 2(\beta - j)} \sup |D^\gamma_\sigma D^j_\tau a_0|$$

$$\leq C_{a, \beta} \sum_{j=0}^\gamma \sum_{j=0}^{\gamma_j} \sup |D^\gamma_\sigma f(y, t)|.$$

To show (i) of Lemma 2.1 for fixed integer $b$ first apply Lemma A.1 and we obtain $\{\bar{a}_0, \bar{a}_1\}$ satisfying (A.6), and next apply Lemma A.2 to $\mathcal{L}_a \{\bar{a}_0, \bar{a}_1\} = \{h_0, h_1\}$ then we have $\{b_0, b_1\}$ verifying

$$\mathcal{L}_a \{b_0, b_1\} = \{h_0, h_1\} - \mathcal{L}_a \{\bar{a}_0, \bar{a}_1\}.$$ 

By using (A.5), (A.6) and (A.9) we have

$$\sum_{j=0}^\gamma \sum_{j=0}^{\gamma_j} \{ |D^\gamma_\sigma D^j_\tau \bar{a}_0(\sigma, y, t) - D^\gamma_\sigma D^j_\tau h_0(\sigma, y, t)| \}$$

$$\leq C_{a, \beta} \sum_{j=0}^\gamma \sum_{j=0}^{\gamma_j} \sum_{|\gamma| < a + 2(\beta - j)} \sup |D^\gamma_\sigma D^j_\tau h_0(\sigma, y, t)|.$$

Moreover it follows form (A.5) and (A.10) that

$$\sum_{|\gamma| < a + 2\beta} \sup |D^\gamma_\sigma((\bar{a}_0 + b_0) - \sqrt{\rho} \; (\bar{a}_1 + b_1))|_{\rho=a}$$

$$\leq C_{a, \beta} \sum_{j=0}^\gamma \sum_{j=0}^{\gamma_j} \sum_{|\gamma| < a + 2(\beta - j)} \sup |D^\gamma_\sigma D^j_\tau h_0(\sigma, y, t)|.$$

Then using Lemma A.3 we have $\{c_0, c_1\}$ verifying
Then we see immediately that \( a_1 = \bar{a}_1 + b_1 + c_1 \), \( l = 0, 1 \) are solutions of the problem (A.1) verifying the boundary condition and they satisfy the estimate of (i) of Lemma 2.1.

References


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